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A VECTOR-VALUED BANACH–STONE THEOREM WITH DISTORTION $\sqrt{2}$

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Let K and S be locally compact Hausdorff spaces and H a real Hilbert space of finite dimension at least two. We prove that if T is an isomorphism from $C_0(K, H)$ onto $C_0(S, H)$ whose distortion $\|T\|\|T^{-1}\|$ is exactly $\sqrt{2}$, then K and S are homeomorphic. This is the vector-valued Banach–Stone theorem via isomorphisms with the largest distortion that is known. It improves a 1976 classical result due to Cambern.

1. Introduction

If K is a locally compact Hausdorff space and X is a Banach space, we denote by $C_0(K, X)$ the Banach space of continuous functions vanishing at infinity on K, taking values in X, and provided with the usual supremum norm. If K is compact, we use the notation $C(K, X)$ to represent this space. Moreover, if $X = \mathbb{R}$ we will denote these spaces by $C_0(K)$ and $C(K)$ respectively. In the present paper, the word "isomorphism" means "linear homeomorphism".

The well-known Banach–Stone theorem states that if K and S are locally compact Hausdorff spaces, then the existence of an isometric isomorphism T of $C_0(K)$ onto $C_0(S)$ implies that K and S are homeomorphic [Banach 1932; Behrends 1979; Stone 1937]. Cambern [1966; 1967] strengthened this theorem by showing that the conclusion holds if the requirement that T be an isometric isomorphism is replaced by the requirement that T be an isomorphism satisfying $\|T\|\|T^{-1}\| < 2$. Amir [1965] established the same result independently for K and S compact. Cambern [1970] showed that 2 is indeed the greatest number for which the formulation of the Banach–Stone theorem given in [Cambern 1967] is valid, by exhibiting a pair of locally compact Hausdorff spaces K and S, with K compact, S noncompact, and an isomorphism T of $C(K)$ onto $C_0(S)$ with $\|T\|\|T^{-1}\| = 2$. Cohen [1975] showed there was such an example where both K and S are compact.

Cambern [1976] was also the first to get a vector-valued Banach–Stone theorem via isomorphisms with distortion $\lambda > 1$. He proved:

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Theorem 1.1. *Let K and S be locally compact Hausdorff spaces and H a finite-dimensional Hilbert space. If there exists an isomorphism T from $C_0(K, H)$ onto $C_0(S, H)$ satisfying $\|T\|\|T^{-1}\| < \sqrt{2}$, then K and S are homeomorphic*

It is still an open question whether the bound $\sqrt{2}$ can be improved. Moreover, after Cambern's theorem, all vector-valued Banach–Stone theorems have been obtained via isomorphisms with distortion $1 \le \lambda < \sqrt{2}$; see [Cidral et al. 2015].

Thus, in view of the above mentioned isomorphisms with distortion 2 between $C_0(K, H)$ spaces constructed independently by Cambern and Cohen in the case where H is the scalar field, it is natural to turn our attention to the isomorphisms with distortion $\sqrt{2}$ between $C_0(K, H)$ spaces in the case where H is an n-dimensional Hilbert space with $n \ge 2$. In other words, the following question arises naturally.

Problem 1.2. *Let K and S be locally compact Hausdorff spaces and H a Hilbert space of finite dimension greater than or equal to 2. Suppose that there exists an isomorphism T from $C_0(K, H)$ onto $C_0(S, H)$ satisfying $\|T\|\|T^{-1}\| = \sqrt{2}$. Does it follow that K and S are homeomorphic*?

The principal purpose of this paper is to show that Problem 1.2 has a positive solution when the scalar field is the real numbers $\mathbb{R}$.

So, henceforward $H = \mathbb{R}^n_2$ the space of n tuples of real numbers with the usual 2 norm and $n \ge 2$. Our main theorem is as follows.

Theorem 1.3. *Let K and S be locally compact Hausdorff spaces. Suppose that there exists an isomorphism T from $C_0(K, H)$ onto $C_0(S, H)$ satisfying*

$$\frac{\|f\|}{\sqrt[4]{2}} \le \|T(f)\| \le \sqrt[4]{2}\|f\|, \tag{1-1}$$

for every $f \in C_0(K, H)$. Then K and S are homeomorphic.

Then, the solution of Problem 1.2 follows immediately from Theorem 1.3 by considering $\tau = T\|T^{-1}\|2^{-1/4}$ and noticing that (1-1) holds for the isomorphism τ. Moreover, Theorem 1.1 in the real case is also a direct consequence of Theorem 1.3. Indeed, put $\|T\|\|T^{-1}\| = \lambda < \sqrt{2}$ and $\tau = T\|T^{-1}\|\lambda^{-1/2}$. Therefore, it suffices to observe that (1-1) again holds for the isomorphism τ.

It is worth mentioning that Theorem 1.3 cannot be extended to infinite dimensional Hilbert spaces. Indeed, let I be an infinite set and write $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \varnothing$ and the cardinalities of I_1 and I_2 equal to the cardinality of I. Let $K_1 = \{1\}$ and $K_2 = \{1, 2\}$ be two discrete compact Hausdorff spaces. Consider the natural isometries

$$\Theta : C(K_2, l_2(I)) \to l_2(I_1) \oplus_\infty l_2(I_2) \quad \text{and} \quad \Upsilon : l_2(I) \to C(K_1, l_2(I)).$$

Now, define $T : l_2(I_1) \oplus_\infty l_2(I_2) \to l_2(I)$ by

$$T((a_i)_{i\in I_1}, (b_i)_{i\in I_2}) = (c_i)_{i\in I},$$

where $c_i = a_i$ if $i \in I_1$ and $c_i = b_i$ if $i \in I_2$. Then, it is easy to check that

$$\|\Upsilon T \Theta\| = \sqrt{2} \quad \text{and} \quad \|(\Upsilon T \Theta)^{-1}\| = 1.$$

But, of course K_1 and K_2 are not homeomorphic.

As we will see, the proof of Theorem 1.3 depends not only on the fact that H has finite dimension but the intrinsic geometry of H as a real Hilbert space. It is divided into five sections.

2. Special sets associated to isomorphisms between $C_0(K, H)$ spaces

We begin by recalling that a bijective map $T : C_0(K, H) \to C_0(S, H)$ is said to be a bijective coarse quasi-isometry if for some constants $M > 0$ and $L \geq 0$ the inequalities

$$\frac{1}{M}\|f - g\| - L \leq \|T(f) - T(g)\| \leq M\|f - g\| + L,$$

are satisfied for all $f, g \in C_0(K, H)$.

In our recent study of these maps ([Galego and Porto da Silva 2016]; henceforth abbreaviated [GPS]) we introduced some classes of subsets $\Gamma_w(k, v)$ and $\Gamma_v(s, w)$ of S and K respectively, where $k \in K$, $s \in S$ and v and w are suitable elements of $\mathbb{R}$. We shall define these sets for $v, w \in H$ instead of $\mathbb{R}$.

In order to prove Theorem 1.3, we will need to state some new properties of these sets in the particular case where T is linear, $M = \sqrt[4]{2}$ and $L = 0$. So, in this short preliminary section we will remember some definitions and results already adapted to the context of Theorem 1.3.

From now on $M = \sqrt[4]{2}$ and T will be an isomorphism of $C_0(K, H)$ onto $C_0(S, H)$ satisfying

$$\frac{\|f\|}{M} \leq \|T(f)\| \leq M\|f\|, \tag{2-1}$$

for every $f \in C_0(K, H)$.

Let $k \in K$, $f \in C_0(K, H)$ and $v \in H$. Following [GPS, Definition 2.2] we set

$$\omega(k, f, v) = \max\{\|f\|, \|f(k) - v\|\}.$$

Moreover, if $v, w \in H$ with $v \neq 0$ satisfy $\|w\| = \|v\|/M$, following [GPS, Definition 3.1], we also set

$$\Gamma_w(k, v) = \{s \in S : \|Tf(s) - w\| \leq M\omega(k, f, v), \forall f \in C_0(K, H)\}.$$

Analogously, for $s \in S$, w and $v \in H$ with $w \neq 0$ and $\|v\| = \|w\|/M$, we set

$$\Gamma_v(s, w) = \{k \in K : \|T^{-1}g(k) - v\| \leq M\omega(s, g, w), \forall g \in C_0(S, H)\}.$$

Let us summarize the results concerning these sets which will be used in the present paper. We will denote by $\langle \cdot , \cdot \rangle$ the usual inner product on H. When the vectors v and w of H are orthogonal we will write $v \perp w$.

Proposition 2.1. *Let $k \in K$ and $v \in H$ with $v \neq 0$.*

(1) *There exists $w \in H$ such that $\Gamma_w(k, v) \neq \varnothing$.*

(2) *For all $t \in \mathbb{R}$ with $t \neq 0$ and $w \in H$ we have $\Gamma_w(k, v) = \Gamma_{tw}(k, tv)$.*

(3) *Let $v', w, w' \in H$ and $k' \in K$ with $k \neq k'$. Suppose that*

$$\Gamma_w(k, v) \cap \Gamma_{w'}(k', v') \neq \varnothing,$$

then $w \perp w'$.

(4) *Let $w \in H$ and suppose that $s \in \Gamma_w(k, v)$. If $\Gamma_z(s, w) \neq \varnothing$ for some $z \in H$ then $\Gamma_z(s, w) = \{k\}$.*

Proof. (1) The proof is essentially the same proof of [GPS, Proposition 3.2]. We leave it to the reader to transpose to the Hilbert context.

(2) It suffices to prove that $\Gamma_w(k, v) \subset \Gamma_{tw}(k, tv)$ for all $t \neq 0$. Let $s \in \Gamma_w(k, v)$. Given $f \in C_0(K, H)$ put $f' = t^{-1} f$. By the definition of $\Gamma_w(k, v)$ it follows that

$$\|Tf'(s) - w\| \leq M\omega(k, f', v),$$

and hence

$$\|Tf(s) - tw\| = |t|\|Tf'(s) - w\| \leq M|t|\omega(k, f', v) = M\omega(k, f, tv).$$

Consequently $s \in \Gamma_{tw}(k, tv)$.

(3) By item (2) of the proposition we may assume that $\|v\| = \|v'\| = 1$. By Urysohn's lemma pick $f \in C_0(K, H)$ such that $\|f\| = \frac{1}{2}$, $f(k) = \frac{v}{2}$ and $f(k') = \frac{v'}{2}$. It is easy to check that $\omega(k, f, v) = \omega(k', f, v') = \frac{1}{2}$. Pick $s \in \Gamma_w(k, v) \cap \Gamma_{w'}(k', v')$. Then, by the definitions of these sets we have

$$\|w - w'\| \leq \|Tf(s) - w\| + \|Tf(s) - w'\| \leq \frac{M}{2} + \frac{M}{2} = M.$$

Now, by applying the law of cosines we see that

$$\langle w, w' \rangle \geq \tfrac{1}{2}(\|w\|^2 + \|w'\|^2 - M^2),$$

Since $\|w\| = \|w'\| = 1/M$ and $M = \sqrt[4]{2}$, it follows that

$$\langle w, w' \rangle \geq \frac{1}{2}\left(\frac{2}{M^2} - M^2\right) = 0.$$

On the other hand, by item (2) of the proposition we have

$$s \in \Gamma_w(k, v) \cap \Gamma_{-w'}(k', -v').$$

So, proceeding as above we obtain that $\langle w, -w' \rangle \geq 0$. Hence $\langle w, w' \rangle = 0$.

(4) According to item (2) of the proposition we may assume that $\|v\| = 1$. By item (1) of the proposition there is $z \in H$ such that $\Gamma_z(s, w) \neq \varnothing$. Fix $m \in \Gamma_z(s, w)$; we need to show that $m = k$. Assume then that $m \neq k$ and choose $h \in C_0(K)$ satisfying

$$\|h\| = \frac{1}{2}, \quad h(k) = \frac{v}{2} \quad \text{and} \quad h(m) = -\frac{1}{2}\frac{z}{\|z\|}.$$

Since $\Gamma_w(k, v)$ and $\Gamma_z(s, w)$ are well defined, we have $\|z\| = 1/M^2 = 1/\sqrt{2}$. Moreover, observe that z is negatively proportional to $h(m)$. Thus, we have

$$\|h(m) - z\| = \|h(m)\| + \|z\| = \frac{1}{2} + \frac{1}{\sqrt{2}}. \tag{2-2}$$

On the other hand, $\omega(k, h, v) = \frac{1}{2}$ and $s \in \Gamma_w(k, v)$ imply that

$$\|Th(s) - w\| \leq \frac{M}{2}.$$

Since $\|Th\| \leq M/2$ it follows that $\omega(s, Th, w) \leq M/2$ and by the definition of $\Gamma_z(s, w)$ (using the function Th and the map T^{-1})

$$\|h(m) - z\| \leq M\omega(s, Th, w) \leq \frac{M^2}{2} = \frac{1}{\sqrt{2}},$$

which by (2-2) lead us to a contradiction. □

Note that since the definitions of $\Gamma_w(k, v)$ and $\Gamma_v(s, w)$ are symmetric the properties proved in Proposition 2.1 on $k \in K$ and $\Gamma_w(k, v)$ are also valid for $s \in S$ and $\Gamma_v(s, w)$.

Henceforth our task will be to construct a homeomorphism $\varphi : K \to S$ using the subsets $\Gamma_w(k, v)$, for every $k \in K$. In fact, we will see that these subsets contain the candidates to be the image of k by φ.

3. On the subsets $\Gamma_w(k, v)$ of K containing irregular points

The purpose of this section is to establish Proposition 3.1. It allows us to relate the vectors v and w involved in the construction of certain special sets $\Gamma_w(k, v)$. For convenience, we introduce the following definition.

A point $s \in S$ is said to be irregular if there exist two different points k and $k' \in K$ such that $s \in \Gamma_w(k, v) \cap \Gamma_{w'}(k', v')$ for some $v, w, v', w' \in H$. Symmetrically, we will say that a point $k \in K$ is irregular if $k \in \Gamma_v(s, w) \cap \Gamma_{v'}(s', w')$ for some different points $s, s' \in S$ and $v, w, v', w' \in H$.

Proposition 3.1. *Suppose that $k \in K$ and s is an irregular point of S.*

(1) *If $s \in \Gamma_{w_1}(k, v_1) \cap \Gamma_{w_2}(k, v_2)$ for some $v_1, v_2, w_1, w_2 \in H$ then*

$$\langle v_1, v_2 \rangle = M^2 \langle w_1, w_2 \rangle.$$

(2) *If* $(v_i)_{1\le i\le l}$ *is a linearly independent set of* H *and* $s \in \Gamma_{w_i}(k, v_i)$, *for some* $w_i \in H$, $1 \le i \le l$, *then* $(w_i)_{1\le i\le l}$ *is a linearly independent set.*

Proof. In virtue of Proposition 2.1(2) we can assume that $\|v_1\| = \|v_2\| = 1$. Hence $\|w_1\| = \|w_2\| = 1/M$. Since s is irregular, there exists $k' \in K$, $k' \neq k$ and vectors $v', w' \in H$ with $\|v'\| = 1$ and $\|w'\| = 1/M$ such that $s \in \Gamma_{w'}(k', v')$. According to Proposition 2.1(3) we have

$$w' \perp w_1 \quad \text{and} \quad w' \perp w_2. \tag{3-1}$$

Since $k \neq k'$ by Urysohn's lemma there exist $f, f' \in C_0(K)$ satisfying:

(i) $f(K), f'(K) \subset [0, 1]$.

(ii) $f(k) = f'(k') = 1$.

(iii) $\operatorname{supp} f \cap \operatorname{supp} f' = \varnothing$.

Put $h_1 = f \cdot (v_1/2)$, $h_2 = f \cdot (v_2/2)$, $h_3 = f' \cdot (v'/2)$ and

$$h = h_1 + h_2 + \|v_1 + v_2\| h_3. \tag{3-2}$$

According to (iii)

$$\|h\| = \tfrac{1}{2}\|v_1 + v_2\|. \tag{3-3}$$

Next we will calculate $\|Th(s)\|$. In order to do this consider the function $h_1 + h_3$. It is easy to see that

$$\omega(k, h_1 + h_3, v_1) = \omega(k', h_1 + h_3, v') = \tfrac{1}{2}.$$

Thus, since $s \in \Gamma_{w_1}(k, v_1) \cap \Gamma_{w'}(k', v')$ it follows by the definition of these sets that

$$\|T(h_1 + h_3)(s) - w_1\| \le \frac{M}{2} \quad \text{and} \quad \|T(h_1 + h_3)(s) - w'\| \le \frac{M}{2}. \tag{3-4}$$

On the other hand, (3-1) gives us that

$$\|w_1 - w'\| = \sqrt{\|w_1\|^2 + \|w'\|^2} = \sqrt{\frac{2}{M^2}} = M. \tag{3-5}$$

By (3-4) and (3-5) we deduce that

$$T(h_1 + h_3)(s) = \frac{w_1 + w'}{2}. \tag{3-6}$$

In the same way we obtain

$$T(h_1 - h_3)(s) = \frac{w_1 - w'}{2}, \tag{3-7}$$

and

$$T(h_2 + h_3)(s) = \frac{w_2 + w'}{2}. \tag{3-8}$$

By combining (3-6), (3-7) and (3-8) we infer that

$$Th_1(s) = \frac{w_1}{2}, \quad Th_2(s) = \frac{w_2}{2} \quad \text{and} \quad Th_3(s) = \frac{w'}{2}.$$

Thus, taking in mind (3-1) and (3-2) we get

$$\|Th(s)\|^2 = \frac{\|w_1 + w_2\|^2}{4} + \|v_1 + v_2\|^2 \frac{\|w'\|^2}{4}.$$

Since that $\|Th\| \le M\|h\|$ and (3-3) holds, it follows that

$$\frac{\|w_1 + w_2\|^2}{4} + \|v_1 + v_2\|^2 \frac{\|w'\|^2}{4} \le M^2 \frac{\|v_1 + v_2\|^2}{4}.$$

Recalling that $\|w'\| = 1/M$, we have

$$\|w_1 + w_2\|^2 \le \left(M^2 - \frac{1}{M^2}\right)\|v_1 + v_2\|^2.$$

But $\|w_1 + w_2\|^2 = 2/M^2 + 2\langle w_1, w_2\rangle$ and $\|v_1 + v_2\|^2 = 2 + 2\langle v_1, v_2\rangle$. Hence

$$\frac{2}{M^2} + 2\langle w_1, w_2\rangle \le \left(M^2 - \frac{1}{M^2}\right)(2 + 2\langle v_1, v_2\rangle).$$

By using that $M^2 = \sqrt{2}$ we conclude

$$M^2\langle w_1, w_2\rangle \le \langle v_1, v_2\rangle.$$

Similarly working with $-v_2$ and $-w_2$ instead of v_2 and w_2 we derive that

$$M^2\langle w_1, -w_2\rangle \le \langle v_1, -v_2\rangle,$$

so the equality holds.

(2) It suffices to notice that item (1) of the proposition implies the following identity of matrices:

$$[\langle v_i, v_j\rangle]_{1 \le i,j \le l} = M^2[\langle w_i, w_j\rangle]_{1 \le i,j \le l}.$$

□

4. The functions $\Phi : K \to \mathcal{P}(S)$ and $\Psi : S \to \mathcal{P}(K)$

Here it is convenient to introduce two functions $\Phi : K \to \mathcal{P}(S)$ and $\Psi : S \to \mathcal{P}(K)$ given by

$$\Phi(k) = \bigcup \Big\{ \Gamma_w(k, v) : v \neq 0 \quad \text{and} \quad \|w\| = \frac{\|v\|}{M} \Big\},$$

and

$$\Psi(s) = \bigcup \Big\{ \Gamma_v(s, w) : w \neq 0 \quad \text{and} \quad \|v\| = \frac{\|w\|}{M} \Big\}.$$

Our next step is to prove that the sets $\Phi(k)$ and $\Psi(s)$ are singletons, see Proposition 5.1. The next proposition works on the assumption that $\Phi(k)$ is not a singleton set. Later, in the proof of Proposition 4.1, we will use it to derive a contradiction.

Proposition 4.1. *Let $k \in K$. Suppose that $\Phi(k)$ is not a singleton set. Then*:

(1) *k is an irregular point of K.*

(2) *$\Phi(k)$ contains only irregular points of S.*

Proof. (1) Pick two different points $s, s' \in \Phi(k)$. So, there are $v, v', w, w' \in H$ such that

$$s \in \Gamma_w(k, v) \quad \text{and} \quad s' \in \Gamma_{w'}(k, v').$$

By Proposition 2.1.4 there exist z and $z' \in H$ satisfying

$$k \in \Gamma_z(s, w) \cap \Gamma_{z'}(s', w'),$$

hence k is an irregular point of K.

(2) First of all notice that by item (1) of the proposition applied to $\Psi(s)$, it suffices to prove that for all $s \in \Phi(k)$, $\Psi(s)$ is not a singleton set.

Assume by contradiction that $\Psi(s)$ is a singleton set for some $s \in \Phi(k)$. Since $s \in \Phi(k)$, there exist $v, w \in H$ such that $s \in \Gamma_w(k, v)$. By Proposition 2.1(4) there exists $z \in H$ satisfying $\Gamma_z(s, w) = \{k\}$. Then $k \in \Psi(s)$ and therefore

$$\Psi(s) = \{k\}. \tag{4-1}$$

Now fix $(w_i)_{1 \le i \le n}$, a basis of H with $\|w_i\| = 1$ for every $1 \le i \le n$. There exist, by Proposition 2.1(1), $(v_i)_{1 \le i \le n}$ in H such that $\Gamma_{v_i}(s, w_i) \neq \varnothing$ for every $1 \le i \le n$. Thus (4-1) implies that

$$\Gamma_{v_i}(s, w_i) = \{k\}, \tag{4-2}$$

for every $1 \le i \le n$.

On the other hand, since by item (1) of the proposition k is an irregular point of K, it follows from (4-2) and Proposition 3.1(2) that $(v_i)_{1 \le i \le n}$ is linearly independent.

Next, since k is an irregular point of K, there exist $s' \in S$, $s' \neq s$ and $w', v' \in H$ such that $k \in \Gamma_{v'}(s', w')$. So, by (4-2) and Proposition 2.1(3) we conclude that

$$v' \perp v_i,$$

for every $1 \leq i \leq n$, a contradiction because the dimension of H is n. □

5. The cardinality of $\Phi(k)$ for every $k \in K$

We are now in position to state the key proposition for proving Theorem 1.3. The span of a subset V of E will be denoted by $[V]$.

Proposition 5.1. *$\Phi(k)$ is a singleton set for every $k \in K$.*

Proof. Assume that there exists $k \in K$ such that $\Phi(k) = \{s_i : i \in I\}$ with cardinality of I greater than or equal two. For all $i \in I$ put

$$V_i = \{v \in H, v \neq 0 : s_i \in \Gamma_w(k, v) \text{ for some } w \in H\}.$$

It follows from the definition of $\Phi(k)$ that $V_i \neq \varnothing$ for every $i \in I$, and according to Proposition 2.1(1)

$$\bigcup_{i \in I} V_i = H \setminus \{0\},$$

and therefore

$$\bigcup_{i \in I} [V_i] = H. \tag{5-1}$$

On the other hand, for all $i \in I$ set

$$Z_i = \{z \in H, z \neq 0 : k \in \Gamma_z(s_i, w) \text{ for some } w \in H\}.$$

Pick $i \in I$. Since $V_i \neq \varnothing$ there exists $v \in H$ such that $s_i \in \Gamma_w(k, v)$ for some $w \in H$. By Proposition 2.1(4), $\Gamma_z(s_i, w) = \{k\}$ for some $z \in H$. Hence $Z_i \neq \varnothing$.

According to Proposition 2.1(2) we can assume that $\|z_i\| = \|z_j\|$ and by the definition of $(Z_i)_{i \in I}$ there are w_i and $w_j \in H$ such that

$$k \in \Gamma_{z_i}(s_i, w_i) \cap \Gamma_{z_j}(s_j, w_j).$$

So by Proposition 2.1(3), $z_i \perp z_j$. Consequently

$$[Z_i] \perp [Z_j]. \tag{5-2}$$

Now we will prove that for all $i \in I$

$$[Z_i] = [V_i]. \tag{5-3}$$

First we will show that $Z_i \subset V_i$. Indeed, let $z \in Z_i$ and take $w \in H$ such that $k \in \Gamma_z(s_i, w)$. By Proposition 2.1(4) there exists $w' \in H$ satisfying $\Gamma_{w'}(k, z) = \{s_i\}$. So $z \in V_i$.

Next we will complete the proof of (5-3) by showing that the dimension of $[V_i]$ is less than or equal to the dimension of $[Z_i]$. Let $\{v_1, \ldots, v_l\} \subset V_i$ be a basis of $[V_i]$. Thus, by the definition of V_i there are $\{w_1, \ldots, w_l\} \subset H$ such that

$$s_i \in \Gamma_{w_j}(k, v_j), \tag{5-4}$$

for every $1 \le j \le l$. Since the cardinality of I is greater than or equal to two, k is an irregular element of K. Thus, according to Proposition 4.1(2), s_i is an irregular element of S. Then, by (5-4) and Proposition 3.1(2) we see that $\{w_1, \ldots, w_l\}$ is linearly independent.

In view of (5-4), Proposition 2.1(4) implies that there are $\{z_1, \ldots, z_l\} \subset H$ such that for all $1 \le j \le l$,

$$\Gamma_{z_j}(s_i, w_j) = \{k\}. \tag{5-5}$$

So, for all $1 \le j \le l$, $z_j \in Z_i$ and by (5-5) and Proposition 3.1(2) we deduce that $\{z_1, \ldots, z_l\}$ is linearly independent. Then, we are done.

Finally, by combining (5-2) and (5-3) it follows that for all $i, j \in I$ with $i \neq j$

$$[V_i] \perp [V_j],$$

a contradiction with (5-1), because H would be a union of nontrivial mutually perpendicular subspaces. $\square$

6. The isomorphisms between $C_0(K, H)$ spaces with distortion $\sqrt{2}$

Proposition 5.1 allows us to define two functions $\varphi : K \to S$ and $\psi : S \to K$ by

$$\Phi(k) = \{\varphi(k)\} \quad \text{and} \quad \Psi(s) = \{\psi(s)\}.$$

Thus, to complete the proof of Theorem 1.3 it remains to prove the following proposition.

Proposition 6.1. *The functions $\varphi : K \to S$ and $\psi : S \to K$ are continuous and $\psi = \varphi^{-1}$.*

Proof. First we will show that $\psi = \varphi^{-1}$. Fix $k \in K$. By the definition of $\Phi(k)$ there are $v, w \in H$ such that

$$\varphi(k) \in \Gamma_w(k, v).$$

Thus, applying the items (1) and (3) of Proposition 2.1, there exists $z \in H$ satisfying

$$\Gamma_z(\varphi(k), w) = \{k\}.$$

Therefore $k \in \Psi(\varphi(k)) = \{\psi(\varphi(k))\}$. That is, $k = \psi(\varphi(k))$. Hence $\psi \circ \varphi = \mathrm{Id}_K$. Analogously we deduce that $\varphi \circ \psi = \mathrm{Id}_S$.

We now prove that φ is continuous. The proof that ψ is continuous is analogous.

Observe that it suffices to prove that each net $(k_j)_{j\in J}$ of K converging to $k \in K$ admits a subnet $(k_{j_p})_{p\in P}$ such that $(\varphi(k_{j_p}))_{p\in P}$ converges to $\varphi(k)$.

Assume then that $(k_j)_{j\in J}$ is a net of K converging to k. By Propositions 2.1(1) and 5.1, for all $j \in J$ take v_j and $w_j \in H$ with $\|v_j\| = 1$ such that

$$\varphi(k_j) \in \Gamma_{w_j}(k_j, v_j). \tag{6-1}$$

Since the nets $(v_j)_{j\in J}$ and $(w_j)_{j\in J}$ are contained in compact sets, we can assume that there are $v, w \in H$ such that $v_j \to v$ and $w_j \to w$.

For each $f \in C_0(K, H)$ we have

$$\omega(k_j, f, v_j) \to \omega(k, f, v), \tag{6-2}$$

and according to (6-1),

$$\|Tf(\varphi(k_j)) - w_j\| \le M\omega(k_j, f, v_j), \quad \forall j \in J. \tag{6-3}$$

Fix $f_1 \in C_0(K, H)$ satisfying $\|f_1\| = \frac{1}{2}$ and $f_1(x) = \frac{v}{2}$. Then (6-2) and (6-3) imply that

$$\|Tf_1(\varphi(k_j))\| \ge \|w_j\| - \|Tf_1(\varphi(k_j)) - w_j\| \ge \frac{1}{M} - M\omega(k_j, f_1, v_j),$$

for every $j \in J$. Notice that $\omega(k, f_1, v) = \frac{\|v\|}{2} = \frac{1}{2}$, so by (6-2) we have

$$\liminf_{j\in J} \|Tf_1(\varphi(k_j))\| \ge \frac{1}{M} - \frac{M}{2} > 0.$$

Since Tf_1 vanishes at infinity, this implies that $(\varphi(k_j))_{j\in J}$ admits a subnet converging to some $s \in S$, so we assume that $\varphi(k_j) \to s$. Hence, by (6-2) and (6-3),

$$\|Tf(s) - w\| \le M\omega(k, f, v), \quad \forall f \in C_0(K, H),$$

which means that $s \in \Gamma_w(k, v) \subset \Phi(k) = \{\varphi(k)\}$, and consequently $s = \varphi(k)$. □

7. Open questions

In view of Theorem 1.3, the following questions arise naturally:

Problem 7.1. *Is Theorem 1.3 optimal, in the sense that* $\sqrt[4]{2}$ *is the best number for formalizing it*?

Problem 7.2. *What are the Banach spaces* X *satisfying the following property*: *whenever* K *and* S *are locally compact Hausdorff spaces and there exists an isomorphism* T *from* $C_0(K, X)$ *onto* $C_0(S, X)$ *with* $\|T\|\|T^{-1}\| = \sqrt{2}$, *then* K *and* S *are homeomorphic*?

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