ON THE CLASSIFICATION
OF POINTED FUSION CATEGORIES
UP TO WEAK MORITA EQUIVALENCE

BERNARDO URIBE

ON THE CLASSIFICATION
OF POINTED FUSION CATEGORIES
UP TO WEAK MORITA EQUIVALENCE

BERNARDO URIBE

A pointed fusion category is a rigid tensor category with finitely many isomorphism classes of simple objects which moreover are invertible. Two tensor categories $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent if there exists an indecomposable right module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ and $\mathcal{D}$ are tensor equivalent. We use the Lyndon–Hochschild–Serre spectral sequence associated to abelian group extensions to give necessary and sufficient conditions in terms of cohomology classes for two pointed fusion categories to be weakly Morita equivalent. This result allows one to classify the equivalence classes of pointed fusion categories of any given global dimension.

Introduction

Pointed fusion categories are rigid tensor categories with finitely many isomorphism classes of simple objects with the property that all simple objects are invertible. Any pointed fusion category $\mathcal{C}$ is equivalent to the fusion category $\mathrm{Vect}(G, \omega)$ of complex vector spaces graded by the finite group G together with the associativity constraint defined by the 3-cocycle $\omega \in Z^3(G, \mathbb{C}^*)$. Whenever we have a right module category $\mathcal{M}$ over $\mathcal{C}$ we can define the dual category $\mathcal{C}_{\mathcal{M}}^* := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ which becomes a tensor category via composition of functors. Whenever $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is an indecomposable fusion category, the dual category $\mathcal{C}_{\mathcal{M}}^*$ is also a fusion category [Ostrik 2003a, §2.2]. An indecomposable module category $\mathcal{M}$ of $\mathrm{Vect}(G, \omega)$ may be defined by $\mathcal{M} = \mathcal{M}(K, \mu)$, where K is the space of cosets $K := A \backslash G$ for A a subgroup of G and $\mu \in C^2(G, \mathrm{Map}(K, \mathbb{C}^*))$ is a cochain that satisfies the equation $\delta_G \mu^{-1} = \omega$. Two tensor categories $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent if there exists an indecomposable right module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathcal{C}_{\mathcal{M}}^*$ and $\mathcal{D}$ are tensor equivalent [Müger 2003, Definition 4.2].

Now, if we have two pointed fusion categories $\mathrm{Vect}(G, \omega)$ and $\mathrm{Vect}(\hat{G}, \hat{\omega})$, what are the necessary and sufficient conditions for them to be weakly Morita equivalent? This question was raised in [Davydov 2000; Movshev 1993], it was answered by Davydov [2000, Corollary 6.2] for the case in which both ω and $\hat{\omega}$ were trivial, and the general case was answered by Naidu [2007, Theorem 5.8] in terms of the properties that A, ω and μ need to satisfy. Nevertheless these conditions were given in equations that a priori had no interpretation in terms of known cohomology classes.

We continue the work started by Naidu [2007] and frame all the calculations done there in the language of the double complex associated to an abelian group extension which induces the Lyndon–Hochschild–Serre (LHS) spectral sequence. By doing so we are able to obtain in Corollary 3.2 cohomological conditions on ω in order for the tensor category $\mathrm{Vect}(G, \omega)^*_{\mathcal{M}(A\backslash G, \mu)}$ to be pointed, namely that ω must be cohomologous to a cocycle appearing in $C^{2,1} \oplus C^{3,0}$ of the double complex which induces the Lyndon–Hochschild–Serre spectral sequence associated to the extension $1 \to A \to G \to K \to 1$.

With the previous result at hand, we construct explicit representatives of ω and μ in terms of coordinates and we determine explicitly the groups $\hat{G}$ and the cocycles $\hat{\omega}$. The main result of this paper is Theorem 3.9, in which we give the necessary and sufficient conditions for the categories $\mathrm{Vect}(H, \eta)$ and $\mathrm{Vect}(\hat{H}, \hat{\eta})$ to be weakly Morita equivalent. We may summarize the conditions as follows: $\mathrm{Vect}(H, \eta)$ and $\mathrm{Vect}(\hat{H}, \hat{\eta})$ are weakly Morita equivalent if and only if there exist isomorphisms of groups $\phi : A \rtimes_F K \xRightarrow{\cong} H$ and $\hat{\phi} : K \ltimes_{\hat{F}} \mathbb{A} \xRightarrow{\cong} \hat{H}$ for some finite group K acting on the abelian group A, with $F \in Z^2(K, A)$ and $\hat{F} \in Z^2(K, \mathbb{A})$ where $\mathbb{A} := \mathrm{Hom}(A, \mathbb{C}^*)$, such that both $[\hat{F}]$ and $[F]$ survive respectively the LHS spectral sequence for the groups $A \rtimes_F K$ and $K \ltimes_{\hat{F}} \mathbb{A}$, and such that $\phi^*\eta$ is cohomologous to

$$\omega((a_1, k_1), (a_2, k_2), (a_3, k_3)) := \hat{F}(k_1, k_2)(a_3)\epsilon(k_1, k_2, k_3)$$

and $\hat{\phi}^*\hat{\eta}$ is cohomologous to

$$\hat{\omega}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)),$$

where $\epsilon : K^3 \to \mathbb{C}^*$ satisfies $\delta_K\epsilon = \hat{F} \wedge F$.

Theorem 3.9 may be used to determine the weak Morita equivalence classes of pointed fusion categories of a given global dimension but the cohomological calculations can become very elaborate and are beyond the scope of this article. Nevertheless in Section 4 we include a calculation in which we show how Theorem 3.9 can be used to prove that there are only seven weak Morita equivalence classes of pointed fusion categories of global dimension four and calculate the pointed fusion categories which are weakly Morita equivalent to $\mathrm{Vect}(Q_8, \eta)$ for the quaternion group Q_8.

1. Preliminaries

1A. *Abelian group extensions.* Consider the short exact sequence of finite groups

(1-1) $$1 \to A \to G \to K \to 1$$

with A abelian. Consider $u : K \to G$ any section of the projection map $p : G \to K$, $p(g) = (Ag)$ such that $u(1_K) = 1_G$ and denote the right G-action on K by

$$k \triangleleft g := p((u(k)g)$$

for $k \in K$ and $g \in G$. The elements $u(k)g$ and $u(k \triangleleft g)$ differ by an element $\kappa_{k,g} \in A$ satisfying the equation

(1-2) $$u(k)g = \kappa_{k,g} u(k \triangleleft g),$$

which furthermore satisfies the relation

$$\kappa_{k,g_1 g_2} = \kappa_{k,g_1} \kappa_{k \triangleleft g_1, g_2}$$

for $k \in K$ and $g_1, g_2 \in G$. Since A is an abelian normal subgroup G, there is an induced K-left action on A by conjugation:

$$^k a := u(k) a u(k)^{-1} \quad \text{for } k \in K \quad \text{and} \quad a \in A.$$

Since the isomorphism class of the extension (1-1) can be classified by the cohomology class of the cocycle $F \in Z^2(K, A)$, i.e., a map $F : K \times K \to A$ such that

$$\delta_K F(k_1, k_2, k_3) = {}^{k_1} F(k_2, k_3) F(k_1 k_2, k_3)^{-1} F(k_1, k_2 k_3) F(k_1, k_2)^{-1} = 1,$$

without loss of generality we will further assume that

$$G := A \rtimes_F K,$$

where the product structure of G is given by the formula

$$(a_1, k_1)(a_2, k_2) := (a_1 ({}^{k_1} a_2) F(k_1, k_2), k_1 k_2).$$

With this explicit choice of the group G, we choose the function $u : K \to G$ to be $u(k) := (1_A, k)$ and therefore we have that

$$\kappa_{k_1, (a, k_2)} = {}^{k_1} a F(k_1, k_2),$$

thus obtaining $F(k_1, k_2) = \kappa_{k_1, (1, k_2)}$. We furthermore have that for $x \in K$ and $g = (a, k) \in G$,

$$x \triangleleft g = x \triangleleft (a, k) = xk.$$

Denote the dual group $\mathbb{A} := \operatorname{Hom}(A, \mathbb{C}^*)$ and note that there is an induced K-right action on $\mathbb{A}$ defined as $\rho^k(a) := \rho({}^k a)$ for $\rho \in \mathbb{A}$ and $k \in K$.

1B. *Cohomology of groups and the LHS spectral sequence.* In what follows we will construct an explicit double complex whose cohomology calculates the co-homology of the group G, and whose associated spectral sequence recovers the Lyndon–Hochschild–Serre (LHS) spectral sequence of the extension (1-1).

Endow the set $\mathrm{Map}(K, \mathbb{C}^*)$ with the left G-action $(g \triangleright f)(k) := f(k \triangleleft g)$, where $g \in G$, $k \in K$ and $f : K \to \mathbb{C}^*$, and consider the complex $C^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ with elements normalized chains

$$C^q(G, \mathrm{Map}(K, \mathbb{C}^*)) := \{ f : K \times G^q \to \mathbb{C}^* \mid f(k; g_1, \ldots, g_q) = 1 \text{ whenever some } g_i = 1 \}$$

and boundary map

$$(1\text{-}3)\ (\delta_G f)(k; g_1, \ldots, g_q)$$

$$= f(k \triangleleft g_1; g_2, \ldots, g_q) \prod_{i=1}^{q-1} f(k; g_1, \ldots, g_i g_{i+1}, \ldots, g_q)^{(-1)^i} f(k; g_1, \ldots, g_{q-1})^{(-1)^q}.$$

Since the natural morphism of groupoids, defined by the inclusion of the group A into the action groupoid defined by the right action of G on K, is an equivalence of categories, we have that the restriction map

$$\psi : C^*(G, \mathrm{Map}(K, \mathbb{C}^*)) \to C^*(A, \mathbb{C}^*), \quad \psi(f)(a_1, \ldots, a_q) := f(1_K; a_1, \ldots, a_q),$$

is a morphism of complexes which induces an isomorphism in cohomology

$$\tilde{\psi} : H^*(G, \mathrm{Map}(K, \mathbb{C}^*)) \xrightarrow{\cong} H^*(A, \mathbb{C}^*).$$

The inverse map can be constructed at the level of cocycles as follows:

Lemma 1.1. *The map* $\varphi : C^q(A, \mathbb{C}^*) \to C^q(G, \mathrm{Map}(K, \mathbb{C}^*))$,

$$\varphi(\alpha)(k; g_1, \ldots, g_q) := \alpha(\kappa_{k, g_1}, \kappa_{k \triangleleft g_1, g_2}, \ldots, \kappa_{k \triangleleft g_1 g_2 \ldots g_{q-1}, g_q}),$$

defines a map of complexes which induces an isomorphism in cohomology $\tilde{\varphi} :$ $H^*(A, \mathbb{C}^*) \xrightarrow{\cong} H^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ *which is the inverse of the map* $\tilde{\psi}$.

Proof. On the one hand we have

$$\delta_G \varphi(\alpha)(k; g_1, \ldots, g_p)$$

$$= \varphi(\alpha)(k \triangleleft g_1; g_2, \ldots, g_q) \prod_{i=1}^{q-1} \varphi(\alpha)(k; g_1, \ldots, g_i g_{i+1}, \ldots, g_q)^{(-1)^i} \varphi(\alpha)(k; g_1, \ldots, g_{q-1})^{(-1)^q}$$

$$= \alpha(\kappa_{k \triangleleft g_1, g_2}, \kappa_{k \triangleleft g_1 g_2, g_3}, \ldots, \kappa_{k \triangleleft g_1 \ldots g_{q-1}, g_q})$$

$$\prod_{i=1}^{q-1} \alpha(\kappa_{k, g_1}, \kappa_{k \triangleleft g_1, g_2}, \ldots, \kappa_{k \triangleleft g_1 \ldots g_{i-1}, g_i g_{i+1}}, \ldots, \kappa_{k \triangleleft g_1 \ldots g_{q-1}, g_q})^{(-1)^i}$$

$$\alpha((\kappa_{k, g_1}, \kappa_{k \triangleleft g_1, g_2}, \ldots, \kappa_{k \triangleleft g_1 \ldots g_{q-2}, g_{q-1}})^{(-1)^q}$$

and on the other

$$\varphi(\delta_G\alpha)(k; g_1, \ldots, g_p)$$
$$= \delta_G\alpha(\kappa_{k,g_1}, \kappa_{k\triangleleft g_1, g_2}, \ldots, \kappa_{k\triangleleft g_1 g_2, \ldots, g_{q-1}, g_q})$$
$$= \alpha(\kappa_{k\triangleleft g_1, g_2}, \kappa_{k\triangleleft g_1 g_2, g_3}, \ldots, \kappa_{k\triangleleft g_1 \ldots g_{q-1}, g_q})$$
$$\prod_{i=1}^{q-1} \alpha(\kappa_{k,g_1}, \kappa_{k\triangleleft g_1, g_2}, \ldots, \kappa_{k\triangleleft g_1 \ldots g_{i-1}, g_i} \kappa_{k\triangleleft g_1 \ldots g_{i-1} g_i, g_{i+1}}, \ldots, \kappa_{k\triangleleft g_1 \ldots g_{q-1}, g_q})^{(-1)^i}$$
$$\alpha((\kappa_{k,g_1}, \kappa_{k\triangleleft g_1, g_2}, \ldots, \kappa_{k\triangleleft g_1 \ldots g_{q-2}, g_{q-1}})^{(-1)^q}.$$

The equality $\delta_G\varphi(\alpha) = \varphi(\delta_G\alpha)$ follows from the identity

$$\kappa_{k\triangleleft g_1 \ldots g_{i-1}, g_i g_{i+1}} = \kappa_{k\triangleleft g_1 \ldots g_{i-1}, g_i} \kappa_{k\triangleleft g_1 \ldots g_{i-1} g_i, g_{i+1}}.$$

Finally, the composition $\psi(\varphi(\alpha)) = \alpha$ follows from $\kappa_{1,a} = a$ for $a \in A$. $\qquad\square$

The complex $C^*(A, \mathbb{C}^*)$ can be endowed with the structure of a right K-module by setting for $\alpha \in C^q(A, \mathbb{C}^*)$ and $k \in K$

$$\alpha^k(a_1, \ldots, a_q) := \alpha(u(k)a_1 u(k)^{-1}, \ldots, u(k)a_q u(k)^{-1}),$$

and the complex $C^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ can also be endowed with the structure of a right K-module by setting for $f \in C^q(G, \mathrm{Map}(K, \mathbb{C}^*))$ and $k \in K$

$$(f \triangleleft k)(x; g_1, \ldots, g_q) := f(kx; g_1, \ldots, g_q).$$

The map φ fails to be a K-module map; nevertheless it induces a K-module map at the level of cohomology:

Lemma 1.2. *The isomorphism $\tilde{\varphi} : H^*(A, \mathbb{C}^*) \xrightarrow{\cong} H^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ is an isomorphism of K-modules.*

Proof. Take $\alpha \in Z^q(A, \mathbb{C}^*)$ and $k \in K$. We claim that $\psi(\varphi(\alpha)\triangleleft k) = \alpha^k$, and since $\psi(\varphi(\alpha^k)) = \alpha^k$, we conclude that $\varphi(\alpha)\triangleleft k$ and $\varphi(\alpha^k)$ are cohomologous. Now, let us calculate

$$\psi(\varphi(\alpha)\triangleleft k)(a_1, \ldots, a_q) = (\varphi(\alpha)\triangleleft k)(1; a_1, \ldots, a_q)$$
$$= \varphi(\alpha)(k; a_1, \ldots, a_q)$$
$$= \alpha(\kappa_{k,a_1}, \kappa_{k\triangleleft a_1, a_2}, \ldots, \kappa_{k\triangleleft a_1 a_2, \ldots, a_{q-1}, a_q})$$
$$= \alpha(\kappa_{k,a_1}, \kappa_{k,a_2}, \ldots, \kappa_{k,a_q})$$
$$= \alpha(u(k)a_1 u(k)^{-1}, u(k)a_2 u(k)^{-1}, \ldots, u(k)a_q u(k)^{-1})$$
$$= \alpha^k(a_1, a_2, \ldots, a_q);$$

the lemma follows. $\qquad\square$

1B1. *Double complex.* Since $C^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ is a complex of right K-modules, we can consider the complexes

$$C^*(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))),$$

with $C^p(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*)))$ consisting of normalized cochains

$$\{f : K^p \to C^q(G, \mathrm{Map}(K, \mathbb{C}^*)) \mid f(k_1, \ldots, k_p) = 1 \text{ whenever some } k_i = 1\}$$

and whose differentials are

$(\delta_K f)(k_1, \ldots, k_p)$

$$= f(k_2, \ldots, k_p) \prod_{i=1}^{p-1} f(k_1, \ldots, k_i k_{i+1}, \ldots, k_p)^{(-1)^i} (f(k_1, \ldots, k_{p-1}) \triangleleft k_p)^{(-1)^p}.$$

These complexes assemble into a double complex

$$C^{p,q} := C^p(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))).$$

Let us denote by $\mathrm{Tot}(C^{*,*})$ the total complex associated to the double complex and let $\delta_{\mathrm{Tot}} := \delta_K \oplus (\delta_G)^{(-1)^p}$ be its differential.

We may filter the total complex by the degree of the G cochains, thus obtaining a spectral sequence whose first page becomes

$$E_1^{p,q} = H^p(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))).$$

Since the K-modules $C^q(G, \mathrm{Map}(K, \mathbb{C}^*))$ are free K-modules, we conclude that the first page localizes on the y-axis,

$$E_1^{0,q} = H^0(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))) = C^q(G, \mathrm{Map}(K, \mathbb{C}^*))^K \cong C^q(G, \mathbb{C}^*)$$

and $E_1^{p,q} = 0$ for $p > 0$. The spectral sequence collapses at the second page, with the only surviving elements on the y-axis

$$E_2^{0,q} = H^q(G, \mathbb{C}^*).$$

Hence we have:

Proposition 1.3. *The inclusion of K-invariant cochains*

$$C^*(G, \mathrm{Map}(K, \mathbb{C}^*))^K \hookrightarrow \mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*))))$$

is a quasi-isomorphism. Therefore the cohomology groups

$$H^*(G, \mathbb{C}^*) \xrightarrow{\cong} H^*(\mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*)))))$$

are canonically isomorphic.

Filtering the double complex by the degree of the K cochains we obtain the Lyndon–Hochschild–Serre spectral sequence associated to the group extension $1 \to A \to G \to K \to 1$; see [Evens 1991, §7.2]. The first page becomes

$$E_1^{p,q} = C^p(K, H^q(G, \mathrm{Map}(K, \mathbb{C}^*))),$$

and the second page becomes

$$E_2^{p,q} = H^p(K, H^q(G, \mathrm{Map}(K, \mathbb{C}^*))).$$

Since the projection map $\tilde{\psi} : H^q(G, \mathrm{Map}(K, \mathbb{C}^*)) \xRightarrow{\cong} H^q(A, \mathbb{C}^*)$ is an isomorphism of K-modules, we conclude:
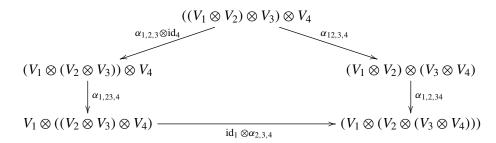
Proposition 1.4 (LHS spectral sequence). *Filtering the total complex by the degree of the K-chains, we obtain a spectral sequence whose second page is*

$$E_2^{p,q} \cong H^p(K, H^q(A, \mathbb{C}^*))$$

and that converges to $H^*(G, \mathbb{C}^*)$.

We will denote by $d_i : E_i^{p,q} \to E_i^{p+i,q-i+1}$ the differentials of this spectral sequence.

1C. *Tensor categories.* Following [Bakalov and Kirillov 2001, §1], a tensor category consists of $(\mathcal{C}, \otimes, 1_\mathcal{C}, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, α is the associativity constraint, i.e., a functorial isomorphism $\alpha_{UVW} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$ of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $1_\mathcal{C} \in \mathrm{Obj}(\mathcal{C})$ is a unit element and λ, ρ are functorial isomorphisms $\lambda_V : 1_\mathcal{C} \otimes V \xrightarrow{\sim} V$, $\rho_V : V \otimes 1_\mathcal{C} \xrightarrow{\sim} V$ satisfying the pentagon axiom

and the triangle axiom

1D. *The fusion category* **Vect**(G, ω)**.** A fusion category over $\mathbb{C}$ is a rigid semisimple $\mathbb{C}$-linear tensor category, with only finitely many isomorphism classes of simple objects, such that the endomorphisms of the unit object is $\mathbb{C}$; see [Etingof et al. 2005].

For G a finite group and a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^*)$, define the category Vect(G, ω) whose objects are G-graded complex vector spaces $V = \bigoplus_{g \in G} V_g$, whose tensor product is $(V \otimes W)_g := \bigoplus_{hk=g} V_h \otimes W_k$, whose associativity constraint is

$$\alpha_{V_g, V_h, V_k} = \omega(g, h, k)\gamma \quad \text{with } \gamma((x \otimes y) \otimes z) = x \otimes (y \otimes z),$$

and whose left and right unit isomorphisms are

$$\lambda_{V_g} = \omega(1, 1, g)^{-1} \, \mathrm{id}_{V_g} \quad \text{and} \quad \rho_{V_g} = \omega(g, 1, 1) \, \mathrm{id}_{V_g} \,.$$

The category Vect(G, ω) is a fusion category where the simple objects are the 1-dimensional vector spaces.

We will assume that all group cochains are normalized, and hence the left and right unit isomorphisms become identities.
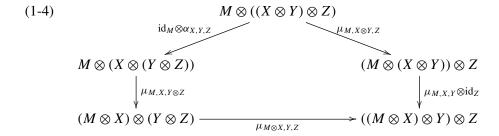
For convenience we will work with a category $\mathcal{V}(G, \omega)$ which is *skeletal*, i.e., one on which isomorphic objects are equal, and which is equivalent to Vect(G, ω). The category $\mathcal{V}(G, \omega)$ has for simple objects the elements g of the group G, the tensor product is $g \otimes h = gh$ and the associativity isomorphisms are $\omega(g, h, k) \, \mathrm{id}_{ghk}$.

A finite tensor category is called *pointed* if all its simple objects are invertible. It is thus easy to see that any finite tensor category which is pointed is equivalent to Vect(G, ω) for some finite group G and some 3-cocycle ω.

1E. *Module categories.* Following [Ostrik 2003b, §2.3], a right *module category* over the tensor category $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, \alpha, \lambda, \rho)$ consists of $(\mathcal{M}, \otimes, \mu, \tau)$, where $\mathcal{M}$ is a category, $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ is an exact bifunctor,

$$\mu_{M,X,Y} : M \otimes (X \otimes Y) \xrightarrow{\sim} (M \otimes X) \otimes Y$$

is a functorial associativity and $\tau_M : M \otimes 1_{\mathcal{C}} \xrightarrow{\sim} M$ is a unit isomorphism for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$, satisfying the pentagon axiom

(1-4)
$$
\begin{array}{ccc}
& M \otimes ((X \otimes Y) \otimes Z) & \\
\mathrm{id}_M \otimes \alpha_{X,Y,Z} \swarrow & & \searrow \mu_{M, X \otimes Y, Z} \\
M \otimes (X \otimes (Y \otimes Z)) & & (M \otimes (X \otimes Y)) \otimes Z \\
\downarrow \mu_{M,X,Y \otimes Z} & & \downarrow \mu_{M,X,Y} \otimes \mathrm{id}_Z \\
(M \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\mu_{M \otimes X, Y, Z}} & ((M \otimes X) \otimes Y) \otimes Z
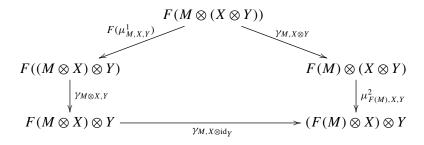\end{array}
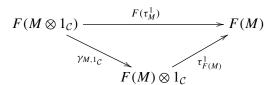$$

and the triangle axiom

$$(1\text{-}5) \qquad M \otimes (1_{\mathcal{C}} \otimes Y) \xrightarrow{\mu_{M,1_{\mathcal{C}},Y}} (M \otimes 1_{\mathcal{C}}) \otimes Y$$

with $\mathrm{id}_M \otimes \lambda_Y$ and $\tau_M \otimes \mathrm{id}_Y$ to $M \otimes Y$.

A *module functor* $(F, \gamma) : (\mathcal{M}_1, \mu^1, \tau^1) \to (\mathcal{M}_2, \mu^2, \tau^2)$ between two module categories consists of a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ and a functorial isomorphism $\gamma_{M,X} : F(M \otimes X) \to F(M) \otimes X$ for any $X \in \mathcal{C}$, $M \in \mathcal{M}$, satisfying the pentagon axiom

$$F(M \otimes (X \otimes Y))$$

with $F(\mu^1_{M,X,Y})$ to $F((M \otimes X) \otimes Y)$ and $\gamma_{M,X\otimes Y}$ to $F(M) \otimes (X \otimes Y)$;

$$F((M \otimes X) \otimes Y) \xrightarrow{\gamma_{M\otimes X,Y}} \quad \downarrow$$

$$F(M \otimes X) \otimes Y \xrightarrow{\gamma_{M,X}\otimes \mathrm{id}_Y} (F(M) \otimes X) \otimes Y$$

with $\mu^2_{F(M),X,Y}$ on the right, from $F(M) \otimes (X \otimes Y)$ to $(F(M) \otimes X) \otimes Y$.

and the triangle axiom

$$F(M \otimes 1_{\mathcal{C}}) \xrightarrow{F(\tau^1_M)} F(M)$$

with $\gamma_{M,1_{\mathcal{C}}}$ and $\tau^1_{F(M)}$ to $F(M) \otimes 1_{\mathcal{C}}$.

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exists a module functor between the two which is moreover an equivalence of categories. The *direct sum* $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the module category with the obvious structure. A module category is *indecomposable* if it is not equivalent to the direct sum of two nontrivial module categories.

A *natural module transformation* $\eta : (F^1, \gamma^1) \to (F^2, \gamma^2)$ consists of a natural transformation $\eta : F^1 \to F^2$ such that the square

$$\begin{array}{ccc} F^1(M \otimes X) & \xrightarrow{\eta_{M\otimes X}} & F^2(M \otimes X) \\ \gamma^1_{M,X} \downarrow & & \downarrow \gamma^2_{M,X} \\ F^1(M) \otimes X & \xrightarrow{\eta_M \otimes \mathrm{id}_X} & F^2(M) \otimes X \end{array}$$

commutes for all $M \in \mathcal{M}$ and $X \in \mathcal{C}$.

1F. *Indecomposable module categories over $\mathcal{V}(G, \omega)$.* Let $\mathcal{M}$ be a skeletal right module category over $\mathcal{V}(G, \omega)$. The set of simple objects of $\mathcal{M}$ is a transitive right G-set and therefore it can be identified with the coset $K := A \setminus G$ for A a subgroup of G. The isomorphisms μ_{k,g_1,g_2} for $k \in K$ and $g_1, g_2 \in G$ are scalars, and we can assemble these scalars as an element

$$\mu \in C^2(G, \mathrm{Map}(K, \mathbb{C}^*)), \quad \mu(k; g_1, g_2) := \mu_{k,g_1,g_2}.$$

The pentagon axiom (1-4) translates into the equation

$$\omega(g_1, g_2, g_3)\mu(k; g_1, g_2 g_3)\mu(k \triangleleft g_1; g_2, g_3) = \mu(k; g_1 g_2, g_3)\mu(k; g_1, g_2),$$

which in view of the definition of the differential δ_G in (1-3) becomes

$$(1\text{-}6) \qquad\qquad\qquad \delta_G \mu^{-1} = \pi^* \omega,$$

where $\pi^* \omega \in C^3(G, \mathrm{Map}(K, \mathbb{C}^*))^K$ is the K-invariant cocycle defined by ω, i.e.,

$$\pi^* \omega(k; g_1, g_2, g_3) := \omega(g_1, g_2, g_3).$$

Since μ is normalized and the unit constraint in $\mathcal{V}(G, \omega)$ is trivial, we have that the triangle axiom (1-5) implies that the unit constraint in $\mathcal{M}$ is trivial.

Denote this skeletal module category $\mathcal{M} = \mathcal{M}(A \setminus G, \mu)$. Note that two $\mathcal{V}(G, \omega)$-module categories $\mathcal{M}_1 = \mathcal{M}(A_1 \setminus G, \mu_1)$ and $\mathcal{M}_2 = \mathcal{M}(A_2 \setminus G, \mu_2)$ are equivalent if and only if there exist a right G-equivariant isomorphism $F : A_1 \setminus G \xrightarrow{\cong} A_2 \setminus G$ and an element $\gamma \in C^1(G, \mathrm{Map}(A_1 \setminus G, \mathbb{C}^*))$ such that

$$\gamma(A_1 g; g_1 g_2)\mu_2(F(A_1 g); g_1, g_2) = \mu_1(A_1 g; g_1, g_2)\gamma(A_1 g g_1; g_2)\gamma(A_1 g; g_1).$$

This information implies that A_1 and A_2 are conjugate subgroups of G and that

$$\delta_G \gamma = \frac{F^* \mu_2}{\mu_1}.$$

In the case that $A = A_1 = A_2$, the G-equivariant isomorphisms are parametrized by the elements of the group $A \setminus N_G(A)$, and the equation $\delta_G \gamma = F^* \mu_2 / \mu_1$ implies that $F^* \mu_2 / \mu_1$ is trivial in $H^2(G, \mathrm{Map}(A \setminus G, \mathbb{C}^*))$. Since we know that $\tilde{\psi} : H^2(G, \mathrm{Map}(A \setminus G, \mathbb{C}^*)) \xrightarrow{\cong} H^2(A, \mathbb{C}^*)$ is an isomorphism, we can conclude that the isomorphism classes of module categories over $\mathcal{V}(G, \omega)$ may be parametrized (in a noncanonical manner) by pairs $([A], [\psi(\mu)])$, where $[A]$ is a conjugacy class of subgroups of G, and $[\psi(\mu)]$ is a representative of a cohomology class in the group of invariants $H^2(A, \mathbb{C}^*)/N_G(A)$.

1G. *Dual category.* Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ an indecomposable right module category. The dual category $\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is the category whose objects are module functors from $\mathcal{M}$ to itself and whose morphisms are natural module transformations.

The category $\mathcal{C}^*_{\mathcal{M}}$ becomes a tensor category by composition of functors; namely for $(\gamma^1, F_1), (\gamma^2, F_2) \in \mathrm{Obj}(\mathcal{C}^*_{\mathcal{M}})$, where γ^1, γ^2 represent the module structures on the functors F_1 and F_2 respectively, we define the tensor structure by

$$(\gamma^1, F_1) \otimes (\gamma^2, F_2) := (\gamma, F_1 \circ F_2),$$

where the module structure γ is defined by $\gamma_{M,X} := \gamma^1_{F_2(M),X} \circ F_1(\gamma^2_{M,X})$ for $M \in \mathcal{M}$ and $X \in \mathcal{C}$. For two morphisms $\eta : (\gamma^1, F_1) \to (\gamma^2, F_2)$ and $\eta' : (\gamma'^1, F'_1) \to (\gamma'^2, F'_2)$ in $\mathcal{C}^*_{\mathcal{M}}$ their tensor product is $(\eta \otimes \eta')(M) := \eta_{F'_2(M)} \circ F_1(\eta'_M)$.

Whenever $\mathcal{C}$ and $\mathcal{M}$ are semisimple, the dual category $\mathcal{C}^*_{\mathcal{M}}$ is semisimple [Ostrik 2003a, §2.2]. Moreover, since $\mathcal{M}$ is itself a left module category over $\mathcal{C}^*_{\mathcal{M}}$ it has been shown in [Ostrik 2003b, Corollary 4.1] that the double dual is tensor equivalent to the original category, i.e., $(\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} \simeq \mathcal{C}$. Furthermore, the module categories of $\mathcal{C}$ and of $\mathcal{C}^*_{\mathcal{M}}$ are in canonical bijection (Proposition 2.1 of the same work) by the following maps. For $\mathcal{M}_1$ a module category over $\mathcal{C}$, the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M})$ of module functors from $\mathcal{M}_1$ to $\mathcal{M}$ is a left module category of $\mathcal{C}^*_{\mathcal{M}} = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ via the composition of functors. Conversely, if $\mathcal{M}_2$ is a left module category over $\mathcal{C}^*_{\mathcal{M}}$, then $\mathrm{Fun}_{\mathcal{C}^*_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}_2)$ is a right module category over $\mathrm{Fun}_{\mathcal{C}^*_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}) = (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} \simeq \mathcal{C}$ via composition of functors. These maps are inverse from each other.

1H. *Center of a tensor category.* The *center* $\mathcal{Z}(\mathcal{C})$ of the tensor category $\mathcal{C}$ is the category whose objects are pairs (X, η), where X is an object in $\mathcal{C}$ and η is a functorial set of isomorphisms $\eta_Y : X \otimes Y \to Y \otimes X$ such that the hexagon diagram

$$
\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{\ \alpha\ } & X \otimes (Y \otimes Z) & \xrightarrow{\ \eta_{Y \otimes Z}\ } & (Y \otimes Z) \otimes X \\
\Big\downarrow{\scriptstyle \eta_Y \otimes 1} & & & & \Big\downarrow{\scriptstyle \alpha} \\
(Y \otimes X) \otimes Z & \xrightarrow{\ \alpha\ } & Y \otimes (X \otimes Z) & \xrightarrow{\ 1 \otimes \eta_Z\ } & Y \otimes (Z \otimes X)
\end{array}
$$

and the triangle diagram

$$
\begin{array}{ccc}
X \otimes 1_{\mathcal{C}} & \xrightarrow{\ \eta_{1_{\mathcal{C}}}\ } & 1_{\mathcal{C}} \otimes X \\
& {\scriptstyle \rho} \searrow \quad \swarrow {\scriptstyle \lambda} & \\
& X &
\end{array}
$$

are commutative. A morphism $f : (X, \eta) \to (Y, \nu)$ consists of a morphism $f : X \to Y$ for which the diagram

$$
\begin{array}{ccc}
X \otimes Z & \xrightarrow{\ \eta_Z\ } & Z \otimes X \\
\Big\downarrow{\scriptstyle f \otimes 1} & & \Big\downarrow{\scriptstyle 1 \otimes f} \\
Y \otimes Z & \xrightarrow{\ \nu_Z\ } & Z \otimes Y
\end{array}
$$

commutes for any object Z in $\mathcal{C}$. The tensor structure is defined as $(X, \eta) \otimes (Y, \nu) :=$ $(X \otimes Y, \gamma)$, where γ_Z is defined as the composition

$$(X \otimes Y) \otimes Z \xrightarrow{\alpha} X \otimes (Y \otimes Z) \xrightarrow{1 \otimes \nu_Z} X \otimes (Z \otimes Y)$$

$$(X \otimes Z) \otimes Y \xrightarrow{\eta_Z \otimes 1} (Z \otimes X) \otimes Y \xrightarrow{\alpha} Z \otimes (X \otimes Y)$$

The center $\mathcal{Z}(\mathcal{C})$ is moreover *braided* and the braiding for the pair $(X, \eta), (Y, \nu)$ is precisely the map η_Y.

The center $\mathcal{Z}(\mathrm{Vect}(G, \omega))$ of the tensor category $\mathrm{Vect}(G, \omega)$ contains the information necessary for constructing the quasi-Hopf algebra that is known as the twisted Drinfeld double $D^\omega(G)$ of the group G twisted by ω (see [Dijkgraaf et al. 1991, §3.2]).

1I. *Weak Morita equivalence of tensor categories.* Two tensor categories $\mathcal{C}$ and $\mathcal{D}$ are *weakly Morita equivalent* if there exists an indecomposable right module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathcal{C}^*_\mathcal{M}$ and $\mathcal{D}$ are tensor equivalent [Müger 2003, Definition 4.2]. In Proposition 4.6 of the same work it is shown that weak Morita equivalence is an equivalence relation, and in [Etingof et al. 2011, Theorem 3.1] it is shown that two tensor categories are weak Morita equivalent if and only if their centers are braided equivalent. In particular we have that for $\mathcal{M}$ an indecomposable module category over $\mathcal{C}$ there is a canonical equivalence of braided tensor categories $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C}^*_\mathcal{M})$ [Ostrik 2003a, Proposition 2.2].

2. The dual of $\mathcal{V}(G, \omega)$ with respect to $\mathcal{M}(A \backslash G, \mu)$

Let us consider the tensor category $\mathcal{C} = \mathcal{V}(G, \omega)$ and the right module category $\mathcal{M} = \mathcal{M}(A \backslash G, \mu)$ described in Section 1F. In this chapter we will review the main results of [Naidu 2007], where explicit conditions are stated under which the dual category $\mathcal{C}^*_\mathcal{M}$ is pointed. For the sake of completeness and clarity we will review the constructions done in §3 and §4 of that work and we will reinterpret the equations given there in the terminology that we have set up in Section 1A and Section 1B.

2A. *Conditions for $\mathcal{C}^*_\mathcal{M}$ to be pointed.* Let us set up some notation for this section: let $K := A \backslash G$, $u : K \to G$ satisfy $p \circ u = 1_G$ and $u(p(1_G)) = 1_G$ for $p : G \to K$ the projection, $\kappa : K \times G \to A$ satisfy $u(k)g = \kappa_{k,g} u(k \triangleleft g)$ and K^A be the elements of K fixed under the conjugation by elements of A. The module category $\mathcal{M}(A \backslash G, \mu)$ is the skeletal category whose simple objects are the elements of $K = A \backslash G$, whose tensor structure is $k \otimes g := k \triangleleft g$ for $k \in K$ and $g \in G$ and whose associativity constraint μ satisfies $\delta_G \mu^{-1} = \pi^* \omega$; see (1-6). In what follows we will focus on parametrizing the invertible objects of $\mathcal{C}^*_\mathcal{M}$.

Following [Naidu 2007, Lemma 3.2] any invertible module functor in $\mathcal{C}^*_\mathcal{M}$ is of the form (F_y, γ), where the functor $F_y : \mathcal{M} \to \mathcal{M}$ is the one that extends the G-equivariant map $f_y : K \to K$, $f_y(k) = p(u(y)u(k))$, for $y \in K^A$, and γ is a functorial isomorphism $\gamma_{k,g} : F_y(k \otimes g) \overset{\cong}{\Longrightarrow} F_y(k) \otimes g$ that satisfies the pentagon axiom. Writing $\gamma_{k,g} := \gamma(k; g) \, \mathrm{id}_{p(u(y)u(k \triangleleft g))}$ for $\gamma \in C^1(G, \mathrm{Map}(K, \mathbb{C}^*))$ we have that the pentagon axiom of a module functor translates into the equation

$$\mu(k; g_1, g_2)\gamma(k \triangleleft g_1; g_2)\gamma(k; g_1) = \gamma(k; g_1 g_2)\mu(f_y(k); g_1, g_2),$$

which can also be written as

$$\delta_G \gamma(k; \gamma_1, \gamma_2) = \frac{\mu(f_y(k); g_1, g_2)}{\mu(k; g_1, g_2)}.$$

The inverse of (F_y, γ) is the module functor $(F_{p(u(y)^{-1})}, \bar{\gamma})$ with

$$\bar{\gamma}(k; g) := \gamma(p(u(y)^{-1}u(x))^{-1}; g)^{-1}.$$

Defining for each $y \in K^A$ the set

$$\mathrm{Fun}_y := \left\{ \gamma \in C^1(G, \mathrm{Map}(K, \mathbb{C}^*)) \,\middle|\, \delta_G \gamma(k; g_1, g_2) = \frac{\mu(f_y(k); g_1, g_2)}{\mu(k; g_1, g_2)} \right\}$$

for all $k \in K$ and $g_1, g_2 \in G$, we have that of invertible objects of $\mathcal{C}^*_\mathcal{M}$ are precisely the module functors (F_y, γ) where $y \in K^A$ and $\gamma \in \mathrm{Fun}_y$. To simplify the notation we will denote such a module functor by the pair (y, γ).

Two invertible module functors (y_1, γ^1) and (y_2, γ^2) in $\mathcal{C}^*_\mathcal{M}$ are isomorphic if and only if $y_1 = y_2$ and if there exists natural transformation parametrized by a map $\eta \in C^0(G, \mathrm{Map}(K, \mathbb{C}^*))$ satisfying the equation

(2-1) $$\gamma^1(k; g)\eta(k) = \eta(k \triangleleft g)\gamma^2(k; g)$$

for all $k \in K$ and $g \in G$. These equations can be rewritten as the equation

$$\delta_G \eta = \frac{\gamma^2}{\gamma^1}$$

in $C^1(G, \mathrm{Map}(K, \mathbb{C}^*))$. Therefore for each $y \in K^A$ we may define an equivalence relation on the elements $\gamma^1, \gamma^2 \in \mathrm{Fun}_y$ by setting $\gamma^2 \simeq \gamma^1$ whenever there exists η such that $\delta_G \eta = \gamma^2 / \gamma^1$; denote by $\overline{\mathrm{Fun}}_y$ the associated set of equivalence classes.

For each $y \in K^A$ let us choose an element $\gamma_y \in \mathrm{Fun}_y$, and note that the maps

$$\mathrm{Fun}_y \to Z^1(G, \mathrm{Map}(K, \mathbb{C}^*)), \quad \beta \mapsto \frac{\beta}{\gamma_y},$$

$$Z^1(G, \mathrm{Map}(k, \mathbb{C}^*)) \to \mathrm{Fun}_y, \qquad\qquad \epsilon \mapsto \epsilon \gamma_y$$

are inverse to each other. Therefore we obtain bijections

$$\overline{\mathrm{Fun}}_y \cong H^1(G, \mathrm{Map}(K, \mathbb{C}^*)) \cong H^1(A, \mathbb{C}^*) = \mathbb{A},$$

which are realized by the maps

(2-2) $\quad \zeta_y : \mathbb{A} \to \mathrm{Fun}_y, \quad \zeta_y(\rho) := \gamma_y \varphi(\rho), \qquad \theta_y : \mathrm{Fun}_y \to \mathbb{A}, \quad \theta_y(\beta) := \psi(\beta/\gamma_y).$

Recall from [Etingof et al. 2005, Definition 2.2] that the *global dimension* $\dim(\mathcal{C})$ of a fusion category $\mathcal{C}$ is the sum of the squared norms of its simple objects, and note that by Theorem 2.15 of the same paper we have $\dim(\mathcal{C}_{\mathcal{M}}^*) = \dim(\mathcal{C})$ whenever $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is an indecomposable module category over $\mathcal{C}$.

Let us suppose now that the dual category $\mathcal{C}_{\mathcal{M}}^* = \mathcal{V}(G, \omega)_{\mathcal{M}(A\backslash G, \mu)}^*$ is pointed. Therefore its global dimension

$$\dim(\mathcal{C}_{\mathcal{M}}^*) = |\mathbb{A}||K^A|$$

must be equal to the number of isomorphic classes of invertible objects, since on pointed categories all simple objects are invertible. On the other hand, by [Etingof et al. 2005, Theorem 2.15] we have $\dim(\mathcal{C}_{\mathcal{M}}^*) = \dim(\mathcal{C})$ and $\dim(\mathcal{C}) = |G|$. Therefore in order for the category $\mathcal{C}_{\mathcal{M}}^*$ to be pointed it is necessary that $|\mathbb{A}||K^A| = |G|$. Since $|G| = |A||K|$, $|\mathbb{A}| \leq |A|$ and $|K^A| \leq |K|$, the equality holds if and only if A is abelian, thus giving that $|\mathbb{A}| = |A|$, and if A is normal in G and $K^A = K$.

On the other hand, if A is abelian and normal on G, then the number of isomorphism classes of invertible objects in $\mathcal{C}_{\mathcal{M}}^*$ is $|\mathbb{A}||K| = |G|$. Since $\dim(\mathcal{C}_{\mathcal{M}}^*) = \dim(\mathcal{C}) = |G|$, we have that the set of isomorphism classes of invertible objects exhausts the set of simple elements, and therefore $\mathcal{C}_{\mathcal{M}}^*$ must be pointed.

Summarizing we have:

Theorem 2.1 [Naidu 2007, Theorem 3.4]. *The tensor category*

$$\mathcal{C}_{\mathcal{M}}^* = \mathcal{V}(G, \omega)_{\mathcal{M}(A\backslash G, \mu)}^*$$

is pointed if and only if A is abelian and normal in G and the cohomology class $[(\mu \triangleleft y)/\mu]$ is trivial in $H^2(G, \mathrm{Map}(K, \mathbb{C}^*))$ for all $y \in K$.

Note that since A is normal in G, we may use the notation introduced in Section 1B so that $\mu(f_y(k); g_1, g_2) = \mu(yk; g_1, g_2) = (\mu \triangleleft y)(k; g_1, g_2)$. Since we have that $\delta_G \mu^{-1} = \pi^* \omega = \delta_G(\mu^{-1} \triangleleft y)$, the quotient $(\mu \triangleleft y)/\mu$ defines a cocycle in $Z^2(G, \mathrm{Map}(K, \mathbb{C}^*))$. The equation $\delta_G \gamma_y = (\mu \triangleleft y)/\mu$ implies that the quotient is trivial in cohomology.

2B. *The Grothendieck ring of the pointed category $\mathcal{C}_{\mathcal{M}}^*$.* From now on we will assume that the dual category $\mathcal{C}_{\mathcal{M}}^*$ is pointed. Therefore we have that A is abelian and normal in G and that we can choose elements $\gamma_y \in C^1(G, \mathrm{Map}(K, \mathbb{C}^*))$ for each $y \in K$ such that $\delta_G \gamma_y = (\mu \triangleleft y)/\mu$.

The Grothendieck ring $K_0(\mathcal{C}^*_{\mathcal{M}})$ of the category $\mathcal{C}^*_{\mathcal{M}}$ is the ring defined by the semiring whose elements are the isomorphism classes of objects and whose product is the one induced by the tensor product. Since $\mathcal{C}^*_{\mathcal{M}}$ is pointed, $K_0(\mathcal{C}^*_{\mathcal{M}})$ is isomorphic to the group ring $\mathbb{Z}[\Lambda]$ for some finite group Λ. In this section we will recall the construction of this isomorphism carried out in [Naidu 2007, Theorem 4.5].

The tensor product of two invertible elements (y_1, γ^1), (y_2, γ^2) in $\mathcal{C}^*_{\mathcal{M}}$ as defined in Section 1G is

$$(y_1, \gamma^1) \otimes (y_2, \gamma^2) = (y_1 y_2, (\gamma^1 \triangleleft y_2)\gamma^2).$$

This tensor product defines a group structure on the set of isomorphism classes of invertible objects

$$\Lambda := \bigcup_{y \in K} \{y\} \times \overline{\mathrm{Fun}}_y$$

by the equation $(y_1, [\gamma^1]) \star (y_2, [\gamma^2]) = (y_1 y_2, [(\gamma^1 \triangleleft y_2)\gamma^2])$, where $[\gamma]$ denotes the equivalence class of γ in Fun_y.

Define the element $\gamma \in C^1(K, C^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$ by the equation

$$\gamma(y) := \gamma_y$$

and note that the equations $\delta_G \gamma_y = (\mu \triangleleft y)/\mu$ are equivalent to the equation

$$\delta_G \gamma = \delta_K \mu.$$

Define the element $\tilde{\nu} := \delta_K \gamma$, i.e., $\tilde{\nu}(y_1, y_2) = (\gamma(y_2)\gamma(y_1) \triangleleft y_2)/\gamma(y_1 y_2)$, and note that

$$\delta_K \tilde{\nu} = \delta_K^2 \gamma = 1 \quad \text{and} \quad \delta_G \tilde{\nu} = \delta_G \delta_K \gamma = \delta_K \delta_G \gamma = \delta_K^2 \mu = 1.$$

Hence $\tilde{\nu} \in Z^2(K, Z^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$ and we may define

(2-3) $$\nu := \psi \circ \tilde{\nu} \in Z^2(K, Z^1(A, \mathbb{C}^*)) = Z^2(K, \mathbb{A}),$$

thus having $\nu(y_1, y_2)(a) := \tilde{\nu}(y_1, y_2)(1; a)$.

With this 2-cocycle ν we may define the crossed product $K \ltimes_\nu \mathbb{A}$ by setting on pairs of elements of the set $K \times \mathbb{A}$

$$(y_1, \rho_1) \cdot (y_2, \rho_2) := (y_1 y_2, \rho_1^{y_2} \rho_2 \nu(y_1, y_2)).$$

Using the notation of (2-2) we have:

Theorem 2.2 [Naidu 2007, Theorem 4.5]. *The map*

$$T : K \ltimes_\nu \mathbb{A} \to \Lambda, \quad T((y, \rho)) = (y, [\zeta_y(\rho)]),$$

is an isomorphism of groups. Hence $K_0(\mathcal{C}^*_{\mathcal{M}}) \cong \mathbb{Z}[K \ltimes_\nu \mathbb{A}]$.

Proof. On the one hand we have

$$T((y_1, \rho_1) \cdot (y_2, \rho_2)) = T((y_1 y_2, \rho_1^{y_2} \rho_2 \nu(y_1, y_2))) = (y_1 y_2, [\zeta_{y_1 y_2}(\rho_1^{y_2} \rho_2 \nu(y_1, y_2))])$$

and on the other

$$\begin{aligned}
T((y_1, \rho_1)) \star T((y_2, \rho_2)) &= (y_1, [\zeta_{y_1}(\rho_1)]) \star (y_2, [\zeta_{y_2}(\rho_2)]) \\
&= (y_1 y_2, [(\zeta_{y_1}(\rho_1) \triangleleft y_2) \zeta_{y_2}(\rho_2)]).
\end{aligned}$$

The result follows if we check the equality

$$\theta_{y_1 y_2}((\zeta_{y_1}(\rho_1) \triangleleft y_2) \zeta_{y_2}(\rho_2)) = \rho_1^{y_2} \rho_2 \nu(y_1, y_2)$$

since this implies that $\zeta_{y_1 y_2}((\rho_1 \triangleleft y_2) \rho_2 \nu(y_1, y_2))$ and $(\zeta_{y_1}(\rho_1) \triangleleft y_2) \zeta_{y_2}(\rho_2)$ are cohomologous; hence we have

$$\begin{aligned}
\theta_{y_1 y_2}((\zeta_{y_1}(\rho_1) \triangleleft y_2) \zeta_{y_2}(\rho_2))(a) &= \frac{((\zeta_{y_1}(\rho_1) \triangleleft y_2)(1; a)) \zeta_{y_2}(\rho_2)(1; a)}{\gamma(y_1 y_2)(1; a)} \\
&= \frac{(\gamma(y_1) \triangleleft y_2 \varphi(\rho_1) \triangleleft y_2)(1; a)(\gamma(y_2) \varphi(\rho_2))(1; a)}{\gamma(y_1 y_2)(1; a)} \\
&= \delta_K \gamma(y_1, y_2)(1; a) \rho_1^{y_2}(a) \rho_2(a) \\
&= (\nu(y_1, y_2) \rho_1^{y_2} \rho_2)(a). \qquad \square
\end{aligned}$$

2C. *A skeleton of the pointed category* $\mathcal{C}_{\mathcal{M}}^*$.

A skeleton $\mathrm{sk}(\mathcal{C}_{\mathcal{M}}^*)$ of $\mathcal{C}_{\mathcal{M}}^*$ is a full subcategory of $\mathcal{C}_{\mathcal{M}}^*$ on which each object of $\mathcal{C}_{\mathcal{M}}^*$ is isomorphic to only one object in $\mathrm{sk}(\mathcal{C}_{\mathcal{M}}^*)$. Let us choose for objects

$$\mathrm{Obj}(\mathrm{sk}(\mathcal{C}_{\mathcal{M}}^*)) := \{(y, \zeta_y(\rho)) \mid (y, \rho) \in K \ltimes_\nu \mathbb{A}\}$$

and define its tensor product $\bullet$ by the one induced by $\star$, i.e.,

$$((y_1, \zeta_{y_1}(\rho_1)) \bullet (y_2, \zeta_{y_2}(\rho_2)) := (y_1 y_2, \zeta_{y_1 y_2}(\nu(y_1, y_2) \rho_1^{y_2} \rho_1)).$$

For each pair of objects, choose isomorphisms in $\mathcal{C}_{\mathcal{M}}^*$

$$f((y_1, \zeta_{y_1}(\rho_1)), (y_2, \zeta_{y_2}(\rho_2))$$
$$: (y_1, \zeta_{y_1}(\rho_1)) \bullet (y_2, \zeta_{y_2}(\rho_2)) \xrightarrow{\sim} (y_1, \zeta_{y_1}(\rho_1)) \otimes (y_2, \zeta_{y_2}(\rho_2)),$$

which by equation (2-1) satisfy

$$((\zeta_{y_1}(\rho_1) \triangleleft y_2) \zeta_{y_1}(\rho_1))(k; g)$$
$$= \frac{f((y_1, \zeta_{y_1}(\rho_1)), (y_2, \zeta_{y_2}(\rho_2))(k \triangleleft g)}{f((y_1, \zeta_{y_1}(\rho_1)), (y_2, \zeta_{y_2}(\rho_2))(k)} \times \zeta_{y_1 y_2}(\nu(y_1, y_2) \rho_1^{y_2} \rho_1)(k; g).$$

The tensor product $\otimes$ in $\mathcal{C}_{\mathcal{M}}^*$ is associative since it is defined by the composition of functors, but the tensor product $\bullet$ in its skeleton $\mathrm{sk}(\mathcal{C}_{\mathcal{M}}^*)$ may fail to be associative.

The associativity constraint for $\mathrm{sk}(\mathcal{C}_{\mathcal{M}}^*)$ is then

$$
\begin{aligned}
\hat{\omega}'&((y_1, \zeta_{y_1}(\rho_1)), (y_2, \zeta_{y_2}(\rho_2)), (y_3, \zeta_{y_3}(\rho_3))) \\
&= \frac{f((y_1, \zeta_{y_1}(\rho_1)), (y_2, \zeta_{y_2}(\rho_2)) \otimes \mathrm{id}_{(\zeta_{y_3}(\rho_3), y_3)}}{f((y_1, \zeta_{y_1}(\rho_1)), (y_2, \zeta_{y_2}(\rho_2)) \bullet (y_3, \zeta_{y_3}(\rho_3)))} \\
&\qquad \times \frac{f((y_1, \zeta_{y_1}(\rho_1)) \bullet (y_2, \zeta_{y_2}(\rho_2)), (y_3, \zeta_{y_3}(\rho_3)))}{\mathrm{id}_{(\zeta_{y_1}(\rho_1), y_1)} \otimes f((y_2, \zeta_{y_2}(\rho_2)), (y_3, \zeta_{y_3}(\rho_3))).}
\end{aligned}
$$

In [Naidu 2007, Theorem 4.9] it is shown that $\hat{\omega}'$ is K-invariant and moreover that it can be given in explicit form by the equation

$$
\hat{\omega}'((y_1, \zeta_{y_1}(\rho_1)), (y_2, \zeta_{y_2}(\rho_2)), (y_3, \zeta_{y_3}(\rho_3))) = \tilde{\nu}(y_1, y_2)(1; u(y_3))\rho_1(\kappa_{y_2, u(y_3)}).
$$

Therefore we may define the 3-cocycle on $K \ltimes_\nu \mathbb{A}$ by the equation

$$
\hat{\omega}((y_1, \rho_1), (y_2, \rho_2), (y_3, \rho_3)) = \tilde{\nu}(y_1, y_2)(1; u(y_3))\rho_1(\kappa_{y_2, u(y_3)}),
$$

and choosing $G = A \rtimes_F K$ and $u(y) = (1, y)$ as was done at the end of Section 1A, the 3-cocycle on $K \ltimes_\nu \mathbb{A}$ becomes

$$
(2\text{-}4) \qquad \hat{\omega}((y_1, \rho_1), (y_2, \rho_2), (y_3, \rho_3)) = \tilde{\nu}(y_1, y_2)(1; (1, y_3))\rho_1(F(y_2, y_3)).
$$

Therefore the skeleton $\mathrm{sk}(\mathcal{C}_{\mathcal{M}}^*)$ of $\mathcal{C}_{\mathcal{M}}^*$ becomes isomorphic to $\mathcal{V}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$, which is equivalent to $\mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$. Therefore we can conclude with:

Theorem 2.3 [Naidu 2007, Theorem 4.9]. *The fusion categories*

$$
\mathcal{C}_{\mathcal{M}}^* = \mathcal{V}(G, \omega)_{\mathcal{M}(A \backslash G, \mu)}^* \quad \text{and} \quad \mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega})
$$

are equivalent.

Applying the results of Section 1I we have:

Corollary 2.4. *The categories* $\mathrm{Vect}(A \rtimes_F K, \omega)$ *and* $\mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$ *are weakly Morita equivalent. Hence their centers*

$$
\mathcal{Z}(\mathrm{Vect}(A \rtimes_F K, \omega)) \simeq \mathcal{Z}(\mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega}))
$$

are canonically equivalent as braided tensor categories.

3. Weak Morita equivalence classes of group-theoretical tensor categories

We are interested in classifying group-theoretical tensor categories of a specific global dimension up to weak Morita equivalence. For this purpose we will fix the group $G = A \rtimes_F K$ with A abelian and normal in G and $F \in Z^2(K, A)$, and we will give an explicit description of the cocycles $\omega \in Z^3(A \rtimes_F K, \mathbb{C}^*)$ and $\hat{\omega} \in Z^3(K \ltimes_\nu \mathbb{A}, \mathbb{C}^*)$ such that the tensor categories $\mathcal{V}(A \rtimes_F K, \omega)$ and $\mathcal{V}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$ are weakly Morita equivalent.

3A. *Description of ω, μ and γ.* In Theorem 2.1 and in Section 2B we have seen the conditions needed for the tensor category $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G, \omega)^*_{\mathcal{M}(A \backslash G, \mu)}$ to be pointed. In particular we have seen that we need the existence of $\gamma \in C^1(K, C^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$ such that

$$\delta_G \gamma = \delta_K \mu.$$

Since we also have that $\delta_G \mu^{-1} = \pi^* \omega$ we can obtain the following lemma:

Lemma 3.1. *In $\mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*))))$, the cocycles $\pi^* \omega$ and $\tilde{\nu}$ are cohomologous.*

Proof. Recall the definition of the double complex $C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*)))$ given in Section 1B1, and note that we have $\pi^* \omega \in C^{0,3}$, $\mu \in C^{0,2}$, $\gamma \in C^{1,1}$ and $\tilde{\nu} = \delta_K \gamma \in C^{2,1}$, satisfying $\pi^* \omega \cdot \delta_G \mu = 1$ and $\delta_K \mu \cdot \delta_G \gamma^{-1} = 1$.

Consider the element $\mu \oplus \gamma \in \mathrm{Tot}^2$ and note that

$$\delta_{\mathrm{Tot}}(\mu \oplus \gamma) = (\delta_K \oplus \delta_G^{(-1)^p})(\mu \oplus \gamma) = \delta_G \mu \oplus \delta_K \mu \cdot \delta_G \gamma^{-1} \oplus \delta_K \gamma.$$

Therefore $\pi^* \omega \cdot \delta_{\mathrm{Tot}}(\mu \oplus \gamma) = \tilde{\nu}$. $\square$

Lemma 3.1 implies further conditions on the cohomology class of ω for the tensor category $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G, \omega)^*_{\mathcal{M}(A \backslash G, \mu)}$ to be pointed.

Corollary 3.2. *If the tensor category $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G, \omega)^*_{\mathcal{M}(A \backslash G, \mu)}$ is pointed then ω is cohomologous to a cocycle that lives in $C^{2,1} \oplus C^{3,0}$ of the double complex that induces the Lyndon–Hochschild–Serre spectral sequence.*

Remark 3.3. Note that this implies that the cohomology class of ω belongs to the subgroup of $H^3(G, \mathbb{C}^*)$ defined as

$$\Omega(G; A) := \ker(\ker(H^3(G, \mathbb{C}^*) \to E_\infty^{0,3}) \to E_\infty^{1,2}),$$

which fits into the short exact sequence

$$1 \to E_\infty^{3,0} \to \Omega(G; A) \to E_\infty^{2,1} \to 1.$$

The cohomology classes in $\Omega(G; A)$ are the only cohomology classes such that $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G, \omega)^*_{\mathcal{M}(A \backslash G, \mu)}$ is pointed.

In what follows we will construct explicit representatives for ω and μ, but to do so we will start by constructing explicit 3-cocycles in $\mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*))))$ which appear in $\Omega(G; A)$. Let us start by determining the second differential $d_2 : E_2^{2,1} \to E_2^{4,0}$.

Lemma 3.4. *The second differential $d_2 : E_2^{2,1} \to E_2^{4,0}$ is isomorphic to the homomorphism*

$$H^2(K, \mathbb{A}) \to H^4(K, \mathbb{C}^*), \quad [\hat{F}] \mapsto [(\hat{F} \wedge F)^{-1}],$$

where $(\hat{F} \wedge F)(k_1, k_2, k_3, k_4) := \hat{F}(k_1, k_2)(F(k_3, k_4))$.

Proof. First recall that

$$E_2^{2,1} = H^2(K, H^1(G, \mathrm{Map}(K, \mathbb{C}^*))) \cong H^2(K, \mathrm{Hom}(A, \mathbb{C}^*)) = H^2(K, \mathbb{A}),$$

$$E_2^{4,0} = H^4(K, H^0(G, \mathrm{Map}(K, \mathbb{C}^*))) = H^4(K, \mathrm{Map}(K, \mathbb{C}^*)^G) \cong H^4(K, \mathbb{C}^*).$$

Take $\hat{F} \in Z^2(K, \mathbb{A})$ and use the map φ of Lemma 1.1 to lift this cocycle to $\varphi(\hat{F}) \in C^2(K, Z^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$; in coordinates:

$$
\begin{aligned}
\varphi(\hat{F})(k_1, k_2)(x_1, (a_2, x_2)) &= \hat{F}(k_1, k_2)(\kappa_{x_1, (a_2, x_2)}) \\
&= \hat{F}(k_1, k_2)(^{x_1}a_2 F(x_1, x_2)) \\
&= \hat{F}(k_1, k_2)(^{x_1}a_2) \hat{F}(k_1, k_2)(F(x_1, x_2)).
\end{aligned}
$$

Its boundary is

$$
\begin{aligned}
\delta_k \varphi(\hat{F})&(k_1, k_2, k_3)(x_1, (a_2, x_2)) \\
&= \hat{F}(k_2, k_3)(^{x_1}a_2 F(x_1, x_2)) \hat{F}(k_1 k_2, k_3)(^{x_1}a_2 F(x_1, x_2))^{-1} \\
&\quad \hat{F}(k_1, k_2 k_3)(^{x_1}a_2 F(x_1, x_2)) \hat{F}(k_1, k_2)(^{k_3 x_1}a_2 F(k_3 x_1, x_2))^{-1} \\
&= \hat{F}(k_1, k_2)^{k_3}(F(x_1, x_2)) \hat{F}(k_1, k_2)(F(k_3 x_1, x_2))^{-1} \\
&= \hat{F}(k_1, k_2) \left(\frac{F(k_3, x_1)}{F(k_3, x_1 x_2)} \right),
\end{aligned}
$$

and we can define $u \in C^3(K, C^0(G, \mathrm{Map}(K, \mathbb{C}^*)))$ as

$$u(k_1, k_2, k_3)(x) := \hat{F}(k_1, k_2)(F(k_3, x)).$$

On the one hand we have

$$
\begin{aligned}
\delta_G u(k_1, k_2, k_3)(x_1, (a_2, x_2)) &= u(k_1, k_2, k_3)(x_1 x_2) u(k_1, k_2, k_3)(x_1)^{-1} \\
&= \hat{F}(k_1, k_2) \left(\frac{F(k_3, x_1 x_2)}{F(k_3, x_1)} \right)
\end{aligned}
$$

and on the other

$$
\begin{aligned}
\delta_K u&(k_1, k_2, k_3, k_4)(x) \\
&= \hat{F}(k_2, k_3)(F(k_4, x)) \hat{F}(k_1 k_2, k_3)(F(k_4, x))^{-1} \hat{F}(k_1, k_2 k_3)(F(k_4, x)) \\
&\quad \hat{F}(k_1, k_2)(F(k_3 k_4, x))^{-1} \hat{F}(k_1, k_2)(F(k_3, k_4 x)) \\
&= \hat{F}(k_1, k_2)^{k_3}(F(k_4, x)) \hat{F}(k_1, k_2)(F(k_3 k_4, x))^{-1} \hat{F}(k_1, k_2)(F(k_3, k_4 x)) \\
&= \hat{F}(k_1, k_2)(F(k_3, k_4)).
\end{aligned}
$$

Since $\delta_G u = \delta_K \varphi(\hat{F})$ we have that

$$\delta_{\mathrm{Tot}}(\varphi(\hat{F}) \oplus u^{-1}) = \delta_K \varphi(\hat{F}) \delta_G u \oplus \delta_k u^{-1} = (\hat{F} \wedge F)^{-1};$$

therefore $d_2[\varphi(\hat{F})] = [(\hat{F} \wedge F)^{-1}]$. $\qquad \square$

Suppose that $d_2[\varphi(\hat{F})] = 0$; hence there is $\epsilon \in C^3(K, \mathbb{C}^*)$ such that $\delta_K \epsilon = \hat{F} \wedge F$. Define $\bar{\epsilon} \in C^3(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ by the equation

$$\bar{\epsilon}(k_1, k_2, k_3)(x) := \epsilon(k_1, k_2, k_3)$$

and note $\delta_K \bar{\epsilon} = \hat{F} \wedge F$ and $\delta_G \bar{\epsilon} = 1$. Hence the class $\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1} \in C^{2,1} \oplus C^{3,0}$ defines a 3-cocycle in the total complex:

$$\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1} \in Z^3 \mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*)))).$$

Define $\beta \in C^2(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ by the equation

$$\beta(k_1, k_2)(x) := \epsilon(k_1, k_2, x)$$

and note that

$$
\begin{aligned}
\delta_K \beta(k_1, k_2, k_3)(x) &= \epsilon(k_2, k_3, x)\epsilon(k_1 k_2, k_3, x)^{-1}\epsilon(k_1, k_2 k_3, x)\epsilon(k_1, k_2, k_3 x)^{-1} \\
&= \delta_K \epsilon(k_1, k_2, k_3, x)\epsilon(k_1, k_2, k_3)^{-1} \\
&= \hat{F}(k_1, k_2)((F(k_3, x))\bar{\epsilon}(k_1, k_2, k_3)(x)^{-1}.
\end{aligned}
$$

Therefore $\delta_K \beta \bar{\epsilon} u^{-1} = 1$; hence we have that the class $\varphi(\hat{F})\delta_G \beta \in C^{2,1}$ is a 3-cocycle in the total complex and moreover that it is cohomologous to the class $\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1}$, in coordinates:

$$
\begin{aligned}
(3\text{-}1) \quad (\varphi(\hat{F})\delta_G \beta)&(k_1, k_2)(x_1, (a_2, x_2)) \\
&= \hat{F}(k_1, k_2)(^{x_1}a_2)\hat{F}(k_1, k_2)(F(x_1, x_2))\epsilon(k_1, k_2, x_1 x_2)\epsilon(k_1, k_2, x_1)^{-1}.
\end{aligned}
$$

Summarizing the previous results:

Proposition 3.5. *Every cohomology class which appears in $\Omega(G; A)$ can be represented by a 3-cocycle $\varphi(\hat{F})\delta_G \beta \in C^{2,1}$ with $\hat{F} \in Z^2(K, \mathbb{A})$, $\beta(k_1, k_2)(x) = \epsilon'(k_1, k_2, x)$ and $\delta_K \epsilon' = \hat{F} \wedge F$.*

Proof. Take $[\omega] \in \Omega(G; A)$ and let $[\hat{F}] \in E_2^{2,1}$ be a representative of the cohomology class of the image of $[\omega]$ in $E_\infty^{2,1}$. Since $d_2[\varphi(\hat{F})] = 0$ we know that the cohomology class $[\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1}]$ constructed above belongs to $\Omega(G; A)$. Therefore we have

$$[\omega^{-1}] \cdot [\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1}] \in E_\infty^{3,0}.$$

Hence we can choose a representative cocycle $[\tau] \in H^3(K, \mathbb{C}^*) \cong E_2^{3,0}$ such that

$$[\omega] = [\varphi(\hat{F}) \oplus \bar{\epsilon}\bar{\tau} u^{-1}],$$

with $\bar{\tau} \in C^3(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ defined as

$$\bar{\tau}(k_1, k_2, k_3)(x) := \tau(k_1, k_2, k_3).$$

Let $\epsilon' := \epsilon\tau$ and define $\beta \in C^2(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ by the equation

$$\beta(k_1, k_2)(x) := \epsilon'(k_1, k_2, x).$$

Equation (3-1) implies that $\delta_K\beta = (\bar{\epsilon}\bar{\tau})^{-1}u$ and therefore the proposition follows from the equation

$$(\varphi(\hat{F}) \oplus \bar{\epsilon}\bar{\tau}u^{-1})\delta_{\mathrm{Tot}}\beta = \varphi(\hat{F})\delta_G\beta \oplus \delta_K\beta\bar{\epsilon}\bar{\tau}u^{-1} = \varphi(\hat{F})\delta_G\beta. \qquad \square$$

Now we need to find an explicit description of $\omega \in Z^3(G, \mathbb{C}^*)$ such that $\pi^*\omega$ and $\varphi(\hat{F})\delta_G\beta$ are cohomologous.

Theorem 3.6. *Let $G = A \rtimes_F K$ and consider $\omega \in C^3(G, \mathbb{C}^*)$, $\mu \in C^{0,2}$ and $\gamma \in C^{1,1}$ defined by the following equations*:

$$\omega((a_1, x_1), (a_2, x_2), (a_3, x_3)) := \hat{F}(x_1, x_2)(a_3)\epsilon(x_1, x_2, x_3),$$

$$\mu(x_1, (a_2, x_2), (a_3, x_3)) = (\hat{F}(x_1, x_2)(a_3)\epsilon(x_1, x_2, x_3))^{-1},$$

$$\gamma(y)(x_1, (a_2, x_2)) = \hat{F}(y, x_1)(a_2)\epsilon(y, x_1, x_2,).$$

Then $\pi^*\omega \cdot (\delta_{\mathrm{Tot}}\mu \oplus \gamma) = \varphi(\hat{F})\delta_G\beta$.

Proof. Let us calculate:

$$\delta_G\mu(x_1, (a_2, x_2), (a_3, x_3), (a_4, x_4))$$
$$= \mu(x_1x_2, (a_3, x_3), (a_4, x_4))\mu(x_1, (a_2^{x_2}a_3F(x_2, x_3), x_2x_3), (a_3, x_3))^{-1}$$
$$\mu(x_1, (a_2, x_2)(a_3^{x_3}a_4F(x_3, x_4), x_3x_4)) \mu(x_1, (a_2, x_2), (a_3, x_3))^{-1}$$
$$= \hat{F}(x_1x_2, x_3)(a_4)^{-1}\hat{F}(x_1, x_2x_3)(a_4)\hat{F}(x_1, x_2)(a_3^{x_3}a_4F(x_3, x_4))^{-1}$$
$$\hat{F}(x_1, x_2)(a_3)\epsilon(x_2, x_3, x_4)^{-1}\delta_K\epsilon(x_1, x_2, x_3, x_4)$$
$$= \hat{F}(x_2, x_3)(a_4)^{-1}\epsilon(x_2, x_3, x_4)^{-1},$$

and

$$\pi^*\omega(x_1, (a_2, x_2), (a_3, x_3), (a_4, x_4)) = \omega((a_2, x_2), (a_3, x_3), (a_4, x_4))$$
$$= \hat{F}(x_2, x_3)(a_4)\epsilon(x_2, x_3, x_4);$$

hence we have that $\delta_G\mu \cdot \pi^*\omega = 1$.

Now

$$\delta_K\mu(y)(x_1, (a_2, x_2), (a_3, x_3)) = \mu(x_1, (a_2, x_2), (a_3, x_3))\mu(yx_1, (a_2, x_2), (a_3, x_3))^{-1}$$
$$= \frac{\hat{F}(yx_1, x_2)(a_3)\epsilon(yx_1, x_2, x_3)}{\hat{F}(x_1, x_2)(a_3)\epsilon(x_1, x_2, x_3)},$$

and

$$\delta_G \gamma(y)(x_1, (a_2, x_2), (a_3, x_3))$$
$$= \gamma(y)(x_1 x_2, (a_3, x_3)) \gamma(y)(x_1, (a_2{}^{x_2} a_3 F(x_2, x_3), x_2 x_3))^{-1} \gamma(y)(x_1, (a_2, x_2))$$
$$= \hat{F}(y, x_1 x_2)(a_3) \hat{F}(y, x_1)(a_2{}^{x_2} a_3 F(x_2, x_3))^{-1} \hat{F}(y, x_1)(a_2)$$
$$\epsilon(y, x_1 x_2, x_3) \epsilon(y, x_1, x_2 x_3)^{-1} \epsilon(y, x_1, x_2)$$
$$= \hat{F}(y x_1, x_2)(a_3) \hat{F}(x_1, x_2)(a_3)^{-1} \epsilon(y x_1, x_2, x_3) \epsilon(x_1, x_2, x_3)^{-1};$$

hence we have that

$$\delta_K \mu \cdot \delta_G \gamma^{-1} = 1.$$

Finally we calculate

$$\delta_K \gamma(k_1, k_2)(x_1, (a_2, x_2))$$
$$= \gamma(k_2)(x_1, (a_2, x_2)) \gamma(k_1 k_2)(x_1, (a_2, x_2))^{-1} \gamma(k_1)(k_2 x_1, (a_2, x_2))$$
$$= \hat{F}(k_2, x_1)(a_2) \hat{F}(k_1 k_2, x_1)(a_2)^{-1} \hat{F}(k_1, k_2 x_2)(a_2)$$
$$\epsilon(k_2, x_1, x_2) \epsilon(k_1 k_2, x_1, x_2)^{-1} \epsilon(k_1, k_2 x_1, x_2)$$
$$= \hat{F}(k_1, k_2)({}^{x_1} a_2) \delta_K \epsilon(k_1, k_2, x_1, x_2) \epsilon(k_1, k_2, x_1 x_2) \epsilon(k_1, k_2, x_1)^{-1}$$
$$= \hat{F}(k_1, k_2)({}^{x_1} a_2) \hat{F}(k_1, k_2)(F(x_1, x_2)) \epsilon(k_1, k_2, x_1 x_2) \epsilon(k_1, k_2, x_1)^{-1},$$

and since by equation (3-1) we have that

$$(\varphi(\hat{F}) \delta_G \beta)(k_1, k_2)(x_1, (a_2, x_2))$$
$$= \hat{F}(k_1, k_2)({}^{x_1} a_2) \hat{F}(k_1, k_2)(F(x_1, x_2)) \epsilon(k_1, k_2, x_1 x_2) \epsilon(k_1, k_2, x_1)^{-1},$$

we have that

$$\delta_K \gamma = \varphi(\hat{F}) \delta_G \beta.$$

Hence $\pi^* \omega \cdot (\delta_{\text{Tot}} \mu \oplus \gamma) = \varphi(\hat{F}) \delta_G \beta.$ $\square$

3B. *Description of $\hat{\omega}$ and v.* Assuming the explicit descriptions of ω, μ and γ given in Theorem 3.6, we see that $\tilde{v} = \varphi(\hat{F}) \delta_G \beta$. Applying this explicit description of $\tilde{v}$ to the definition of v given in (2-3) and of $\hat{\omega}$ given in (2-4) we obtain

$$v(k_1, k_2)(a) := \tilde{v}(k_1, k_2)(1, (a, 1)) = \hat{F}(k_1, k_2)(a),$$

which implies that $v = \hat{F}$ and

$$\hat{\omega}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \tilde{v}(k_1, k_2)(1; (1, k_3)) \rho_1(F(k_2, k_3))$$
$$= \epsilon(k_1, k_2, k_3) \rho_1(F(k_2, k_3)).$$

After applying Corollary 2.4 to the previous explicit construction of $\hat{\omega}$ we obtain the following theorem:

Theorem 3.7. *Let K be a finite group acting on the finite abelian group A. Consider cocycles $F \in Z^2(K, A)$ and $\hat{F} \in Z^2(K, \mathbb{A})$ such that $\hat{F} \wedge F$ is trivial in cohomology, i.e., there exists $\epsilon \in C^3(K, \mathbb{C}^*)$ such that $\delta_K \epsilon = \hat{F} \wedge F$. Define the 3-cocycles $\omega \in Z^3(A \rtimes_F K, \mathbb{C}^*)$ and $\hat{\omega} \in Z^3(K \ltimes_{\hat{F}} \mathbb{A}, \mathbb{C}^*)$ by the equations*:

$$\omega((a_1, k_1), (a_2, k_2), (a_3, k_3)) := \hat{F}(k_1, k_2)(a_3)\epsilon(k_1, k_2, k_3)$$

$$\hat{\omega}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)).$$

Then the tensor categories $\mathrm{Vect}(A \rtimes_F K, \omega)$ and $\mathrm{Vect}(K \ltimes_{\hat{F}} \mathbb{A}, \hat{\omega})$ are weakly Morita equivalent, and therefore their centers are braided equivalent:

$$\mathcal{Z}(\mathrm{Vect}(A \rtimes_F K, \omega)) \simeq \mathcal{Z}(\mathrm{Vect}(K \ltimes_{\hat{F}} \mathbb{A}, \hat{\omega})).$$

Note that we may have taken a different choice of μ and γ in Section 3A thus producing different $\tilde{\nu}$ and $\hat{\omega}$. The description of $\hat{\omega}$ depends on the choice of cohomology class $[\hat{F}] \in H^2(K, \mathbb{A}) \cong E_2^{2,1}$ in the second page representing the image of $[\omega]$ in $E_3^{2,1} = E_\infty^{2,1}$. This choice may be changed by elements in the image of the second differential $d_2 : E_2^{0,2} \to E_2^{2,1}$.

Changing ω by a coboundary $\omega' = \omega\delta_G\alpha$, and writing ω' explicitly as

(3-2) $$\omega'((a_1, x_1), (a_2, x_2), (a_3, x_3)) := \hat{F}'(x_1, x_2)(a_3)\epsilon'(x_1, x_2, x_3)$$

produces a $\hat{\omega}'$ which becomes

(3-3) $$\hat{\omega}'((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon'(k_1, k_2, k_3)\rho_1(F(k_2, k_3)).$$

Applying Theorem 3.7 and using the equivalence of categories

$$\mathrm{Vect}(A \rtimes_F K, \omega) \simeq \mathrm{Vect}(A \rtimes_F K, \omega')$$

we obtain that the tensor categories $\mathrm{Vect}(A \rtimes_F K, \omega)$ and $\mathrm{Vect}(K \ltimes_{\hat{F}'} \mathbb{A}, \hat{\omega}')$ are also weakly Morita equivalent. The previous argument permits us to conclude the following corollary:

Corollary 3.8. *Suppose that the fusion category $\mathcal{C}_\mathcal{M}^* = \mathcal{V}(A \rtimes_F K, \omega)_{\mathcal{M}(K,\mu)}^*$ is pointed. Then it is equivalent to the category $\mathrm{Vect}(K \ltimes_{\hat{F}'} \mathbb{A}, \hat{\omega}')$, where $\hat{\omega}'$ and ω' are the cocycles defined in (3-2) and (3-3) respectively and ω' is cohomologous to ω.*

3C. *Classification theorem.* Now we are ready to state the key result in order to establish the weak Morita equivalence classes of group theoretical tensor categories.

Theorem 3.9. *Let H and $\hat{H}$ be finite groups, $\eta \in Z^3(H, \mathbb{C}^*)$ and $\hat{\eta} \in Z^3(\hat{H}, \mathbb{C}^*)$. Then the tensor categories $\mathrm{Vect}(H, \eta)$ and $\mathrm{Vect}(\hat{H}, \hat{\eta})$ are weakly Morita equivalent if and only if the following conditions are satisfied*:

- *There exist isomorphisms of groups*

$$\phi : G = A \rtimes_F K \xrightarrow{\cong} H, \quad \hat{\phi} : \hat{G} = K \ltimes_{\hat{F}} \mathbb{A} \xrightarrow{\cong} \hat{H}$$

  for some finite group K acting on the abelian group A, with $F \in Z^2(K, A)$ and $\hat{F} \in Z^2(K, \mathbb{A})$ where $\mathbb{A} := \mathrm{Hom}(A, \mathbb{C}^*)$.

- *There exists $\epsilon : K^3 \to \mathbb{C}^*$ such that $\hat{F} \wedge F = \delta_K \epsilon$.*

- *The cohomology classes satisfy the equations $[\phi^* \eta] = [\omega]$ and $[\hat{\phi}^* \hat{\eta}] = [\hat{\omega}]$ with*

$$\omega((a_1, k_1), (a_2, k_2), (a_3, k_3)) := \hat{F}(k_1, k_2)(a_3)\epsilon(k_1, k_2, k_3),$$

$$\hat{\omega}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)).$$

Proof. Suppose that $\mathrm{Vect}(H, \eta)$ and $\mathrm{Vect}(\hat{H}, \hat{\eta})$ are weakly Morita equivalent. Then $\mathrm{Vect}(\hat{H}, \hat{\eta})$ is equivalent to the dual category $\mathcal{V}(H, \eta)^*_{\mathcal{M}(A \backslash H, \mu)}$ with $K := A \backslash H$, $\phi : G = A \rtimes_F K \xrightarrow{\cong} H$ and $\mathcal{M}(A \backslash H, \mu)$ some module category of $\mathcal{V}(H, \eta)$. By Corollary 3.8 the tensor category $\mathrm{Vect}(\hat{H}, \hat{\eta})$ is furthermore equivalent to $\mathrm{Vect}(K \ltimes_{\hat{F}'} \mathbb{A}, \hat{\omega}')$, where ω' and $\hat{\omega}'$ are the cocycles defined in equations (3-2) and (3-3) respectively, and such that ω' is cohomologous to $\phi^* \eta$. In particular we have that $\hat{\phi} : \hat{G} = K \ltimes_{\hat{F}} \mathbb{A} \xrightarrow{\cong} \hat{H}$ and that $\hat{\phi}^* \hat{\eta}$ is cohomologous to $\hat{\omega}'$.

The converse is the statement of Theorem 3.7. $\qquad\qquad\qquad\qquad\square$

In the case that both ω and $\hat{\omega}$ are cohomologically trivial, we conclude that $\mathrm{Vect}(A \rtimes_F K, 1)$ and $\mathrm{Vect}(K \ltimes_{\hat{F}} \mathbb{A}, 1)$ are weakly Morita equivalent if and only if the cohomology class $[\hat{F}] \in H^2(K, \mathbb{A})$ lies in the image of the second differential of the spectral sequence $d_2 : H^2(A, \mathbb{C}^*)^K \to H^2(K, \mathbb{A})$. This result was originally proved in [Davydov 2000, Corollary 6.2].

4. Examples

4A. *Pointed fusion categories of global dimension 4.* We can now calculate the weakly Morita equivalence classes of pointed fusion categories of global dimension 4.

For $G = \mathbb{Z}/4$ we have that $H^*(\mathbb{Z}/4, \mathbb{Z}) \cong \mathbb{Z}[u]/4u$ with $|u| = 2$ and that the nontrivial automorphism of $\mathbb{Z}/4$ maps u to $-u$; therefore

$$H^4(\mathbb{Z}/4, \mathbb{Z})/\mathrm{Aut}(\mathbb{Z}/4) = \langle u^2 \rangle = \mathbb{Z}/4.$$

For $G = (\mathbb{Z}/2)^2$ we have that

$$H^4((\mathbb{Z}/2)^2, \mathbb{Z}) \cong \ker(\mathrm{Sq}^1 : H^4((\mathbb{Z}/2)^2, \mathbb{F}_2) \to H^5((\mathbb{Z}/2)^2, \mathbb{F}_2)) = \langle x^4, x^2 y^2, y^4 \rangle,$$

where $H^*((\mathbb{Z}/2)^2, \mathbb{F}_2) = \mathbb{F}_2[x, y]$ and Sq^1 is the Steenrod operation, and up to automorphisms of $(\mathbb{Z}/2)^2$ we get

$$H^4((\mathbb{Z}/2)^2, \mathbb{Z})\,/\,\mathrm{Aut}((\mathbb{Z}/2)^2) = \begin{cases} 0, \\ (x^4) = \{x^4, y^4, x^4 + y^4\}, \\ (x^2 y^2) = \{x^2 y^2, x^2 y^2 + x^4, x^2 y^2 + y^4\}, \\ (x^4 + x^2 y^2 + y^4) = \{x^4 + x^2 y^2 + y^4\}. \end{cases}$$

Since we have a clear description for a base of $H^4((\mathbb{Z}/2)^2, \mathbb{Z})$, we will abuse notation and denote with the symbols of $H^4((\mathbb{Z}/2)^2, \mathbb{Z})$ the elements of $H^3((\mathbb{Z}/2)^2, \mathbb{C}^*)$. With this clarification, the relevant terms of the second page of the LHS spectral sequence of the extension $1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$ become

3	$\mathbb{Z}/2 = \langle y^4 \rangle$				
2	0	0			
1	$\mathbb{Z}/2$	$\mathbb{Z}/2 = \langle yx \rangle$	$\mathbb{Z}/2 = \langle yx^2 \rangle$		
0	$\mathbb{C}^*$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 = \langle x^4 \rangle$	0
	0	1	2	3	4

where the second differential is defined by the assignment $d_2(yx^k) = \mathrm{Sq}^1(x^{k+2})$ with the class x^2 classifying the extension. We conclude that the only weak Morita equivalence that appears, which does not come from an automorphism of a group, is

$$\mathrm{Vect}(\mathbb{Z}/4, 0) \simeq \mathrm{Vect}((\mathbb{Z}/2)^2, x^2 y^2).$$

Therefore we see that there are exactly seven weak Morita equivalence classes of pointed fusion categories of global dimension 4, namely the three for $\mathbb{Z}/4$:

$$\mathrm{Vect}(\mathbb{Z}/4, u^2), \quad \mathrm{Vect}(\mathbb{Z}/4, 2u^2), \quad \mathrm{Vect}(\mathbb{Z}/4, 3u^2);$$

the three for $(\mathbb{Z}/2)^2$:

$$\mathrm{Vect}((\mathbb{Z}/2)^2, 0), \quad \mathrm{Vect}((\mathbb{Z}/2)^2, x^4), \quad \mathrm{Vect}((\mathbb{Z}/2)^2, x^4 + y^4 + x^2 y^2);$$

and the one that we have just constructed

$$\mathrm{Vect}(\mathbb{Z}/4, 0) \simeq_M \mathrm{Vect}((\mathbb{Z}/2)^2, x^2 y^2).$$

4B. *Nontrivial action of $\mathbb{Z}/2$ on $\mathbb{Z}/4$.* Consider the nontrivial action of $\mathbb{Z}/2$ on $\mathbb{Z}/4$ and the abelian extension $1 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/2 \to 1$. The group G is either the dihedral group D_8 in the case that the extension is a split extension or the quaternion group Q_8 in the case that the extension is a nonsplit extension.

In the case of D_8 the relevant elements of the second page of the LHS spectral sequence associated to the extension are

3	$\mathbb{Z}/4 = \langle a \rangle$			
2	0	0		
1	$\mathbb{Z}/2$	$\mathbb{Z}/2 = \langle e \rangle$	$\mathbb{Z}/2 = \langle b \rangle$	
0	$\mathbb{C}^*$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 = \langle c \rangle$	0
	0	1	2	3	4

and they all survive to the page at infinity. Since $H^3(D_8, \mathbb{C}^*) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ we may say that $H^3(D_8, \mathbb{C}^*) \cong \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$, and since $D_8 \cong \mathbb{Z}/4 \rtimes \mathbb{Z}/2$ we have that $F = 0$. The element $b \in H^2(\mathbb{Z}/2, \mathbb{Z}/4)$ defines the nontrivial extension $Q_8 \cong \mathbb{Z}/2 \ltimes_b \mathbb{Z}/4$.

The second page of the LHS spectral sequence of the extension $Q_8 \cong \mathbb{Z}/2 \ltimes_b \mathbb{Z}/4$ becomes

3	$\mathbb{Z}/4 = \langle \alpha \rangle$			
2	0	0		
1	$\mathbb{Z}/2$	$\mathbb{Z}/2 = \langle e \rangle$	$\mathbb{Z}/2 = \langle 4\alpha \rangle$	
0	$\mathbb{C}^*$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 = \langle c \rangle$	0
	0	1	2	3	4

where $d_2 : E_2^{1,1} \xrightarrow{\cong} E_2^{3,0}$ is an isomorphism and $H^3(Q_8, \mathbb{C}^*) = \langle \alpha \rangle = \mathbb{Z}/8$.

Therefore for these extensions we only have the weak Morita equivalences

$$\mathrm{Vect}(D_8, b) \simeq_M \mathrm{Vect}(Q_8, 0) \simeq_M \mathrm{Vect}(D_8, b \oplus c),$$

where the equivalence of the right is obtained from the fact that c does not survive the spectral sequence for the group Q_8, and the self-Morita equivalence

$$\mathrm{Vect}(Q_8, 4\alpha) \simeq_M \mathrm{Vect}(Q_8, 4\alpha).$$

4C. *Extension of $\mathbb{Z}/2 \times \mathbb{Z}/2$ by $\mathbb{Z}/2$.* Consider the nonabelian extensions of the form

$$1 \to \mathbb{Z}/2 \to G \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1,$$

namely D_8 and Q_8.

The second page of the LHS spectral sequence for these extensions becomes

3	$\mathbb{Z}/2$				
2	0	0			
1	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$		
0	$\mathbb{C}^*$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$
	0	1	2	3	4

and we need only to concentrate on the differentials $d_2 : E_2^{p,1} \to E_2^{p+2,0}$ between the first two rows since we know that $E_2^{0,3} = \mathbb{Z}/2$ survives the spectral sequence in all the groups.

First we will determine the differential $\bar{d}_2^G$ in the LHS spectral sequence for coefficients in the field of two elements $\mathbb{F}_2$. In this case

$$E_2 \cong H^*(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(\mathbb{Z}/2, \mathbb{F}_2) \cong \mathbb{F}_2[x, y, e],$$

and $\bar{d}_2^G e \in H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{F}_2)$ represents the class that defines the extension G. It is known that the class $x^2 + xy + y^2$ defines Q_8 [Adem and Milgram 1994, Lemma 2.10], the classes $x^2 + xy$, $xy + y^2$, xy define D_8 (p. 130 of the same book) and the classes x^2, y^2, $x^2 + y^2$ define $\mathbb{Z}/2 \times \mathbb{Z}/4$.

Second we use the fact that for the group $(\mathbb{Z}/2)^2$ we have the isomorphism

$$H^j((\mathbb{Z}/2)^2, \mathbb{Z}) \cong \ker(\mathrm{Sq}^1 : H^j((\mathbb{Z}/2)^2, \mathbb{Z}/2) \to H^{j+1}((\mathbb{Z}/2)^2, \mathbb{Z}/2)),$$

where Sq^1 is the first Steenrod square. This implies that the canonical map

$$H^j((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \to H^j((\mathbb{Z}/2)^2, \mathbb{C}^*)$$

can be seen as the map

$$H^j((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \xrightarrow{\mathrm{Sq}^1} \ker(\mathrm{Sq}^1 : H^{j+1}((\mathbb{Z}/2)^2, \mathbb{Z}/2) \longrightarrow$$
$$H^{j+2}((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \cong H^{j+1}((\mathbb{Z}/2)^2, \mathbb{Z}) \cong H^j((\mathbb{Z}/2)^2, \mathbb{C}^*).$$

Therefore the second differential

$$d_2^G : H^{p-2}((\mathbb{Z}/2)^2, \mathbb{Z}/2) \to H^p((\mathbb{Z}/2)^2, \mathbb{C}^*)$$

is isomorphic to the composite map

$$d_2^G : H^{p-2}((\mathbb{Z}/2)^2, \mathbb{Z}/2) \longrightarrow \ker(\mathrm{Sq}^1 : H^{p+1}((\mathbb{Z}/2)^2, \mathbb{Z}/2) \longrightarrow$$
$$H^{p+2}((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \cong H^{p+1}((\mathbb{Z}/2)^2, \mathbb{Z}) \cong H^p((\mathbb{Z}/2)^2, \mathbb{C}^*)$$

taking z to $\mathrm{Sq}^1(z \cup \bar{d}_2^G e)$.

Without loss of generality we may choose $\bar{d}_2^G e = xy + x^2$ for calculating the LHS spectral sequence for D_8. Applying the differential d_2^G to the elements 1, x, y, x^2, xy, y^2 we obtain that the surviving terms in the infinite page of the LHS spectral sequence for D_8 become

3	$\mathbb{Z}/2$				
2	0	0			
1	0	$\mathbb{Z}/2=\langle e(y)\rangle$	$\mathbb{Z}/2=\langle e(xy+x^2)\rangle$		
0	$\mathbb{C}^*$	$(\mathbb{Z}/2)^2=\langle x^2, y^2\rangle$	0	$(\mathbb{Z}/2)^2=\dfrac{\langle x^4, x^2y^2, y^4\rangle}{\langle x^2y^2+x^4\rangle}$	0
	0	1	2	3	4

Here we are abusing the notation and we are using the explicit base of $H^4((\mathbb{Z}/2)^2, \mathbb{Z})$ to denote the elements in $H^3((\mathbb{Z}/2)^2, \mathbb{C}^*)$. Since $E_3^{2,1} = \langle e(xy+x^2)\rangle$, the weak Morita equivalences that we obtain in the extension are

$$\mathrm{Vect}(D_8, 0) \simeq_M \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(xy+x^2))),$$
$$\mathrm{Vect}(D_8, x^4) \simeq_M \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(xy+x^2)) + x^4),$$
$$\mathrm{Vect}(D_8, y^4) \simeq_M \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(xy+x^2)) + y^4),$$

and the self-equivalence

$$\mathrm{Vect}(D_8, e(xy+x^2)) \simeq \mathrm{Vect}(D_8, e(xy+x^2)).$$

The surviving terms for Q_8 with $\bar{d}_2^G e = x^2 + xy + y^2$ are

3	$\mathbb{Z}/2$				
2	0	0			
1	0	0	$\mathbb{Z}/2 = \langle e(x^2+xy+y^2)\rangle$		
0	$\mathbb{C}^*$	$(\mathbb{Z}/2)^2 = \langle x^2, y^2\rangle$	0	$\mathbb{Z}/2 = \langle x^2y^2\rangle$	0
	0	1	2	3	4

with $E_\infty^{0,3} = \mathbb{Z}/2 = \langle\alpha\rangle$, $\langle x^2+xy+y^2\rangle = \langle 2\alpha\rangle$ and $\langle x^2y^2\rangle = \langle 4\alpha\rangle$, where α is a generator $\langle\alpha\rangle = H^3(Q_8, \mathbb{C}^*)$ that was defined in section Section 4B.

Hence the only Morita equivalences that we obtain are

$$\mathrm{Vect}(Q_8, 0) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(x^2 + xy + y^2)))$$
$$\mathrm{Vect}(Q_8, 4\alpha) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(x^2 + xy + y^2)) + x^2y^2)$$

and the self-Morita equivalences

$$\mathrm{Vect}(Q_8, 2\alpha) \simeq_M \mathrm{Vect}(Q_8, 2\alpha) \quad \text{and} \quad \mathrm{Vect}(Q_8, 6\alpha) \simeq_M \mathrm{Vect}(Q_8, 6\alpha).$$

Bundling up the previous results for the group Q_8 we obtain the following result:

Proposition 4.1. *Let us suppose that* $\mathrm{Vect}(Q_8, k\alpha)$ *is weakly Morita equivalent to* $\mathrm{Vect}(G, \eta)$. *Then*:

- *For k odd or* $k = 2, 6$, *the group G must be isomorphic to* Q_8 *and* η *must correspond to* $j\alpha$ *with j odd or* $j = 2, 6$.
- *For* $k = 4$, *G must be isomorphic to* Q_8 *or* $(\mathbb{Z}/2)^3$.
- *For* $k = 0$, *G must be isomorphic to* Q_8, D_8 *or* $(\mathbb{Z}/2)^3$.

Proof. First note the action of $\mathrm{Aut}(Q_8)$ on $H^3(Q_8, \mathbb{C}^*)$ is trivial. Second note the only normal subgroups of Q_8 are its center and the cyclic ones generated by roots of unity and that they all fit into the central extension $1 \to \mathbb{Z}/2 \to Q_8 \to (\mathbb{Z}/2)^2 \to 1$ or the nonsplit extension $1 \to \mathbb{Z}/4 \to Q_8 \to \mathbb{Z}/2 \to 1$ that we have studied before. Since any weak Morita equivalence between pointed fusion categories comes from a normal and abelian subgroup of Q_8, the classification that we have done before exhausts all possibilities. For k odd we know that $k\alpha$ survives to the restriction to the center and to the cyclic subgroups isomorphic to $\mathbb{Z}/4$ and therefore G can only be Q_8. The classes 2α and 6α trivialize on the center of Q_8 but these classes define extensions of $(\mathbb{Z}/2)^2$ by $\mathbb{Z}/2$ which are isomorphic to Q_8 and define cohomology classes which are precisely 2α and 6α. The class 4α trivializes in all normal and abelian subgroups; in the case of the subgroup $\mathbb{Z}/4$ the only group that may appear is Q_8, and in the case of the center we may obtain the weak Morita equivalence

$$\mathrm{Vect}(Q_8, 4\alpha) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \quad \mathrm{Sq}^1(e(x^2 + xy + y^2)) + x^2y^2).$$

Finally, the trivial class produces only the group D_8 in the case of the subgroup $\mathbb{Z}/4$ and $(\mathbb{Z}/2)^3$ in the case of the center; some weak Morita equivalences are

$$\mathrm{Vect}(Q_8, 0) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \quad \mathrm{Sq}^1(e(x^2 + xy + y^2))) \simeq_M \mathrm{Vect}(D_8, b). \qquad \square$$

References

[Adem and Milgram 1994] A. Adem and R. J. Milgram, *Cohomology of finite groups*, Grundlehren der Math. Wissenschaften **309**, Springer, Berlin, 1994. MR Zbl

[Bakalov and Kirillov 2001] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series **21**, American Mathematical Society, Providence, RI, 2001. MR Zbl

[Davydov 2000] A. A. Davydov, "Finite groups with the same character tables, Drinfeld algebras and Galois algebras", pp. 99–111 in *Algebra* (Moscow, 1998), edited by Y. Bahturin, de Gruyter, Berlin, 2000. MR Zbl arXiv

[Dijkgraaf et al. 1991] R. Dijkgraaf, V. Pasquier, and P. Roche, "Quasi Hopf algebras, group cohomology and orbifold models", *Nuclear Phys. B Proc. Suppl.* **18B** (1991), 60–72. MR Zbl

[Etingof et al. 2005] P. Etingof, D. Nikshych, and V. Ostrik, "On fusion categories", *Ann. of Math.* (2) **162**:2 (2005), 581–642. MR Zbl

[Etingof et al. 2011] P. Etingof, D. Nikshych, and V. Ostrik, "Weakly group-theoretical and solvable fusion categories", *Adv. Math.* **226**:1 (2011), 176–205. MR Zbl

[Evens 1991] L. Evens, *The cohomology of groups*, Oxford Univ. Press, 1991. MR Zbl

[Movshev 1993] M. V. Movshev, "Twisting in group algebras of finite groups", *Funktsional. Anal. i Prilozhen.* **27**:4 (1993), 17–23. In Russian; translated in *Funct. Anal. Appl.* **27**:4 (1993), 240–244. MR Zbl

[Müger 2003] M. Müger, "From subfactors to categories and topology, I: Frobenius algebras in and Morita equivalence of tensor categories", *J. Pure Appl. Algebra* **180**:1-2 (2003), 81–157. MR Zbl

[Naidu 2007] D. Naidu, "Categorical Morita equivalence for group-theoretical categories", *Comm. Algebra* **35**:11 (2007), 3544–3565. MR Zbl

[Ostrik 2003a] V. Ostrik, "Module categories over the Drinfel'd double of a finite group", *Int. Math. Res. Not.* **2003**:27 (2003), 1507–1520. MR Zbl

[Ostrik 2003b] V. Ostrik, "Module categories, weak Hopf algebras and modular invariants", *Transform. Groups* **8**:2 (2003), 177–206. MR Zbl

BERNARDO URIBE
DEPARTAMENTO DE MATEMÁTICAS Y ESTADÍSTICA
UNIVERSIDAD DEL NORTE
BARRANQUILLA
COLOMBIA
bjongbloed@uninorte.edu.co

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu