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Let $P : \Sigma \to S$ be a finite degree covering map between surfaces. Rafi and Schleimer showed that there is an induced quasi-isometric embedding $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ between the associated curve complexes. We define an operation on curves in $\mathcal{C}(\Sigma)$ using minimal intersection number conditions and prove that it approximates a nearest point projection to $\Pi(\mathcal{C}(S))$. We also approximate hulls of finite sets of vertices in the curve complex, together with their corresponding nearest point projections, using intersection numbers.

1. Overview

Let S be a closed, orientable, connected surface of genus $g \geq 0$ with $m \geq 0$ marked points whose *complexity* $\xi(S) := 3g - 3 + m$ is at least 2. The *curve complex* of S, denoted $\mathcal{C}(S)$, is the simplicial complex whose vertices are free isotopy classes of simple closed curves on S and whose simplices are spanned by multicurves. The curve complex has seen much activity in recent years due to its connections to mapping class groups, Teichmüller theory, and the geometry of hyperbolic 3-manifolds.

Given a finite degree covering map $P : \Sigma \to S$ and a simple closed curve $a \in \mathcal{C}(S)$, the preimage $P^{-1}(a)$ is a disjoint union of simple closed curves on Σ. Rafi and Schleimer, using techniques from Teichmüller theory, proved that the (one-to-many) *lifting operation* $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ defined by setting $\Pi(a) = P^{-1}(a)$ is a quasi-isometric embedding. In [Tang 2012], we give a new proof using results from hyperbolic 3-manifold geometry.

Theorem 1.1 [Rafi and Schleimer 2009]. *Let $P : \Sigma \to S$ be a finite degree covering map. Then the map $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ defined above is a Λ-quasi-isometric embedding, where Λ depends only on $\xi(\Sigma)$ and* deg P.

The primary aim of this paper is to give a combinatorial approximation of the nearest point projection map to the image of Π. We define an operation $\pi : \mathcal{C}(\Sigma) \to \Pi(\mathcal{C}(S)) \subseteq \mathcal{C}(\Sigma)$ as follows: Given a curve $\alpha \in \mathcal{C}(\Sigma)$, let $\pi(\alpha) = \Pi(b)$

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where b is a curve which has minimal intersection number with $P(\alpha)$ among all curves in $\mathcal{C}(S)$.

Theorem 6.1. *Let $P : \Sigma \to S$ be a finite degree covering map and suppose $\alpha \in \mathcal{C}(\Sigma)$ is a curve. Then $\pi(\alpha)$ is a uniformly bounded distance from any nearest point projection of α to $\Pi(\mathcal{C}(S))$ in $\mathcal{C}(\Sigma)$, where the bounds depend only on $\xi(\Sigma)$ and the degree of P.*

Proposition 6.2. *Assume further that P is regular, with deck group G. Then $\pi(\alpha)$ is a uniformly bounded distance from any circumcenter for the G-orbit of α in $\mathcal{C}(\Sigma)$. Moreover, the bounds depend only on $\xi(\Sigma)$ and the degree of P.*

The main tools we develop in order to prove our main results are descriptions of hulls in the curve complex using intersection number conditions, which may be of independent interest. These generalize Bowditch's [2006b] approximation of quasigeodesics in $\mathcal{C}(S)$ using intersection numbers, and Masur and Minsky's [1999] notion of balance time for a curve on a Teichmüller geodesic. Our results rely on the geometry of singular Euclidean surfaces used to estimate weighted intersection numbers. We state simplified versions of the relevant propositions below — see Section 5 for more precise formulations.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of distinct curves in $\mathcal{C}(S)$, where $n \geq 2$. Given a nonzero vector $\mathbf{t} = (t_1, \ldots, t_n)$ of nonnegative reals, let $\gamma_{\mathbf{t}} \in \mathcal{C}(S)$ be a curve which minimizes the weighted intersection number $\sum_i t_i i(\alpha_i, \cdot)$. Define the *hyperbolic hull* $\mathrm{Hull}(\boldsymbol{\alpha})$ to be the union of all geodesic segments in $\mathcal{C}(S)$ connecting a pair of points in $\boldsymbol{\alpha}$ (viewed as a vertex set in $\mathcal{C}(S)$).

Proposition 5.2. *The sets $\mathrm{Hull}(\boldsymbol{\alpha})$ and $\bigcup_{\mathbf{t}} \gamma_{\mathbf{t}}$ agree up to a uniformly bounded Hausdorff distance in $\mathcal{C}(S)$, where the union is taken over all nonzero $\mathbf{t} \in \mathbb{R}^n_{\geq 0}$. Moreover, the bound depends only on $\xi(S)$ and n.*

Proposition 5.5. *Suppose $\beta \in \mathcal{C}(S)$ is a curve satisfying $i(\alpha_i, \beta) \neq 0$ for all i. Let the **balance vector** $\mathbf{t}_\beta = (t_1, \ldots, t_n)$ of β with respect to $\boldsymbol{\alpha}$ be given by $t_i = i(\alpha_i, \beta)^{-1}$ for each i. Then $\gamma_{\mathbf{t}_\beta}$ is a uniformly bounded distance from any nearest point projection of β to $\mathrm{Hull}(\boldsymbol{\alpha})$ in $\mathcal{C}(S)$, where the bound depends only on $\xi(S)$ and n.*

Organization. We review the curve complex in Section 2, and some coarse geometric notions in Section 3, placing a particular emphasis on δ-hyperbolic spaces.

In Section 4, we introduce a generalization of Bowditch's [2006b] construction of singular Euclidean structures on surfaces on which the geodesic lengths of curves estimate suitable weighted intersection numbers. We verify in Section 7 that these surfaces satisfy a quadratic isoperimetric inequality and then apply a theorem of Bowditch to establish the existence of wide annuli.

In Section 5, we introduce two notions of hulls for finite sets in $\mathcal{C}(S)$: one arising geometrically in $\mathcal{C}(S)$; the other defined using intersection number conditions. We give proofs of Propositions 5.2 and 5.5 assuming bounded diameter properties for sets of curves satisfying certain bounded weighted intersection number conditions (Lemma 5.1) — a key fact whose proof we defer to Section 8. In Section 6, we utilize the results from Sections 3 and 5 to give proofs of the main theorems.

2. The curve complex

Let $S = (S, \Omega)$ denote a closed, orientable, connected surface of genus $g \geq 0$ together with a set Ω of $m \geq 0$ marked points. A *curve* on S is a continuous map $a : S^1 \to S - \Omega$. We will also write a for its image on S. A curve a is *simple* if it is an embedded copy of S^1. We call a curve *trivial* or *peripheral* if it is freely homotopic to a curve bounding a disc or a disc with exactly one marked point, respectively. A simple closed curve which is nontrivial and nonperipheral is called *essential*. A *multicurve* on S is a finite collection of nonparallel essential simple closed curves which can be realized disjointly simultaneously.

Let $\mathcal{C}^0(S)$ denote the set of free homotopy classes of essential simple closed curves on S. Unless explicitly stated otherwise, we will blur the distinction between curves and their free homotopy classes. In this paper, we assume that S has *complexity* $\xi(S) := 3g - 3 + m$ at least 2; modifications to the following definition are required for low-complexity cases but we shall not deal with them here. For an introduction to the curve complex, see [Schleimer 2005].

Definition 2.1. The *curve complex* of S, denoted $\mathcal{C}(S)$, is a simplicial complex whose vertex set is $\mathcal{C}^0(S)$ and whose simplices are spanned by multicurves. In particular, two distinct simple closed curves are connected by an edge in $\mathcal{C}(S)$ if and only if they have disjoint representatives on S.

For our purposes, it suffices to study the 1-skeleton $\mathcal{C}^1(S)$ of the curve complex, known as the *curve graph*. Indeed $\mathcal{C}^1(S)$ equipped with the induced path metric, denoted d_S, is naturally *quasi-isometric* to $\mathcal{C}(S)$ with the standard simplicial metric. To simplify notation, we shall write $\mathcal{C}(S)$ in place of $\mathcal{C}^1(S)$ and $\alpha \in \mathcal{C}(S)$ to denote a curve (or multicurve).

A finite collection of curves *fills* S if their complement is a disjoint union of discs each with at most one marked point. Note that a pair $\alpha, \beta \in \mathcal{C}(S)$ fill S if and only if $d_S(\alpha, \beta) \geq 3$. Given free homotopy classes of curves α and β, not necessarily simple, define their *(geometric) intersection number* $i(\alpha, \beta)$ to be the minimal value of $|a \cap b|$ over all representatives $a \in \alpha$ and $b \in \beta$ in general position on S.

Lemma 2.2 [Hempel 2001], [Schleimer 2005]. *Suppose α and β are curves in $\mathcal{C}(S)$. Then*

$$d_S(\alpha, \beta) \leq 2 \log_2 i(\alpha, \beta) + 2$$

whenever $i(\alpha, \beta) \neq 0$.

As an immediate corollary, we see that $\mathcal{C}(S)$ is connected (this was originally observed by Harvey [1981]). The curve graph is also locally infinite and has infinite diameter [Kobayashi 1988]. Masur and Minsky [1999] proved the following celebrated theorem regarding the large scale geometry of the curve graph:

Theorem 2.3. *Given any surface S with $\xi(S) \geq 2$, there exists $\delta > 0$ so that the curve graph $\mathcal{C}(S)$ is δ-hyperbolic.*

Bowditch [2006b] gives a combinatorial proof of hyperbolicity using intersection numbers. We will be extending many of the results established in his paper in Sections 4 and 5.

Theorem 2.4 [Bowditch 2014], [Aougab 2013], [Clay et al. 2014], [Hensel et al. 2015]. *The constant $\delta > 0$ in Theorem 2.3 can be chosen independently of S.*

Hensel, Przytycki and Webb in particular show that all geodesic triangles in $\mathcal{C}(S)$ possess 17-centers.

3. Coarse geometry

We now recall some basic notions concerning Gromov hyperbolic spaces. Most of the statements and results are either well known in the literature or are relatively straightforward to deduce. We refer the reader to [Bridson and Haefliger 1999], [Gromov 1987], [Alonso et al. 1991], and [Bowditch 2006a] for more background, and to [Tang 2013] for most of the proofs.

3A. ***Notation.*** Let $(\mathcal{X}, d)$ be a metric space. Given any subset $A \subseteq \mathcal{X}$ and a point $x \in \mathcal{X}$, we define $d(x, A) := \inf\{d(x, a) \mid a \in A\}$. For $\mathsf{r} \geq 0$, let

$$\mathcal{N}_\mathsf{r}(A) = \{x \in \mathcal{X} \mid d(x, A) \leq \mathsf{r}\}$$

denote the r-*neighborhood* of A in $\mathcal{X}$. For subsets $A, B \subseteq \mathcal{X}$ and $\mathsf{r} \geq 0$, write

$$A \subseteq_\mathsf{r} B \iff A \subseteq \mathcal{N}_\mathsf{r}(B)$$

and

$$A \approx_\mathsf{r} B \iff A \subseteq_\mathsf{r} B \text{ and } B \subseteq_\mathsf{r} A.$$

Define the *Hausdorff distance* between A and B to be

$$\operatorname{HausDist}(A, B) = \inf\{\mathsf{r} \geq 0 \mid A \approx_\mathsf{r} B\}.$$

To simplify notation, we will often write $a \in \mathcal{X}$ in place of a singleton set $\{a\} \subseteq \mathcal{X}$. If a and b are real numbers then write

$$a \approx_\mathsf{r} b \iff |a - b| \leq \mathsf{r}.$$

The *diameter* of $A \subseteq \mathcal{X}$ is defined to be

$$\operatorname{diam}(A) := \sup\{d(x, y) \mid x, y \in A\}.$$

We will also abbreviate $d(x, y)$ to xy if there is no chance of confusion.

3B. *Geodesics, quasiconvexity and quasi-isometries.* Let $I \subseteq \mathbb{R}$ be an interval. A *geodesic* is a map $\gamma : I \to \mathcal{X}$ so that $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in I$. A *geodesic segment* connecting points x and y in $\mathcal{X}$ is the image of a geodesic $\gamma : [0, d(x, y)] \to \mathcal{X}$ such that $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. A metric space $\mathcal{X}$ is called a *geodesic space* if every pair of points can be connected by a geodesic segment. A subset $U \subseteq \mathcal{X}$ is Q-*quasiconvex* if every geodesic segment connecting any pair of points in U lies in $\mathcal{N}_{\mathsf{Q}}(U)$. We say a subset is *quasiconvex* if it is Q-quasiconvex for some $\mathsf{Q} \geq 0$.

A (one-to-many) map $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces is a Λ-*quasi-isometric embedding* if for all $x_1, x_2 \in \mathcal{X}$ and $y_1 \in f(x_1),\ y_2 \in f(x_2)$ we have

$$d_{\mathcal{Y}}(y_1, y_2) \leq \Lambda d_{\mathcal{X}}(x_1, x_2) + \Lambda \quad \text{and} \quad d_{\mathcal{X}}(x_1, x_2) \leq \Lambda d_{\mathcal{Y}}(y_1, y_2) + \Lambda.$$

In addition, if $\mathcal{N}_{\Lambda}(f(\mathcal{X})) = \mathcal{Y}$ then f is called a Λ-*quasi-isometry* and we say that $\mathcal{X}$ and $\mathcal{Y}$ are Λ-*quasi-isometric*. If $\mathcal{X}$ and $\mathcal{Y}$ are Λ-quasi-isometric for some $\Lambda \geq 1$ then we may simply say that they are *quasi-isometric*.

3C. *Gromov hyperbolic spaces.* Throughout this paper, we shall use the *thin triangles* definition of δ-hyperbolicity which we now describe.

Let $(\mathcal{X}, d)$ be a geodesic space. Let $T = [x, y] \cup [y, z] \cup [z, x]$ be a geodesic triangle in $\mathcal{X}$ with corners at $x, y, z \in \mathcal{X}$. There exist unique *internal points* $o_x \in [y, z]$, $o_y \in [x, z]$, and $o_z \in [x, y]$ such that $xo_y = xo_z$, $yo_x = yo_z$, and $zo_x = zo_y$. The internal points cut T into three pairs of geodesic segments; each pair consists of two segments of equal length emanating from the same corner of T. We say that T is δ-*thin* if each pair of segments δ-fellow travel: for all $u \in [x, o_y]$ and $v \in [x, o_z]$ satisfying $xu = xv$, we have $uv \leq \delta$ (and similarly for the other two pairs). We say $\mathcal{X}$ is δ-*hyperbolic* if every geodesic triangle in $\mathcal{X}$ is δ-thin.

We will also use some equivalent notions of Gromov hyperbolicity:

Lemma 3.1 (four point condition, [Bridson and Haefliger 1999] Proposition 1.22). *Let $\mathcal{X}$ be a geodesic space. If $\mathcal{X}$ is δ-hyperbolic then*

$$xy + zw \leq \max\{xz + yw, xw + yz\} + 2\delta$$

for all $x, y, z, w \in \mathcal{X}$. Conversely, if this inequality holds for all points x, y, z and w in $\mathcal{X}$, then $\mathcal{X}$ is δ'-hyperbolic for some $\delta' \geq 0$ depending only on δ.

Suppose $\mathsf{k} \geq 0$. A k-*center* for a geodesic triangle $T \subseteq \mathcal{X}$ is a point in $\mathcal{X}$ which lies within a distance k of each side of T.

Lemma 3.2 [Bowditch 2006a, Proposition 6.13]. *Any geodesic triangle in a δ-hyperbolic space possesses a δ-center, namely, any of its internal points. Conversely, if $\mathcal{X}$ is a geodesic space for which there is some $\mathsf{k} \geq 0$ such that all geodesic triangles in $\mathcal{X}$ possess k-centers then $\mathcal{X}$ is δ-hyperbolic for some $\delta \geq 0$ depending only on k.*

3D. ***Nearest point projections to quasiconvex sets.*** Let $\mathcal{X}$ be a δ-hyperbolic space, and $U \subseteq \mathcal{X}$ be a closed, nonempty Q-quasiconvex subset. The set of *nearest point projections* of a point $x \in \mathcal{X}$ to U in $\mathcal{X}$ is

$$\mathrm{proj}_U(x) := \{p \in U \mid xp = d(x, U)\}.$$

Since U is closed, $\mathrm{proj}_U(x)$ is nonempty.

Lemma 3.3. *For all $x \in \mathcal{X}$, we have* $\mathrm{diam}(\mathrm{proj}_U(x)) \leq 2\delta + 2\mathsf{Q}$.

Lemma 3.4. *Given $x \in \mathcal{X}$, let $p \in \mathrm{proj}_U(x)$ be any nearest point projection. Then for all $u \in U$, $[x, u] \approx_{2\delta+\mathsf{Q}} [x, p] \cup [p, u]$ and $xu \approx_{2\delta+2\mathsf{Q}} xp + pu$.*

For $\mathsf{r} \geq 0$, call $q \in U$ an r-*entry point* of x to U if for every $u \in U$, all geodesics from x to u intersect $\mathcal{N}_\mathsf{r}(q)$. Let $\mathrm{entry}_U(x, \mathsf{r})$ denote the set of such points.

Lemma 3.5. *Let $\mathsf{r} \geq 0$. Then for all $x \in \mathcal{X}$, we have* $\mathrm{entry}_U(x, \mathsf{r}) \subseteq_{2\mathsf{r}} \mathrm{proj}_U(x)$. *Furthermore, if $\mathsf{r} \geq 2\delta + \mathsf{Q}$ then* $\mathrm{entry}_U(x, \mathsf{r}) \approx_{2\mathsf{r}} \mathrm{proj}_U(x)$.

We shall also need the fact that nearest point projections to quasiconvex sets are well behaved under quasi-isometric embeddings.

Lemma 3.6. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a Λ-quasi-isometric embedding of geodesic spaces, where $\mathcal{X}'$ is δ'-hyperbolic. Let C be a Q-quasiconvex subset of $\mathcal{X}$ and let $C' = f(C)$. Given a point $x \in \mathcal{X}$, let x' be a point in $f(x)$. Let p and q' be nearest point projections of x to C and x' to C' respectively. Let $q \in \mathcal{X}$ be a point satisfying $q' \in f(q)$. Then $p \approx_\mathsf{K} q$, where K depends only on δ', Λ and Q.*

Proof. First, note that $\mathcal{X}$ is δ-hyperbolic and C' is Q'-quasiconvex in $\mathcal{X}'$ for some constants $\delta = \delta(\Lambda, \delta')$ and $\mathsf{Q}' = \mathsf{Q}'(\mathsf{Q}, \Lambda, \delta)$. Let $c \in \mathcal{X}$ be a k-center for x, p and q, where $\mathsf{k} = \delta$. Any point $c' \in f(c)$ is then a k'-center for x', p' and q', where $\mathsf{k}' = \mathsf{k}'(\mathsf{k}, \Lambda)$ and $p' \in f(p)$. One can check that $xp \approx_{2\mathsf{k}} xc + cp$. By quasiconvexity of C, there is some point $y \in C$ satisfying $cy \leq \mathsf{k} + \mathsf{Q}$. Since p is a nearest point projection of x to C, we obtain

$$xc + cp - 2\mathsf{k} \leq xp \leq xy \leq xc + cy \leq xc + \mathsf{k} + \mathsf{Q},$$

which implies $cp \leq \mathsf{Q} + 3\mathsf{k}$. Similarly, we can deduce $c'q' \leq \mathsf{Q}' + 3\mathsf{k}'$. Since f is a Λ-quasi-isometric embedding, it follows that $cq \leq \Lambda \times c'q' + \Lambda$ and hence

$$pq \leq pc + cq \leq \mathsf{K},$$

where $\mathsf{K} = \mathsf{Q} + 3\mathsf{k} + \Lambda(\mathsf{Q}' + 3\mathsf{k}') + \Lambda$. □

3E. *Hyperbolic hulls.* Let $\mathcal{X}$ be a δ-hyperbolic space and suppose $U \subseteq \mathcal{X}$ is nonempty. The *hyperbolic hull* of U, denoted $\mathrm{Hull}(U)$, is the union of all geodesic segments in $\mathcal{X}$ connecting a pair of points in U.

Example 3.7. Let U be a finite subset of $\mathbb{H}^n$, where $n \geq 1$. Then $\mathrm{Hull}(U)$ is a uniformly bounded Hausdorff distance away from the convex hull of U in $\mathbb{H}^n$.

Lemma 3.8. *The hyperbolic hull* $\mathrm{Hull}(U)$ *is* 2δ*-quasiconvex. Furthermore, if* $C \subseteq \mathcal{X}$ *is a* Q*-quasiconvex set which contains* U *then* $\mathrm{Hull}(U) \subseteq_{\mathsf{Q}} C$.

In fact, these properties characterize $\mathrm{Hull}(U)$ up to finite Hausdorff distance.

Corollary 3.9. *Let* $C \subseteq \mathcal{X}$ *be a* Q*-quasiconvex set containing* U *with the following property: for any* Q'*-quasiconvex set* $C' \subseteq \mathcal{X}$ *also containing* U*, we have* $C \subseteq_{\mathsf{r}} C'$ *for some* $\mathsf{r} = \mathsf{r}(\mathsf{Q}, \mathsf{Q}') \geq 0$*. Then* $\mathrm{HausDist}(C, \mathrm{Hull}(U)) \leq \max\{\mathsf{Q}, \mathsf{r}(\mathsf{Q}, 2\delta)\}$.

3F. *Circumcenters.* Let U be a nonempty finite subset of a δ-hyperbolic space $\mathcal{X}$. The *radius* of U is

$$\mathrm{rad}(U) := \min\{\mathsf{r} \geq 0 \mid \text{there exists } x \in \mathcal{X}, U \subseteq \mathcal{N}_{\mathsf{r}}(x)\}.$$

Call $x \in \mathcal{X}$ a *circumcenter* of U if $U \subseteq \mathcal{N}_{\mathsf{r}}(x)$, where $\mathsf{r} = \mathrm{rad}(U)$, and write $\mathrm{circ}(U)$ for the set of circumcenters of U.

Lemma 3.10. *Suppose* $x \in \mathcal{X}$ *satisfies* $U \subseteq \mathcal{N}_{\mathsf{r}+\epsilon}(x)$*, where* $\mathsf{r} = \mathrm{rad}(U)$ *and* $\epsilon \geq 0$*. Then for any* $c \in \mathrm{circ}(U)$*, we have* $cx \leq 2\delta + 2\epsilon$ *and hence* $\mathrm{diam}(\mathrm{circ}(U)) \leq 2\delta$.

Lemma 3.11. *Let* $x, y \in U$ *be points such that* $xy \geq \mathrm{diam}(U) - 2\epsilon$*, for some* $\epsilon \geq 0$*. Let* m *be the midpoint of a geodesic segment* $[x, y]$*. Then* $c \approx_{2\delta+\epsilon} m$*, where* c *is any circumcenter of* U*. Furthermore, we have* $\mathrm{diam}(U) \leq 2\,\mathrm{rad}(U) \leq \mathrm{diam}(U) + 2\delta$.

We also give the following characterization of circumcenters of orbits under finite group actions on δ-hyperbolic spaces:

Lemma 3.12. *Assume* G *is a finite group acting by isometries on a* δ*-hyperbolic space* $\mathcal{X}$*. Fix a point* $x_0 \in \mathcal{X}$ *and let* c *be a circumcenter for* Gx_0*. Given a point* $z \in \mathcal{X}$*, let* p *be any of its nearest point projection to* $\mathrm{Hull}(Gx_0)$*. Then*

$$pc \leq \mathrm{rad}(Gz) + 7\delta$$

and hence

$$zc \leq \mathrm{rad}(Gz) + d(z, \mathrm{Hull}(Gx_0)) + 7\delta.$$

Proof. We first claim that p lies within a distance δ of a geodesic segment $[u, v]$, where $u, v \in Gx_0$ are points such that $uv \geq \mathrm{diam}(Gx_0) - 2\delta$. Suppose p lies on a geodesic segment $[x, y]$ for some x and y in Gx_0. There exist some x' and y' in Gx_0 such that $xx' = yy' = \mathrm{diam}(Gx_0)$. If $x' = y'$ then the claim follows from hyperbolicity. Now assume $x' \neq y'$. By Lemma 3.1, we have

$$2\,\mathrm{diam}(Gx_0) = xx' + yy' \geq \max\{xy + x'y', xy' + x'y\} \geq 2\,\mathrm{diam}(Gx_0) - 2\delta.$$

If $xy + x'y' \geq 2\,\mathrm{diam}(Gx_0) - 2\delta$, then $xy \geq \mathrm{diam}(Gx_0) - 2\delta$, which implies the claim. If not, then $xy' \geq \mathrm{diam}(Gx_0) - 2\delta$. The claim then follows by considering a geodesic triangle with x, y and y' as its vertices.

Now suppose $q \in [u, v]$ is a point such that $pq \leq \delta$. Then

$$d(z, [u, v]) \leq zq \leq zp + pq \leq d(z, \mathrm{Hull}(Gx_0)) + \delta \leq d(z, [u, v]) + \delta.$$

By considering a geodesic triangle with vertices u, v, and z, one can show that $q \approx_{3\delta} o$, where $o \in [u, v]$ is the internal point opposite z. Observe that

$$d(z, x_0) = d(gz, gx_0) \approx_{2\mathsf{D}} d(z, gx_0)$$

for all $g \in G$, where $\mathsf{D} := \mathrm{rad}(Gz) \geq \frac{1}{2}\,\mathrm{diam}(Gz)$. Therefore $zu \approx_{2\mathsf{D}} zv$ which implies $uo \approx_{2\mathsf{D}} ov$. It follows that $o \approx_{\mathsf{D}} m$, where m is the midpoint of $[u, v]$. Finally, applying Lemma 3.11 gives $p \approx_{\delta} q \approx_{3\delta} o \approx_{\mathsf{D}} m \approx_{3\delta} c$ and we are done. □

3G. *Almost fixed point sets.* Let G be a finite group acting by isometries on a δ-hyperbolic space $\mathcal{X}$. Given $\mathsf{R} \geq 0$, let

$$\mathrm{Fix}_{\mathcal{X}}(G, \mathsf{R}) := \{x \in \mathcal{X} \mid \mathrm{diam}(Gx) \leq \mathsf{R}\}$$

be the set of R-*almost fixed points* of G in $\mathcal{X}$.

Lemma 3.13. *The set* $\mathrm{Fix}_{\mathcal{X}}(G, 2\delta)$ *is nonempty. Moreover, if* $\mathsf{R} \geq \delta$ *then*

$$\mathrm{Fix}_{\mathcal{X}}(G, 2\mathsf{R}) \approx_{\mathsf{R}+\delta} \mathrm{Fix}_{\mathcal{X}}(G, 2\delta).$$

Proof. For any $x \in \mathcal{X}$, it is straightforward to check that $\mathrm{Fix}_{\mathcal{X}}(G, 2\delta)$ contains $\mathrm{circ}(Gx) \neq \varnothing$. Furthermore, if $x \in \mathrm{Fix}_{\mathcal{X}}(G, 2\mathsf{R})$ then, by Lemma 3.11, we have

$$xc \leq \mathrm{rad}(Gx) \leq \tfrac{1}{2}\,\mathrm{diam}(Gx) + \delta \leq \mathsf{R} + \delta,$$

and hence $\mathrm{Fix}_{\mathcal{X}}(G, 2\mathsf{R}) \subseteq_{\mathsf{R}+\delta} \mathrm{Fix}_{\mathcal{X}}(G, 2\delta)$. The reverse inclusion is immediate. □

Thus, to understand the geometry of $\mathrm{Fix}_{\mathcal{X}}(G, 2\mathsf{R})$, for $\mathsf{R} \geq \delta$, it suffices to study that of $\mathrm{Fix}_{\mathcal{X}}(G, 2\delta)$. One can also show that $\mathrm{Fix}_{\mathcal{X}}(G, 2\mathsf{R})$ is quasiconvex for $\mathsf{R} \geq \delta$.

Lemma 3.14. *Let* $\mathsf{R} \geq \delta$. *For any point* $x \in \mathcal{X}$, *let* c *be a circumcenter for its* G*-orbit, and* p *be any nearest point projection to* $\mathrm{Fix}_{\mathcal{X}}(G, 2\mathsf{R})$. *Then* $cp \leq 2\delta + 4\mathsf{R}$.

Proof. For all $g \in G$, we have

$$d(gx, p) \leq d(gx, gp) + d(gp, p) \leq d(x, p) + 2\mathsf{R} \leq d(x, c) + 2\mathsf{R} \leq \mathrm{rad}(Gx) + 2\mathsf{R}.$$

Applying Lemma 3.10 completes the proof. □

It is worth noting that when $\mathsf{R} < \delta$, Lemmas 3.13 and 3.14 need not hold: it is possible for $\mathrm{Fix}_{\mathcal{X}}(G, 2\mathsf{R})$ to lie very deeply inside $\mathrm{Fix}_{\mathcal{X}}(G, 2\delta)$, as shown by the example below. Recall that the point $(r, \theta, t) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}$ in cylindrical coordinates on $\mathbb{R}^3$ represents $(r\cos\theta, r\sin\theta, t) \in \mathbb{R}^3$ in Cartesian coordinates.

Example 3.15 (Rocketship). A *rocketship* of length $l > 0$ with $n \geq 2$ fins, denoted $\mathfrak{R} = \mathfrak{R}(n, l)$, is the union of the following sets defined using cylindrical coordinates:

- the *nose* $\mathfrak{N} = \{(t, \theta, t) \mid 0 \leq t \leq 1,\ \theta \in \mathbb{R}\}$, a right circular cone of height 1 and base radius 1;
- the *shaft* $\mathfrak{S} = \{(1, \theta, t) \mid 1 \leq t \leq l + 1,\ \theta \in \mathbb{R}\}$, a right circular cylinder of height l and base radius 1; and
- the *fins* $\mathfrak{F}_n = \{(1, 2k\pi/n, t) \mid t \geq l + 1,\ k \in \mathbb{Z}\}$, a disjoint union of n closed rays.

We endow $\mathfrak{R}$ with the induced path metric from $\mathbb{R}^3$ with the Euclidean metric. One can show that $\mathfrak{R}$ is quasi-isometric to a tree and hence δ-hyperbolic for some $\delta > 0$; this can be done by collapsing the radial component of the nose and shaft. Moreover, $\delta \geq \pi/2$ for l sufficiently large. The group $G = \mathbb{Z}/n\mathbb{Z}$ acts isometrically on $\mathfrak{R}$ by rotations about the t-axis through integral multiples of $2\pi/n$. For any $x \in \mathfrak{F}_n$, the circumcenters of Gx are points of the form $(1, (4k+1)\pi/2n, l+1)$, where $k \in \mathbb{Z}$. For $\mathsf{R} \geq 0$ sufficiently small, $\mathrm{Fix}_{\mathfrak{R}}(G, 2\mathsf{R})$ is contained in $\mathfrak{N}$. Therefore, $\mathrm{circ}(Gx)$ is at least a distance l away from $\mathrm{Fix}_{\mathfrak{R}}(G, 2\mathsf{R})$. Furthermore, $\mathrm{Fix}_{\mathfrak{R}}(G, 2\delta)$ contains both $\mathfrak{N}$ and $\mathfrak{S}$, and so its Hausdorff distance from $\mathrm{Fix}_{\mathfrak{R}}(G, 2\mathsf{R})$ is at least l.

4. Singular Euclidean structures

We now generalize Bowditch's [2006b] construction of singular Euclidean surfaces which are used to estimate weighted intersection numbers. Suppose $S = (S, \Omega)$ is a closed surface of genus g with a set of m marked points Ω such that $\xi(S) \geq 2$. Throughout this section, fix an n-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ of distinct multicurves in $\mathcal{C}(S)$. A vector $\mathbf{t} = (t_1, \ldots, t_n) \neq \mathbf{0}$ of nonnegative real numbers shall be referred to as a *weight vector*. Write $\mathbf{t} \cdot \boldsymbol{\alpha}$ for the formal sum $\sum_i t_i \alpha_i$. For simplicity, assume that $\boldsymbol{\alpha}$ fills S and that all entries of $\mathbf{t}$ are positive. The appropriate modifications for the nonfilling case shall be dealt with in Section 7A.

4A. ***Construction of $S(\mathbf{t} \cdot \boldsymbol{\alpha})$.*** Realize the multicurves α_i on S so that they intersect generally and pairwise minimally. The union of the α_i is a connected 4-valent graph Υ on S. The closure of each component of $S - \Upsilon$ is a polygon with at most one marked point. The polygons together with Υ give S the structure of a 2-dimensional cell complex. By taking the dual 2-cell structure, we obtain a tiling of S by rectangles which are in bijection with the self-intersection points of $\boldsymbol{\alpha}$. We will insist that any marked point of S coincides with a vertex of this tiling.

Each rectangle R corresponding to an intersection of α_i with α_j is isometrically identified with a Euclidean rectangle of side lengths t_i and t_j so that α_i is transverse to the two sides of length t_i. Each vertex in this tiling meeting $k \neq 4$ corners of rectangles becomes a singular point with cone angle $k\pi/2$. This gives a singular

Euclidean metric on S. We may arrange for each α_i to be locally geodesic by requiring $\alpha_i \cap R$ to be a straight line connecting the midpoints of opposite sides of R, for every rectangle R meeting α_i. Thus, each component of α_i is the core curve of an annulus of width t_i formed by taking the union of all rectangles R it meets.

The singular Euclidean surface defined above shall be denoted $S(\mathbf{t} \cdot \boldsymbol{\alpha})$. We remark that the metric depends on the realization of $\boldsymbol{\alpha}$ on S up to isotopy, however, any such choice will work equally well for our purposes.

We will allow representatives of a curve $\gamma \in \mathcal{C}(S)$ to meet marked points to speak of (locally) geodesic representatives. Say c is a *representative* of γ if there exists an embedded curve c' representing γ and a homotopy $\mathbf{F} : S^1 \times [0, 1] \to S$ such that $\mathbf{F}(\theta, 0) = c'(\theta)$, $\mathbf{F}(\theta, 1) = c(\theta)$ and $\mathbf{F}(S^1 \times \{t\}) \subseteq S - \Omega$ for all $0 \le t < 1$. A locally geodesic representative c of γ on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ may not necessarily be embedded. In these cases, there is a decomposition of the circle $S^1 = \cup I_k$ into a finite union of closed intervals with disjoint interiors so that $c : S^1 \to S(\mathbf{t} \cdot \boldsymbol{\alpha})$ sends each I_k to a straight line segment with endpoints at singular points or marked points.

By a *geodesic representative* of γ, we mean a curve representing γ attaining the minimal length among all representatives of γ. Geodesic representatives exist: there is a lower bound on the injectivity radius and distance between singular points on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$, and therefore there are only finitely many locally geodesic representatives of γ with length less than any given constant. We will use $l(\gamma)$ to denote the length of a geodesic representative of γ on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$.

For notational convenience, define a function on weight vectors by setting

$$\|\mathbf{t}\|_{\boldsymbol{\alpha}} := \sqrt{i(\mathbf{t} \cdot \boldsymbol{\alpha})},$$

where

$$i(\mathbf{t} \cdot \boldsymbol{\alpha}) = \sum_{j<k} t_j t_k i(\alpha_j, \alpha_k)$$

is the *self-intersection number* of $\mathbf{t} \cdot \boldsymbol{\alpha}$. This serves as a rescaling factor for the singular Euclidean surface $S(\mathbf{t} \cdot \boldsymbol{\alpha})$. We will extend intersection number linearly:

$$i(\mathbf{t} \cdot \boldsymbol{\alpha}, \gamma) := \sum_i t_i i(\alpha_i, \gamma).$$

Proposition 4.1. *The singular Euclidean surface $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ has the following properties*:

(1) $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ *has area* $\|\mathbf{t}\|_{\boldsymbol{\alpha}}^2 = \sum_{j<k} t_j t_k i(\alpha_j, \alpha_k)$.

(2) *For all curves $\gamma \in \mathcal{C}(S)$, we have*

$$l(\gamma) \le i(\mathbf{t} \cdot \boldsymbol{\alpha}, \gamma) \le \sqrt{2} l(\gamma).$$

(3) *There exists an essential annulus on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ whose width is at least* $\mathsf{W}_0 \|\mathbf{t}\|_{\boldsymbol{\alpha}}$, *where* $\mathsf{W}_0 > 0$ *is a constant depending only on* $\xi(S)$.

Here, the *width* of an annulus is the length of a shortest arc connecting its two boundary components. The first claim is immediate from the construction. The second claim shall be proven in Section 4B below; and the third in Section 7. It is worth mentioning that the third claim holds for a larger class of metrics satisfying a suitable isoperimetric inequality. The metric on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ can be approximated by a nonsingular Riemannian metric but we shall not need to do so.

4B. ***A grid structure on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$.*** A *quarter-translation surface* is a topological surface S with a finite set of singularities ς, together with an atlas of charts from $S-\varsigma$ to $\mathbb{R}^2$ whose transition maps are translations of $\mathbb{R}^2$ possibly composed with rotations through integral multiples of $\pi/2$. The singular points have cone angles which are integral multiples of $\pi/2$ and at least π.

Given a quarter-translation surface S, we may pull back the standard Euclidean metric on $\mathbb{R}^2$ to give a singular Euclidean metric on S. Geodesics which do not meet any singular points or marked points with respect to this metric can only self-intersect orthogonally. We can also define an L^1-metric on S by pulling back the metric given infinitesimally by $|dx|+|dy|$ on $\mathbb{R}^2$. We will work with the singular Euclidean metric unless otherwise specified. The following is immediate:

Lemma 4.2. *Let $l^2(\eta)$ and $l^1(\eta)$ denote, respectively, the Euclidean and L^1-lengths of a path η on S. Then $l^2(\eta)\le l^1(\eta)\le\sqrt{2}l^2(\eta)$.*

We may pull back the horizontal and vertical directions on $\mathbb{R}^2$ to give a preferred (unordered) pair of orthogonal directions on S defined away from the singular points. These shall be referred to as the *grid directions*. Geodesics which run parallel to a grid direction will be called *grid arcs*. Every nonsingular point on S has an open rectangular neighborhood, with sides parallel to the grid directions, on which the grid leaves restrict to give a pair of transverse foliations. Such a rectangle will be called an *open grid rectangle*.

It is straightforward to check that $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ is a quarter-translation surface. We will assume that the grid directions on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ run parallel to the sides of the rectangles used in its construction.

Lemma 4.3. *Given a curve $\gamma\in\mathcal{C}(S)$, let c be any of its geodesic representatives on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ with respect to the Euclidean metric. Then $l^1(c)=i(\mathbf{t}\cdot\boldsymbol{\alpha},\gamma)$.*

Proof. If c is embedded then we can isotope it to another geodesic representative meeting at least one singularity. Thus we can assume that S^1 decomposes as a finite union of intervals $\cup I_k$ with disjoint interiors such that $c:S^1\to S(\mathbf{t}\cdot\boldsymbol{\alpha})$ embeds each I_k as a straight line segment connecting singularities or marked points.

We can homotope c to a closed path $c':S^1\to S(\mathbf{t}\cdot\boldsymbol{\alpha})$ so that each $c'(I_k)$ is an edge-path in the 1-skeleton of $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ with the same endpoints as $c(I_k)$. The homotopy can be performed in a way which preserves the l^1-length of the path and without creating new intersection points with any of the α_i. One can check that c intersects each α_i minimally and thus the same is also true of c'. Finally, we deduce $l^1(c'(S^1)) = i(\mathbf{t}\cdot\boldsymbol{\alpha}, \gamma)$ by observing that every edge in the 1-skeleton of $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ transverse to α_i has length t_i. □

The second claim of Proposition 4.1 follows from the previous two lemmas.

5. Hulls in the curve complex

Let $S = (S, \Omega)$ be a connected compact surface S without boundary with a finite set of marked points Ω satisfying $\xi(S) \geq 2$. Throughout this section, we will fix an n-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ of distinct multicurves in $\mathcal{C}(S)$, where $n \geq 2$. We will assume that no pair α_i and α_j has a common component. We shall establish a coarse equality between two subsets of $\mathcal{C}(S)$ determined by $\boldsymbol{\alpha}$ — its hyperbolic hull $\mathrm{Hull}(\boldsymbol{\alpha})$, defined purely in terms of the geometry of $\mathcal{C}(S)$; and $\mathrm{Short}(\boldsymbol{\alpha}, \mathsf{L})$ which is defined using only intersection numbers. We also give a combinatorial method of approximating nearest point projections to $\mathrm{Hull}(\boldsymbol{\alpha})$.

5A. ***Short curve sets.*** Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of distinct multicurves in $\mathcal{C}^0(S)$, and $\mathbf{t} = (t_1, \ldots, t_n)$ be a weight vector. Given $\mathsf{L} \geq 0$, define

$$\mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L}) := \{\gamma \in \mathcal{C}(S) \mid i(\mathbf{t}\cdot\boldsymbol{\alpha}, \gamma) \leq \mathsf{L}\|\mathbf{t}\|_{\boldsymbol{\alpha}}\}.$$

If $\|\mathbf{t}\|_{\boldsymbol{\alpha}} = 0$ then this set is contained in the 1-neighborhood of $\boldsymbol{\alpha}$. Note that $\mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})$ remains invariant under multiplying $\mathbf{t}$ by a positive scalar. When $\boldsymbol{\alpha}$ fills S, the geodesic length of a curve γ on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ approximates its intersection number with $\mathbf{t}\cdot\boldsymbol{\alpha}$ (Proposition 4.1). (The same is also true in the nonfilling case — see Section 7A). Thus, we can view $\mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})$ as the set of bounded length curves on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ rescaled to have unit area.

Lemma 5.1. *There exists a constant* $\mathsf{L}_0 > 0$ *depending only on* $\xi(S)$ *such that, for any* $\mathsf{L} \geq \mathsf{L}_0$, *the set* $\mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})$ *is nonempty. Moreover,*

$$\mathrm{diam}_{\mathcal{C}(S)}(\mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})) \leq 4\log_2 \mathsf{L} + \mathsf{k}_0,$$

where k_0 *is a constant depending only on* $\xi(S)$.

Consequently, up to bounded error, we can view $\mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})$ as a single curve in $\mathcal{C}(S)$ which has minimal intersection number with $\mathbf{t}\cdot\boldsymbol{\alpha}$. The proof of this result will be given in Section 8, and largely follows the proof of Lemma 4.1 in [Bowditch 2006b].

5B. ***A hull via intersection numbers.*** For $\mathsf{L} \geq 0$, define the L-*short curve hull* of $\boldsymbol{\alpha}$ to be

$$\mathrm{Short}(\boldsymbol{\alpha}, \mathsf{L}) := \bigcup_{\mathbf{t}} \mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L}),$$

where the union is taken over all weight vectors $\mathbf{t} \in \mathbb{R}^n_{\geq 0}$ (or, equivalently, by choosing one representative from each projective class). Write $\mathrm{Hull}(\boldsymbol{\alpha}) \subseteq \mathcal{C}(S)$ for the hyperbolic hull of $\boldsymbol{\alpha}$ considered as a set of vertices in $\mathcal{C}(S)$.

Proposition 5.2. *Let $\mathsf{L} \geq \mathsf{L}_0$. Then for any n-tuple of multicurves $\boldsymbol{\alpha}$ in $\mathcal{C}(S)$,*

$$\mathrm{Short}(\boldsymbol{\alpha}, \mathsf{L}) \approx_{\mathsf{k}_1} \mathrm{Hull}(\boldsymbol{\alpha}),$$

where k_1 depends only on $\xi(S)$, n and L.

This is essentially an extension of Bowditch's coarse description of geodesics in $\mathcal{C}(S)$ using intersection numbers, which we now reformulate:

Lemma 5.3 [Bowditch 2006b, Proposition 6.2]. *Let $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2)$ be a pair of multicurves in $\mathcal{C}(S)$. Let $[\alpha_1, \alpha_2]$ denote any geodesic segment connecting α_1 and α_2 in $\mathcal{C}(S)$. Then for all $\mathsf{L} \geq \mathsf{L}_0$, we have*

$$\mathrm{Short}(\boldsymbol{\alpha}', \mathsf{L}) \approx_{\mathsf{k}_1'} [\alpha_1, \alpha_2],$$

where $\mathsf{k}_1' \geq 0$ depends only $\xi(S)$ and L.

Proof of Proposition 5.2. Applying Lemma 5.3 to all pairs of multicurves (α_i, α_j) in $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, we obtain the inclusion:

$$\mathrm{Hull}(\boldsymbol{\alpha}) \subseteq_{\mathsf{k}_1'} \mathrm{Short}(\boldsymbol{\alpha}, \mathsf{L}).$$

Let $\mathbf{t} = (t_1, \ldots, t_n)$ be a weight vector and assume, without loss of generality, that the quantity $t_j t_k i(\alpha_j, \alpha_k)$ is maximized when $\{j, k\} = \{1, 2\}$. Let $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2)$ and $\mathbf{t}' = (t_1, t_2)$. Since there are $n(n-1)/2$ distinct unordered pairs of indices $\{j, k\}$, it follows that

$$\|\mathbf{t}\|_{\boldsymbol{\alpha}}^2 = \sum_{j<k} t_j t_k i(\alpha_j, \alpha_k) \leq \frac{n(n-1)}{2} t_1 t_2 i(\alpha_1, \alpha_2) = \frac{n(n-1)}{2} \|\mathbf{t}'\|_{\boldsymbol{\alpha}'}^2.$$

Now let γ be a curve in $\mathrm{short}(\mathbf{t} \cdot \boldsymbol{\alpha}, \mathsf{L})$. Then

$$i(\mathbf{t}' \cdot \boldsymbol{\alpha}', \gamma) \leq i(\mathbf{t} \cdot \boldsymbol{\alpha}, \gamma) \leq \mathsf{L}\|\mathbf{t}\|_{\boldsymbol{\alpha}} \leq \mathsf{L}\sqrt{\frac{n(n-1)}{2}\|\mathbf{t}'\|_{\boldsymbol{\alpha}'}^2} \leq \frac{n\mathsf{L}}{\sqrt{2}}\|\mathbf{t}'\|_{\boldsymbol{\alpha}'},$$

which implies

$$\mathrm{short}(\mathbf{t} \cdot \boldsymbol{\alpha}, \mathsf{L}) \subseteq \mathrm{short}\Big(\mathbf{t}' \cdot \boldsymbol{\alpha}', \frac{n\mathsf{L}}{\sqrt{2}}\Big).$$

Invoking Lemma 5.3, we have

$$\mathrm{short}\Big(\mathbf{t}' \cdot \boldsymbol{\alpha}', \frac{n\mathsf{L}}{\sqrt{2}}\Big) \subseteq_{\mathsf{r}} [\alpha_1, \alpha_2] \subseteq \mathrm{Hull}(\boldsymbol{\alpha}),$$

where $\mathsf{r} \geq 0$ is some constant depending on n, L and $\xi(S)$. □

We can describe the above proof in terms of the geometry of $S(\mathbf{t}\cdot\boldsymbol{\alpha})$. Assume $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ has unit area. One can obtain $S(\mathbf{t}'\cdot\boldsymbol{\alpha}')$ by homotoping the annuli consisting of rectangles traversed by α_i to the core curve α_i for each $i \neq 1, 2$. The maximality assumption on α_1 and α_2 ensures that the total area of the remaining rectangles is at least $2/(n(n-1))$. Scale $S(\mathbf{t}'\cdot\boldsymbol{\alpha}')$ by a factor of at most $n/\sqrt{2}$ to give it unit area. This process scales the length of a curve γ on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ by a factor of at most $n/\sqrt{2}$.

5C. ***Nearest point projections to hulls.*** In this section, we approximate nearest point projections to short curve hulls using only intersection number conditions.

Definition 5.4. Let $\beta \in \mathcal{C}(S)$ be a multicurve. A weight vector $\mathbf{t} = (t_1, \dots, t_n)$ satisfying

$$t_j i(\alpha_j, \beta) = t_k i(\alpha_k, \beta)$$

for all j, k is called a *balance vector* for β with respect to $\boldsymbol{\alpha}$.

If β intersects all α_i then setting $t_i = i(\alpha_i, \beta)^{-1}$ yields the unique balance vector up to positive scale. If not, we can set $t_i = 1$ whenever $i(\alpha_i, \beta) = 0$ and $t_i = 0$ otherwise to produce a balance vector. Let $\mathbf{t}_\beta$ denote any balance vector for β. We also remark that the above definition is analogous to the notion of *balance time* for quadratic differentials as described by Masur and Minsky [1999].

The proof of the following will be given at the end of this section.

Proposition 5.5. *Assume* $\mathsf{L} \geq \mathsf{L}_0$. *Given a multicurve* $\beta \in \mathcal{C}(S)$, *let* γ *be any nearest point projection of* β *to* $\mathrm{Hull}(\boldsymbol{\alpha})$. *Then*

$$\gamma \approx_{\mathsf{k}_2} \mathrm{short}(\mathbf{t}_\beta \cdot \boldsymbol{\alpha}, \mathsf{L}),$$

where $\mathsf{k}_2 \geq 0$ *depends only on* $\xi(S)$, n *and* L.

As was the case with Proposition 5.2, this is an extension of a result of Bowditch. His result was originally phrased in terms of centers for geodesic triangles; however, our statement agrees with it up to uniformly bounded error.

Lemma 5.6 [Bowditch 2006b, Proposition 3.1 and Section 4]. *Let* α_1, α_2 *and* β *be multicurves in* $\mathcal{C}(S)$. *Let* $\mathbf{t}'_\beta$ *be a balance vector for* β *with respect to* $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2)$. *Let* γ *be a nearest point projection of* β *to* $[\alpha_1, \alpha_2]$. *Then*

$$\gamma \approx_{\mathsf{k}'_2} \mathrm{short}(\mathbf{t}'_\beta \cdot \boldsymbol{\alpha}', \mathsf{L}),$$

where k'_2 *depends only on* $\xi(S)$ *and* L.

If β is disjoint from some α_i then Proposition 5.5 follows immediately from Lemma 2.2. We will henceforth assume this is not the case. We reduce the problem of finding a nearest point projection to a hyperbolic hull to that of projecting to a suitable geodesic.

Lemma 5.7. *Let U be a subset of a δ-hyperbolic space $\mathcal{X}$. Fix a point $w \in \mathcal{X}$. Assume there exist $x, y \in U$ and $\mathsf{R} \geq 0$ such that*

$$d_{\mathcal{X}}([x, y], [z, w]) \leq \mathsf{R}$$

for all $z \in U$. Let p and q be nearest point projections of w to $\mathrm{Hull}(U)$ and $[x, y]$ respectively. Then

$$p \approx_{\mathsf{R}'} q,$$

where R' depends only on R and δ.

Proof. By Lemma 3.5, it suffices to show that for all $u \in \mathrm{Hull}(U)$, any geodesic $[w, u]$ must pass within a bounded distance of q. If u lies on a geodesic segment $[z, z']$ for some $z, z' \in U$ then $[w, u]$ must lie inside the 2δ-neighborhood of $[w, z]$ or $[w, z']$. Hence, we only need to bound $d(q, [w, z])$ for all $z \in U$ in terms of δ and R. Recall that geodesic segments are δ-quasiconvex. Choose points $v \in [x, y]$ and $v' \in [z, w]$ so that $vv' = d_{\mathcal{X}}([x, y], [z, w]) \leq \mathsf{R}$. Then

$$q \subseteq_{3\delta} [w, v] \subseteq_{\mathsf{R}+\delta} [w, v'] \subseteq [w, z],$$

where we have applied Lemma 3.4 for the first comparison. □

In order to exploit the above result, we recall yet another lemma of Bowditch:

Lemma 5.8 [Bowditch 2006b, Proposition 6.3]. *Suppose $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{C}(S)$ are multicurves which satisfy*

$$i(\alpha_1, \alpha_4)i(\alpha_2, \alpha_3) \leq \mathsf{r}\, i(\alpha_1, \alpha_2)i(\alpha_3, \alpha_4)$$

for some $\mathsf{r} > 0$. Then

$$d_S([\alpha_1, \alpha_2], [\alpha_3, \alpha_4]) \leq \mathsf{R},$$

where $\mathsf{R} \geq 0$ depends only on r and $\xi(S)$.

Proof of Proposition 5.5. Let $\mathbf{t}_\beta$ be a balance vector for β with respect to $\boldsymbol{\alpha}$. To simplify notation, assume $t_j t_k i(\alpha_j, \alpha_k)$ is maximized when $\{j, k\} = \{1, 2\}$. Then

$$t_2 t_j i(\alpha_2, \alpha_j) \leq t_1 t_2 i(\alpha_1, \alpha_2)$$

for any $j = 1, \ldots, n$. As β is assumed to intersect all the α_i, we have $t_i = i(\alpha_i, \beta)^{-1}$ (after rescaling) and so

$$i(\alpha_1, \beta)i(\alpha_2, \alpha_j) \leq i(\alpha_1, \alpha_2)i(\alpha_j, \beta).$$

Invoking Lemma 5.8 gives

$$d_S([\alpha_1, \alpha_2], [\alpha_j, \beta]) \leq \mathsf{R}.$$

Let γ_{12} and γ be nearest point projections of β to $[\alpha_1, \alpha_2]$ and $\mathrm{Hull}(\boldsymbol{\alpha})$ respectively. Applying Lemma 5.7 with $U = \boldsymbol{\alpha},\ x = \alpha_1,\ y = \alpha_2$, and $w = \beta$ gives

$$d_S(\gamma_{12}, \gamma) \le \mathsf{R}',$$

where R' depends only on $\xi(S)$.

Now suppose γ' is a curve in $\mathrm{short}(\mathbf{t}_\beta \cdot \boldsymbol{\alpha}, \mathsf{L})$. Using the same reasoning as for the proof of Proposition 5.2, we see that

$$\gamma' \in \mathrm{short}(\mathbf{t}_\beta \cdot \boldsymbol{\alpha}, \mathsf{L}) \subseteq \mathrm{short}\Big(\mathbf{t}'_\beta \cdot \boldsymbol{\alpha}', \frac{n\mathsf{L}}{\sqrt{2}}\Big),$$

where $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2)$ and $\mathbf{t}'_\beta = (t_1, t_2)$. By Lemma 5.6, we deduce that

$$d_S(\gamma', \gamma_{12}) \le \mathsf{k}'_2$$

for some k'_2 depending only on n, L and $\xi(S)$. The preceding inequalities give

$$d_S(\gamma', \gamma) \le \mathsf{R}' + \mathsf{k}'_2,$$

which concludes the proof of the proposition. □

6. Covering maps

6A. ***Operations on curves arising from covering maps.*** We first recall some definitions and notation. Let $P : \Sigma \to S$ be a finite degree covering map of surfaces. The preimage $P^{-1}(a)$ of a simple closed curve a on S under P is a multicurve on Σ. This induces a one-to-many *lifting map* $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ between curve complexes by setting $\Pi(a) := P^{-1}(a)$. Recall the following theorem of Rafi and Schleimer:

Theorem 1.1 [Rafi and Schleimer 2009]. *Let $P : \Sigma \to S$ be a finite degree covering map. Then the map $\Pi : \mathcal{C}(S) \to \mathcal{C}(\Sigma)$ defined above is a Λ-quasi-isometric embedding, where Λ depends only on $\xi(\Sigma)$ and* $\deg P$.

It immediately follows that $\Pi(\mathcal{C}(S))$ is quasiconvex in $\mathcal{C}(\Sigma)$. This naturally leads to the question of understanding nearest point projections to $\Pi(\mathcal{C}(S))$. Define an operation $\pi : \mathcal{C}(\Sigma) \to \Pi(\mathcal{C}(S))$ as follows: given a curve $\alpha \in \mathcal{C}(\Sigma)$, let $b \in \mathcal{C}(S)$ be a curve which has minimal intersection number with $P(\alpha)$ on S and set $\pi(\alpha) = \Pi(b)$.

Theorem 6.1. *Let $P : \Sigma \to S$ be a finite degree covering map, and let Π and π be as above. Given a curve $\alpha \in \mathcal{C}(\Sigma)$, let γ be a nearest point projection of α to $\Pi(\mathcal{C}(S))$ in $\mathcal{C}(\Sigma)$. Then $\pi(\alpha) \approx_{\mathsf{k}_3} \gamma$, where k_3 depends only on* $\deg P$ *and* $\xi(\Sigma)$.

Consequently, the operation $\alpha \mapsto \pi(\alpha)$ is coarsely well defined. The above will be proven in Section 6B, and the following in Section 6C.

Proposition 6.2. *Suppose further that P is regular, and let G be its group of deck transformations. Let γ' be a circumcenter of the G-orbit of a curve α in $\mathcal{C}(\Sigma)$. Then $\pi(\alpha) \approx_{\mathsf{k}_4} \gamma'$, where k_4 is some constant depending only on* $\deg P$ *and* $\xi(\Sigma)$.

Recall that the *deck transformation group* $\mathrm{Deck}(P)$ of a covering map $P : \Sigma \to S$ is the group of all homeomorphisms $f \in \mathrm{Homeo}(\Sigma)$ satisfying $P \circ f = P$. In order for the above statement to make sense, we must check that $\mathrm{Deck}(P)$ can be identified with its image in the *mapping class group* $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}(\Sigma)/\mathrm{Homeo}_0(\Sigma)$.

Lemma 6.3. *Suppose S has negative Euler characteristic, and let $P : \Sigma \to S$ be a finite degree covering map. Then the natural map* $\mathrm{Deck}(P) \to \mathrm{Mod}(\Sigma)$ *is injective.*

Proof. We will only give a sketch proof. Endow $\mathrm{int}(S)$ with a hyperbolic metric and pull it back to $\mathrm{int}(\Sigma)$ via P. The group $\mathrm{Deck}(P)$ then acts on $\mathrm{int}(\Sigma)$ by isometries. The result follows since any isometry of a hyperbolic surface isotopic to the identity must in fact coincide with the identity. □

Note, however, that this lemma does not hold for covers of the torus or annulus.

6B. *Nearest point projections.*

6B.1. *Regular covers.* We shall first deal with the case where $P : \Sigma \to S$ is a regular cover. Let $G = \mathrm{Deck}(P)$. Given a curve $\alpha \in \mathcal{C}(\Sigma)$, observe that the set of lifts of $P(\alpha)$ to Σ via P is exactly $G\alpha$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of curves whose entries are the lifts of $P(\alpha)$ in any order. Note that $n \geq 1$ is some divisor of $\deg P$. Let $\mathbf{1}$ denote the vector of length n with all entries equal to 1.

Lemma 6.4. *Let α and $\boldsymbol{\alpha}$ be as above. Then $\pi(\alpha) \in \mathrm{short}(\mathbf{1} \cdot \boldsymbol{\alpha}, \mathsf{L}_0|G|)$ where L_0 is a constant depending only on $\xi(\Sigma)$.*

Proof. Let b be a closed curve on S. Each point of $b \cap P(\alpha)$ on S lifts to exactly $|G| = \deg P$ points of $P^{-1}(b) \cap G\alpha$ on Σ via P, hence

$$i(P^{-1}(b), \boldsymbol{\alpha}) = |G| i(b, P(\alpha)).$$

By Lemma 5.1, there exists a curve $\gamma \in \mathcal{C}(\Sigma)$ such that

$$i(\gamma, \alpha) \leq \mathsf{L}_0 \|\mathbf{1}\|_{\boldsymbol{\alpha}}$$

for some constant $\mathsf{L}_0 = \mathsf{L}_0(\xi(\Sigma))$. Now assume b has minimal intersection with $P(\alpha)$ out of all curves on S. It follows that

$$i(P^{-1}(b), \boldsymbol{\alpha}) = |G| i(b, P(\alpha)) \leq |G| i(P(\gamma), P(\alpha)) = i(G\gamma, \boldsymbol{\alpha}) \leq |G| i(\gamma, \boldsymbol{\alpha}).$$

Finally, by combining the preceding inequalities, we see that

$$i(\pi(\alpha), \boldsymbol{\alpha}) = i(P^{-1}(b), \boldsymbol{\alpha}) \leq |G| i(\gamma, \boldsymbol{\alpha}) \leq |G| \mathsf{L}_0 \|\mathbf{1}\|_{\boldsymbol{\alpha}}.$$

Thus $\pi(\alpha) \in \mathrm{short}(\mathbf{1} \cdot \boldsymbol{\alpha}, \mathsf{L})$ for $\mathsf{L} = \mathsf{L}_0|G|$. □

Lemma 6.5. *Given $\gamma \in \Pi(\mathcal{C}(S))$, let β be any of its nearest point projections to* $\mathrm{Hull}(\boldsymbol{\alpha})$. *Then $d_\Sigma(\pi(\alpha), \beta) \le \mathsf{k}_5$, where k_5 depends only on* $\deg P$ *and* $\xi(\Sigma)$.

Proof. We may replace γ with the multicurve $G\gamma$ since their nearest point projections to $\mathrm{Hull}(\boldsymbol{\alpha})$ are a uniformly bounded distance apart. Since G acts transitively on $G\alpha$, it follows that $i(G\gamma, \alpha_i) = i(G\gamma, \alpha_j)$ for all i, j. Thus, $\mathbf{1}$ serves as a balance vector for $G\gamma$ with respect to $\boldsymbol{\alpha}$. By Proposition 5.5, we deduce that

$$\beta \approx_{\mathsf{k}_2} \mathrm{short}(\mathbf{1} \cdot \boldsymbol{\alpha}, \mathsf{L}),$$

where k_2 depends only on $\xi(\Sigma)$, n and $\mathsf{L} \ge \mathsf{L}_0$. Applying the previous lemma completes the proof. □

Proof of Theorem 6.1 for regular covers. Let α and $\boldsymbol{\alpha}$ be as above. Let γ be any curve in $\Pi(\mathcal{C}(S))$. Since $\mathrm{Hull}(\boldsymbol{\alpha})$ is quasiconvex, Lemmas 3.4 and 6.5 imply that any geodesic connecting α to γ in $\mathcal{C}(\Sigma)$ must pass within a distance r of $\pi(\alpha)$, where r depends only on $\deg P$ and $\xi(\Sigma)$. Therefore $\pi(\alpha)$ is an r-entry point of α to $\Pi(\mathcal{C}(S))$. Since $\Pi(\mathcal{C}(S))$ is also quasiconvex, Lemma 3.5 implies $\pi(\alpha)$ is a uniformly bounded distance away from any nearest point projection of α to $\Pi(\mathcal{C}(S))$. □

6B.2. *The general case.* The main obstacle in proving Theorem 6.1 for a nonregular cover $P : \Sigma \to S$ is the following: given a simple closed curve $\alpha \in \mathcal{C}(\Sigma)$ there may be some lifts of $P(\alpha)$ to Σ which are not simple. To address this issue, we pass to a suitable finite cover of Σ using a standard group theoretic argument.

Lemma 6.6. *Let $P : \Sigma \to S$ be a covering map of finite degree. Then there exists a cover $Q : \widehat{\Sigma} \to \Sigma$ such that $F := P \circ Q$ is regular and* $\deg F \le (\deg P)!$.

Proof. Let H be the finite index subgroup of $\Gamma = \pi_1(S)$ corresponding to the covering map P, and let H_0 be the intersection of all Γ-conjugates of H. It is straightforward to check that H_0 is exactly the kernel of the action of Γ on the set of left cosets of H by left multiplication. The desired result then follows. □

The covering map F defined above is universal in the sense that any regular cover of S which factors through P must also factor through F.

Lemma 6.7. *Let $P : \Sigma \to S$ and $F : \widehat{\Sigma} \to S$ be as above. If α is a simple closed curve on Σ then all lifts of $P(\alpha)$ to $\widehat{\Sigma}$ via F are simple.*

Proof. Any lift of α to $\widehat{\Sigma}$ via Q is also a simple lift of $P(\alpha)$ via F. Since F is regular, it follows that all other lifts of $P(\alpha)$ to $\widehat{\Sigma}$ are simple. □

Let $\Phi : \mathcal{C}(S) \to \mathcal{C}(\widehat{\Sigma})$ and $\Psi : \mathcal{C}(\Sigma) \to \mathcal{C}(\widehat{\Sigma})$ be the lifting maps induced by the covering maps F and Q respectively. Let $\phi : \mathcal{C}(\widehat{\Sigma}) \to \Phi(\mathcal{C}(S))$ be the projection map associated to F as described in Section 6A. We may assume $\phi \circ \Psi = \Psi \circ \pi$.

Proof of Theorem 6.1. Given $\alpha \in \mathcal{C}(\Sigma)$, let $\hat{\alpha}$ be any of its lifts to $\widehat{\Sigma}$ via Q. Note that $\phi(\hat{\alpha}) = \Psi(\pi(\alpha))$. Let $\hat{\gamma}$ be a nearest point projection of $\hat{\alpha}$ to $\Phi(\mathcal{C}(S))$ in $\mathcal{C}(\widehat{\Sigma})$ and let $\gamma = Q(\hat{\gamma}) \in \Pi(\mathcal{C}(S))$. Since F is regular, we can apply Theorem 6.1 for regular covers to deduce that

$$d_{\widehat{\Sigma}}(\phi(\hat{\alpha}), \hat{\gamma}) \leq \hat{\mathsf{k}}_3,$$

where $\hat{\mathsf{k}}_3$ depends only on deg F and $\xi(\widehat{\Sigma})$ which can in turn be bounded in terms of deg P and $\xi(\Sigma)$. By Theorem 1.1, Ψ is a Λ-quasi-isometric embedding, where $\Lambda = \Lambda(\deg F, \xi(\widehat{\Sigma}))$, and so

$$d_{\Sigma}(\pi(\alpha), \gamma) \leq \Lambda\hat{\mathsf{k}}_3 + \Lambda.$$

By Lemma 3.6, γ is a uniformly bounded distance away from any nearest point projection of α to $\Pi(\mathcal{C}(S))$ in $\mathcal{C}(\Sigma)$ and we are done. □

6C. *Circumcenters for regular covers.* Let $P : \Sigma \to S$ be a regular cover, and G its deck group. Given $\alpha \in \mathcal{C}(\Sigma)$, we show $\pi(\alpha)$ approximates circ$(G\alpha)$ in $\mathcal{C}(\Sigma)$.

Proof of Proposition 6.2. Since $\pi(\alpha)$ is a G-invariant multicurve, we deduce that rad$(G\pi(\alpha)) \leq 1$. Proposition 5.2 and Lemma 6.4 together give

$$d_{\Sigma}(\pi(\alpha), \mathrm{Hull}(G\alpha)) \leq \mathsf{k}_4',$$

where k_4' depends only on $\xi(\Sigma)$ and deg P. Finally, combining the above with Lemma 3.12 yields $d_{\Sigma}(\pi(\alpha), \mathrm{circ}(G\alpha)) \leq \mathsf{k}_4' + 7\delta + 1$ as desired. □

Observe that the vertices in $\mathrm{Fix}_{\mathcal{C}(\Sigma)}(G, 1)$ coincide exactly with those of $\Pi(\mathcal{C}(S))$. An immediate corollary of Theorem 6.1 and Proposition 6.2 is the following:

Corollary 6.8. *Any circumcenter for the G-orbit of a curve $\alpha \in \mathcal{C}(S)$ is within a uniformly bounded distance of any nearest point projection of α to $\Pi(\mathcal{C}(S))$.*

Therefore Lemma 3.14 still holds for $\mathrm{Fix}_{\mathcal{C}(\Sigma)}(G, 1)$, albeit with weaker control over the constants. As Example 3.15 demonstrates, this cannot be proven using purely synthetic methods assuming only δ-hyperbolicity of $\mathcal{C}(\Sigma)$. In conclusion: "There are no rocketships in the curve complex."

7. An isoperimetric inequality on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$

7A. *Constructing $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ for nonfilling curves.* We now generalize the construction of $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ to encompass nonfilling curves. Assume $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of distinct multicurves and $\mathbf{t} = (t_1, \ldots, t_n) \neq \mathbf{0}$ is a weight vector satisfying $\|\mathbf{t}\|_{\boldsymbol{\alpha}} \neq 0$. Realize $\boldsymbol{\alpha}$ minimally on S to form a 4-valent graph Υ on S.

Let $\Sigma \subseteq S$ be the (possibly disconnected) subsurface filled by Υ. This can be obtained by taking a closed regular neighborhood of Υ on S and then attaching all complementary regions which are discs with at most one marked point. If $\boldsymbol{\alpha}$ fills S then $\Sigma = S$. In general, Σ will be a disjoint union of surfaces $\Sigma_1 \cup \ldots \cup \Sigma_s$. Observe that $s \leq \xi(S)$ since we can find a multicurve on S so that exactly one component is contained in each Σ_k (by choosing a suitable subset of all curves appearing in $\boldsymbol{\alpha}$, for example). Some of these components may be annuli — this occurs precisely when a multicurve α_i has a component disjoint from all other α_j. All other components will have genus at least one, or are spheres where the sum of the number of marked points and boundary components is at least four.

We now define a 2-dimensional complex $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ as a quotient of S. Suppose Σ_k is an annular component of Σ whose core curve is a component of α_i. Identify Σ_k with $S^1 \times [0, t_i]$, then collapse the first coordinate to give a closed interval I_k of length t_i. Next, collapse every complementary component of Σ in S to a marked point. These marked points will be called *essential*. We then apply the construction from Section 4A to the image of each nonannular component of Σ in the quotient space. The resulting space is a finite collection of singular Euclidean surfaces and closed intervals identified along appropriate essential marked points. Note that this construction agrees with the one given in Section 4A for the case of filling curves. For brevity, call the image of a component of Σ a *component* of $S(\mathbf{t} \cdot \boldsymbol{\alpha})$.

Let c be a representative of a curve $\gamma \in \mathcal{C}(S)$ on S. Its image $\bar{c}$ on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ will be a closed curve or a union of paths connecting essential marked points. Define $l(\gamma)$ to be the minimal length of $\bar{c}$ over all representatives c of γ.

Proposition 7.1. *Suppose $\boldsymbol{\alpha}$ and $\mathbf{t}$ satisfy $\|\mathbf{t}\|_{\boldsymbol{\alpha}} > 0$. Then the first two claims of Proposition 4.1 hold for $S(\mathbf{t} \cdot \boldsymbol{\alpha})$.*

The proof of the above proceeds in the same manner as for the case of filling curves. It remains to prove an analogue of the third claim.

7B. ***An isoperimetric inequality.*** Let $S = (S, \Omega)$ be a closed singular Riemannian surface S with a finite set of marked points Ω. Let Δ be a closed disc and suppose $\iota : \Delta \to S$ is a piecewise smooth immersion which restricts to an embedding on its interior. Let D denote the image $\iota(\mathrm{int}(\Delta))$.

Definition 7.2. An open disc D arising in the above manner is called a *trivial region* on S if it contains at most one marked point. The boundary ∂D is an embedded Eulerian graph on S whose edges are piecewise smooth arcs. Define length(∂D) to be the sum of the lengths of these arcs using the metric on S.

Bowditch defines trivial regions as open discs on S containing at most one marked point without any conditions concerning piecewise smooth embeddings. Nevertheless, his proof of the following proposition still holds with our definition:

Proposition 7.3 [Bowditch 2006b]. *Suppose $f : [0, \infty) \to [0, \infty)$ is a homeomorphism. Let ρ be a singular Riemannian metric on an orientable closed surface S*

with unit area. Let Ω *be a finite set of marked points on* S. *We will assume* $|\Omega| \geq 5$ *whenever* S *is a* 2*-sphere. If* $\operatorname{area}(D) \leq f(\operatorname{length}(\partial D))$ *for any trivial region* D *then there is an essential annulus* $A \subseteq S - \Omega$ *such that* $\operatorname{width}(A) \geq \mathsf{W}_0$, *where* $\mathsf{W}_0 > 0$ *depends only on* $\xi(S)$ *and* f.

This section will be devoted to proving the following lemma which, together with the above proposition, implies the third claim of Proposition 4.1:

Lemma 7.4. *Suppose* D *is a trivial region on* $S(\mathbf{t} \cdot \boldsymbol{\alpha})$. *Then*

$$\operatorname{area}(D) \leq 4 \operatorname{length}(\partial D)^2.$$

Before launching into the details of the proof, we briefly outline our argument. First, we reduce the problem to that of studying embedded closed discs on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ whose boundary is a finite union of grid arcs. We then show that such a disc D can be tiled by grid rectangles. This tiling is dual to a collection of arcs on D, where each arc is parallel to a component of some $\alpha_i \cap D$. We call the union of all rectangles meeting a given arc a *band*. The key step is to observe that any two arcs in the collection intersect at most twice. Thus, the intersection of two distinct bands is the union of at most two rectangles arising from the tiling. Conversely, any rectangle from the tiling is contained in the intersection of two such bands. We can then bound the area of the rectangles in terms of the length of ∂D.

7B.1. *Technical adjustments.* Let us first make a couple of observations to simplify the problem.

Lemma 7.5. *Any trivial region* D *on* $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ *can be perturbed to a trivial region* D' *whose boundary is a finite union of grid leaves. Moreover*, D' *can be chosen so that* $\operatorname{area}(D') \geq \operatorname{area}(D)$ *and* $\operatorname{length}(\partial D') \leq \sqrt{2} \operatorname{length}(\partial D)$.

We will henceforth assume that the boundary of any trivial region on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ is a finite union of grid leaves.

Let $\iota : \Delta \to S(\mathbf{t} \cdot \boldsymbol{\alpha})$ be a piecewise smooth immersion whose restriction to $\operatorname{int}(\Delta)$ is an embedding with image D. Observe that $\iota : \partial\Delta = S^1 \to \partial D$ is an immersion of a circle which runs over each edge of ∂D at most twice. We will metrize Δ by pulling back the metric on $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ via ι.

Lemma 7.6. *Suppose* D *and* Δ *are as given above. Then* $\operatorname{area}(\Delta) = \operatorname{area}(D)$ *and* $\operatorname{length}(\partial D) \leq \operatorname{length}(\partial\Delta) \leq 2 \operatorname{length}(\partial D)$.

7B.2. *Tiling* Δ *by rectangles.* The disc Δ inherits grid directions from $S(\mathbf{t} \cdot \boldsymbol{\alpha})$ via ι away from the preimage of the singular points. The boundary decomposes as a finite union $\partial\Delta = \cup I_k$ of closed grid arcs with disjoint interiors. We may assume that this decomposition is minimal, that is, it cannot be obtained from any other such decomposition by subdividing arcs. An endpoint of any grid arc I_k will be called a *corner point* of $\partial\Delta$. A corner point which does not coincide with a singularity or a marked point must be an orthogonal intersection point of two grid arcs.

It is worth noting that $\partial\Delta$ must contain at least two corner points and at least three if D contains no marked points. To see this, recall that the grid leaves on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ are parallel to some α_i. Any of the forbidden cases will imply that some α_i is trivial, peripheral, self-intersects or does not intersect some α_j minimally.

Let us refer to marked points, corner points and singularities collectively as *bad points*. Let $Z \subset \Delta$ be the union of $\partial\Delta$ with all grid arcs in Δ which have a bad point for at least one of their endpoints. Since there are finitely many bad points in Δ, it follows that Z is a finite embedded graph on Δ. A vertex $v \in \operatorname{int}\Delta \cap Z$ has valence k if and only if the cone angle at v is $k\pi/2$. If v is a vertex which lies on $\partial\Delta$ then it has valence $k+1$ if and only if the cone angle at v inside Δ is $k\pi/2$. It follows that every vertex v of Z has valence at least 2, and at least 3 if v is not a marked point.

Lemma 7.7. *There exists a tiling of Δ by finitely many grid rectangles with Z as its* 1*-skeleton.*

Proof. First note that there are finitely many connected components of $\Delta - Z$ since Z is a finite graph. Let R be such a component and let $\bar{R}$ be its completion with respect to its induced path metric. Observe that $\bar{R}$ is a closed planar region admitting a Euclidean metric with piecewise geodesic boundary, where the interior angle between adjacent edges of $\partial\bar{R}$ is $\pi/2$. By the Gauss–Bonnet formula, the sum of its interior angles must equal $2\pi\chi(R)$. Since the frontier of R in Δ meets at least one vertex of Z, the angle sum must be strictly positive. As R is planar, it follows that $\chi(R) = 1$ and therefore $\bar{R}$ is a Euclidean rectangle. Also note that Z is connected, for otherwise there would exist some component of $\Delta - Z$ with disconnected frontier.

The inclusion $R \hookrightarrow \Delta$ can be extended continuously to a map $\bar{R} \to \Delta$, sending each edge of $\partial\bar{R}$ isometrically to an edge of Z meeting the frontier of R. Thus R is a grid rectangle since the edges of Z, by construction, are parallel to the grid directions. Finally, the closures of distinct rectangles R and R' can only intersect in a union of vertices and edges of Z. □

7B.3. *Controlling the area.* Let $\mathcal{A}$ be the set of maximal grid arcs in Δ which intersect Z only at midpoints of edges of Z. This is a collection of arcs dual to the rectangular tiling of Δ as described in Lemma 7.7. For any arc $a \in \mathcal{A}$, there is curve $\alpha \in \boldsymbol{\alpha}$ such that a can be properly isotoped in Δ to a component of $\iota^{-1}(\alpha \cap D)$ without passing through any singular points or marked points. (There cannot be any closed curves in Δ dual to the tiling as this would imply some α_i is not essential.) Let $B = B(a)$ be the union of all rectangles in the tiling which meet a. We will call B a *band* and a a *core arc* of B. Define $\mathrm{width}(B)$ to be the length of any edge of Z crossed by a. The set of bands in Δ is in bijection with $\mathcal{A}$.

Lemma 7.8. *The intersection of two distinct bands B and B' is the union of at most two rectangles whose side lengths are* $\mathrm{width}(B)$ *by* $\mathrm{width}(B')$. *Conversely, each rectangle in the tiling lies in the intersection of a unique pair of distinct bands.*

Proof. Let a and a' be core arcs of B and B' respectively. If a and a' intersect at least 3 times then they must bound a bigon in Δ containing no marked points. We can properly isotope a and a' in Δ to components of $\iota^{-1}(\alpha_i \cap D)$ and $\iota^{-1}(\alpha_j \cap D)$, for some α_i and α_j respectively, without passing through any singular points or marked points. Since any right-angled bigon on Δ must contain at least one singularity, it follows that α_i and α_j also bound a bigon in D, contradicting minimality.

For the converse, simply take the bands corresponding to the unique pair of arcs which have an intersection point inside the given rectangle. □

We will refer to an edge of Z lying in $\partial\Delta$ simply as an *edge* of $\partial\Delta$.

Lemma 7.9. *Let Δ be as above. Then*

$$\mathrm{area}(\Delta) \le \tfrac{1}{2}\,\mathrm{length}(\partial\Delta)^2.$$

Proof. By Lemma 7.8, Δ is a union of rectangles, each of which lies in the intersection of a pair of distinct bands. Thus

$$\mathrm{area}(\Delta) = \mathrm{area}\Big(\bigcup_{B\neq B'} B\cap B'\Big) = \sum_{B\neq B'} \mathrm{area}(B\cap B').$$

Since the intersection of two distinct bands is the union of at most two rectangles whose side lengths are equal to the widths of the bands, we have

$$\mathrm{area}(B\cap B') \le 2\,\mathrm{width}(B)\times\mathrm{width}(B'),$$

and hence

$$\mathrm{area}(\Delta) \le 2\sum_{B\neq B'} \mathrm{width}(B)\times\mathrm{width}(B') \le 2\Big(\sum_{B}\mathrm{width}(B)\Big)^2.$$

Finally, the desired result follows from observing that

$$\mathrm{length}(\partial\Delta) = 2\sum_{B}\mathrm{width}(B),$$

where the sum is taken over all bands B in Δ. □

Combining this with Lemmas 7.5 and 7.6 completes the proof of Lemma 7.4.

8. Proof of Lemma 5.1

Fix an n-tuple of distinct multicurves $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and a weight vector $\mathbf{t} = (t_1, \dots, t_n) \neq \mathbf{0}$. We show that for all $\mathsf{L} \geq \mathsf{L}_0$, where L_0 is to be determined,

$$\mathrm{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L}) = \{\gamma \in \mathcal{C}(S) \mid i(\mathbf{t}\cdot\boldsymbol{\alpha}, \gamma) \leq \mathsf{L}\|\mathbf{t}\|_{\boldsymbol{\alpha}}\}$$

is nonempty and has uniformly bounded diameter in $\mathcal{C}(S)$. If $\|\mathbf{t}\|_{\boldsymbol{\alpha}} = 0$ then short$(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})$ contains the α_i and is contained in the 1-neighborhood of $\boldsymbol{\alpha}$ in $\mathcal{C}(S)$, and we are done.

Assume $\|\mathbf{t}\|_{\boldsymbol{\alpha}} > 0$, and let Y be a component of $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ with maximal area. Since $S(\mathbf{t}\cdot\boldsymbol{\alpha})$ has at most $\xi(S)$ components, we have $\mathrm{area}(Y) \geq \|\mathbf{t}\|_{\boldsymbol{\alpha}}^2/\xi(S)$. Note that Y cannot be an interval since $\|\mathbf{t}\|_{\boldsymbol{\alpha}} > 0$. To simplify the exposition, we first prove Lemma 5.1 when Y has genus at least 1, or at least 5 marked points — the case where Y is a sphere with 4 marked points shall be dealt with in Section 8B.

8A. ***Case 1: Y has genus at least 1 or at least 5 marked points.*** By Proposition 7.3 and Lemma 7.4, there exists an essential annulus A on Y with

$$\mathrm{width}(A) \geq \mathsf{W}_0\sqrt{\mathrm{area}(Y)} \geq \frac{\mathsf{W}_0\|\mathbf{t}\|_{\boldsymbol{\alpha}}}{\sqrt{\xi(S)}},$$

where $\mathsf{W}_0 = \mathsf{W}_0(\xi(Y))$. Let $\gamma \in \mathcal{C}(S)$ be the core curve of A. Setting

$$\mathsf{W} = \mathsf{W}(\xi(S)) := \min_{1\leq k\leq \xi(S)} \frac{\mathsf{W}_0(k)}{\sqrt{\xi(S)}},$$

we have $\mathrm{width}(A) \geq \mathsf{W}\|\mathbf{t}\|_{\boldsymbol{\alpha}}$. Applying the Besicovitch Lemma [1952] (see Lemma $4.5\frac{1}{2}$ in [Gromov 1999] for a proof), we have

$$\mathrm{width}(A) \times \mathrm{length}(A) \leq \mathrm{area}(A),$$

where length(A) is the length of a shortest core curve on A. Since area(A) is at most $\mathrm{area}(S(\mathbf{t}\cdot\boldsymbol{\alpha})) = \|\mathbf{t}\|_{\boldsymbol{\alpha}}^2$, it follows that $l(\gamma) \leq \mathrm{length}(A) \leq \|\mathbf{t}\|_{\boldsymbol{\alpha}}/\mathsf{W}$.

Now let b be a geodesic representative of a curve $\beta \in \mathcal{C}(S)$ on $S(\mathbf{t}\cdot\boldsymbol{\alpha})$. Each essential intersection of b with A contributes at least width(A) to length(b), and so

$$\mathrm{width}(A) \times i(\gamma, \beta) \leq \mathrm{length}(b) = l(\beta).$$

Combining the above with Proposition 7.1, we deduce

$$i(\mathbf{t}\cdot\boldsymbol{\alpha}, \gamma) \leq \sqrt{2}\,l(\gamma) \leq \frac{\sqrt{2}\|\mathbf{t}\|_{\boldsymbol{\alpha}}}{\mathsf{W}} \quad \text{and} \quad i(\gamma, \beta) \leq \frac{l(\beta)}{\mathrm{width}(A)} \leq \frac{i(\mathbf{t}\cdot\boldsymbol{\alpha}, \beta)}{\mathsf{W}\|\mathbf{t}\|_{\boldsymbol{\alpha}}}.$$

Set $\mathsf{L}_0 = \sqrt{2}/\mathsf{W}$. The curve γ satisfies $i(\mathbf{t}\cdot\boldsymbol{\alpha}, \gamma) \le \mathsf{L}_0\|\mathbf{t}\|_{\boldsymbol{\alpha}}$, and so $\operatorname{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L}) \ne \varnothing$ for all $\mathsf{L} \ge \mathsf{L}_0$. Furthermore, if $\beta \in \operatorname{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})$ then

$$i(\gamma, \beta) \le \frac{i(\mathbf{t}\cdot\boldsymbol{\alpha}, \beta)}{\mathsf{W}\|\mathbf{t}\|_{\boldsymbol{\alpha}}} \le \frac{\mathsf{L}\|\mathbf{t}\|_{\boldsymbol{\alpha}}}{\mathsf{W}\|\mathbf{t}\|_{\boldsymbol{\alpha}}} = \frac{\mathsf{L}}{\mathsf{W}}.$$

Applying Lemma 2.2 and the triangle inequality gives

$$\operatorname{diam}(\operatorname{short}(\mathbf{t}\cdot\boldsymbol{\alpha}, \mathsf{L})) \le 2\Big[2\log_2\Big(\frac{\mathsf{L}}{\mathsf{W}}\Big) + 2\Big] = 4\log_2 \mathsf{L} + \mathsf{k}_0,$$

where k_0 is a constant depending only on $\xi(S)$. □

8B. ***Case 2: Y is a sphere with 4 marked points.*** For our purposes, it suffices to find a wide annulus on a suitable double branched cover of Y. Identify Y with the quotient of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ under a hyperelliptic involution $h : \mathbb{T}^2 \to \mathbb{T}^2$ given by $h(x, y) = (-x, -y)$ modulo $\mathbb{Z}^2$, so that the marked points coincide with the branch points. Metrize $\mathbb{T}^2$ by pulling back the singular Euclidean metric on Y. This metric can also be obtained by taking the preimages of the α_i contained in Y to $\mathbb{T}^2$ and then applying the construction as described in Section 4A. It follows that $\mathbb{T}^2$ enjoys the isoperimetric inequality stated in Lemma 7.4, and so applying Proposition 7.3 gives the following:

Lemma 8.1. *There exists an essential annulus on $\mathbb{T}^2$ of width at least $\mathsf{W}'\|\mathbf{t}\|_{\boldsymbol{\alpha}}/\sqrt{\xi(S)}$ for some universal constant $\mathsf{W}' > 0$.*

Remark 8.2. By following the proof of Proposition 7.3 in [Bowditch 2006b] for the case of the torus, one can take $\mathsf{W}' = \frac{1}{3\sqrt{2}}$.

Observe that $h(\tilde{\gamma})$ is homotopic to $\tilde{\gamma}$ for any simple closed curve $\tilde{\gamma}$ on $\mathbb{T}^2$. Thus, any simple closed on $\mathbb{T}^2$ descends to a simple closed curve on Y (up to homotopy).

Lemma 8.3. *Let A be an essential annulus on $\mathbb{T}^2$ with core curve $\tilde{\gamma}$. Let $\gamma \in \mathcal{C}(S)$ be the image of $\tilde{\gamma}$ on Y under the quotient map $H : \mathbb{T}^2 \to Y$. Then*

$$i(\gamma, \beta) \le \frac{2i(\mathbf{t}\cdot\boldsymbol{\alpha}, \beta)}{\operatorname{width}(A)}$$

for all $\beta \in \mathcal{C}(S)$.

Proof. Recall that $\beta \cap Y$ is either a simple closed curve or a union of paths connecting marked points of Y. The preimage $H^{-1}(\beta)$ is a finite union of (not necessarily disjoint) essential curves on $\mathbb{T}^2$. By perturbing γ to an embedded curve which misses the marked points of Y, we see that each point of $\gamma \cap \beta$ lifts to exactly two points on $\mathbb{T}^2$ under H, and so

$$i(\gamma, \beta) = \frac{i(H^{-1}(\gamma), H^{-1}(\beta))}{2} \le i(\tilde{\gamma}, H^{-1}(\beta)).$$

By observing that each intersection of $H^{-1}(\beta)$ with A contributes at least width(A) to its length, and applying Proposition 7.1, we deduce

$$\mathrm{width}(A) \times i(\tilde{\gamma}, H^{-1}(\beta)) \leq l(H^{-1}(\beta)) = 2l(\beta \cap Y) \leq 2l(\beta) \leq 2i(\mathbf{t} \cdot \boldsymbol{\alpha}, \beta).$$

The result follows. □

We may use the previous lemmas and argue as in Case 1 to bound the diameter of short$(\mathbf{t} \cdot \boldsymbol{\alpha}, \mathsf{L})$. Finally, short$(\mathbf{t} \cdot \boldsymbol{\alpha}, \mathsf{L})$ is nonempty since $\boldsymbol{\alpha}$ does not fill S.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 291 No. 1 November 2017

Chain transitive homeomorphisms on a space: all or none 1
Ethan Akin and Juho Rautio

Spinorial representation of submanifolds in Riemannian space forms 51
Pierre Bayard, Marie-Amélie Lawn and Julien Roth

Compact composition operators with nonlinear symbols on the H^2 space of Dirichlet series 81
Frédéric Bayart and Ole Fredrik Brevig

A local relative trace formula for PGL(2) 121
Patrick Delorme and Pascale Harinck

Regularity of the analytic torsion form on families of normal coverings 149
Bing Kwan So and GuangXiang Su

Thick subcategories over isolated singularities 183
Ryo Takahashi

Projections in the curve complex arising from covering maps 213
Robert Tang

The local Ginzburg–Rallis model over the complex field 241
Chen Wan