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GOULNARA ARZHANTSEVA AND MARKUS STEENBOCK



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**Given a finitely presented group $Q$, we produce a short exact sequence $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$ such that $G$ is a torsion-free hyperbolic group without the unique product property and $N$ is without the unique product property and has Kazhdan's Property (T). Varying $Q$ yields a wide diversity of concrete examples of hyperbolic groups without the unique product property. We also note, as an application of Ol'shanskiĭ's construction of torsion-free Tarski monsters, the existence of torsion-free Tarski monster groups without the unique product property.**

## 1. Introduction

A group $G$ has the *unique product property*, or is said to be a *unique product group*, whenever for all pairs of nonempty finite subsets $A$ and $B$ of $G$ the set of products $AB$ has an element $g \in G$ with a unique representation of the form $g = ab$ with $a \in A$ and $b \in B$. Unique product groups are torsion-free. They satisfy the Kaplansky zero-divisor conjecture [1957; 1970], which states that the group ring of a torsion-free group over an integral domain has no zero-divisors. Rips and Segev [1987] gave the first examples of torsion-free groups without the unique product property. In [Steenbock 2015], the second author proved that the (generalized) Rips–Segev groups are hyperbolic, and gave an uncountable family of nonunique product groups. Other examples of torsion-free groups without the unique product property are in [Promislow 1988; Carter 2014; Soelberg 2018].

Our goal is to provide new concrete examples of nonunique product groups with diverse algebraic and geometric properties. In fact, we produce a variety of strongly nonamenable examples.

**Theorem 1.1.** *Let $Q$ be a finitely generated group. Then there exists a short exact sequence* $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$ *such that*

- *$G$ is a torsion-free group without the unique product property which is a direct limit of hyperbolic groups*,

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- *$N$ is a finitely generated subgroup of $G$ with Kazhdan's Property ($T$) and without the unique product property.*

*If, in addition, $Q$ is finitely presented, then $G$ is hyperbolic.*

Theorem 1.1 extends the result on Rips short exact sequence with Kazhdan's Property (T) kernel from [Ollivier and Wise 2007]. An alternative construction is in [Belegradek and Osin 2008].

Varying $Q$ in Theorem 1.1 yields many new groups without the unique product property that have various algebraic and algorithmic properties, see Section 4. The examples obtained using Theorem 1.1 contrast the torsion-free groups without the unique product property from [Promislow 1988; Carter 2014; Soelberg 2018], which are infinite groups with the Haagerup property (= a-T-menable groups, in Gromov's terminology, see [Cherix et al. 2001]), and, hence, groups which do not have Kazhdan's Property (T). Indeed, the group in [Promislow 1988] is solvable, hence, a-T-menable; groups in [Carter 2014] are a-T-menable as they have $\mathbb{Z}^k \times \mathbb{F}_m$ as a finite index subgroup; the group in [Soelberg 2018] is a-T-menable as it has a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$ as a finite index subgroup, see [Soelberg 2018, p. 24].

## 2. Small cancellation theory over hyperbolic groups

A useful way to get novel nonunique product groups is to take quotients of free products of hyperbolic nonunique product groups with other suitably chosen groups. We will apply the following result:

**Theorem 2.1** [Ol'shanskiĭ 1993, Theorem 2]. *Let $G = H_1 * H_2$ be the free product of two nonelementary torsion-free hyperbolic groups and $M \subseteq H_1$ be a finite subset. Then $G$ has a nonelementary torsion-free hyperbolic quotient $\overline{G}$ such that the canonical projection $G \twoheadrightarrow \overline{G}$ is injective on $M$ and restricts to a surjection $H_2 \twoheadrightarrow \overline{G}$.*

Theorem 2.1, together with [Steenbock 2015, Theorem 2], yields first examples of Kazhdan's Property (T) groups without the unique product property.

**Corollary 2.2.** *There are torsion-free hyperbolic groups with Kazhdan's Property ($T$) and without the unique product property.*

*Proof.* Take for $H_1$ a torsion-free hyperbolic group without the unique product property such that the unique product property fails for the sets $A$ and $B$, see [Steenbock 2015]. Take for $H_2$ a hyperbolic group with Property (T) (e.g., a discrete subgroup of finite covolume in $\mathrm{Sp}(n, 1)$) and for $M$ a finite subset of $H_1$ containing $A$, $B$, and $AB$. By Theorem 2.1, there exists a torsion-free hyperbolic quotient $\overline{G}$ of $H_1 * H_2$ with Property (T) such that $M$ injects into this quotient. It follows that $\overline{G}$ is without the unique product property. □

**Remark 2.3.** The group $H_1$ can be generated by two letters, say, $a_1$ and $a_2$, and it can be defined using a finite set of relators that we denote by $\mathcal{RS}$. A procedure to obtain such a set of relators follows from [Rips and Segev 1987; Steenbock 2015]. Thus, $H_1$ can be given by an explicit presentation $H_1 = \langle a_1, a_2 \mid \mathcal{RS} \rangle$.

Let $H_2 = \langle Y \mid \mathcal{R}_T \rangle$, where $\mathcal{R}_T$ is a fixed finite set of relators. An explicit presentation of an infinite torsion hyperbolic group with Property (T) with 16 relators is given, for example, in [Caprace 2018]. To get a required torsion-free $H_2$, one can then take a subgroup of sufficiently large index in this Property (T) group. A finite presentation of such $H_2$ can be obtained from the group presentation given in [Caprace 2018], using Schreier's method.

Let $g$ and $h$ be hyperbolic elements of $H_2$ that do not generate an elementary subgroup. Let $q$, $s$, and $t$ denote natural numbers. Let

$$\mathcal{R}_{q,s,t} := \{a_1^{-1} g^q h^s g^q h^{2s} \cdots g^q h^{ts},\ a_2^{-1} g^q h^{(t+1)s} g^q h^{(t+2)s} \cdots g^s h^{2ts}\}.$$

Following [Ol'shanskiĭ 1993], there are $s_0 > 0$, $t_0 > 0$, and $q_0 > 0$ such that

$$\overline{G} := \langle a_1, a_2, Y \mid \mathcal{RS} \sqcup \mathcal{R}_T \sqcup \mathcal{R}_{q_0,s_0,t_0} \rangle$$

defines a group, as required by Corollary 2.2. The numbers $q_0$, $s_0$, and $t_0$ depend only on $A$ and $B$, the hyperbolicity constant and the size of the balls in the Cayley graph of $H_2$.

Moreover, we obtain torsion-free Tarski monster groups without the unique product property. These are the first examples of torsion-free groups without the unique product property, all of whose proper subgroups are unique product groups.

**Corollary 2.4.** *There are torsion-free Kazhdan's Property* (*T*) *groups G without the unique product property such that all proper subgroups of G are cyclic. Moreover, these groups have explicit recursive presentations.*

*Proof.* Let $G$ be a noncyclic torsion-free hyperbolic group, and let $M$ be a finite subset of $G$. It follows from [Ol'shanskiĭ 1993, Theorem 2] that there exists a nonabelian torsion-free quotient $\widetilde{G}$ such that all proper subgroups of $\widetilde{G}$ are cyclic, and such that $G \twoheadrightarrow \widetilde{G}$ is injective on $M$ [Ol'shanskiĭ 1993, Corollary 1]. Moreover, an explicit presentation of $G$ yields an explicit recursive presentation of $\widetilde{G}$. Applied to a finite subset containing $A$, $B$, and $AB$ in a torsion-free hyperbolic group $G$ without the unique product property for $A$ and $B$ from [Steenbock 2015], this immediately yields Tarski monster groups without the unique product property, that have explicit recursive presentations. □

## 3. Rips construction via small cancellation over hyperbolic groups

We now prove Theorem 1.1. The idea is to adapt [Belegradek and Osin 2008] by using Theorem 2.1 as in Remark 2.3. Recall that $H_1 := \langle a_1, a_2 \mid \mathcal{RS} \rangle$ is our torsion-free hyperbolic group without the unique product property for sets $A$ and $B$ (see [Steenbock 2015], we set $a_2 := b$) and $M$ is a finite subset of $H_1$ containing $A$, $B$, and $AB$. Recall that $H_2 := \langle y_1, \dots, y_l \mid \mathcal{R}_T \rangle$ is a torsion-free hyperbolic group with Property (T).

Let $Q := \langle x_1, \dots, x_m \mid r_1, \dots, r_n, \dots \rangle$ be a finitely generated group. We produce the required $G$ as a suitable quotient of the free product $H_1 * H_2 * \langle x_1, \dots, x_m \rangle$.

Let $g, h \in H_2$ be hyperbolic elements that do not generate an elementary subgroup. Let $s, t, q, q_1, \dots, q_i, \dots$ denote natural numbers, let $\bar{q} = \{q, q_1, \dots\}$, and let $\mathcal{R}_{\bar{q},s,t}$ be the set of words:

$$a_1^{-1} g^q h^s g^q h^{2s} \cdots g^q h^{ts} \text{ and } a_2^{-1} g^q h^{(t+1)s} g^q h^{(t+2)s} \cdots g^q h^{2ts}, \tag{1}$$

$$\begin{aligned}
x_j a_1 x_j^{-1} g^q h^{((j+1)t+1)s} g^q h^{((j+1)t+2)s} \cdots g^q h^{(j+2)ts} \quad & \forall\, 1 \leqslant j \leqslant m, \\
x_j a_2 x_j^{-1} g^q h^{((j+m+1)t+1)s} g^q h^{((j+m+1)t+2)s} \cdots g^q h^{(j+m+2)ts} \quad & \forall 1 \leqslant j \leqslant m, \\
x_j^{-1} a_1 x_j g^q h^{((j+2m+1)t+1)s} g^q h^{((j+2m+1)t+2)s} \cdots g^q h^{(j+2m+2)ts} \quad & \forall\, 1 \leqslant j \leqslant m, \\
x_j^{-1} a_2 x_j g^q h^{((j+3m+1)t+1)s} g^q h^{((j+3m+1)t+2)s} \cdots g^q h^{(j+3m+2)ts} \quad & \forall\, 1 \leqslant j \leqslant m,
\end{aligned} \tag{2}$$

$$\begin{aligned}
& x_j y_k x_j^{-1} g^q h^{((j+(k-1)m+4m+1)t+1)s} g^q h^{((j+(k-1)m+4m+1)t+2)s} \\
& \qquad \cdots g^q h^{(j+(k-1)m+4m+2)ts} \quad \forall\, 1 \leqslant j \leqslant m,\ \forall\, 1 \leqslant k \leqslant l, \\
& x_j^{-1} y_k x_j g^q h^{((j+(k+l+3)m+1)t+1)s} g^q h^{((j+(k+l+3)m+1)t+2)s} \\
& \qquad \cdots g^q h^{(j+(k+l+3)m+2)ts} \quad \forall\, 1 \leqslant j \leqslant m,\ \forall\, 1 \leqslant k \leqslant l,
\end{aligned} \tag{3}$$

$$r_i g^{q_i} h^s g^{q_i} h^{s+1} \cdots g^{q_i} h^{ts} \quad \forall\, i = 1, 2, \dots . \tag{4}$$

Following [Ol'shanskiĭ 1993, Lemma 4.2], there exist $s_0 > 0$, $t_0 > 0$, and $\bar{q}_0$ such that $\mathcal{R}_{\bar{q}_0,s_0,t_0}$ satisfies the $C_1$-condition of [Ol'shanskiĭ 1993, Section 4] with respect to $H_1 * H_2 * \langle x_1, \dots, x_m \rangle$. It follows from the proof of Theorem 2 of [Ol'shanskiĭ 1993] that the quotient

$$G := \langle a_1, a_2, y_1, \dots, y_l, x_1, \dots, x_m \mid \mathcal{RS} \sqcup \mathcal{R}_T \sqcup \mathcal{R}_{\bar{q}_0,s_0,t_0} \rangle$$

is a direct limit of torsion-free hyperbolic groups, that $G$ is torsion-free, and that $M$ injects into $G$. In particular, $G$ does not have the unique product property.

Let $N$ be the subgroup generated by $a_1, a_2, y_1, \dots, y_n$. By the relators (2) and (3), $N$ is normal. By the relators (4), the map defined by sending the generators $x_i$ onto themselves, and the $a_1, a_2, y_k$ onto 1 is a projection onto $Q$, the kernel of which is the group $N$.

As $M$ consists of words in $a_1$ and $a_2$, the set $M$ injects into $N$ as well, so that $N$ does not have the unique product property. By the relators (1), $N$ is a quotient of $H_2$, hence $N$ has Property (T).

This finishes the proof of Theorem 1.1 and gives presentations of the groups $G$.

**Remark 3.1.** If $Q$ is the trivial group, we recover Corollary 2.2 and the conclusion of Remark 2.3.

## 4. More examples of torsion-free groups without unique product

We now vary the quotient group $Q$. All examples of groups $G$ below are not isomorphic to a free product. The following results are immediate generalizations of [Rips 1982]:

**Proposition 4.1.** *For each of the following, there exists a torsion-free hyperbolic group $G$ without the unique product property and such that*:

(1) *$G$ has unsolvable generalized word problem*;

(2) *there are finitely generated subgroups $P_1$ and $P_2$ of $G$ such that $P_1 \cap P_2$ is not finitely generated*;

(3) *there is a finitely generated, but not finitely presented, subgroup of $G$*;

(4) *for any $r \geqslant 3$, there is an infinite strictly increasing sequence of $r$-generated subgroups of $G$.*

More algorithmic properties in the context of Rips construction are investigated in [Baumslag et al. 1994]. Applied to our situation they yield the following:

**Proposition 4.2.** *There is no algorithm to determine each of the following*:

(1) *the rank of a torsion-free hyperbolic group without unique product*;

(2) *whether an arbitrary finitely generated subgroup of a torsion-free hyperbolic group without unique product has finite index*;

(3) *whether an arbitrary finitely generated subgroup of a torsion-free hyperbolic group without unique product is normal*;

(4) *whether an arbitrary finitely generated subgroup of a torsion-free hyperbolic group without unique product is finitely presented*;

(5) *whether an arbitrary finitely generated subgroup $S$ of a torsion-free hyperbolic group without unique product has a finitely generated second integral homology group $H_2(S, \mathbb{Z})$.*

The proofs of (2)–(5) are by choosing a group $Q$ with the required property, which then allows to pullback the property to the group $G$, see [Baumslag et al. 1994, Theorem 4]. To prove (1), one produces a family of groups $G$ with the required properties as in the proof of [Baumslag et al. 1994, Theorem 2].

**Remark 4.3.** As pointed out by a referee, groups satisfying Proposition 4.1 or assertion (1), (4), or (5) of Proposition 4.2 could also be produced by taking free products of a hyperbolic group without the unique product property with a hyperbolic group with the respective properties, or more generally, by embedding them as peripheral subgroups in a relatively hyperbolic group.

## 5. Further remarks

We first proved Theorem 1.1 by a completely different method of graphical small cancellation theory over free products. The interested reader can find this proof in the arXiv version of this article, [Arzhantseva and Steenbock 2014]. It provides a variant of Theorem 1.1, where the group $G$ has, moreover, a graphical presentation that satisfies the graphical $\mathrm{Gr}'_*\left(\frac{1}{6}\right)$-small cancellation condition over the free product.

This initial approach is independent of prior results from [Ol'shanskiĭ 1993; Belegradek and Osin 2008]. It combines, under this novel free product viewpoint, the Rips construction [1982], the construction by Rips and Segev [1987] of groups without the unique product property, and Gromov's construction [2003, Section 1.2.A and item (3) in Section 4.8] of graphical small cancellation groups with Property (T), based on his spectral characterization of this property [Silberman 2003; Ollivier and Wise 2007].

We observe, in particular, that Gromov's probabilistic construction of graph labelings defining groups with Property (T) is flexible under taking edge subdivisions.

**Theorem 5.1** [Arzhantseva and Steenbock 2014, Theorem 4]. *For all $m > 64$, there exists a finite connected graph $\mathcal{T}$ labeled by $\{a_1, \ldots, a_m\}$ such that the labeling satisfies the $\mathrm{Gr}'_*\left(\frac{1}{6}\right)$-small cancellation condition over the free product $\langle a_1\rangle * \cdots * \langle a_m\rangle$, the labeling satisfies the $\mathrm{Gr}'\left(\frac{1}{6}\right)$-small cancellation condition with respect to the word length metric, and the group with $a_1, \ldots, a_m$ as generators and the labels of the cycles of $\mathcal{T}$ as relators has Kazhdan's Property (T).*

One can take $\mathcal{T}$ of arbitrarily large girth. Following the strategy of Ollivier and Wise [2007], the graph $\mathcal{T}$ is produced by assigning to every edge of an expander graph a letter and an orientation independently uniformly at random.

The intuition behind Theorem 5.1 is that the free product length in $\langle a_1\rangle * \cdots * \langle a_m\rangle$ approximates the word length on the free group on $a_1, \ldots, a_m$ as $m \to \infty$. Indeed, the minimal cycle length in the free product length bounds the length of the minimal cycles in the word length from below. Pieces are words of finite length chosen uniformly at random. Let us evaluate the probability that the word length and the free product length of such a random word in letters $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$ coincide. Such a word is of word length equal to $n$ if it is $a_{i_1}^{P_1} a_{i_2}^{P_2} \ldots a_{i_j}^{P_j}$ with all coefficients $P_i \neq 0$, $a_{i_j} \neq a_{i_{j+1}}$, and $\sum_{i=1}^{j} |P_i| = n$. Its free product length is equal to $n$ if, in addition, all exponents $P_i = \pm 1$. The probability that all $P_i = \pm 1$ in such a word is given by $((2m-2)/2m)^{n-1}$, which tends to 1 as $m \to \infty$.

For further details on the genericity aspects underlying Theorem 1.1 see [Arzhantseva and Steenbock 2014].

## 6. Open problems

Our constructions are motivated by two open problems.

**Open problem 6.1.** Do the Rips–Segev groups without the unique product property satisfy the Kaplansky zero-divisor conjecture?

Combining [Schreve 2014; Linnell et al. 2012; Agol 2013], we observe that the Kaplansky zero-divisor conjecture holds for all torsion-free CAT(0)-cubical[1] hyperbolic groups, over the field of complex numbers. The groups from Corollary 2.2 are not CAT(0)-cubical as they are infinite Property (T) groups. Thus, the CAT(0)-cubulation cannot solve the conjecture for all hyperbolic groups without the unique product property.

It is unknown whether or not any of the hyperbolic groups without the unique product from [Rips and Segev 1987; Steenbock 2015; Gruber et al. 2015] is CAT(0)-cubical [Martin and Steenbock 2017] or, more generally, a-T-menable.

**Open problem 6.2.** Is every hyperbolic group residually finite?

We mention this question as every residually finite hyperbolic group has a finite index subgroup with the unique product property by a result of Delzant [1997]. If $Q$ is finite, then $N$ in our construction is normal of finite index and without the unique product property. Then the following questions arise naturally:

- Does there exist a hyperbolic group all of whose normal finite index subgroups are without the unique product property?
- Does there exist a hyperbolic group all of whose subgroups of index at most $k$, for a given $k \geqslant 2$, are without the unique product property?

After we first announced our results in 2014, our last question has been answered in the affirmative [Gruber et al. 2015].

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[1] A group is *CAT(0)-cubical* if it admits a proper cocompact action on a CAT(0)-cubical complex.

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GOULNARA ARZHANTSEVA
FACULTY OF MATHEMATICS
UNIVERSITY OF VIENNA
VIENNA
AUSTRIA
goulnara.arzhantseva@univie.ac.at

MARKUS STEENBOCK
FACULTY OF MATHEMATICS
UNIVERSITY OF VIENNA
VIENNA
AUSTRIA
markus.steenbock@univie.ac.at

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