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The J^k space of k-jets of a real function of one real variable x admits the structure of a sub-Riemannian manifold, which then has an associated Hamiltonian geodesic flow, and it is integrable. As in any Hamiltonian flow, a natural question is the existence of periodic solutions. Does J^k have periodic geodesics? This study will find the action-angle coordinates in T^*J^k for the geodesic flow and demonstrate that geodesics in J^k are never periodic.

1. Introduction

This paper is the first attempt to prove that Carnot groups do not have periodic sub-Riemannian geodesics; Enrico Le Donne made this conjecture. Here, we will establish the first case we found, which also has a simple and elegant proof.

This work is the continuation of that done in [4; 5]. In [4], J^k was presented as a sub-Riemannian manifold, the sub-Riemannian geodesic flow was defined, and its integrability was verified. In [5], the sub-Riemannian geodesics in J^k were classified, and some of their minimizing properties were studied. The main goal of this paper is to prove:

Theorem A. *J^k does not have periodic geodesics.*

Following the classification of geodesics from [5, p. 5], the only candidates to be periodic are the ones called x-periodic (the other geodesics are not periodic on the x-coordinate); so we are focusing on the x-periodic geodesics.

An essential tool during this work is the bijection made by Monroy-Perez and Anzaldo-Meneses [2; 8; 9], also described in [5, p. 4], between geodesics on J^k and the pair (F, I) (module translation $F(x) \to F(x-x_0)$), where $F(x)$ is a polynomial of degree bounded by k and I is a closed interval, called the hill interval. Let us formalize its definition.

Definition 1. A closed interval I is called a hill interval of $F(x)$, if for each x inside I, then $F^2(x) < 1$ and $F^2(x) = 1$ if x is in the boundary of I.

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By definition, the hill interval I of a constant polynomial $F^2(x) = c^2 < 1$ is $\mathbb{R}$, while the hill interval I of the constant polynomial $F(x) = \pm 1$ is a single point. Also, I is compact if and only if $F(x)$ is not a constant polynomial; in this case, if I is of the form $[x_0, x_1]$, then $F^2(x_1) = F^2(x_0) = 1$. This terminology comes from celestial mechanics, and I is the region where the dynamics governed by the fundamental equation (3-5) take place.

Geodesics corresponding to constant polynomials are called horizontal lines since their projection to (x, θ_0)-planes are lines. In particular, geodesics corresponding to $F(x) = \pm 1$ are abnormal geodesics (see [6], [10], or [11]). Then this work will be restricted to geodesics associated with nonconstant polynomials. Further, x-periodic geodesics correspond to the pair $(F, [x_0, x_1])$, where x_0 and x_1 are regular points of $F(x)$, which implies they are simple roots of $1 - F^2(x)$.

Outline of the paper. In Section 2, Proposition 2 is introduced and Theorem A is proved. The main purpose of Section 3 is to prove Proposition 2. In Section 3.1, the sub-Riemannian structure and the sub-Riemannian Hamiltonian geodesic function are introduced. In Section 3.2, a generating function is presented and a canonical transformation from traditional coordinates in T^*J^k to action-angle coordinates (μ, ϕ) for the Hamiltonian systems is shown. In Section 3.3, Proposition 2 is proved.

2. Proof of Theorem A

Throughout this work, the alternate coordinates $(x, \theta_0, \dots, \theta_k)$ will be used, the meaning of which is introduced in Section 3 and described in more detail in [2], [9], or [5]. Further, x-periodic geodesics have the property that the change undergone by the coordinates θ_i after one x-period is finite and does not depend on the initial point. We summarize the above discussion with the following proposition:

Proposition 2. *Let $\gamma(t) = (x(t), \theta_0(t), \dots, \theta_k(t))$ in J^k be an x-periodic geodesic corresponding to the pair (F, I). Then the x-period is*

$$L(F, I) = 2\int_I \frac{dx}{\sqrt{1-F^2(x)}}. \tag{2-1}$$

Moreover, it is twice the time it takes for the x-curve to cross its hill interval exactly once. After one period, the changes $\Delta\theta_i := \theta_i(t_0 + L) - \theta_i(t_0)$ for $i = 0, 1, \dots, k$ undergone by θ_i are given by

$$\Delta\theta_i(F, I) = \frac{2}{i!}\int_I \frac{x^i F(x)dx}{\sqrt{1 - F^2(x)}}. \tag{2-2}$$

In [5], a sub-Riemannian manifold $\mathbb{R}^3_F$, called magnetic space, was introduced, and a similar statement like Proposition 2 was proved, see [5, Proposition 4.1], with an argument of classical mechanics, see [7, (11.5)].

Proposition 2 implies that a x-periodic geodesic $\gamma(t)$ corresponding to the pair (F, I) is periodic if and only if $\Delta\theta_i(F, I) = 0$ for all i.

Because that period L from (2-1) is finite, we can define an inner product in the space of polynomials of degree bounded by k in the following way:

$$\langle P_1(x), P_2(x)\rangle_F := \int_I \frac{P_1(x)P_2(x)dx}{\sqrt{1-F^2(x)}}. \tag{2-3}$$

This inner product is nondegenerate and will be the key to the proof of Theorem A.

2.1. *Proof of Theorem A.*

Proof. We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on J^k corresponding to the pair (F, I), where $F(x)$ is not constant, then $\Delta\theta_i(F, I) = 0$ for all i in $0, \dots, k$.

In the context of the space of polynomials of degree bounded by k with inner product $\langle\ ,\ \rangle_F$, the condition $\Delta\theta_i(F, I) = 0$ is equivalent to $F(x)$ being perpendicular to x^i $\big(0 = \Delta\theta_i(F, I) = \langle x^i, F(x)\rangle_F\big)$, so $F(x)$ being perpendicular to x^i for all i in $0, 1, \dots, k$. However, the set $\{x^i\}$, with $0 \le i \le k$, is a base for the space of polynomials with degree bounded by k. Then $F(x)$ is perpendicular to any vector, so $F(x)$ is zero since the inner product is nondegenerate. However, $F(x)$ equals 0 contradicts the assumption that $F(x)$ is not a constant polynomial. ☐

Coming work: The proof of the conjecture in the meta-abelian group $\mathbb{G}$, that is, $\mathbb{G}$ is such that $0 = [[\mathbb{G}, \mathbb{G}], [\mathbb{G}, \mathbb{G}]]$.

3. Proof of Proposition 2

3.1. *J^k as a sub-Riemannian manifold.* The sub-Riemannian structure on J^k will be described here briefly. For more details, see [4; 5]. We see J^k as $\mathbb{R}^{k+2}$, using $(x, \theta_0, \dots, \theta_k)$ as global coordinates, then J^k is endowed with a natural rank 2 distribution $D \subset TJ^k$ characterized by the k Pfaffian equations

$$0 = d\theta_i - \frac{1}{i!}x^i d\theta_0, \quad i = 1, \dots, k. \tag{3-1}$$

D is globally framed by two vector fields

$$X_1 = \frac{\partial}{\partial x} \quad \text{and} \quad X_2 = \sum_{i=0}^{k} \frac{x^i}{i!}\frac{\partial}{\partial\theta_i}. \tag{3-2}$$

A sub-Riemannian structure on $\mathcal{J}^k$ is defined by declaring these two vector fields to be orthonormal. In these coordinates, the sub-Riemannian metric is given by restricting $ds^2 = dx^2 + d\theta_0^2$ to D.

3.1.1. *Sub-Riemannian geodesic flow.* Here it is emphasized that the projections of the solution curves for the Hamiltonian geodesic flow are geodesics, that is, if $(p(t),\gamma(t))$ is a solution for the Hamiltonian geodesic flow, then $\gamma(t)$ is a geodesic on J^k.

Let $(p_x, p_{\theta_0},\dots,p_{\theta_k}, x,\theta_0,\dots,\theta_k)$ be the traditional coordinates on T^*J^k, or (p,q) for short. Let $P_1, P_2 : T^*J^k \to \mathbb{R}$ be the momentum functions of the vector fields X_1 and X_2, see [10, p. 8] or [1], in terms of the coordinates (p,q) given by

$$P_1(p,q) := p_x \quad \text{and} \quad P_2(p,q) := \sum_{i=0}^{k} p_{\theta_i}\frac{x^i}{i!}. \tag{3-3}$$

Then the Hamiltonian governing the geodesic on J^k is

$$H_{sR}(p,q) := \frac{1}{2}(P_1^2 + P_2^2) = \frac{1}{2}p_x^2 + \frac{1}{2}\left(\sum_{i=0}^{k} p_{\theta_i}\frac{x^i}{i!}\right)^2. \tag{3-4}$$

It is noteworthy that $h=\frac{1}{2}$ implies that the geodesic is parameterized by arc-length. It can be noticed that if H does not depend on θ_i for all i, then the p_θ define $k+1$ constants of motion.

Lemma 3. *The sub-Riemannian geodesic flow in J^k is integrable. If $(p(t),\gamma(t))$ is a solution, then*

$$\dot\gamma(t) = P_1(t)X_1 + P_2(t)X_2 \quad \textit{and} \quad (P_1(t),P_2(t)) = (p_x(t), F(x(t))),$$

where $p_{\theta_i} = i!\,a_i$ and $F(x) = \sum_{i=0}^{k} a_i x^i$.

Proof. H does not depend on t and θ_i for all i, so $h := H_{sR}$ and p_{θ_i} are constants of motion, thus the Hamiltonian system is integrable. A consequence of the first equation from Lemma 3 is that P_1 and P_2 are linear in p_x and p_θ. We denote by $(a_0,\dots,a_k)$ the level set $i!\,a_i = p_{\theta_i}$, then the result follows by the definitions of P_1 and P_2 given by (3-3). ☐

3.1.2. *Fundamental equation.* The level set $(a_0,\dots,a_k)$ defines a fundamental equation

$$H_F(p_x,x) := \tfrac{1}{2}p_x^2 + \tfrac{1}{2}F^2(x) = H|_{(a_0,\dots,a_k)}(p,q) = \tfrac{1}{2}. \tag{3-5}$$

Here, $H_F(p_x,x)$ is a Hamiltonian function in the phase plane (p_x,x), where the dynamic of $x(s)$ takes place in the hill region $I=[x_0,x_1]$ and its solution $(p_x(t),x(t))$ with energy $h=\frac{1}{2}$ lies in an algebraic curve or loop given by

$$\alpha_{(F,I)} := \left\{(p_x,x) : \tfrac{1}{2} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}F^2(x) \text{ and } x_0 \le x \le x_1\right\}, \tag{3-6}$$

and $\alpha_{(F,I)}$ is closed and simple.

Lemma 4. *$\alpha(F, I)$ is smooth if and only if x_0 and x_1 are regular points of $F(x)$, in other words, $\alpha(F, I)$ is smooth if and only if the corresponding geodesic $\gamma(t)$ is x-periodic.*

Proof. A point $\alpha = (p_x, x)$ in $\alpha(F, I)$ is smooth if and only if

$$0 \neq \nabla H_F(p_x, x)|_{\alpha(F,I)} = \bigl(p_x,\, F(x)F'(x)\bigr).$$

Then α is smooth for all $p_x \neq 0$, and the points $\alpha(F, I)$ such that $p_x = 0$ correspond to endpoints of the hill interval I, since the condition $p_x = 0$ implies $F^2(x) = 1$. The point $\alpha = (0, x_0)$ is smooth if $F'(x_0) \neq 0$, and the point $\alpha = (0, x_1)$ is smooth if $F'(x_1) \neq 0$. Then $\alpha(F, I)$ is smooth if and only if x_0 and x_1 are regular points of $F(x)$. Also, $\alpha(F, I)$ is smooth is equivalent to $H_F(p_x, x)|_{\alpha(F,I)}$ is never zero, which is equivalent to the Hamiltonian vector field is never zero on $\alpha(F, I)$. □

3.1.3. *Arnold–Liouville manifold.* The Arnold–Liouville manifold $M|_F$ is given by

$$M_F := \bigl\{(p, q) \in T^*J^k : \tfrac{1}{2} = H_F(p_x, x),\ p_{\theta_i} = i!\,a_i\bigr\}.$$

In the case $\gamma(t)$ is x-periodic, M_F is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{k+1}$, where $\mathbb{S}^1$ is the simple, closed, and smooth curve $\alpha(F, I)$.

The curve $\alpha(F, I)$ has two natural charts using x as coordinates and is given by solving the equation $H_F = \frac{1}{2}$ with respect to p_x, namely $(p_x, x) = \bigl(\pm\sqrt{1-F^2(x)},\, x\bigr)$. With this in mind:

Lemma 5. *Let $d\phi_t$ be the closed one-form on $M_F \subset T^*J^k$ given by*

$$d\phi_h := \frac{p_x}{\Pi(F, I)}|_{M_F} dx = \frac{\sqrt{1 - F^2(x)}}{\Pi(F, I)} dx, \tag{3-7}$$

where $\Pi(F, I)$ is the area enclosed by $\alpha(F, I)$. Then,

$$\int_{\alpha(F,I)} d\phi_h = 1 \quad \textit{and} \quad \frac{\partial}{\partial h}\Pi(F, I) = L(F, I),$$

and as a consequence the inverse function $h(\Pi)$ exists.

Proof. Let $\Omega(F, I)$ be the closed region by $\alpha(F, I)$, then $d\phi_h$ can be extended to $\Omega(F, I)$ and Stokes' theorem implies

$$\Pi(F, I) := \int_{\alpha(F,I)} p_x\, dx = \int_{\Omega(F,I)} dp_x \wedge dx = 2\int_I \sqrt{2h - F^2(x)}\,|_{h=1/2}\, dx. \tag{3-8}$$

This shows that $\int_{\alpha(F,I)} d\phi_h = 1$, thus $d\phi_h$ is not exact.

Since $\Pi(F, I)$ is a function of h,

$$\frac{\partial}{\partial h}\Pi(F, I) = \frac{\partial}{\partial h}\int_I d\phi_h = \int_I \frac{2\, dx}{\sqrt{1-F^2(x)}}. \tag{3-9}$$

□

We note that $\Pi(F, I)$ is also called an adiabatic invariant, see [3, p. 297]. We will use Π when we use it as a variable, and we will use $\Pi(F, I)$ for the adiabatic invariant.

3.2. ***Action-angle variables in T^*J^k.*** We consider the action $\mu = (\Pi, a_0, \dots, a_k)$ and find its angle coordinates $\phi = (\phi_h, \phi_0, \dots, \phi_k)$, such that the set (μ, ϕ) of coordinates are action-angle coordinates in T^*J^k.

Lemma 6. *There exist a canonical transformation $\Phi(p, q) = (\mu, \phi)$, where ϕ_h is the local function defined by the close form $d\phi_h$ from Lemma 5 and*

$$\phi_i = -\int^x \frac{\tilde{x}^i F(\tilde{x})\, d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} + i!\,\theta_i, \quad x \in I \text{ and } i = 0, \dots, k.$$

To construct the canonical transformation $\Phi(p, q)$, we will look for its generating function $S(\mu, q)$ of the second type that satisfies the three following conditions:

$$p = \frac{\partial S}{\partial q}, \quad \phi = \frac{\partial S}{\partial \mu}, \quad H\Big(\frac{\partial S}{\partial q}, q\Big) = h(\Pi) = \frac{1}{2}, \tag{3-10}$$

where $h(\Pi)$ is the function defined in Lemma 5. For more details on the definition of $S(\mu, q)$, see [3, Section 50] or [7].

To find $S(\mu, q)$, we will solve the sub-Riemannian Hamilton–Jacobi equation associated with the sub-Riemannian geodesic flow. For more details about the definition of this equation in sub-Riemannian geometry and its relation to the Eikonal equation, see [10, p. 8] or [5].

Proof. The sub-Riemannian Hamilton–Jacobi equation is given by

$$h|_{1/2} = \frac{1}{2}\Big(\frac{\partial S}{\partial x}\Big)^2 + \frac{1}{2}\Big(\sum_{i=0}^{k} \frac{x^i}{i!}\frac{\partial S}{\partial \theta_i}\Big)^2. \tag{3-11}$$

Take the ansatz

$$S(\mu, q) := f(x) + \sum_{i=0}^{k} i!\, a_i \theta_i$$

as a solution. The equation (3-11) becomes (3-5), and then the generating function is given by

$$S(\mu, q) = \int_{x_0}^{x} \sqrt{2h(\Pi) - F^2(\tilde{x})}\, d\tilde{x} + \sum_{i=0}^{n} i!\, a_i \theta_i. \tag{3-12}$$

Here, $h(\Pi) = \frac{1}{2}$ and $S(\mu, q)$ is a local function, since x must lay in the hill region I, that is, $S(\mu, q)$ is defined in the subset $\mu \times I \times \mathbb{R}^{k+1}$.

We can see that conditions 1 and 3 of (3-10) are satisfied: $p(\mu, q) = \partial S/\partial q$ and $H(p(\mu, q), q) = h$. To find the new coordinates ϕ, we use condition 2:

$$\frac{\partial S}{\partial h} = \int^{x} \frac{d\tilde{x}}{\sqrt{1-F^2(\tilde{x})}} = \phi_h,$$

$$\frac{\partial S}{\partial a_i} = -\int^{x} \frac{\tilde{x}^i F(\tilde{x})\, d\tilde{x}}{\sqrt{1-F^2(\tilde{x})}} + i!\,\theta_i = \phi_i. \qquad \square$$

Note that in [5] a projection $\pi_F : J^k \to \mathbb{R}^3_F$ was built, and the solution to the sub-Riemannian Hamilton–Jacobi equation on the magnetic space $\mathbb{R}^3_F$ was found. The solution given by (3-12) is the pull-back by π_F of the solution previously found in $\mathbb{R}_F$, where π_F is, in fact, a sub-Riemannian submersion.

Corollary 7. *The coordinates* (μ, ϕ) *are action-angle coordinates.*

Proof. Using the Hamilton equations for the new coordinates (μ, ϕ), we have $\phi_t = t$ and $\phi_i = \text{const}$. ☐

Note that h and ϕ_t are action-angles coordinates for the Hamiltonian H_F.

3.2.1. *Horizontal derivative.* A horizontal derivative ∇_{hor} of a function $S : J^k \to \mathbb{R}$ is the unique horizontal vector field that satisfies; for every q in J^k,

$$\langle \nabla_{\text{hor}} S, v\rangle_q = dS(v), \quad \text{for } v \in D_q, \tag{3-13}$$

where $\langle\ ,\ \rangle_q$ is the sub-Riemannian metric in D_q. For further details, see [10, pp. 14–15] or [1].

Lemma 8. *Let* $\gamma(t)$ *be a geodesic parameterized by arc length corresponding to the pair* (F, I) *and* S_F *be the solution given by* (3-12), *then*

$$dS_F(\dot{\gamma})(t) = 1.$$

Proof. Let us prove that $\dot{\gamma}(t) = (\nabla_{\text{hor}} S_F)_{\gamma(t)}$, which is just a consequence of S_F being a solution to the Hamilton–Jacobi equation, that is,

$$X_1(S_F)|_{\gamma(t)} = \frac{\partial S}{\partial x}\Big|_{\gamma(t)} = p_x(t).$$

However, Lemma 3 implies that $P_1(t) = p_x(t)$, so $P_1(t) = X_1(S_F)|_{\gamma(t)}$. As well,

$$X_2(S_F)|_{\gamma(t)} = \sum_{i=0}^{k} \frac{x^i(t)}{i!} \frac{\partial S}{\partial \theta_i}\Big|_{\gamma(t)} = \sum_{i=0}^{k} a_i x^i(t) = F(x(t)).$$

Also, Lemma 3 implies that $P_2(t) = F(x(t))$, so $P_2(t) = X_2(S_F)|_{\gamma(t)}$. As a consequence,

$$\nabla_{\text{hor}} S|_{\gamma(t)} := X_1(S_F)|_{\gamma(t)} X_1 + X_2(S_F)|_{\gamma(t)} X_2 = P_1(t)X_1 + P_2(t)X_2.$$

Lemma 3 implies $P_1(t)X_1 + P_2(t)X_2 = \dot{\gamma}(t)$. Thus, $\nabla_{\text{hor}} S = \dot{\gamma}(t)$ and $dS_F(v)|_q = \langle \nabla_{\text{hor}} S_F, v \rangle$ for all D_q. In particular,

$$dS_F(\dot{\gamma}) = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 1,$$

since t is the arc length parameter. □

3.3. *Proof of Proposition 2.*

Proof. It is well known that the fundamental system H_F with energy $\frac{1}{2}$ has period $L(F, I)$ given by (2-1) and the relation between $\Pi(F, I)$ and $L(F, I)$ is given by Lemma 5, see [3, p. 281]. Let $\gamma(t)$ be an x-periodic corresponding to (F, I), we are interested in seeing the change suffered by the coordinates θ_i after one $L(I, F)$. For that, we consider the change in $S(\mu, q)$ after $\gamma(t)$ travel from t to $t + L(F, I)$, in other words,

$$L(F, I) = \int_t^{t+L(F,I)} dS(\dot{\gamma}(t))\,dt = \Pi(F, I) + \sum_{i=0}^{n} i!\,a_i \Delta\theta_i(F, I). \tag{3-14}$$

The left side of the equation is a consequence of Lemma 8, and the right side is the integration term by term. Taking the derivative of (3-14) with respect to a_i to find $-(\partial/\partial a_i)\Pi(F, I) = i!\,\Delta\theta_i$, which is equivalent to (2-2).

We differentiate $\Delta\theta_i := \theta_i(t + L) - \theta_i(t)$, with respect to t, to see that $\Delta\theta_i(F, I)$ is independent of the initial point. The derivative is

$$\frac{x^i(t+L)F(x(t+L))}{\sqrt{1 - F^2(x(t+L))}} - \frac{x^i(t)F(x(t))}{\sqrt{1 - F^2(x(t))}},$$

but $x(t + L) = x(t)$. □

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