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QUASILINEAR SCHRÖDINGER EQUATIONS: GROUND STATE AND INFINITELY MANY NORMALIZED SOLUTIONS

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We study the normalized solutions for the following quasilinear Schrödinger equations:

$$-\Delta u - u\Delta u^2 + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

with prescribed mass

$$\int_{\mathbb{R}^N} u^2 = a^2.$$

We first consider the mass-supercritical case $p > 4 + \frac{4}{N}$, which has not been studied before. By using a perturbation method, we succeed to prove the existence of ground state normalized solutions, and by applying the index theory, we obtain the existence of infinitely many normalized solutions. We also obtain new existence results for the mass-critical case $p = 4 + \frac{4}{N}$ and remark on a concentration behavior for ground state solutions.

1. Introduction

We consider the equation

$$\begin{cases} i\partial_t\phi = -\Delta\phi - \sigma|\phi|^{p-2}\phi - \kappa\phi\Delta(|\phi|^2) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \phi(0,x) = \phi_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1-1}$$

where $N \geq 1$ is the space dimension, $2 < p < 2N/(N-2)^+$ and σ, κ are constants.

Equation (1-1) arises in the study of superfluid helium films (see [28; 46]), which describes the thickness and superfluid velocity of the helium films. More precisely, consider a superfluid helium film adsorbed on a substrate. Let $\psi(t,x)$ denote the condensate wave function, which is chosen proportionally so that the film thickness d and the superfluid velocity v can be defined by

$$n_0 \cdot d(t,x) = a + |\psi(t,x)|^2, \quad v(t,x) = \mathrm{Re}\left[\frac{\hbar}{M}\frac{\psi^*\nabla\psi}{|\psi(t,x)|^2}\right], \tag{1-2}$$

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where n_0 is the density number, M is the mass of helium atoms and a is the density of solid layer. Then the energy density of this quantum state consists of

$$\text{kinetic term} = \tfrac{1}{2} i\hbar(\psi^*\dot{\psi} - \dot{\psi}^*\psi),$$

the potential terms:

$$\text{bending energy term} = \frac{\hbar^2}{2M}|\nabla\psi|^2, \quad \text{chemical potential term} = -\mu|\psi|^2,$$

the van der Waals force term [2; 3]

$$\text{van der Waals term} \propto \frac{1}{d^2} - \frac{1}{d_{\min}^2} \propto \frac{1}{(a+|\psi|^2)^2} - \frac{1}{a^2},$$

and finally the surface energy term [46]

$$\text{surface term} \propto |\nabla d|^2 \propto \left|\nabla|\psi|^2\right|^2.$$

The Lagrangian density is the sum of these terms (we omit the constant $-1/a^2$, since it is irrelevant for our discussion):

$$L = \frac{1}{2} i\hbar(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{\hbar^2}{2M}|\nabla\psi|^2 + \mu|\psi|^2 - \frac{A}{2(a+|\psi|^2)^2} - \frac{B}{2}\left|\nabla|\psi|^2\right|^2.$$

From the variational principle

$$\delta \int dt \int dx\, L = 0,$$

we write the equation of motion of the condensate wavefunction, which is a Schrödinger equation describing the nonlinear dynamics of the superfluid condensate

$$i\hbar\partial_t\phi = -\frac{\hbar^2}{2M}\Delta\phi - \mu\phi - \frac{A\phi}{(1+|\phi|^2)^3} - B\phi\Delta(|\phi|^2). \tag{1-3}$$

Equation (1-3) was already obtained in [28; 46]. To solve (1-3), expanding the van der Waals term in $|\psi|^2$ to the lowest order, and simplifying as in [28], we obtain the following special case of (1-1):

$$i\partial_t\phi = -\Delta\phi - \sigma|\phi|^2\phi - \kappa\phi\Delta(|\phi|^2), \tag{1-4}$$

where σ, κ are constants.

Except superfluid helium films, equation (1-4) also appears in plasmas, see [30; 52] for more physical information. If $\kappa = 0$, equation (1-4) reduces essentially to the ordinary nonlinear Schrödinger equation, which arises in the study of standing wave solutions of the nonlinear Gross–Pitaevskii equations proposed by Gross [22] and Pitaevskii [44], and its soliton solutions have been studied widely in physics and mathematics. But when $\kappa \neq 0$, the term $\kappa(\Delta|\phi|^2)\phi$ brings new difficulties to the theoretical analysis of soliton solution of (1-4). In [28; 46], the numerical simulations of soliton solutions to (1-4) and (1-3) was given, but the theoretical research is far from clear due to the appearance of the term $\kappa(\Delta|\phi|^2)\phi$. So in this paper, we focus on the theoretical research. In the following, we will analyze the reason why the term $\kappa(\Delta|\phi|^2)\phi$ is hard to handle, and we will use some techniques to overcome these difficulties to study soliton solutions.

We set $\sigma = 1$ and $\kappa = 1$. By considering soliton wave solutions, substituting $\phi(t, x) = e^{i\lambda t}u(x)$ into (1-1), we obtain

$$-\Delta u - u\Delta u^2 + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1-5}$$

which is usually called the modified nonlinear Schrödinger equation. Usually, to study (1-5) one always considers this equation for a given parameter λ. But now we introduce a second approach.

From (1-2), we know that $|\phi(t, x)|^2$ represents the superfluid film thickness and the total quasiparticle number

$$M \propto \int_{\mathbb{R}^N} |\phi(t, x)|^2 \, \mathrm{d}x.$$

Multiplying (1-1) with ϕ^*, subtracting the complex conjugate, and integrating over space, we find

$$\partial_t M = 0,$$

which means that the total quasiparticle number remains the same constant as t changes, i.e., the law of conservation of mass. So it is natural to assume

$$\int_{\mathbb{R}^N} |\phi(t, x)|^2 \, \mathrm{d}x = \text{constant}, \tag{1-6}$$

when considering soliton wave solutions. Combining (1-5) and (1-6), we obtain

$$\begin{cases} -\Delta u - u\Delta u^2 + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = a, \end{cases} \tag{1-7}$$

and the aim is to find $u \in \mathcal{H}$ with a $\lambda \in \mathbb{R}$ such that (u, λ) satisfies (1-7) for a given $a > 0$. Here

$$\mathcal{H} = \left\{ u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 < +\infty \right\}.$$

Solutions of (1-7) are often referred to as normalized solutions, and the search for such solutions has became a hot direction in recent years. We have to admit that although the physical motivation of searching for such solutions is described as above, we don't know much about its physical meaning and application. We point out that the barrier exponent $4 + \frac{4}{N}$ is also the threshold of the stability and instability of soliton solutions. Roughly speaking, it was shown in [17] that the standing wave of (1-1) is stable for $p < 4 + \frac{4}{N}$, while it is unstable for $p \geq 4 + \frac{4}{N}$. Later in [15] the results about stability was extended to equations with $u\Delta u^2$ replaced by general quasilinear terms $u^{\alpha-1}\Delta u^{\alpha}$. Now we give the mathematical motivation of normalized solutions. Formally, to obtain the normalized solutions of (1-5), one needs to consider the corresponding energy functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \tag{1-8}$$

on a L^2 sphere

$$\tilde{\mathcal{S}}(a) := \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^N} |u|^2 = a \right\}, \tag{1-9}$$

which has particular difficulties. To derive the Palais–Smale sequence, one needs new variational methods. The derived Palais–Smale sequence may not be bounded; even if the Palais–Smale sequence is bounded, the weak limit may not be contained in the L^2 sphere (even in the radial case). Such difficulties make the study of normalized solutions of (1-7) much more complicated than the study of (1-5) with prescribed $\lambda \in \mathbb{R}$. So the search for normalized solutions is a challenging and interesting problem, and needs new variational methods.

We introduce some results about the existence of normalized solutions to the semilinear Schrödinger equation

$$-\Delta u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1-10}$$

L. Jeanjean [24] obtained a normalized solution of (1-10) using an auxiliary functional and a minimax theorem from [19]. The existence of infinitely many normalized solutions of (1-10) was later proved by T. Bartsch and S. de Valeriola [4] using a new linking geometry for the auxiliary functional. After that, N. Ikoma and K. Tanaka [23] constructed a deformation theorem suitable for the auxiliary functional, and then obtained infinitely many normalized solutions of (1-10) through Krasnoselskii index under a weaker condition on $g(u)$. Soon later, L. Jeanjean and S. S. Lu [25] obtained infinitely many normalized solutions of (1-10) under a totally different assumption on $g(u)$ which permits $g(u)$ to be just continuous. As for the least energy normalized solutions, N. Soave [48; 49] obtained the existence of ground state normalized solutions with $g(u) = |u|^{p-2}u + \mu|u|^{q-2}u$ by restraining the energy functional on a smaller manifold. For more results on normalized solutions for scalar equations and systems, we refer to [5; 6; 7; 8; 9; 20; 21; 31].

Now back to the modified nonlinear Schrödinger equation (1-5), we analyze the difficulties induced by the term $\kappa(\Delta|\phi|^2)\phi$. When considering (1-5) with $\lambda \in \mathbb{R}$ fixed, one would always study the functional

$$E_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda |u|^2) + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \tag{1-11}$$

on the space $\mathcal{H}$. It is easy to check that u is a weak solution of (1-5) if and only if

$$E'_\lambda(u)\phi = \lim_{t \to 0^+} \frac{E_\lambda(u + t\phi) - E_\lambda(u)}{t} = 0$$

for every $\phi \in C_0^\infty(\mathbb{R}^N)$. We recall, see [37] for example, that the value 22^* with

$$2^* := \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ +\infty, & N \leq 2 \end{cases}$$

corresponds to a critical exponent. Compared to (1-10), the search for solutions of (1-5) presents a major difficulty: the functional associated with the term $u\Delta u^2$

$$V(u) = \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2$$

is nondifferentiable in $\mathcal{H}$ when $N \geq 2$. To overcome this difficulty, various arguments have been developed, such as the minimization methods [35] where the nondifferentiability of E_λ does not come into play, the methods of a Nehari manifold approach [38; 39], the methods of changing variables [16; 37] which transform problem (1-5) into a semilinear one (1-10), and a perturbation method in a series of papers [36; 40; 41] which recovers the differentiability by considering a perturbed functional on a smaller function space.

However, when considering the normalized solution problem (1-7), one would find that the methods of Nehari manifold approach and changing variables are no longer applicable, since the parameter λ is unknown and the L^2-norm $\|u\|_2$ must be equal to a given number. So there are very few results on problem (1-7). Formally, a normalized solution of (1-7) can be obtained as a critical point of $I(u)$ defined by (1-8) on the set $\tilde{\mathcal{S}}(a)$. That is, a normalized solution of (1-7) is a $u \in \tilde{\mathcal{S}}(a)$ such that there exists a $\lambda \in \mathbb{R}$ satisfying

$$\text{(1-12)} \quad \int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi + 2\int_{\mathbb{R}^N} (u\phi|\nabla u|^2 + |u|^2 \nabla u \cdot \nabla \phi) + \lambda \int_{\mathbb{R}^N} u\phi - \int_{\mathbb{R}^N} |u|^{p-2} u\phi = 0$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. To proceed our paper, we introduce a sharp Gagliardo–Nirenberg inequality [1]:

$$\text{(1-13)} \qquad \int_{\mathbb{R}^N} |u|^{\frac{p}{2}} \leq \frac{C(p,N)}{\|Q_p\|_1^{(p-2)/(N+2)}} \Big(\int_{\mathbb{R}^N} |u|\Big)^{\frac{4N-(N-2)p}{2(N+2)}} \Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^{\frac{N(p-2)}{2(N+2)}}$$

for all $u \in \mathcal{E}^1$ where $2 < p < 22^*$,

$$C(p,N) = \frac{p(N+2)}{[4N-(N-2)p]^{\frac{4-N(p-2)}{2(N+2)}} [2N(p-2)]^{\frac{N(p-2)}{2(N+2)}}},$$

and the space $\mathcal{E}^q$ for $q \geq 1$ is defined by

$$\mathcal{E}^q := \{u \in L^q(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\},$$

with norm $\|u\|_{\mathcal{E}^q} := \|\nabla u\|_2 + \|u\|_q$. For embedding theorems and related properties of $\mathcal{E}^q$, we refer to [29]. Moreover, Q_p optimizes (1-13) and the unique nonnegative radially symmetric solution of the following equation [47]:

$$-\Delta u + 1 = u^{\frac{p}{2}-1} \quad \text{in } \mathbb{R}^N. \tag{1-14}$$

Strictly speaking, it has been proved in [47, Theorem 1.3] that Q_p has a compact support in $\mathbb{R}^N$ and it exactly satisfies a Dirichlet–Neumann free boundary problem. Namely, there exists an $R > 0$ such that Q_p is the unique positive solution of

$$\begin{cases} -\Delta u + 1 = u^{\frac{p}{2}-1} & \text{in } B_R, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B_R. \end{cases} \tag{1-15}$$

In what follows, if we say that u is a nonnegative solution of (1-14), then we mean that u is a solution of (1-15). By replacing u with u^2 in (1-13), one immediately obtains the following Gagliardo–Nirenberg-type inequality:

$$\int_{\mathbb{R}^N} |u|^p \le \frac{C(p,N)}{\|Q_p\|_1^{(p-2)/(N+2)}} \left(\int_{\mathbb{R}^N} |u|^2 \right)^{\frac{4N-p(N-2)}{2(N+2)}} \left(4 \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \right)^{\frac{N(p-2)}{2(N+2)}}. \tag{1-16}$$

Now we collect some known results about normalized solutions of (1-7). First, to avoid the nondifferentiability of $V(u)$, M. Colin, L. Jeanjean and M. Squassina [17] (see also [15] for general quasilinear terms) and L. Jeanjean and T. J. Luo [26] considered the minimization problem

$$\tilde{m}(a) = \inf_{u \in \tilde{S}(a)} I(u),$$

with $2 < p \le 4 + \frac{4}{N}$. Using inequality (1-16), one can find that $\tilde{m}(a) > -\infty$ when $2 < p < 4 + \frac{4}{N}$ and $\tilde{m}(a) = -\infty$ when $p > 4 + \frac{4}{N}$, since

$$\frac{N(p-2)}{2(N+2)} < 1 \quad \text{if and only if } p < 4 + \frac{4}{N}.$$

These considerations show that the exponent $4 + \frac{4}{N}$ for (1-7) plays the role of $2 + \frac{4}{N}$ in (1-10). After that, X. Y. Zeng and Y. M. Zhang [53] studied the existence and asymptotic behavior of the minimizers to

$$\inf_{u \in \tilde{S}(a)} I(u) + \int_{\mathbb{R}^N} a(x)|u|^2,$$

where $a(x)$ is an infinite potential well. In addition to these minimization approaches, L. Jeanjean, T. J. Luo and Z. Q. Wang [27] obtained another mountain-pass-type normalized solution of (1-7) through the perturbation method. We remark that all of these results on normalized solution of (1-7) have considered either the mass-subcritical or mass-critical case, i.e., $2 < p \le 4 + \frac{4}{N}$.

In this paper, we consider the mass-critical and mass-supercritical cases, i.e., $p \ge 4 + \frac{4}{N}$. To the best of our knowledge, the case of mass-supercritical has not been considered before. Actually, we obtain:

Theorem 1.1. *Assume that one of the following conditions holds*:

(H1) $N = 1, 2,\ p > 4 + \frac{4}{N},\ a > 0.$

(H2) $N = 3,\ 4 + \frac{4}{N} < p < 2^*,\ a > 0.$

Then there exists a radially symmetric positive ground state normalized solution $u \in W^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of* (1-7) *in the sense that*

$$I(u) = \inf\{I(v) : v \in \tilde{\mathcal{S}}(a),\ I|'_{\tilde{\mathcal{S}}(a)}(v) = 0,\ v \neq 0\}.$$

Theorem 1.2. *Assume that one of the following conditions holds*:

(H1′) $N = 2,\ p > 4 + \frac{4}{N},\ a > 0.$

(H2) $N = 3,\ 4 + \frac{4}{N} < p < 2^*,\ a > 0.$

Then there exists a sequence of normalized solutions $u^j \in W^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of* (1-7) *with increasing energy* $I(u^j) \to +\infty$.

Remark 1.3. (1) We state that the dimension is limited due to a lemma limitation used to control the Lagrange multipliers, see Lemma 2.2 and Remark 4.2.

(2) The difference between Theorems 1.1 and 1.2 is that we cannot prove the existence of infinitely many solutions when $N = 1$, because the failure of the compact embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for $2 < q < 2^*$. When considering the ground state, however, we are able to recover the compactness of bounded sequences using the symmetric decreasing arrangement, due to the advantage of the associated minimization $m_\mu(a)$ defined in (3-8).

Now we turn to the mass-critical case, i.e., $p = 4 + \frac{4}{N}$. Let $a_* = \|Q_{4+\frac{4}{N}}\|_1$.

Theorem 1.4. *Assume that one of the following conditions holds*:

(H3) $N \le 3,\ p = 4 + \frac{4}{N},\ a > a_*$;

(H4) $N \ge 4,\ p = 4 + \frac{4}{N},\ a_* < a < \left(\frac{N-2}{N-2-(4/N)}\right)^{\frac{N}{2}} a_*,$

Then there exists a radially symmetric positive ground state normalized solution $u \in W^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of* (1-7) *in the sense that*

$$I(u) = \inf\{I(v) : v \in \tilde{\mathcal{S}}(a),\ I|'_{\tilde{\mathcal{S}}(a)}(v) = 0,\ v \neq 0\}.$$

Remark 1.5. Recently H. Y. Ye and Y. Y. Yu [51] obtained the existence of ground state normalized solution of (1-7) under assumption (H3). As one can see, although Theorem 1.4 contains their existence result, the method we used in the current paper is totally different from theirs, while as they said in [51, Remark 1.3], they are unable to handle the case $N \ge 4$. Moreover, they also consider an asymptotic behavior, but our Theorem 1.8 is more accurate, since we give a description of u_n when $a \to a_*$.

We observe that when $p = 4 + \frac{4}{N}$, the value a_* is a threshold of the existence of normalized solution of (1-7). Actually, we have:

Proposition 1.6. *Let* $p = 4 + \frac{4}{N}$ *and* $N \geq 1$. *Then*:

(1) $\tilde{m}(a) = \begin{cases} 0, & 0 < a \leq a_*, \\ -\infty, & a > a_*. \end{cases}$

(2) *Equation* (1-7) *has no solutions for any* $0 < a \leq a_*$.

(3) *Equation* (1-7) *has at least one radially symmetric positive solution for* $a > a_*$ *and* a *is close to* a_*.

Remark 1.7. We state that (1) is a direct conclusion of [17, Theorem 1.9] and (3) is a direct conclusion of Theorem 1.4 above. Now we prove (2). Since u is a solution of (1-7), there holds (see Lemma 2.1)

$$\int_{\mathbb{R}^N} |\nabla u|^2 + (2+N)\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \frac{N(2+N)}{4(N+1)}\int_{\mathbb{R}^N} |u|^{4+\frac{4}{N}} = 0.$$

Combining with (1-16), we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 + (2+N)\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 \leq (2+N)\Big(\frac{a}{a_*}\Big)^{\frac{2}{N}}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2,$$

from which we get $u = 0$ for any $0 < a \leq a_*$, a contradiction since $\|u\|_2 = a$.

Inspired by Proposition 1.6, we enlighten a concentration behavior of the radially symmetric positive solution of (1-7) when $p = 4 + \frac{4}{N}$ and $a \to a_*$.

Theorem 1.8. *Let* $p = 4 + \frac{4}{N}$, $N \geq 1$, *and let* u_n *be a radially symmetric positive solution of* (1-7) *for* $a = a_n$ *with* $a_n > a_*$ *and* $a_n \to a_*$. *Then there exists a sequence* $y_n \in \mathbb{R}^N$ *such that up to a subsequence*, *we have*

$$\Big[\Big(\frac{Na_*}{N}\Big)^{\frac{1}{2+N}}\varepsilon_n\Big]^N u_n^2\Big(\Big(\frac{Na_*}{N}\Big)^{\frac{1}{2+N}}\varepsilon_n x + \varepsilon_n y_n\Big) \to Q_{4+\frac{4}{N}} \quad \text{in } L^q(\mathbb{R}^N) \tag{1-17}$$

for $1 \leq q < 2^*$, *where*

$$\varepsilon_n = \Big(\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\Big)^{-(2+N)} \to 0.$$

Remark 1.9. Theorem 1.8 gives a description of radially symmetric positive solution of (1-7) as the mass a_n approaches to a_* from above. Roughly speaking, it shows that for n large enough, we have

$$u_n(x) = \Big[\Big(\frac{Na_*}{N}\Big)^{\frac{1}{2+N}}\varepsilon_n\Big]^{-\frac{N}{2}} Q_{4+\frac{4}{N}}\Big(\Big(\frac{Na_*}{N}\Big)^{-\frac{1}{2+N}}\varepsilon_n^{-1}(x - \varepsilon_n^{-1}y_n)\Big).$$

The paper is organized as follows. In Section 2, we give perturbation settings and an important lemma. In Section 3A, we give some properties of the associated Pohozaev manifold. In Sections 3B and 3C, we prove the existence of ground state and infinitely many critical points for perturbed functional. In Section 4, we study the convergence of the critical points for the perturbed functional as $\mu \to 0^+$. And Theorem 1.1 for $N = 1$ is proved in Section 3B; Theorem 1.1 for $N \geq 2$ and Theorem 1.2 are proved in Section 4. Finally, in Section 5, we study the mass-critical case, and prove Theorems 1.4 and 1.8. In the Appendix, we prove some valuable results.

Throughout the paper, we use standard notations. For simplicity, we write $\int_{\mathbb{R}^N} f$ to mean the Lebesgue integral of $f(x)$ over $\mathbb{R}^N$ and $\|\cdot\|_p$ denotes the standard norm of $L^p(\mathbb{R}^N)$. We use $\to$ and $\rightharpoonup$, respectively, to denote the strong and weak convergences in the related function spaces. By $C, C_1, C_2, \ldots$ we denote positive constants unless specified otherwise.

2. Preliminary

2A. ***Perturbation setting.*** Let $I(u)$ be defined by (1-8). Observe that when $N = 1$, $I(u)$ is of class $\mathcal{C}^1$ in $W^{1,2}(\mathbb{R})$, so there is no need to perturb $I(u)$, and in this case the proof will be stated separately in the last of part Section 3B. Thus we assume $N \geq 2$. To avoid the nondifferentiability, we take the perturbation method, which has been applied firstly to unconstrained situation in [40; 41] and then to constrained situation in [27]. For $\mu \in (0, 1]$, we define

$$I_\mu(u) := \frac{\mu}{\theta}\int_{\mathbb{R}^N} |\nabla u|^\theta + I(u) \tag{2-1}$$

on the space $\mathcal{X} := W^{1,\theta}(\mathbb{R}^N) \cap W^{1,2}(\mathbb{R}^N)$ for some fixed θ satisfying

$$\frac{4N}{N+2} < \theta < \min\left\{\frac{4N+4}{N+2}, N\right\}, \quad \text{when } N \geq 3 \qquad \text{and} \qquad 2 < \theta < 3, \quad \text{when } N = 2.$$

Then $\mathcal{X}$ is a reflexive Banach space. And Lemma A.1 implies $I_\mu \in \mathcal{C}^1(\mathcal{X})$. We will consider I_μ on the constraint

$$\mathcal{S}(a) := \left\{u \in \mathcal{X} : \int_{\mathbb{R}^N} |u|^2 = a\right\}. \tag{2-2}$$

Recalling the L^2-norm preserved transform [24]

$$u \in \mathcal{S}(a) \mapsto s \star u(x) = e^{\frac{N}{2}s} u(e^s x) \in \mathcal{S}(a), \tag{2-3}$$

we define

$$\begin{aligned} Q_\mu(u) &:= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} I_\mu(s \star u) \\ &= (1+\gamma_\theta)\mu\int_{\mathbb{R}^N} |\nabla u|^\theta + \int_{\mathbb{R}^N} |\nabla u|^2 + (2+N)\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 - \gamma_p\int_{\mathbb{R}^N} |u|^p, \end{aligned}$$

where $\gamma_p = N(p-2)/2p$. And again Lemma A.1 implies $Q_\mu \in \mathcal{C}^1(\mathcal{X})$. Then we define the manifold

$$\mathcal{Q}_\mu(a) := \{u \in \mathcal{S}(a) : Q_\mu(u) = 0\}. \tag{2-4}$$

We observe that:

Lemma 2.1. *Any critical point u of $I_\mu|_{\mathcal{S}(a)}$ is contained in $\mathcal{Q}_\mu(a)$.*

Proof. By [11, Lemma 3], there exists a $\lambda \in \mathbb{R}$ such that

$$I'_\mu(u) + \lambda u = 0 \quad \text{in } \mathcal{X}^*. \tag{2-5}$$

On one hand, testing (2-5) with $x \cdot \nabla u$ (see [10, Proposition 1] for details), we obtain

$$\begin{aligned} 0 = \frac{\theta - N}{\theta}\mu \int_{\mathbb{R}^N} |\nabla u|^\theta + \frac{2-N}{2}\int_{\mathbb{R}^N}|\nabla u|^2 \\ + (2-N)\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 + \frac{N}{p}\int_{\mathbb{R}^N}|u|^p - \frac{N}{2}\lambda\int_{\mathbb{R}^N}|u|^2. \end{aligned} \tag{2-6}$$

On the other hand, testing (2-5) with u, we obtain

$$0 = \mu\int_{\mathbb{R}^N}|\nabla u|^\theta + \int_{\mathbb{R}^N}|\nabla u|^2 + 4\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 - \int_{\mathbb{R}^N}|u|^p + \lambda\int_{\mathbb{R}^N}|u|^2. \tag{2-7}$$

Combining (2-6) and (2-7), we have $Q_\mu(u) = 0$. Then $u \in \mathcal{Q}_\mu(a)$. □

2B. *An important lemma.* We need the following result, which is crucially used to control the possible values of the Lagrange parameters.

Lemma 2.2. *Suppose $u \neq 0$ is a critical point of $I_\mu|_{\mathcal{S}(a)}$ with $0 \leq \mu \leq 1$, that is, there exists a $\lambda \in \mathbb{R}$ such that*

$$I'_\mu(u) + \lambda u = 0 \quad \text{in } \mathcal{X}^*.$$

And assume that one of the following conditions holds:

(a) $1 \leq N \leq 2,\ p \geq 4 + \frac{4}{N},\ a > 0.$

(b) $N = 3,\ 4 + \frac{4}{N} \leq p \leq 2^*,\ a > 0.$

(c) $N \geq 4,\ p = 4 + \frac{4}{N},\ 0 < a < \left(\frac{N-2}{N-2-(4/N)}\right)^{\frac{N}{2}} a_*.$

Then $\lambda > 0$.

Proof. By combining $Q_\mu(u) = 0$ and (2-7), we obtain

$$\begin{aligned} \frac{\lambda N(p-2)}{2p}a = \left(1 + \frac{N(p-\theta)}{p\theta}\right)\mu\int_{\mathbb{R}^N}|\nabla u|^\theta \\ + \frac{2N-(N-2)p}{2p}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{4N-(N-2)p}{2p}\int_{\mathbb{R}^N}u^2|\nabla u|^2. \end{aligned}$$

So if condition (a) holds, we immediately get $\lambda > 0$. Now suppose condition (b) holds. Again from $Q_\mu(u) = 0$ and (2-7), and using inequality (1-16), we obtain

$$\begin{aligned}\lambda a &= \frac{N(\theta-2)}{2\theta}\mu\int_{\mathbb{R}^N}|\nabla u|^\theta + (N-2)\int_{\mathbb{R}^N}u^2|\nabla u|^2 - \frac{N^2-2N-4}{4(N+1)}\int_{\mathbb{R}^N}|u|^{4+\frac{4}{N}}\\ &\geq \Big[(N-2) - \Big(N-2-\frac{4}{N}\Big)\Big(\frac{a}{a_*}\Big)^{\frac{2}{N}}\Big]\int_{\mathbb{R}^N}u^2|\nabla u|^2 > 0,\end{aligned}$$

which gives $\lambda > 0$. □

3. The critical points of perturbed functional

Throughout this section we assume $p > 4 + \frac{4}{N}$.

3A. *Properties of $\mathcal{Q}_\mu(a)$.*

Lemma 3.1. *Let $0 < \mu \leq 1$, then $\mathcal{Q}_\mu(a)$ is a $\mathcal{C}^1$-submanifold of codimension 1 in $\mathcal{S}(a)$, and hence a $\mathcal{C}^1$-submanifold of codimension 2 in $\mathcal{X}$.*

Proof. As a subset of $\mathcal{X}$, the set $\mathcal{Q}_\mu(a)$ is defined by the two equations $G(u) = 0$ and $Q_\mu(u) = 0$, where

$$G(u) = a - \int_{\mathbb{R}^N}|u|^2,$$

and clearly $G \in \mathcal{C}^1(\mathcal{X})$. We have to check that

$$\mathrm{d}(Q_\mu, G) : \mathcal{X} \to \mathbb{R}^2 \quad \text{is surjective.} \tag{3-1}$$

If this is not true, $\mathrm{d}Q_\mu(u)$ and $\mathrm{d}G(u)$ are linearly dependent, i.e., there exists $\nu \in \mathbb{R}$ such that

$$\begin{aligned}\theta(1+\gamma_\theta)\mu\int_{\mathbb{R}^N}|\nabla u|^{\theta-2}\nabla u\cdot\nabla\phi + 2\int_{\mathbb{R}^N}\nabla u\cdot\nabla\phi\\ + (2+N)2\int_{\mathbb{R}^N}(|u|^2\nabla u\cdot\nabla\phi + u\phi|\nabla u|^2) - p\gamma_p\int_{\mathbb{R}^N}|u|^{p-2}u\phi = 2\nu\int_{\mathbb{R}^N}u\phi\end{aligned} \tag{3-2}$$

for any $\phi \in \mathcal{X}$. Similar to Lemma 2.1, taking $\phi = x\cdot\nabla u$ and $\phi = u$, we obtain

$$\begin{aligned}\theta(1+\gamma_\theta)^2\mu\int_{\mathbb{R}^N}|\nabla u|^\theta\\ + 2\int_{\mathbb{R}^N}|\nabla u|^2 + (2+N)^2\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 - p\gamma_p^2\int_{\mathbb{R}^N}|u|^p = 0.\end{aligned} \tag{3-3}$$

Since $Q_\mu(u) = 0$, we get

$$\begin{aligned}(p\gamma_p - \theta - \theta\gamma_\theta)(1+\gamma_\theta)\mu\int_{\mathbb{R}^N}|\nabla u|^\theta + (p\gamma_p - 2)\int_{\mathbb{R}^N}|\nabla u|^2\\ + (p\gamma_p - 2 - N)(2+N)\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 = 0,\end{aligned} \tag{3-4}$$

which means $u = 0$ since $p\gamma_p > \theta + \theta\gamma_\theta$ and $p\gamma_p > 2 + N$. That contradicts with $u \in \mathcal{S}(a)$. □

Lemma 3.2. *For any $0 < \mu \le 1$ and any $u \in \mathcal{X} \setminus \{0\}$, the following statements hold.*

(1) *There exists a unique number $s_\mu(u) \in \mathbb{R}$ such that $Q_\mu(s_\mu(u) \star u) = 0$.*

(2) *$I_\mu(s \star u)$ is strictly increasing in $s \in (-\infty, s_\mu(u))$ and is strictly decreasing in $s \in (s_\mu(u), +\infty)$, then*

$$\lim_{s\to-\infty} I_\mu(s \star u) = 0^+, \qquad \lim_{s\to+\infty} I_\mu(s \star u) = -\infty, \qquad I_\mu(s_\mu(u) \star u) > 0.$$

(3) *$s_\mu(u) < 0$ if and only if $Q_\mu(u) < 0$.*

(4) *The map $u \in \mathcal{X} \setminus \{0\} \mapsto s_\mu(u) \in \mathbb{R}$ is of class $\mathcal{C}^1$.*

(5) *$s_\mu(u)$ is an even function with respect to $u \in \mathcal{X} \setminus \{0\}$.*

Proof. (1) By direct computation, one can check that

$$\begin{aligned} Q_\mu(s\star u) :=& \frac{\mathrm{d}}{\mathrm{d}s} I_\mu(s\star u) \\ =& (1+\gamma_\theta)\mu e^{\theta(1+\gamma_\theta)s}\int_{\mathbb{R}^N}|\nabla u|^\theta + e^{2s}\int_{\mathbb{R}^N}|\nabla u|^2 \\ &+ (2+N)\, e^{(2+N)s}\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 - \gamma_p e^{p\gamma_p s}\int_{\mathbb{R}^N}|u|^p \\ =& e^{p\gamma_p s}\Big[(1+\gamma_\theta)\mu e^{-(p\gamma_p-\theta-\theta\gamma_\theta)s}\int_{\mathbb{R}^N}|\nabla u|^\theta + e^{-(p\gamma_p-2)s}\int_{\mathbb{R}^N}|\nabla u|^2 \\ &+ (2+N)\, e^{-(p\gamma_p-2-N)s}\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 - \gamma_p\int_{\mathbb{R}^N}|u|^p\Big]. \end{aligned} \tag{3-5}$$

Since $p\gamma_p > \theta + \theta\gamma_\theta$ and $p\gamma_p > 2 + N$ when $p > 4 + \frac{4}{N}$, $Q_\mu(s \star u) = 0$ has only one solution $s_\mu(u) \in \mathbb{R}$.

(2) From (1), $Q_\mu(s \star u) > 0$ when $s < s_\mu(u)$ and $Q_\mu(s \star u) < 0$ when $s > s_\mu(u)$. So $I_\mu(s \star u)$ is strictly increasing in $s \in (-\infty, s_\mu(u))$ and is strictly decreasing in $s \in (s_\mu(u), +\infty)$. Obviously,

$$\lim_{s\to-\infty} I_\mu(s \star u) = 0^+, \qquad \lim_{s\to+\infty} I_\mu(s \star u) = -\infty,$$

which implies that

$$I_\mu(s_\mu(u) \star u) = \max_{s\in\mathbb{R}} I_\mu(s \star u) > 0.$$

(3) It can be obtained directly from (2).

(4) Let $\Phi_\mu(s, u) = Q_\mu(s \star u)$. Then $\Phi_\mu(s_\mu(u), u) = 0$. Moreover,

$$\begin{aligned} \frac{\partial}{\partial s}\Phi_\mu(s, u) =& \theta(1+\gamma_\theta)^2\mu e^{\theta(1+\gamma_\theta)s}\int_{\mathbb{R}^N}|\nabla u|^\theta + 2e^{2s}\int_{\mathbb{R}^N}|\nabla u|^2 \\ &+ (2+N)^2 e^{(2+N)s}\int_{\mathbb{R}^N}|u|^2|\nabla u|^2 - p\gamma_p^2 e^{p\gamma_p s}\int_{\mathbb{R}^N}|u|^p. \end{aligned} \tag{3-6}$$

Combining with $Q_\mu(s_\mu(u)\star u)=0$, we obtain

$$\begin{aligned}(3\text{-}7)\quad &\frac{\partial}{\partial s}\Phi_\mu(s_\mu(u),u)\\ &=-(p\gamma_p-\theta-\theta\gamma_\theta)(1+\gamma_\theta)\mu\int_{\mathbb{R}^N}|\nabla u|^\theta-(p\gamma_p-2)\int_{\mathbb{R}^N}|\nabla u|^2\\ &\qquad -(p\gamma_p-2-N)(2+N)\int_{\mathbb{R}^N}|u|^2|\nabla u|^2\\ &<0.\end{aligned}$$

Then the implicit function theorem [14] implies that the map $u\mapsto s_\mu(u)$ is of class $\mathcal{C}^1$.

(5) Since

$$Q_\mu(s_\mu(u)\star(-u))=Q_\mu(-s_\mu(u)\star u)=Q_\mu(s_\mu(u)\star u)=0,$$

by the uniqueness, there is $s_\mu(-u)=s_\mu(u)$. □

3B. *Ground state critical point of $I_\mu|_{\mathcal{S}(a)}$.* In this subsection, we consider a minimization problem

$$m_\mu(a):=\inf_{u\in\mathcal{Q}_\mu(a)} I_\mu(u). \tag{3-8}$$

From Lemma 2.1, we know that if $m_\mu(a)$ is achieved, then the minimizer is a ground state critical point of $I_\mu|_{\mathcal{S}(a)}$. We have:

Lemma 3.3. (1) $\mathcal{D}(a):=\inf_{0<\mu\le 1,u\in\mathcal{Q}_\mu(a)}\int_{\mathbb{R}^N}|u|^2|\nabla u|^2>0$ *is independent of* μ.

(2) *If* $\sup_{n\ge1}I_\mu(u_n)<+\infty$ *for* $u_n\in\mathcal{Q}_\mu(a)$, *then*

$$\sup_{n\ge1}\max\Big\{\mu\int_{\mathbb{R}^N}|\nabla u_n|^\theta,\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2,\int_{\mathbb{R}^N}|\nabla u_n|^2\Big\}<+\infty.$$

Proof. (1) For any $u\in\mathcal{Q}_\mu(a)$, by the inequality (1-16), there holds

$$\begin{aligned}(3\text{-}9)\quad &(2+N)\int_{\mathbb{R}^N}|u|^2|\nabla u|^2\\ &\le\gamma_p\int_{\mathbb{R}^N}|u|^p\le K(p,N)\gamma_p\,a^{\frac{4N-p(N-2)}{2(N+2)}}\Big(\int_{\mathbb{R}^N}|u|^2|\nabla u|^2\Big)^{\frac{N(p-2)}{2(N+2)}}.\end{aligned}$$

Since $\frac{N(p-2)}{2(N+2)}>1$, we obtain $\mathcal{D}(a)>0$.

(2) For any $u\in\mathcal{Q}_\mu(a)$, there is

$$\begin{aligned}(3\text{-}10)\quad I_\mu(u)&=I_\mu(u)-\frac{1}{p\gamma_p}Q_\mu(u)\\ &=\frac{p\gamma_p-\theta-\theta\gamma_\theta}{\theta p\gamma_p}\mu\int_{\mathbb{R}^N}|\nabla u|^\theta+\frac{p\gamma_p-2}{2p\gamma_p}\int_{\mathbb{R}^N}|\nabla u|^2\\ &\qquad+\frac{p\gamma_p-2-N}{p\gamma_p}\int_{\mathbb{R}^N}|u|^2|\nabla u|^2.\end{aligned}$$

So the conclusion holds. □

Remark 3.4. Form (3-10), we see that

$$m_\mu(a) \geq \mathcal{D}_0(a) := \frac{p\gamma_p - 2 - N}{p\gamma_p}\mathcal{D}(a) > 0 \quad \text{for all } \mu \in (0, 1].$$

Then we have:

Lemma 3.5. *There exists a small $\rho > 0$ independent of μ such that for any $0 < \mu \leq 1$, we have that*

$$0 < \sup_{u \in B_\mu(\rho,a)} I_\mu(u) < \mathcal{D}_0(a) \qquad \text{and} \qquad I_\mu(u),\, Q_\mu(u) > 0 \quad \text{for all } u \in B_\mu(\rho, a),$$

where

$$B_\mu(\rho, a) = \Big\{ u \in \mathcal{S}(a) : \mu \int_{\mathbb{R}^N} |\nabla u|^\theta + \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \leq \rho \Big\}.$$

Proof. From the definition of I_μ, we have

$$\sup_{u \in B_\mu(\rho,a)} I_\mu(u) \leq \max\Big\{\frac{1}{\theta}, \frac{1}{2}, 1\Big\}\rho < \mathcal{D}_0(a), \tag{3-11}$$

where $\rho > 0$ is small and is independent of μ. On the other hand, by inequality (1-16), for any $u \in \partial B_\mu(r, a)$ with $0 < r < \rho$ for a smaller $\rho > 0$, we have

$$\begin{aligned} \inf_{\partial B_\mu(r,a)} I_\mu(u) &\geq \frac{\mu}{\theta} \int_{\mathbb{R}^N} |\nabla u|^\theta + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \\ &\qquad - \frac{K(p, N)}{p} a^{\frac{4N - p(N-2)}{2(N+2)}} \Big(\int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \Big)^{\frac{N(p-2)}{2(N+2)}} \\ &\geq \frac{\mu}{\theta} \int_{\mathbb{R}^N} |\nabla u|^\theta + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + C \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 \\ &\geq C_1(a, \theta, p, N) r > 0, \\ \inf_{\partial B_\mu(r,a)} Q_\mu(u) &\geq C_2(a, \theta, p, N) r > 0. \end{aligned}$$

□

To find a Palais–Smale sequence, we consider an auxiliary functional as the one in [24]:

$$J_\mu(s, u) := I_\mu(s \star u) : \mathbb{R} \times \mathcal{X} \to \mathbb{R}. \tag{3-12}$$

We study J_μ on the radial space $\mathbb{R} \times \mathcal{S}_r(a)$ with

$$\mathcal{S}_r(a) := \mathcal{S}(a) \cap \mathcal{X}_r, \quad \mathcal{X}_r = W^{1,\theta}_{\text{rad}}(\mathbb{R}^N) \cap W^{1,2}_{\text{rad}}(\mathbb{R}^N).$$

Notice that J_μ is of class $\mathcal{C}^1$. By the symmetric critical point principle [43], a Palais–Smale sequence for $J_\mu|_{\mathbb{R} \times \mathcal{S}_r(a)}$ is also a Palais–Smale sequence for $J_\mu|_{\mathbb{R} \times \mathcal{S}(a)}$. Denoting the closed sublevel set by

$$I^c_\mu = \{u \in \mathcal{S}(a) : I_\mu(u) \leq c\}, \tag{3-13}$$

we introduce the minimax class

$$\Gamma_\mu := \{\gamma = (\alpha, \beta) \in \mathcal{C}([0,1], \mathbb{R} \times \mathcal{S}_r(a)) : \gamma(0) \in \{0\} \times B_\mu(\rho, a), \gamma(1) \in \{0\} \times I_\mu^0\},$$

with the associated minimax level

$$\sigma_\mu(a) := \inf_{\gamma \in \Gamma_\mu} \sup_{t \in [0,1]} J_\mu(\gamma(t)). \tag{3-14}$$

Lemma 3.6. *For any* $0 < \mu \le 1$, *we have* $m_\mu(a) = \sigma_\mu(a)$.

Proof. For any $\gamma = (\alpha, \beta) \in \Gamma_\mu$, let us consider the function

$$f_\gamma(t) := Q_\mu(\alpha(t) \star \beta(t)).$$

We have $f_\gamma(0) = Q_\mu(\beta(0)) > 0$ by Lemma 3.5. We *claim* that $f_\gamma(1) = Q_\mu(\beta(1)) < 0$: indeed, since $I_\mu(\beta(1)) < 0$, we have that $s_\mu(\beta(1)) < 0$, which by Lemma 3.2 means that $Q_\mu(\beta(1)) < 0$. Moreover, f_γ is continuous, and hence we deduce that there exists $t_\gamma \in (0,1)$ such that $f_\gamma(t_\gamma) = 0$, namely $\alpha(t_\gamma) \star \beta(t_\gamma) \in \mathcal{Q}_\mu(a)$. So

$$\max_{t \in [0,1]} J_\mu(\gamma(t)) \ge I_\mu(\alpha(t_\gamma) \star \beta(t_\gamma)) \ge m_\mu(a)$$

and consequently $\sigma_\mu(a) \ge m_\mu(a)$.

On the other hand, if $u \in \mathcal{Q}_\mu(a) \cap \mathcal{X}_r$, then

$$\gamma_u(t) := \big(0, ((1-t)s_0 + ts_1) \star u\big) \in \Gamma_\mu,$$

where $s_0 \ll -1$ and $s_1 \gg 1$. Since

$$I_\mu(u) \ge \max_{t \in [0,1]} I_\mu\big(((1-t)s_0 + ts_1) \star u\big) \ge \sigma_\mu(a),$$

there holds

$$m_\mu^r(a) := \inf_{u \in \mathcal{Q}_\mu(a) \cap \mathcal{X}_r} I_\mu(u) \ge \sigma_\mu(a).$$

Finally the inequality $m_\mu(a) \ge m_\mu^r(a)$ can be obtained easily by using the symmetric decreasing rearrangement, see [33]. $\square$

Remark 3.7. For any $0 < \mu_1 < \mu_2 \le 1$, since $I_{\mu_2}(u) \ge I_{\mu_1}(u)$ and $\Gamma_{\mu_2} \subset \Gamma_{\mu_1}$, there holds

$$\begin{aligned} \sigma_{\mu_2}(a) = \inf_{\gamma \in \Gamma_{\mu_2}} \sup_{t \in [0,1]} J_{\mu_2}(\gamma(t)) &\ge \inf_{\gamma \in \Gamma_{\mu_2}} \sup_{t \in [0,1]} J_{\mu_1}(\gamma(t)) \\ &\ge \inf_{\gamma \in \Gamma_{\mu_1}} \sup_{t \in [0,1]} J_{\mu_1}(\gamma(t)) = \sigma_{\mu_1}(a), \end{aligned}$$

i.e., $\sigma_\mu(a)$ is nondecreasing with respect to $\mu \in (0,1]$.

Definition A [19, Definition 3.1]. Let B be a closed subset of X. We say that a class $\mathcal{F}$ of compact subsets of X is a homotopy stable family with boundary B provided:

(a) Every set in $\mathcal{F}$ contains B.

(b) For any set A in $\mathcal{F}$ and any $\eta \in \mathcal{C}([0,1] \times X, X)$ satisfying $\eta(t,x) = x$ for all (t,x) in $(\{0\} \times X) \cup ([0,1] \times B)$ we have that $\eta(1,A) \subset \mathcal{F}$.

We remark that the case $B = \varnothing$ is admissible.

Theorem B [19, Theorem 5.2]. *Let ϕ be a $\mathcal{C}^1$-functional on a complete connected C^1-Finsler manifold X and consider a homotopy stable family $\mathcal{F}$ with an extended closed boundary B. Set $c = c(\phi, \mathcal{F})$ and let F be a closed subset of X satisfying*

$$A \cap F \setminus B \neq \varnothing \quad \textit{for all } A \in \mathcal{F} \tag{3-15}$$

and

$$\sup \phi(B) \le c \le \inf \phi(F). \tag{3-16}$$

Then for any sequence of sets $A_n \subset \mathcal{F}$ such that $\lim_{n\to\infty} \sup_{A_n} \phi = c$, there exists a sequence $x_n \subset X \setminus B$ such that

(1) $\lim_{n\to\infty} \phi(x_n) = c$,

(2) $\lim_{n\to\infty} \|\mathrm{d}\phi(x_n)\| = 0$,

(3) $\lim_{n\to\infty} \mathrm{dist}(x_n, F) = 0$,

(4) $\lim_{n\to\infty} \mathrm{dist}(x_n, A_n) = 0$.

Now we establish a technical result showing the existence of a Palais–Smale sequence of $\sigma_\mu(a)$ with an additional property.

Lemma 3.8. *For any fixed $\mu \in (0,1]$, there exists a sequence $u_n \in \mathcal{S}_r(a)$ such that*

$$I_\mu(u_n) \to \sigma_\mu(a), \quad I_\mu|'_{\mathcal{S}(a)}(u_n) \to 0, \quad Q_\mu(u_n) \to 0 \quad \textit{and} \quad u_n^- \to 0 \textit{ a.e. in } \mathbb{R}^N.$$

Proof. Using Definition A, it is easy to check that $\mathcal{F} = \{A = \gamma([0,1]) : \gamma \in \Gamma_\mu\}$ is a homotopy stable family of compact subsets of $X = \mathbb{R} \times \mathcal{S}_\mu^r$ with boundary $B = (\{0\} \times B_\mu(\rho, a)) \cup (\{0\} \times I_\mu^0)$. Set $F = \{J_\mu \ge \sigma_\mu(a)\}$, then the assumptions (3-15) and (3-16) with $\phi = J_\mu$ and $c = \sigma_\mu(a)$ are satisfied. Therefore, taking a minimizing sequence $\{\gamma_n = (0, \beta_n)\} \subset \Gamma_\mu$ with $\beta_n \ge 0$ a.e. in $\mathbb{R}^N$, there exists a Palais–Smale sequence $\{(s_n, w_n)\} \subset \mathbb{R} \times \mathcal{S}_r(a)$ for $J_\mu|_{\mathbb{R}\times\mathcal{S}_r(a)}$ at level $\sigma_\mu(a)$, that is,

$$\partial_s J_\mu(s_n, w_n) \to 0 \quad \text{and} \quad \partial_u J_\mu(s_n, w_n) \to 0 \qquad \text{as } n \to \infty, \tag{3-17}$$

with the additional property that

$$|s_n| + \mathrm{dist}_{\mathcal{X}}(w_n, \beta_n([0,1])) \to 0 \quad \text{as } n \to \infty. \tag{3-18}$$

Let $u_n = s_n \star w_n$. The first condition in (3-17) reads $Q_\mu(u_n) \to 0$, while the second condition gives

$$
\begin{aligned}
\text{(3-19)} \qquad \|dI_\mu|_{\mathcal{S}(a)}(u_n)\| &= \sup_{\psi \in T_{u_n}\mathcal{S}(a), \|\psi\|_{\mathcal{X}} \le 1} |dI_\mu(u_n)[\psi]| \\
&= \sup_{\psi \in T_{u_n}\mathcal{S}(a), \|\psi\|_{\mathcal{X}} \le 1} |dI_\mu(s_n \star w_n)[s_n \star (-s_n) \star \psi]| \\
&= \sup_{\psi \in T_{u_n}\mathcal{S}(a), \|\psi\|_{\mathcal{X}} \le 1} |\partial_u J_\mu(s_n, w_n)[(-s_n) \star \psi]| \\
&\le \|\partial_u J_\mu(s_n, w_n)\| \sup_{\psi \in T_{u_n}\mathcal{S}(a), \|\psi\|_{\mathcal{X}} \le 1} |(-s_n) \star \psi| \\
&\le C\|\partial_u J_\mu(s_n, w_n)\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Finally, (3-18) implies that $u_n^- \to 0$ a.e. in $\mathbb{R}^N$. □

Now we show the compactness of the Palais–Smale sequence obtained in Lemma 3.8.

Lemma 3.9. *For any fixed* $\mu \in (0, 1]$, *let* u_n *be a sequence obtained in Lemma 3.8. Then there exists a* $u_\mu \in \mathcal{X} \setminus \{0\}$ *and a* $\lambda_\mu \in \mathbb{R}$ *such that up to a subsequence*,

$$\text{(3-20)} \qquad u_n \rightharpoonup u_\mu \ge 0 \quad \textit{in } \mathcal{X},$$

$$\text{(3-21)} \qquad I_\mu(u_\mu) = \sigma_\mu(a) \quad \textit{and} \quad I'_\mu(u_\mu) + \lambda_\mu u_\mu = 0.$$

Moreover, *if* $\lambda_\mu \ne 0$, *we have that*

$$u_n \to u_\mu \quad \textit{in } \mathcal{X}.$$

Proof. From Lemma 3.3 and Remark 3.7, we know that u_n is bounded in $\mathcal{X}_r$. Thus by [13, Propositon 1.7.1], we conclude that up to a subsequence, there exists a $u_\mu \in \mathcal{X}_r$ such that

$$
\begin{aligned}
u_n &\rightharpoonup u_\mu && \text{in } \mathcal{X} \text{ and in } L^2(\mathbb{R}^N), \\
u_n &\to u_\mu && \text{in } L^q(\mathbb{R}^N) \text{ for all } q \in (2, 2^*), \\
u_n &\to u_\mu \ge 0 && \text{a.e. in } \mathbb{R}.
\end{aligned}
$$

By interpolation and inequality (1-16), we have that

$$u_n \to u_\mu \quad \text{in } L^q(\mathbb{R}^N) \text{ for all } q \in (2, 22^*).$$

We claim that $u_\mu \ne 0$. Assume $u_\mu = 0$. Then as $n \to \infty$, we write

$$
\begin{aligned}
(1+\gamma_\theta)\mu \int_{\mathbb{R}^N} |\nabla u_n|^\theta + \int_{\mathbb{R}^N} |\nabla u_n|^2 + (2+N) \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \\
= Q_\mu(u_n) + \gamma_p \int_{\mathbb{R}^N} |u_n|^p \to 0,
\end{aligned}
$$

which implies that $I_\mu(u_n) \to 0$, in contradiction with Remark 3.4. So $u_\mu \neq 0$. By [11, Lemma 3], it follows from $I_\mu|'_{\mathcal{S}(a)}(u_n) \to 0$ that there exists a sequence $\lambda_n \in \mathbb{R}$ such that

$$I'_\mu(u_n) + \lambda_n u_n \to 0 \quad \text{in } \mathcal{X}^*. \tag{3-22}$$

Hence $\lambda_n = \frac{1}{a} I'_\mu(u_n)[u_n] + o_n(1)$ is bounded in $\mathbb{R}$, and we assume, up to a subsequence, $\lambda_n \to \lambda_\mu$. Since u_n is bounded, we have $I'_\mu(u_n) + \lambda_\mu u_n \to 0$. From Lemma A.2, we see that

$$I'_\mu(u_\mu) + \lambda_\mu u_\mu = 0. \tag{3-23}$$

Then testing (3-23) with $x \cdot \nabla u$ and u, we obtain $Q_\mu(u_\mu) = 0$. It follows that

$$Q_\mu(u_n) + \gamma_p \int_{\mathbb{R}^N} |u_n|^p \to Q_\mu(u_\mu) + \gamma_p \int_{\mathbb{R}^N} |u_\mu|^p.$$

Then using the weak lower semicontinuous property (see [17, Lemma 4.3]) there must be

$$\mu \int_{\mathbb{R}^N} |\nabla u_n|^\theta \to \mu \int_{\mathbb{R}^N} |\nabla u_\mu|^\theta, \tag{3-24}$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \to \int_{\mathbb{R}^N} |\nabla u_\mu|^2, \tag{3-25}$$

$$\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \to \int_{\mathbb{R}^N} |u_\mu|^2 |\nabla u_\mu|^2. \tag{3-26}$$

That gives $I_\mu(u_\mu) = \lim_{n\to\infty} I_\mu(u_n) = \sigma_\mu(a)$. Moreover, from (3-24)–(3-26):

$$I'_\mu(u_n)[u_n] \to I'_\mu(u_\mu)[u_\mu]. \tag{3-27}$$

Thus combining (3-27) with (3-22) and (3-23), there holds $\lambda_\mu \|u_n\|_2^2 \to \lambda_\mu \|u_\mu\|_2^2$. So $\lambda_\mu \neq 0$ implies that $u_n \to u_\mu$ in $\mathcal{X}$. □

Based on the above preliminary works, we conclude that:

Theorem 3.10. *For any fixed $\mu \in (0, 1]$, there exists a $u_\mu \in \mathcal{X}_r \setminus \{0\}$ and a $\lambda_\mu \in \mathbb{R}$ such that*

$$I'_\mu(u_\mu) + \lambda_\mu u_\mu = 0,$$
$$I_\mu(u_\mu) = m_\mu(a), \quad Q_\mu(u_\mu) = 0, \qquad 0 < \|u_\mu\|_2^2 \le a, \quad u_\mu \ge 0.$$

Moreover, if $\lambda_\mu \neq 0$, we have that $\|u_\mu\|_2^2 = a$, i.e., $m_\mu(a)$ is achieved, and u_μ is a ground state critical point of $I_\mu|_{\mathcal{S}(a)}$.

Proof of Theorem 1.1 for $N = 1$. When $N = 1$, there is $W^{1,2}(\mathbb{R}) \hookrightarrow \mathcal{C}^{0,\alpha}(\mathbb{R})$, so $V(u)$ and hence $I(u)$ is of class $\mathcal{C}^1(W^{1,2}(\mathbb{R}))$. Then one can follow the process in this subsection to prove Theorem 1.1 by taking $\mu = 0$, but we claim that there needs some modifications, since the compact embedding $W^{1,2}_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $2 < q < 2^*$ does not hold when $N = 1$. However, the compactness still holds for bounded sequences of radially decreasing functions (see, e.g., [13, Propositon 1.7.1]). So we need to confirm that the Palais–Smale sequence obtained in Lemma 3.8 consists of radially decreasing functions. Then it is natural to replace the minimizing sequence $\gamma_n = (0, \beta_n)$ chosen in Lemma 3.8 with $\bar{\gamma}_n := (0, \bar{\beta}_n)$, where $\bar{\beta}_n(t) = |\beta_n(t)|^*$ is the symmetric decreasing rearrangement of $\beta_n(t)$ at every $t \in [0, 1]$. This is a natural candidate to be minimizing sequence, with $\bar{\beta}_n(t) \geq 0$, radially symmetric and decreasing for every $t \in [0, 1]$. In order to check that $\bar{\gamma}_n \in \Gamma_0$, we have to check that each $\bar{\beta}_n$ is continuous on $[0, 1]$, which has been proved in [18] (for more argument we refer to [48, Remark 5.2]). As a result, Theorem 3.10 with $\mu = 0$ holds, and combining with Lemma 2.2, we obtain Theorem 1.1 immediately. □

3C. *Infinitely many critical points of $I_\mu|_{\mathcal{S}(a)}$.* This subsection concerns the existence of infinitely many radial critical points of $I_\mu|_{\mathcal{S}(a)}$. Denote $\tau(u) = -u$ and let $Y \subset \mathcal{X}$. A set $A \subset Y$ is called τ-invariant if $\tau(A) = A$. A homotopy $\eta : [0, 1] \times Y \to Y$ is τ-equivariant if $\eta(t, \tau(u)) = \tau(\eta(t, u))$ for all $(t, u) \in [0, 1] \times Y$.

Definition C [19, Definition 7.1]. Let B be a closed τ-invariant subset of Y. A class $\mathcal{G}$ of compact subsets of Y is said to be a τ-homotopy stable family with boundary B provided:

(a) Every set in $\mathcal{G}$ is τ-invariant.

(b) Every set in $\mathcal{G}$ contains B.

(c) For any set $A \in \mathcal{G}$ and any τ-equivariant homotopy $\eta \in \mathcal{C}([0, 1] \times Y, Y)$ satisfying $\eta(t, x) = x$ for all (t, x) in $(\{0\} \times Y) \cup ([0, 1] \times B)$ we have that $\eta(1, A) \subset \mathcal{G}$.

Following [25, Section 5], we consider the functional $K_\mu : \mathcal{X} \setminus \{0\} \to \mathbb{R}$ defined by

$$\begin{aligned} K_\mu(u) := I_\mu(s_\mu(u) \star u) = \frac{\mu}{\theta} e^{\theta(1+\gamma_\theta) s_\mu(u)} \int_{\mathbb{R}^N} |\nabla u|^\theta + \frac{1}{2} e^{2 s_\mu(u)} \int_{\mathbb{R}^N} |\nabla u|^2 \\ + e^{(2+N) s_\mu(u)} \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 - \frac{1}{p} e^{p \gamma_p s_\mu(u)} \int_{\mathbb{R}^N} |u|^p, \end{aligned} \tag{3-28}$$

where $s_\mu(u)$ is given by Lemma 3.2. Then we see that $K_\mu(u)$ is τ-invariant. Moreover, inspired by [50, Proposition 2.9], there holds:

Lemma 3.11. *The functional K_μ is of class $\mathcal{C}^1$ and*

$$\begin{aligned} K'_\mu(u)[\phi] &= \mu e^{\theta(1+\gamma_\theta) s_\mu(u)} \int_{\mathbb{R}^N} |\nabla u|^{\theta-2} \nabla u \cdot \nabla \phi + e^{2 s_\mu(u)} \int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi \\ &\quad + 2 e^{(2+N) s_\mu(u)} \int_{\mathbb{R}^N} (u \phi |\nabla u|^2 + |u|^2 \nabla u \cdot \nabla \phi) - e^{p \gamma_p s_\mu(u)} \int_{\mathbb{R}^N} |u|^{p-2} u \phi \\ &= I'_\mu(s_\mu(u) \star u)[s_\mu(u) \star \phi] \end{aligned}$$

for any $u \in \mathcal{X} \setminus \{0\}$ and $\phi \in \mathcal{X}$.

Proof. Let $u \in \mathcal{X} \setminus \{0\}$ and $\phi \in \mathcal{X}$. We estimate the term

$$K_\mu(u_t) - K_\mu(u) = I_\mu(s_t \star u_t) - I_\mu(s_0 \star u),$$

where $u_t = u + t\phi$ and $s_t = s_\mu(u_t)$ with $|t|$ small enough. By the mean value theorem, we have

$$\begin{aligned} &I_\mu(s_t \star u_t) - I_\mu(s_0 \star u) \\ &\quad \le I_\mu(s_t \star u_t) - I_\mu(s_t \star u) \\ &\quad = \mu e^{\theta(1+\gamma_\theta)s_t} \int_{\mathbb{R}^N} |\nabla u_{\eta_t}|^{\theta-2} (\nabla u \cdot \nabla\phi + \eta_t |\nabla\phi|^2)\, t + e^{2s_t} \int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla\phi + \frac{t}{2}|\nabla\phi|^2\Big) t \\ &\qquad + 2e^{(2+N)s_t} \int_{\mathbb{R}^N} \big(u_{\eta_t}\phi |\nabla u_{\eta_t}|^2 + |u_{\eta_t}|^2 (\nabla u \cdot \nabla\phi + \eta_t |\nabla\phi|^2)\big) t \\ &\qquad - e^{p\gamma_p s_t} \int_{\mathbb{R}^N} |u_{\eta_t}|^{p-2} \Big(u\phi + \frac{\eta_t}{2}\phi^2\Big) t, \end{aligned}$$

where $|\eta_t| \in (0, |t|)$. Similarly,

$$\begin{aligned} &I_\mu(s_t \star u_t) - I_\mu(s_0 \star u) \\ &\quad \ge I_\mu(s_0 \star u_t) - I_\mu(s_0 \star u) \\ &\quad = \mu e^{\theta(1+\gamma_\theta)s_0} \int_{\mathbb{R}^N} |\nabla u_{\xi_t}|^{\theta-2} (\nabla u \cdot \nabla\phi + \xi_t |\nabla\phi|^2)\, t + e^{2s_0} \int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla\phi + \frac{t}{2}|\nabla\phi|^2\Big) t \\ &\qquad + 2e^{(2+N)s_0} \int_{\mathbb{R}^N} \big(u_{\xi_t}\phi |\nabla u_{\xi_t}|^2 + |u_{\xi_t}|^2 (\nabla u \cdot \nabla\phi + \xi_t |\nabla\phi|^2)\big) t \\ &\qquad - e^{p\gamma_p s_0} \int_{\mathbb{R}^N} |u_{\xi_t}|^{p-2} \Big(u\phi + \frac{\xi_t}{2}\phi^2\Big) t, \end{aligned}$$

where $|\xi_t| \in (0, |t|)$. Since $s_t \to s_0$ as $t \to 0$, it follows from the last two inequalities that

$$\begin{aligned} &\lim_{t\to 0} \frac{K_\mu(u_t) - K_\mu(u)}{t} \\ &\quad = \mu e^{\theta(1+\gamma_\theta)s_\mu(u)} \int_{\mathbb{R}^N} |\nabla u|^{\theta-2} \nabla u \cdot \nabla\phi + e^{2s_\mu(u)} \int_{\mathbb{R}^N} \nabla u \cdot \nabla\phi \\ &\qquad + 2e^{(2+N)s_\mu(u)} \int_{\mathbb{R}^N} (u\phi|\nabla u|^2 + |u|^2 \nabla u \cdot \nabla\phi) - e^{p\gamma_p s_\mu(u)} \int_{\mathbb{R}^N} |u|^{p-2} u\phi. \end{aligned}$$

Then similarly as Lemma A.1, we see that the Gâteaux derivative of K_μ is bounded linear and continuous. Therefore K_μ is of class $\mathcal{C}^1$, see [14]. In particular, by changing variables in the integrals, we have

$$K'_\mu(u)[\phi] = I'_\mu(s_\mu(u) \star u)[s_\mu(u) \star \phi]. \qquad \square$$

To get the particular Palais–Smale sequence of $I_\mu|_{\mathcal{S}(a)}$ as in Lemma 3.8, we need:

Lemma 3.12. *Let $\mathcal{G}$ be a τ-homotopy stable family of compact subsets of $Y = \mathcal{S}_r(a)$ with boundary $B = \varnothing$, and set*

$$d := \inf_{A \in \mathcal{G}} \max_{u \in A} K_\mu(u).$$

If $d > 0$, then there exists a sequence $u_n \in \mathcal{S}_r(a)$ such that

$$I_\mu(u_n) \to d, \quad I_\mu|'_{\mathcal{S}(a)}(u_n) \to 0, \quad Q_\mu(u_n) = 0.$$

Proof. Let $A_n \in \mathcal{G}$ be a minimizing sequence of d. We define the mapping

$$\eta : [0, 1] \times \mathcal{S}(a) \to \mathcal{S}(a), \quad \eta(t, u) = (ts_\mu(u)) \star u,$$

which is continuous and satisfies $\eta(t, u) = u$ for all $(t, u) \in \{0\} \times \mathcal{S}(a)$. Thus, by the definition of $\mathcal{G}$, one has

$$D_n := \eta(1, A_n) = \{s_\mu(u) \star u : u \in A_n\} \in \mathcal{G}.$$

In particular, $D_n \subset \mathcal{Q}_\mu(a)$ for any $n \in \mathbb{N}^+$. For any $u \in \mathcal{S}(a)$ and $s \in \mathbb{R}$, we see that

$$Q_\mu\big((s_\mu(u) - s) \star (s \star u)\big) = Q_\mu\big((s_\mu(u) \star u)\big) = 0,$$

that is, $s_\mu(s \star u) = s_\mu(u) - s$, which gives $K_\mu(s \star u) = K_\mu(u)$. Then it is clear that $\max_{D_n} K_\mu = \max_{A_n} K_\mu \to d$ and thus D_n is another minimizing sequence of d. Now, using the minimax principle [19, Theorem 7.2], we obtain a Palais–Smale sequence $v_n \in \mathcal{S}(a)$ for K_μ at the level d such that

$$\text{dist}_{\mathcal{X}}(v_n, D_n) \to 0.$$

Finally, a similar argument as the one in Lemma 3.8 gives $u_n = s_n \star v_n$ satisfying that

$$I_\mu(u_n) \to d, \quad I_\mu|'_{\mathcal{S}(a)}(u_n) \to 0, \quad Q_\mu(u_n) = 0. \qquad \square$$

To construct a sequence of τ-homotopy stable families of compact subsets of $\mathcal{S}_r(a)$ with boundary $B = \varnothing$, we proceed as in [11, Section 8]. Since $\mathcal{X}$ is separable, there exists a nested sequence of finite dimensional subspaces of $\mathcal{X}$, $W_1 \subset W_2 \subset \cdots \subset W_i \subset W_{i+1} \subset \cdots \subset \mathcal{X}$ such that $\dim(W_i) = i$ and the closure of $\bigcup_{i \in \mathbb{N}^+} W_i$ in $\mathcal{X}$ is equal to $\mathcal{X}$. Note that since $\mathcal{X}$ is dense in $W^{1,2}(\mathbb{R}^N)$, the closure in $W^{1,2}(\mathbb{R}^N)$ is also equal to $W^{1,2}(\mathbb{R}^N)$. Since $W^{1,2}(\mathbb{R}^N)$ is a Hilbert space, we denote by P_i the orthogonal projection from $W^{1,2}(\mathbb{R}^N)$ onto W_i. We also recall the definition of the genus of τ-invariant sets due to M. A. Krasnoselskii and refer the reader to [45, Section 7].

Definition D (Krasnoselskii genus). For any nonempty closed τ-invariant set $A \subset \mathcal{X}$, the genus of A is defined by

$$\text{Ind}(A) := \min\big\{k \in \mathbb{N}^+ : \exists \phi : A \to \mathbb{R}^k \setminus \{0\}, \phi \text{ is odd and continuous}\big\}.$$

We set $\text{Ind}(A) = +\infty$ if such ϕ does not exist, and set $\text{Ind}(A) = 0$ if $A = \varnothing$.

Let $\mathcal{A}(a)$ be the family of compact τ-invariant subsets of $\mathcal{S}_r(a)$. For each $j \in \mathbb{N}^+$:

$$\mathcal{A}_j(a) := \{A \in \mathcal{A}(a) : \text{Ind}(A) \geq j\} \quad \text{and} \quad c_\mu^j(a) := \inf_{A \in \mathcal{A}_j(a)} \max_{u \in A} K_\mu(u).$$

Concerning $\mathcal{A}_j(a)$ and $c_\mu^j(a)$, we have:

Lemma 3.13. (1) *$\mathcal{A}_j(a) \neq \varnothing$ for any $j \in \mathbb{N}^+$, and $\mathcal{A}_j(a)$ is a τ-homotopy stable family of compact subsets of $\mathcal{S}_r(a)$ with boundary $B = \varnothing$.*

(2) *$c_\mu^{j+1}(a) \geq c_\mu^j(a) \geq \mathcal{D}_0(a) > 0$ for any $\mu \in (0, 1]$ and $j \in \mathbb{N}^+$.*

(3) *$c_\mu^j(a)$ is nondecreasing with respect to $\mu \in (0, 1]$ for any $j \in \mathbb{N}^+$.*

(4) *$b_j(a) := \inf_{0<\mu\leq 1} c_\mu^j(a) \to +\infty$ as $j \to +\infty$.*

Proof. (1) For any $j \in \mathbb{N}^+$, $\mathcal{S}_r(a) \cap W_j \in \mathcal{A}(a)$. By the basic properties of the genus, one has

$$\operatorname{Ind}(\mathcal{S}_r(a) \cap W_j) = j$$

and thus $\mathcal{A}_j(a) \neq \varnothing$. The rest is clear by the properties of the genus.

(2) For any $A \in \mathcal{A}_j(a)$, using the fact that $s_\mu(u) \star u \in \mathcal{Q}_\mu(a)$ for all $u \in A$, we have

$$\max_{u\in A} K_\mu(u) = \max_{u\in A} I_\mu(s_\mu(u) \star u) \geq m_\mu(a) \geq \mathcal{D}_0(a)$$

and thus $c_\mu^j(a) \geq \mathcal{D}_0(a) > 0$. Since $\mathcal{A}_{j+1}(a) \subset \mathcal{A}_j(a)$, it is clear that $c_\mu^{j+1}(a) \geq c_\mu^j(a)$.

(3) For any $0 < \mu_1 < \mu_2 \leq 1$ and $u \in A \in \mathcal{A}_j(a)$, there holds

$$K_{\mu_2}(u) = I_{\mu_2}(s_{\mu_2}(u) \star u) \geq I_{\mu_2}(s_{\mu_1}(u) \star u) > I_{\mu_1}(s_{\mu_1}(u) \star u) = K_{\mu_1}(u),$$

which means $c_{\mu_2}^j(a) \geq c_{\mu_1}^j(a)$, i.e., $c_\mu^j(a)$ is nondecreasing with respect to $\mu \in (0, 1]$.

(4) The proof is inspired by that of [11, Theorem 9]. First, we claim that:

Claim. *For any $M > 0$, there exists a small $\delta_0 = \delta_0(a, M) > 0$, a small $r_0 = r_0(a, M) > 0$ and a large $k_0 = k_0(a, M) \in \mathbb{N}^+$ such that for any $0 < \mu < \delta_0$ and any $k \geq k_0$, one has*

$$I_\mu(u) \geq M \quad \text{if } \|P_k u\|_{\mathcal{X}} \leq r_0 \text{ and } u \in \mathcal{Q}_\mu^r(a).$$

Now we check it. By contradiction, we assume that there exists $M_0 > 0$ such that for any $0 < \delta \leq 1$, any $r > 0$ and any $k \in \mathbb{N}^+$ one can always find $\mu \in (0, \delta]$, $l \geq k$ and $u \in \mathcal{Q}_\mu^r(a)$ such that

$$\|P_k u\|_{\mathcal{X}} \leq r \quad \text{but} \quad I_\mu(u) < M_0.$$

As a result, one can obtain the sequences $\mu_n \to 0^+$, $k_n \to +\infty$ and $u_n \in \mathcal{Q}_{\mu_n}^r(a)$ such that

$$\|P_{k_n} u_n\|_{\mathcal{X}} \leq \tfrac{1}{n} \quad \text{and} \quad I_{\mu_n}(u_n) < M_0$$

for any $n \in \mathbb{N}^+$. From Lemma 3.3, we know that u_n is bounded in $W^{1,2}(\mathbb{R}^N)$. Since $P_{k_n} u_n$ is also bounded in $\mathcal{X}$, we assume that up to a subsequence

$$u_n \rightharpoonup u \quad \text{in } W^{1,2}(\mathbb{R}^N) \qquad \text{and} \qquad P_{k_n} u_n \rightharpoonup v \quad \text{in } \mathcal{X}.$$

We show that $u = v$. Indeed, one also has $P_{k_n} u_n \rightharpoonup v$ in $W^{1,2}(\mathbb{R}^N)$ and

$$\begin{aligned}\|u-v\|^2_{W^{1,2}(\mathbb{R}^N)} &= \lim_{n\to\infty} \langle u_n - P_{k_n} u_n, u-v\rangle_{W^{1,2}(\mathbb{R}^N)} \\ &= \lim_{n\to\infty} \langle u_n, u-v\rangle_{W^{1,2}(\mathbb{R}^N)} - \lim_{n\to\infty} \langle P_{k_n} u_n, u-v\rangle_{W^{1,2}(\mathbb{R}^N)} \\ &= \langle u, u-v\rangle_{W^{1,2}(\mathbb{R}^N)} - \lim_{n\to\infty} \langle u_n, P_{k_n} u - P_{k_n} v\rangle_{W^{1,2}(\mathbb{R}^N)} \\ &= \langle u, u-v\rangle_{W^{1,2}(\mathbb{R}^N)} - \langle u, u-v\rangle_{W^{1,2}(\mathbb{R}^N)} = 0,\end{aligned}$$

where we use the fact that $P_{k_n} u \to u$ and $P_{k_n} v \to v$ in $W^{1,2}(\mathbb{R}^N)$. Therefore $u = v$ and $u \in \mathcal{X}$. Since $\|P_{k_n} u_n\|_{\mathcal{X}} \to 0$, there must be $u = 0$. Then combining the interpolation inequality and the fact that $\sup_{n\in\mathbb{N}^+} \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 < +\infty$, we obtain $\|u_n\|_p \to 0$. Further, $u_n \in \mathcal{Q}_{\mu_n}(a)$ gives that

$$\mu_n \int_{\mathbb{R}^N} |\nabla u_n|^\theta \to 0, \quad \int_{\mathbb{R}^N} |\nabla u_n|^2 \to 0, \quad \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \to 0,$$

which is in contradiction with Lemma 3.3. So we prove the claim.

Then we can prove the conclusion (4). By contradiction, we assume that

$$\liminf_{j\to\infty} b_j < M \quad \text{for some } M > 0.$$

Then there exist $\mu \in (0, \delta_0)$ for $k > k_0$ such that $c^k_\mu(a) < M$. By the definition of $c^k_\mu(a)$, one can find $A \in \mathcal{A}_k(a)$ such that

$$\max_{u\in A} I_\mu(s_\mu(u) \star u) = \max_{u\in A} K_\mu(u) < M.$$

As Lemma 3.2 implies that the mapping $\varphi : A \to \mathcal{Q}^r_\mu(a)$ defined by $\varphi(u) = s_\mu(u) \star u$ is odd and continuous, we have $\bar{A} := \varphi(A) \subset \mathcal{Q}^r_\mu(a)$, $\max_{u\in\bar{A}} I_\mu(u) < M$ and

$$\operatorname{Ind}(\bar{A}) \ge \operatorname{Ind}(A) \ge k > k_0. \tag{3-29}$$

On the other hand, it follows from the claim that $\inf_{u\in\bar{A}} \|P_{k_0} u_n\|_{\mathcal{X}} \ge r_0 > 0$. Setting

$$\psi(u) = \frac{P_{k_0} u}{\|P_{k_0} u_n\|_{\mathcal{X}}} \quad \text{for any } u \in \bar{A},$$

we obtain an odd continuous mapping $\psi : \bar{A} \to \psi(\bar{A}) \subset W_{k_0} \setminus \{0\}$ and thus

$$\operatorname{Ind}(\bar{A}) \le \operatorname{Ind}(\psi(\bar{A})) \le k_0,$$

which contradicts (3-29). Therefore we have $b_j(a) \to +\infty$ as $j \to +\infty$. □

For any fixed $\mu \in (0, 1]$ and any $j \in \mathbb{N}^+$, by Lemmas 3.12 and 3.13, one can find a sequence $u_n \in \mathcal{S}_r(a)$ such that

$$I_\mu(u_n) \to c^j_\mu(a), \quad I_\mu|'_{\mathcal{S}(a)}(u_n) \to 0, \quad Q_\mu(u_n) = 0.$$

Then similar to Lemma 3.9, we have:

Lemma 3.14. *There exists a* $u^j_\mu \in \mathcal{X} \setminus \{0\}$ *and a* $\lambda^j_\mu \in \mathbb{R}$ *such that up to a subsequence,*

$$u^j_n \rightharpoonup u^j_\mu \quad \text{in } \mathcal{X},$$

$$I_\mu(u^j_\mu) = c^j_\mu(a) \quad \text{and} \quad I'_\mu(u^j_\mu) + \lambda^j_\mu u^j_\mu = 0.$$

Moreover, if $\lambda^j_\mu \neq 0$, *we have that*

$$u^j_n \to u^j_\mu \quad \text{in } \mathcal{X}.$$

Based on the above preliminary works, we conclude that:

Theorem 3.15. *For any fixed* $\mu \in (0, 1]$ *and any* $j \in \mathbb{N}^+$, *there exists a* $u^j_\mu \in \mathcal{X}_r \setminus \{0\}$ *and a* $\lambda^j_\mu \in \mathbb{R}$ *such that*

$$I'_\mu(u^j_\mu) + \lambda^j_\mu u^j_\mu = 0, \quad I_\mu(u^j_\mu) = c^j_\mu(a), \quad Q_\mu(u^j_\mu) = 0, \quad 0 < \|u^j_\mu\|^2_2 \le a.$$

Moreover, if $\lambda^j_\mu \neq 0$, *we have that* $\|u^j_\mu\|^2_2 = a$, *i.e.,* $\{u^j_\mu : j \in \mathbb{N}^+\}$ *are infinitely many critical points of* $I_\mu|_{\mathcal{S}(a)}$ *with increasing energy.*

4. Convergence issues as $\mu \to 0^+$

In this section, letting $\mu \to 0^+$, we show that the sequences of critical points of $I_\mu|_{\mathcal{S}(a)}$ obtained in Section 3 converge to critical points of $I|_{\tilde{\mathcal{S}}(a)}$.

Theorem 4.1. *Let* $N \ge 2$. *Suppose that* $\mu_n \to 0^+$, $I'_{\mu_n}(u_{\mu_n}) + \lambda_{\mu_n} u_{\mu_n} = 0$ *with* $\lambda_{\mu_n} \ge 0$ *and* $I_{\mu_n}(u_{\mu_n}) \to c \in (0, +\infty)$ *for* $u_{\mu_n} \in \mathcal{S}_r(a_n)$ *with* $0 < a_n \le a$. *Then there exists a subsequence* $u_{\mu_n} \rightharpoonup u$ *in* $W^{1,2}(\mathbb{R}^N)$ *with* $u \neq 0$, $u \in W^{1,2}_{rad}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and there exists a* $\lambda \in \mathbb{R}$ *such that*

$$I'(u) + \lambda u = 0, \quad I(u) = c \quad \text{and} \quad 0 < \|u\|^2_2 \le a.$$

Moreover:

(1) *If* $u_{\mu_n} \ge 0$ *for each* $n \in \mathbb{N}^+$, *then* $u \ge 0$,

(2) *If* $\lambda \neq 0$, *we have that* $\|u\|^2_2 = \lim_{n\to\infty} a_n$.

Remark 4.2. We note that the condition $\lambda_{\mu_n} \ge 0$ is only used in the following Step 1 to realize the Morse iteration. If one can prove the conclusion in Step 1 without this condition, then the conclusion in Theorem 1.1 can be extended to $N = 3, 4$ with $4 + \frac{4}{N} < p < 22^*$.

Proof of Theorem 4.1. The proof is inspired by [27; 32]. First, by Lemma 2.1, $I'_{\mu_n}(u_{\mu_n}) + \lambda_{\mu_n} u_{\mu_n} = 0$ implies that

$$Q_{\mu_n}(u_{\mu_n}) = 0 \quad \text{for each } n \in \mathbb{N}^+.$$

Then from Lemma 3.3, we see that

$$\sup_{n\geq 1}\max\Big\{\mu_n\int_{\mathbb{R}^N}|\nabla u_n|^\theta,\ \int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2,\ \int_{\mathbb{R}^N}|\nabla u_n|^2\Big\}<+\infty, \tag{4-1}$$

and hence u_{μ_n} is bounded in $W^{1,2}(\mathbb{R}^N)$. We claim that $\liminf_{n\to\infty} a_n>0$ and hence $\lambda_{\mu_n}=\frac{1}{a_n}I'_{\mu_n}(u_{\mu_n})[u_{\mu_n}]$ is also bounded in $\mathbb{R}$. Indeed, if $a_n\to 0$, then $\|u_{\mu_n}\|_p\to 0$, and it follows from $\mathcal{Q}_{\mu_n}(u_n)=0$ that $I_{\mu_n}(u_{\mu_n})\to 0$ which contradicts $c>0$. Thus, up to a subsequence, $\lambda_{\mu_n}\to\lambda$ in $\mathbb{R}$, $u_{\mu_n}\rightharpoonup u$ in $W^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$, $u_{\mu_n}\to u$ in $L^q(\mathbb{R}^N)$ for $2<q<22^*$, and $u_{\mu_n}\to u$ a.e. on $\mathbb{R}^N$. So if $u_{\mu_n}\geq 0$ for each $n\in\mathbb{N}^+$, we have that $u\geq 0$. Moreover, a similar argument as in Lemma A.2 tells that $u_n\nabla u_n\to u\nabla u$ in $(L^2_{\mathrm{loc}}(\mathbb{R}^N))^N$ and $\nabla u_{\mu_n}\to\nabla u$ a.e. on $\mathbb{R}^N$. Now we prove the conclusion in several steps.

Step 1: We prove that $\|u_{\mu_n}\|_\infty\leq C$ and $\|u\|_\infty\leq C$ for some positive constant C.

We just prove the case $N\geq 3$; the case $N=2$ can be obtained similarly. Set $T>2$, $r>0$ and

$$v_n=\begin{cases} T, & u_n\geq T,\\ u_n, & |u_n|\leq T,\\ -T, & u_n\leq -T.\end{cases}$$

Let $\phi=u_{\mu_n}|v_n|^{2r}$, then $\phi\in\mathcal{X}$. From $I'_{\mu_n}(u_{\mu_n})+\lambda_{\mu_n}u_{\mu_n}=0$ and $\lambda_{\mu_n}\geq 0$,we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}|u_{\mu_n}|^{p-2}u_{\mu_n}\phi&=\mu_{\mu_n}\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^{\theta-2}\nabla u_{\mu_n}\cdot\nabla\phi+\int_{\mathbb{R}^N}\nabla u_{\mu_n}\cdot\nabla\phi\\ &\qquad+2\int_{\mathbb{R}^N}(u_{\mu_n}\phi|\nabla u_{\mu_n}|^2+|u_{\mu_n}|^2\nabla u_{\mu_n}\cdot\nabla\phi)+\lambda_{\mu_n}\int_{\mathbb{R}^N}u_{\mu_n}\phi\\ &\geq 2\int_{\mathbb{R}^N}|u_{\mu_n}|^2\nabla u_{\mu_n}\cdot\nabla\phi\\ &=2\int_{\mathbb{R}^N}|u_{\mu_n}|^2|\nabla u_{\mu_n}|^2|v_n|^{2r}+|u_{\mu_n}|^2 2r|v_n|^{2r-2}u_{\mu_n}v_n\nabla u_{\mu_n}\cdot\nabla v_n\\ &=\frac{1}{2}\int_{\mathbb{R}^N}|v_n|^r|\nabla u_{\mu_n}^2|^2+\frac{4}{r}\int_{\mathbb{R}^N}\big|u_{\mu_n}^2\nabla|v_n|^r\big|^2\\ &\geq\frac{1}{r+4}\int_{\mathbb{R}^N}\big|\nabla(u_{\mu_n}^2|v_n|^2)\big|^2\geq\frac{C}{(r+2)^2}\Big(\int_{\mathbb{R}^N}\big|u_{\mu_n}^2|v_n|^2\big|^{2^*}\Big)^{\frac{2}{2^*}}.\end{aligned}$$

On the other hand, by the interpolation inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|u_{\mu_n}|^{p-2}u_{\mu_n}\phi&=\int_{\mathbb{R}^N}|u_{\mu_n}|^p|v_n|^{2r}\\ &\leq\Big(\int_{\mathbb{R}^N}|u_{\mu_n}|^{22^*}\Big)^{\frac{p-4}{22^*}}\Big(\int_{\mathbb{R}^N}(|v_n|^r|u_{\mu_n}|^2)^{\frac{42^*}{22^*-p+4}}\Big)^{\frac{22^*-p+4}{22^*}}\\ &\leq C\Big(\int_{\mathbb{R}^N}(|v_n|^r|u_{\mu_n}|^2)^{\frac{42^*}{22^*-p+4}}\Big)^{\frac{22^*-p+4}{22^*}}.\end{aligned}\tag{4-2}$$

Combining these inequalities, one has

$$\left(\int_{\mathbb{R}^N}|u_{\mu_n}^2|v_n|^2|^{2^*}\right)^{\frac{2}{2^*}} \le C(r+2)^2\left(\int_{\mathbb{R}^N}(|v_n|^r|u_{\mu_n}|^2)^{\frac{42^*}{22^*-p+4}}\right)^{\frac{22^*-p+4}{22^*}}. \tag{4-3}$$

Let $r_0 : (r_0+2)q = 22^*$ and $d = \frac{2^*}{q} > 1$ where $q = \frac{42^*}{22^*-p+4}$. Taking $r = r_0$ in (4-3), and letting $T \to +\infty$, we obtain

$$\|u_{\mu_n}\|_{(2+r_0)qd} \le (C(r_0+2))^{\frac{1}{r_0+2}}\|u_{\mu_n}\|_{(2+r_0)q}. \tag{4-4}$$

Set $2+r_{i+1} = (2+r_i)\,d$ for $i \in \mathbb{N}$. Then inductively, we have

$$\|u_{\mu_n}\|_{(2+r_0)qd^{i+1}} \le \prod_{k=0}^{i}(C(r_k+2))^{\frac{1}{r_k+2}}\|u_{\mu_n}\|_{(2+r_0)q} \le C_\infty\|u_{\mu_n}\|_{(2+r_0)q}, \tag{4-5}$$

where C_∞ is a positive constant. Taking $i \to \infty$ in (4-5), we get

$$\|u_{\mu_n}\|_\infty \le C \quad \text{and} \quad \|u\|_\infty \le C.$$

Step 2: We prove that $I'(u) + \lambda u = 0$.

Take $\phi = \psi e^{-u_{\mu_n}}$ with $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \ge 0$. We have

$$\begin{aligned}
0 &= (I'_{\mu_n}(u_{\mu_n}) + \lambda_{\mu_n}u_{\mu_n})[\phi] \\
&= \mu_n\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^{\theta-2}\nabla u_{\mu_n}(\nabla\psi e^{-u_{\mu_n}} - \psi e^{-u_{\mu_n}}\nabla u_{\mu_n}) \\
&\quad + \int_{\mathbb{R}^N}\nabla u_{\mu_n}(\nabla\psi e^{-u_{\mu_n}} - \psi e^{-u_{\mu_n}}\nabla u_{\mu_n}) \\
&\quad + 2\int_{\mathbb{R}^N}|u_{\mu_n}|^2\nabla u_{\mu_n}(\nabla\psi e^{-u_{\mu_n}} - \psi e^{-u_{\mu_n}}\nabla u_{\mu_n}) + 2\int_{\mathbb{R}^N}u_{\mu_n}\psi e^{-u_{\mu_n}}|\nabla u_{\mu_n}|^2 \\
&\quad + \lambda_{\mu_n}\int_{\mathbb{R}^N}u_{\mu_n}\psi e^{-u_{\mu_n}} - \int_{\mathbb{R}^N}|u_{\mu_n}|^{p-2}u_{\mu_n}\psi e^{-u_{\mu_n}} \\
&\le \mu_n\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^{\theta-2}\nabla u_{\mu_n}\nabla\psi e^{-u_{\mu_n}} + \int_{\mathbb{R}^N}(1+2u_{\mu_n}^2)\nabla u_{\mu_n}\nabla\psi e^{-u_{\mu_n}} \\
&\quad - \int_{\mathbb{R}^N}(1+2u_{\mu_n}^2 - 2u_{\mu_n})\psi e^{-u_{\mu_n}}|\nabla u_{\mu_n}|^2 \\
&\quad + \lambda_{\mu_n}\int_{\mathbb{R}^N}u_{\mu_n}\psi e^{-u_{\mu_n}} - \int_{\mathbb{R}^N}|u_{\mu_n}|^{p-2}u_{\mu_n}\psi e^{-u_{\mu_n}}.
\end{aligned}$$

Since $\mu_n \to 0^+$ and $\|u_{\mu_n}\|_\infty \le C$, equation (4-1) implies

$$\mu_n\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^{\theta-2}\nabla u_{\mu_n}\nabla\psi e^{-u_{\mu_n}} \to 0.$$

By the weak convergence of u_{μ_n}, the Hölder inequality and by the Lebesgue's dominated convergence theorem we know that

$$\int_{\mathbb{R}^N}(1+2u_{\mu_n}^2)\nabla u_{\mu_n}\nabla\psi e^{-u_{\mu_n}} \to \int_{\mathbb{R}^N}(1+2u^2)\nabla u\nabla\psi e^{-u},$$
$$\lambda_{\mu_n}\int_{\mathbb{R}^N}u_{\mu_n}\psi e^{-u_{\mu_n}} \to \lambda\int_{\mathbb{R}^N}u\psi e^{-u},$$

and

$$\int_{\mathbb{R}^N} |u_{\mu_n}|^{p-2} u_{\mu_n} \psi e^{-u_{\mu_n}} \to \int_{\mathbb{R}^N} |u|^{p-2} u \psi e^{-u}.$$

Moreover, by Fatou's lemma, there holds

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} (1+2u_{\mu_n}^2 - 2u_{\mu_n}) \psi e^{-u_{\mu_n}} |\nabla u_{\mu_n}|^2 \geq \int_{\mathbb{R}^N} (1+2u^2-2u)\psi e^{-u} |\nabla u|^2.$$

Consequently, one has

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^N} \nabla u(\nabla \psi e^{-u} - \psi e^{-u}\nabla u) + 2\int_{\mathbb{R}^N} |u|^2 \nabla u(\nabla \psi e^{-u} - \psi e^{-u} \nabla u) \\ + 2\int_{\mathbb{R}^N} u\psi e^{-u} |\nabla u|^2 + \lambda_{\mu_n} \int_{\mathbb{R}^N} u\psi e^{-u} - \int_{\mathbb{R}^N} |u|^{p-2} u\psi e^{-u}. \end{aligned} \tag{4-6}$$

For any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$, choose a sequence of nonnegative functions $\psi_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\psi_n \to \varphi e^u$ in $W^{1,2}(\mathbb{R}^N)$, $\psi_n \to \varphi e^u$ a.e. in $\mathbb{R}^N$, and that ψ_n is uniformly bounded in $L^\infty(\mathbb{R}^N)$. Then we obtain from (4-6) that

$$0 \leq \int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi + 2\int_{\mathbb{R}^N} (|u|^2 \nabla u \cdot \nabla \varphi + u\varphi |\nabla u|^2) + \lambda \int_{\mathbb{R}^N} u\varphi - \int_{\mathbb{R}^N} |u|^{p-2} u\varphi. \tag{4-7}$$

Similarly by choosing $\phi = \psi e^{u_{\mu_n}}$, we get an opposite inequality. Notice $\varphi = \varphi^+ - \varphi^-$ for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we get $I'(u) + \lambda u = 0$.

Step 3: Here we complete the proof.

Similar to Lemma 2.1, we get from $I'(u) + \lambda u = 0$ that

$$Q(u) := Q_0(u) = 0.$$

It follows that

$$Q_{\mu_n}(u_{\mu_n}) + \gamma_p \int_{\mathbb{R}^N} |u_{\mu_n}|^p \to Q(u) + \gamma_p \int_{\mathbb{R}^N} |u|^p.$$

Then using the weak lower semicontinuous property, there must be

$$\begin{aligned} \mu_n \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^\theta \to 0, \quad \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^2 \to \int_{\mathbb{R}^N} |\nabla u|^2, \\ \int_{\mathbb{R}^N} |u_{\mu_n}|^2 |\nabla u_{\mu_n}|^2 \to \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2. \end{aligned} \tag{4-8}$$

That gives $I(u) = \lim_{n\to\infty} I_\mu(u_{\mu_n}) = c$. Moreover, from (4-8), we obtain

$$I'_{\mu_n}(u_{\mu_n})[u_{\mu_n}] \to I'(u)[u]. \tag{4-9}$$

Thus there holds $\lambda \|u_{\mu_n}\|_2^2 \to \lambda \|u\|_2^2$. So if $\lambda \neq 0$, we have $\|u\|_2^2 = \lim_{n\to\infty} a_n$. □

Now we are able to complete the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 for $N \geq 2$. From Remarks 3.4 and 3.7, we see that

$$d^*(a) := \lim_{\mu\to 0^+} m_\mu(a) \in (0, +\infty).$$

By Theorem 3.10, we can take

$$\mu_n \to 0^+, \quad I'_{\mu_n}(u_{\mu_n}) + \lambda_{\mu_n} u_{\mu_n} = 0, \quad I_{\mu_n}(u_{\mu_n}) \to d^*(a)$$

for $u_{\mu_n} \in \mathcal{S}_r(a_n)$ with $0 < a_n \le a$ and $u_{\mu_n} \ge 0$. Then Lemma 2.2 implies that $\lambda_{\mu_n} > 0$. Now Theorem 4.1 gives that there exist $v \neq 0,\ v \ge 0,\ v \in W^{1,2}_{\text{rad}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\lambda_0 \in \mathbb{R}$ such that

$$I'(v) + \lambda_0 v = 0, \quad I(v) = d^*(a) \quad \text{and} \quad 0 < \|v\|_2^2 \le a.$$

Thus by Lemma 2.2, there is $\lambda_0 > 0$. Since $\lambda_{\mu_n} \to \lambda_0$, we may say that $\lambda_{\mu_n} \neq 0$ for n large. Then $a_n = a$ and $\|v\|_2^2 = a$. That is, v is a nontrivial nonnegative solution of (1-7). To consider the ground state normalized solution, we define

$$d(a) := \inf\{I(v) : v \in \tilde{\mathcal{S}}(a),\ I|'_{\tilde{\mathcal{S}}(a)}(v) = 0,\ v \neq 0\}.$$

Then $d(a) \le I(v) = d^*(a)$. Further, a similar approach to Lemma 3.3 tells that $d(a) > 0$. We take a sequence $v_n \in \tilde{\mathcal{S}}(a),\ I|'_{\tilde{\mathcal{S}}(a)}(v_n) = 0,\ v_n \neq 0$ and $v_n \ge 0$ such that $I(v_n) \to d(a)$. We can show that (the proof is similar to that of Theorem 4.1, so we omit it), up to a subsequence, there exist $u \neq 0,\ u \ge 0,\ u \in W^{1,2}_{\text{rad}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ such that

$$I'(u) + \lambda u = 0 \quad \text{and} \quad I(u) = d(a).$$

Again by Lemma 2.2, there is $\lambda \neq 0$, and hence $\|u\|_2^2 = a$. That is, u is a minimizer of $d(a)$. Finally, by [41, Lemma 2.6], u is classical and strictly positive since $u \in L^\infty(\mathbb{R}^N)$. □

Proof of Theorem 1.2. From Lemma 3.13, we see that

$$b_j(a) = \lim_{\mu \to 0^+} c^j_\mu(a) \in (0, +\infty) \quad \text{and} \quad b_j(a) \to +\infty.$$

By Theorem 3.15, for each $j \in \mathbb{N}^+$ we can take

$$\mu^j_n \to 0^+, \quad I'_{\mu^j_n}(u^j_{\mu^j_n}) + \lambda^j_{\mu^j_n} u^j_{\mu^j_n} = 0, \quad I_{\mu^j_n}(u^j_{\mu^j_n}) \to b_j(a)$$

for $u_{\mu^j_n} \in \mathcal{S}_r(a^j_n)$ with $0 < a^j_n \le a$. And Lemma 2.2 implies that $\lambda^j_{\mu^j_n} > 0$. Now Theorem 4.1 gives that there exist

$$u^j \neq 0, \quad u^j \in W^{1,2}_{\text{rad}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \quad \text{and} \quad \lambda^j \in \mathbb{R}$$

such that

$$I'(u^j) + \lambda^j u^j = 0, \quad I(u^j) = b_j(a) \quad \text{and} \quad 0 < \|u^j\|_2^2 \le a.$$

Thus by Lemma 2.2, there is $\lambda^j > 0$. Going back since $\lambda^j_{\mu^j_n} \to \lambda^j$, we may say that $\lambda^j_{\mu^j_n} \neq 0$ for n large. Then $a^j_n = a$ and $\|u^j\|_2^2 = a$. That is, $\{u^j : j \in \mathbb{N}^+\}$ is a sequence of normalized solutions of (1-7). Moreover, $I(u^j) = b_j \to +\infty$. □

5. The mass critical case $p = 4 + \frac{4}{N}$

In this section we denote $p_* = 4 + \frac{4}{N}$ and assume that $p = p_*$. We still consider I_μ, but on an open subset of $\mathcal{X}$. Let

$$\mathcal{O} := \left\{ u \in \mathcal{X} : \int_{\mathbb{R}^N} u^2 |\nabla u|^2 < \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |u|^{p_*} \right\} \tag{5-1}$$

and for simplicity, we still denote

$$\mathcal{S}(a) := \left\{ u \in \mathcal{O} : \int_{\mathbb{R}^N} u^2 = a \right\}, \quad \mathcal{Q}_\mu(a) := \{u \in \mathcal{S}(a) : Q_\mu(u) = 0\},$$
$$\mathcal{S}_r(a) := \mathcal{S}(a) \cap \mathcal{X}_r, \qquad \mathcal{Q}_\mu^r(a) := \mathcal{Q}_\mu(a) \cap \mathcal{X}_r.$$

Lemma 5.1. *$\mathcal{S}(a)$ is nonempty when $a > a^*$.*

Proof. Let $u = Q_{p_*}^{\frac{1}{2}}$, then from (1-13), we have

$$\int_{\mathbb{R}^N} |u|^{p_*} = \frac{4(N+1)}{N} \int_{\mathbb{R}^N} u^2 |\nabla u|^2. \tag{5-2}$$

Let $w_a = \left(\frac{a}{a_*}\right)^{\frac{1}{2}} u$, then $\|w_a\|_2^2 = a$ and (5-2) implies that

$$\int_{\mathbb{R}^N} w_a^2 |\nabla w_a|^2 = \frac{N}{4(N+1)} \left(\frac{a}{a_*}\right)^{-\frac{2}{N}} \int_{\mathbb{R}^N} |w_a|^{p_*} < \frac{N}{4(N+1)} \int_{\mathbb{R}^N} |w_a|^{p_*}, \tag{5-3}$$

that is, $w_a \in \mathcal{S}(a)$. □

So from now on, we assume $a > a^*$. Then noting that when $p = p_*$, there is $p_*\gamma_{p_*} > \theta + \theta\gamma_\theta$ and $p_*\gamma_{p_*} = 2 + N$, we still have:

Lemma 5.2. *Let $0 < \mu \le 1$, then $\mathcal{Q}_\mu(a)$ is a $\mathcal{C}^1$-submanifold of codimension 1 in $\mathcal{S}(a)$, and hence a $\mathcal{C}^1$-submanifold of codimension 2 in $\mathcal{X}$.*

Lemma 5.3. *For any $0 < \mu \le 1$ and $u \in \mathcal{O} \setminus \{0\}$, the following statements hold.*

(1) *There exists a unique number $s_\mu(u) \in \mathbb{R}$ such that $Q_\mu(s_\mu(u) \star u) = 0$.*

(2) *$I_\mu(s \star u)$ is strictly increasing in $s \in (-\infty, s_\mu(u))$ and is strictly decreasing in $s \in (s_\mu(u), +\infty)$, and*

$$\lim_{s \to -\infty} I_\mu(s \star u) = 0^+, \quad \lim_{s \to +\infty} I_\mu(s \star u) = -\infty, \quad I_\mu(s_\mu(u) \star u) > 0.$$

(3) *$s_\mu(u) < 0$ if and only if $Q_\mu(u) < 0$.*

(4) *The map $u \in \mathcal{X} \setminus \{0\} \mapsto s_\mu(u) \in \mathbb{R}$ is of class $\mathcal{C}^1$.*

(5) *$s_\mu(u)$ is an even function with respect to $u \in \mathcal{X} \setminus \{0\}$.*

Similar to Lemma 3.3, there also holds:

Lemma 5.4. (1) *$\mathcal{D}(a) := \inf_{0<\mu\le 1, u \in \mathcal{Q}_\mu(a)} \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 > 0$ is independent of μ.*

(2) *If* $\sup_{n\geq 1} I_\mu(u_n) < +\infty$ *for* $u_n \in \mathcal{Q}_\mu(a)$, *then*

$$\sup_{n\geq 1} \max\Big\{\mu\int_{\mathbb{R}^N}|\nabla u_n|^\theta,\ \int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2,\ \int_{\mathbb{R}^N}|\nabla u_n|^2\Big\} < +\infty.$$

Proof. The proof is different from the one of Lemma 3.3.

(1) For any $u \in \mathcal{Q}_\mu(a)$, using the equality $Q_\mu(u) = 0$ and (1-16) we obtain

$$\int_{\mathbb{R}^N}|\nabla u|^2 \leq (N+2)\Big[\Big(\frac{a}{a_*}\Big)^{\frac{2}{N}} - 1\Big]\int_{\mathbb{R}^N}u^2|\nabla u|^2. \tag{5-4}$$

On the one hand, when $N \leq 3$, there holds $p_* < 2^*$. Therefore, the classical Gagliardo–Nirenberg inequality [42] tells that

$$\int_{\mathbb{R}^N}|\nabla u|^2 \leq \gamma_{p_*}\int_{\mathbb{R}^N}|u|^{p_*} \leq C(N)\, a^{1+\frac{2}{N}-\frac{N}{2}}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^{\frac{N+2}{2}}, \tag{5-5}$$

following which there is

$$\int_{\mathbb{R}^N}|\nabla u|^2 \geq \frac{C(N)}{a^{\frac{4}{N^2}+\frac{2}{N}-1}}.$$

Combining with (5-4), one obtains

$$\inf_{0<\mu\leq 1, u\in\mathcal{Q}_\mu(a)} \int_{\mathbb{R}^N}|u|^2|\nabla u|^2 > 0.$$

On the other hand, when $N \geq 4$, there is $p_* > 2^*$. But using interpolation inequality and Young's inequality we have

$$\begin{aligned}(N+2)&\int_{\mathbb{R}^N}u^2|\nabla u|^2 + \int_{\mathbb{R}^N}|\nabla u|^2 \\ &\leq \gamma_{p_*}\int_{\mathbb{R}^N}|u|^{p_*} \leq \Big(\int_{\mathbb{R}^N}|u|^{2^*}\Big)^{\frac{22^*-p_*}{2^*}}\Big(\int_{\mathbb{R}^N}|u|^{22^*}\Big)^{\frac{p_*-2^*}{2^*}} \\ &\leq C(N)\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^{\frac{22^*-p_*}{2}}\Big(\int_{\mathbb{R}^N}u^2|\nabla u|^2\Big)^{\frac{p_*-2^*}{2}} \\ &\leq (N+2)\int_{\mathbb{R}^N}u^2|\nabla u|^2 + C(N)\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^{\frac{22^*-p_*}{2^*+2-p_*}},\end{aligned} \tag{5-6}$$

which gives that $\int_{\mathbb{R}^N}|\nabla u|^2 \geq C(N)$ and again

$$\inf_{0<\mu\leq 1, u\in\mathcal{Q}_\mu(a)} \int_{\mathbb{R}^N}|u|^2|\nabla u|^2 > 0.$$

(2) Since $p_*\gamma_{p_*} = 2 + N$, we see from (3-10) that

$$\sup_{n\geq 1}\max\Big\{\mu\int_{\mathbb{R}^N}|\nabla u_n|^\theta,\ \int_{\mathbb{R}^N}|\nabla u_n|^2\Big\} < +\infty.$$

On the one hand, when $N \leq 3$, we obtain from (5-5) that

$$\sup_{n\geq 1}\int_{\mathbb{R}^N}|u_n|^{p_*} \leq C\sup_{n\geq 1}\Big(\int_{\mathbb{R}^N}|\nabla u_n|^2\Big)^{\frac{N+2}{2}} < +\infty,$$

which in turn combining with $Q_\mu(u_n) = 0$ implies $\sup_{n\ge 1}\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2 < +\infty$. On the other hand, when $N \ge 4$, for any $n \ge 1$ we obtain from (5-6) that

$$(N+2)\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2 \le \int_{\mathbb{R}^N} |u_n|^{p_*} \le C\Big(\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2\Big)^{\frac{p_*-2^*}{2}},$$

which gives $\sup_{n\ge 1}\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2 < +\infty$ since $0 < p_* - 2^* < 2$ for $N \ge 4$. □

First, we will consider a minimization problem:

$$m_\mu(a) := \inf_{u\in\mathcal{Q}_\mu(a)} I_\mu(u). \tag{5-7}$$

Remark 5.5. It is easy to see from Lemma 5.4 and (3-10) that

$$\inf_{0\le\mu\le 1} m_\mu(a) \ge \frac{N}{2(2+N)} \inf_{0\le\mu\le 1, u\in\mathcal{Q}_\mu(a)} \int_{\mathbb{R}^N} |\nabla u|^2 > 0. \tag{5-8}$$

On the other hand, to use the convergence Theorem 4.1, we need to give an uniform upper bound of $m_\mu(a)$. Indeed for any fixed $a > a^*$, recalling the function

$$w_a = \Big(\frac{a}{a_*}\Big)^{\frac12} Q_{p_*}^{\frac12} \in \mathcal{S}(a)$$

in Lemma 5.1, and letting $s_\mu := s_\mu(w_a)$, from $Q_\mu(s_\mu \star w_a) = 0$ we obtain

$$\begin{aligned}(1+\gamma_\theta)\mu e^{-(2+N-\theta-\theta\gamma_\theta)s_\mu}\Big(\frac{a}{a_*}\Big)^{\frac{\theta}{2}}\int_{\mathbb{R}^N}|\nabla Q_{p_*}^{\frac12}|^\theta + e^{-Ns_\mu}\Big(\frac{a}{a_*}\Big)\int_{\mathbb{R}^N}|\nabla Q_{p_*}^{\frac12}|^2 \\ = (1+\gamma_\theta)\mu e^{-(2+N-\theta-\theta\gamma_\theta)s_\mu}\int_{\mathbb{R}^N}|\nabla w_a|^\theta + e^{-Ns_\mu}\int_{\mathbb{R}^N}|\nabla w_a|^2 \\ = \gamma_{p_*}\int_{\mathbb{R}^N}|w_a|^{p_*} - (2+N)\int_{\mathbb{R}^N}|w_a|^2|\nabla w_a|^2 \\ = \frac{N(2+N)}{4(N+1)}\Big(1-\Big(\frac{a}{a_*}\Big)^{-\frac{2}{N}}\Big)\Big(\frac{a}{a_*}\Big)^{2+\frac{2}{N}}\|Q_{p_*}^{\frac12}\|_1 > 0,\end{aligned} \tag{5-9}$$

it follows that $\sup_{0\le\mu\le 1} s_\mu < +\infty$. Therefore,

$$\begin{aligned}\sup_{0\le\mu\le 1} m_\mu(a) &\le \sup_{0\le\mu\le 1} I_\mu(s_\mu\star w_a) = \sup_{0\le\mu\le 1} I_\mu(s_\mu\star w_a) - Q_\mu(s_\mu\star w_a) \\ &= \sup_{0\le\mu\le 1} \frac{2+N-\theta-\theta\gamma_\theta}{\theta(2+N)}\mu e^{\theta(1+\gamma_\theta)s_\mu}\int_{\mathbb{R}^N}|\nabla Q_{p_*}^{\frac12}|^\theta \\ &\qquad + \frac{N}{2(2+N)}e^{2s_\mu}\int_{\mathbb{R}^N}|\nabla Q_{p_*}^{\frac12}|^2 \\ &< +\infty.\end{aligned} \tag{5-10}$$

Now we construct a special Palais–Smale sequence of $I_\mu|_{\mathcal{S}(a)}$ at level $m_\mu(a)$. But different from the one in Section 3B, in mass-critical case there is no result as Lemma 3.5, and hence there is no mountain-pass-type result as Lemma 3.6. So we will not consider I_μ directly. Instead, we study the auxiliary functional $K_\mu(u)$

defined by (3-28) and we point out that our approach is inspired by [6] (see also [12]). Similar to [6, Lemma 3.7], we have:

Lemma 5.6. *Let a sequence $u_n \in \mathcal{S}(a)$ with $u_n \to u$ in $\mathcal{X}$ as $n \to \infty$. Then if $u \in \partial\mathcal{O}$, we have $K_\mu(u_n) \to \infty$ as $n \to \infty$.*

Proof. If $u_n \to u$ in $\mathcal{X}$, then there are

$$\int_{\mathbb{R}^N} |\nabla u_n|^\theta \to \int_{\mathbb{R}^N} |\nabla u|^\theta > 0, \qquad \int_{\mathbb{R}^N} |\nabla u_n|^2 \to \int_{\mathbb{R}^N} |\nabla u|^2 > 0,$$
$$\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \to \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 > 0, \qquad \int_{\mathbb{R}^N} |u_n|^{p_*} \to \int_{\mathbb{R}^N} |u|^{p_*} > 0.$$

Let $s_n = s_\mu(u_n)$. Since $Q_\mu(s_n \star u_n) = 0$, we obtain

$$\begin{aligned}(1+\gamma_\theta)\mu e^{-(2+N-\theta-\theta\gamma_\theta)s_n} \int_{\mathbb{R}^N} |\nabla u_n|^\theta + e^{-Ns_n} \int_{\mathbb{R}^N} |\nabla u_n|^2 \\ = \gamma_{p_*} \int_{\mathbb{R}^N} |u_n|^{p_*} - (2+N) \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \\ \to \gamma_{p_*} \int_{\mathbb{R}^N} |u|^{p_*} - (2+N) \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 = 0,\end{aligned} \tag{5-11}$$

where the last equality comes from $u \in \partial\mathcal{O}$. It follows that $s_n \to +\infty$. So

$$\begin{aligned} K_\mu(u_n) &= I_\mu(s_n \star u_n) = I_\mu(s_n \star u_n) - Q_\mu(s_n \star u_n) \\ &= \frac{2+N-\theta-\theta\gamma_\theta}{\theta(2+N)} \mu e^{\theta(1+\gamma_\theta)s_n} \int_{\mathbb{R}^N} |\nabla u_n|^\theta + \frac{N}{2(2+N)} e^{2s_n} \int_{\mathbb{R}^N} |\nabla u_n|^2 \\ &\to +\infty. \end{aligned}$$

□

Recalling Definition A, we give directly the following results without a proof, since the proof is very similar to the one of [6, Proposition 3.9] (see also [12]).

Lemma 5.7. *Let $\mathcal{G}$ be a homotopy stable family of compact subsets of $Y = \mathcal{S}_r(a)$ with boundary $B = \varnothing$, and set*

$$d := \inf_{A \in \mathcal{G}} \max_{u \in A} K_\mu(u). \tag{5-12}$$

If $d > 0$, then there exists a sequence $u_n \in \mathcal{S}_r(a)$ such that as $n \to \infty$,

$$I_\mu(u_n) \to d, \quad I_\mu|'_{\mathcal{S}(a)}(u_n) \to 0, \quad Q_\mu(u_n) = 0.$$

Moreover, if one can find a minimizing sequence A_n for d with the property that $u \ge 0$ a.e. for any $u \in A_n$, then one can find the sequence u_n satisfying the additional condition

$$u_n^- \to 0, \quad \text{a.e. in } \mathbb{R}^N.$$

Remark 5.8. As pointed out in [6], the set $\mathcal{O}$ is neither complete nor connected, and hence in principle the assumptions of the minimax theorem (such as [19, Theorem 3.2]) are not satisfied. However, the connectedness assumption can be avoided considering the restriction of K_μ on the connected component of $\mathcal{O}$

(if $B \neq \varnothing$, we need to assume that B is contained in a connected component of $\mathcal{Q}_\mu(a)$). Regarding the completeness, what is really used in the deformation lemma [19, Lemma 3.7] is that the sublevel sets $K_\mu^c := \{u \in \mathcal{S}(a) : K_\mu(u) \leq c\}$ are complete for every $c \in \mathbb{R}$. This follows by Lemma 5.6. Hence the minimax theorem [19, Theorem 3.2] can be used to obtain the Palais–Smale sequence. The rest of the process is similar to Lemma 3.12.

Lemma 5.9. *For any fixed $\mu \in (0, 1]$, there exists a sequence $u_n \in \mathcal{S}_r(a)$ such that*

$$I_\mu(u_n) \to m_\mu(a), \quad I_\mu|'_{\mathcal{S}(a)}(u_n) \to 0, \quad Q_\mu(u_n) = 0 \quad \text{and} \quad u_n^- \to 0 \text{ a.e. in } \mathbb{R}^N.$$

Proof. We use Lemma 5.7 by taking the set $\mathcal{G}$ of all singletons belonging to $\mathcal{S}_r(a)$. It is clearly a homotopy stable family of compact subsets of $\mathcal{S}_r(a)$ with boundary $B = \varnothing$. Observe that

$$\alpha_\mu(a) = \inf_{A\in\mathcal{G}} \max_{u\in A} K_\mu(u) = \inf_{u\in\mathcal{S}_r(a)} \max_{s\in\mathbb{R}} I_\mu(s \star u).$$

We claim that

$$\alpha_\mu(a) = m_\mu(a).$$

Indeed, on one hand, for any $u \in \mathcal{S}_r(a)$ there exists a $s_\mu(u)$ such that

$$s_\mu(u) \star u \in \mathcal{Q}_\mu(a) \quad \text{and} \quad I_\mu(s_\mu(u) \star u) = \max_{s\in\mathbb{R}} I_\mu(s \star u).$$

This implies that

$$\alpha_\mu(a) = \inf_{u\in\mathcal{S}_r(a)} \max_{s\in\mathbb{R}} I_\mu(s \star u) \geq \inf_{u\in\mathcal{Q}_\mu(a)} I_\mu(u) = m_\mu(a).$$

On the other hand, for any $u \in \mathcal{Q}_\mu^r(a)$, $I_\mu(u) = \max_{s\in\mathbb{R}} I_\mu(s \star u)$, so

$$m_\mu^r(a) := \inf_{u\in\mathcal{Q}_\mu^r(a)} I_\mu(u) \geq \inf_{u\in\mathcal{S}_r(a)} \max_{s\in\mathbb{R}} I_\mu(s \star u) = \alpha_\mu(a).$$

Finally, the inequality $m_\mu(a) \geq m_\mu^r(a)$ can be obtained easily by the symmetric decreasing rearrangement. So, the conclusion follows directly from Lemma 5.7. □

Then as in Section 3B, we have:

Theorem 5.10. *Let $p = p_*$. For any fixed $\mu \in (0, 1]$, there exists a $u_\mu \in \mathcal{X}_r \setminus \{0\}$ and a $\lambda_\mu \in \mathbb{R}$ such that*

$$I'_\mu(u_\mu) + \lambda_\mu u_\mu = 0,$$
$$I_\mu(u_\mu) = m_\mu(a), \quad Q_\mu(u_\mu) = 0, \qquad 0 < \|u_\mu\|_2^2 \leq a, \quad u_\mu \geq 0.$$

Moreover, if $\lambda_\mu \neq 0$, we have that $\|u_\mu\|_2^2 = a$, i.e., $m_\mu(a)$ is achieved, and u_μ is a ground state critical point of $I_\mu|_{\mathcal{S}(a)}$.

Proof of Theorem 1.4. The proof is exactly the same as the one of Theorem 1.1, so we omit the details. □

Remark 5.11. We are not able to obtain multiple solutions as in Section 3C. Indeed, if we consider an open subset $\mathcal{O}$ and follow the strategy in Section 3C, we need to prove a result like Lemma 3.13. However, for any finite dimensional subspace W_j of $\mathcal{X}$, using the equivalence of norms in finite dimensional spaces, we can only obtain that for any $j > 0$, there exists a $a(j) > 0$ large enough such that

$$\{u \in W_j : \|u\|_2^2 = a\} \subset \mathcal{O} \quad \text{when } a > a(j),$$

which is necessary to prove the nonemptiness of the sets of type $\mathcal{A}_j$. And another difficulty is that as $\mu \to 0^+$, we are unable to distinguish the energy

$$b_j(a) := \lim_{\mu \to 0^+} c_\mu^j(a) \quad \text{and} \quad b_k(a) := \lim_{\mu \to 0^+} c_\mu^k(a)$$

for $j \neq k$. As a result, we cannot distinguish the solutions related to $b_j(a)$ and $b_k(a)$.

Recalling Proposition 1.6, we prove the concentration theorem.

Proof of Theorem 1.8. Let u_n be a radially symmetric positive solution of (1-7) for $a = a_n$ with $a_n > a_*$ and $a_n \to a_*$. From Lemma 5.4, we see that

$$\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2 \geq \frac{C}{\left(\frac{a_n}{a_*}\right)^{2/N} - 1} \to +\infty, \tag{5-13}$$

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2}{\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2} \leq C\left(\left(\frac{a_n}{a_*}\right)^{2/N} - 1\right) \to 0. \tag{5-14}$$

Since $Q_\mu(u_n) = 0$, we know that

$$\frac{\int_{\mathbb{R}^N} |u_n|^{p_*}}{\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2} \to \frac{4(N+1)}{N}. \tag{5-15}$$

Let $v_n(x) := \varepsilon_n^{N/2} u_n(\varepsilon_n x)$ with

$$\varepsilon_n = \left(\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2\right)^{-\frac{1}{2+N}} \to 0^+.$$

Direct calculations show that

$$\|v_n\|_2^2 = a_n \to a_*, \quad \int_{\mathbb{R}^N} v_n^2 |\nabla v_n|^2 = 1, \quad \|v_n\|_{p_*}^{p_*} \to \frac{4(N+1)}{N} \quad \text{and} \quad \varepsilon_n^N \|\nabla v_n\|_2^2 \to 0.$$

Then v_n^2 is bounded in $\mathcal{E}^{p_*}$. Moreover, using [34, Lemma I.1], we deduce that there exist $\delta > 0$ and a sequence $y_n \in \mathbb{R}^N$ such that for some $R > 0$,

$$\int_{B_R(y_n)} v_n^2 \geq \delta.$$

Observing that $\mathcal{E}^q$ is a reflexive Banach space when $1 < q < \infty$, we know that there exists a nonnegative radially symmetric function $v \neq 0$ with $v^2 \in \mathcal{E}^{p_*} \cap L^2(\mathbb{R}^N)$

such that

$$\begin{aligned} v_n^2(\cdot + y_n) &\rightharpoonup v^2 && \text{in } \mathcal{E}^{p_*}, \\ v_n(\cdot + y_n) &\rightharpoonup v && \text{in } L^2(\mathbb{R}^N), \\ v_n^2(\cdot + y_n) &\to v^2 && \text{in } L^q(\mathbb{R}^N) \text{ for } 1 < q < 2^*, \\ v_n(\cdot + y_n) &\to v && \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Since u_n solves

$$-\Delta u_n - u_n \Delta u_n^2 + \lambda_n u_n = u_n^{p_*-1},$$

where the Lagrange multiplier is given by

$$\lambda_n = \frac{1}{a_n}\Big(\int_{\mathbb{R}^N} |u_n|^{p_*} - \int_{\mathbb{R}^N} |\nabla u_n|^2 - \int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2\Big),$$

v_n satisfies

$$-\varepsilon_n^N \Delta v_n - v_n \Delta v_n^2 + \varepsilon_n^{2+N} \lambda_n v_n = v_n^{p_*-1}.$$

Combining (5-14) and (5-15), we deduce that $\varepsilon_n^{2+N}\lambda_n \to \frac{4}{Na^*}$. Then a similar approach as Lemma A.2 tells that

$$-v\Delta v^2 + \varepsilon_n^{2+N}\lambda_n v = v^{p_*-1}. \tag{5-16}$$

Now setting

$$\begin{aligned} w_n(x) &:= \Big(\frac{Na^*}{4}\Big)^{\frac{N}{2+N}} v_n^2\Big(\Big(\frac{Na^*}{4}\Big)^{\frac{1}{2+N}} x + y_n\Big) \\ &= \Big[\Big(\frac{Na^*}{4}\Big)^{\frac{1}{2+N}} \varepsilon_n\Big]^N u_n^2\Big(\Big(\frac{Na^*}{4}\Big)^{\frac{1}{2+N}} \varepsilon_n x + \varepsilon_n y_n\Big), \end{aligned} \tag{5-17}$$

$$w(x) := \Big(\frac{Na^*}{4}\Big)^{\frac{N}{2+N}} v^2\Big(\Big(\frac{Na^*}{4}\Big)^{\frac{1}{2+N}} x\Big), \tag{5-18}$$

it is easily seen that $w_n \rightharpoonup w$ in $\mathcal{E}^{p_*}$ and $\|w_n\|_1 = \|v_n\|_2^2 = a_n$. Moreover, it follows from (5-16) that w is a solution of (1-14). Thus $w = Q_{p_*}$, and hence $\|w\|_1 = \|v\|_2^2 = a_*$. So we have $v_n \to v$ in $L^2(\mathbb{R}^N)$, which finishes the proof. □

Appendix

Lemma A.1. *In the setting of Section 2A, $V(u) \in \mathcal{C}^1(\mathcal{X})$.*

Proof. The proof is elementary. When $N = 2$, since $W^{1,\theta}(\mathbb{R}^2) \hookrightarrow \mathcal{C}^{0,\alpha}(\mathbb{R}^2)$, it is easy to check that $V(u) \in \mathcal{C}^1(\mathcal{X})$. Now we set $N \geq 3$. For any $u, \phi \in \mathcal{X}$,

$$\frac{V(u+t\phi)-V(u)}{t} = At + Bt^2 + Ct^3 + 2\int_{\mathbb{R}^N} u\phi|\nabla u|^2 + u^2 \nabla u \cdot \nabla\phi, \tag{A-1}$$

where

$$A=\int_{\mathbb{R}^N}u^2|\nabla\phi|^2+\phi^2|\nabla u|^2+4u\phi\nabla u\cdot\nabla\phi,$$
$$B=\int_{\mathbb{R}^N}\phi^2\nabla u\cdot\nabla\phi+u\phi|\nabla\phi|^2 \quad\text{and}\quad C=\int_{\mathbb{R}^N}\phi^2|\nabla\phi|^2.$$

We need to prove that A, B, C are finite numbers. Indeed, since $\frac{4N}{N+2}<\theta<\frac{4N+4}{N+2}<4$, there is $\theta<\frac{2\theta}{\theta-2}<\frac{\theta N}{N-\theta}$ and hence

$$\text{(A-2)}\qquad \begin{aligned}\int_{\mathbb{R}^N}u^2|\nabla\phi|^2&\le\Big(\int_{\mathbb{R}^N}|u|^{2\theta/(\theta-2)}\Big)^{(\theta-2)/\theta}\Big(\int_{\mathbb{R}^N}|\nabla\phi|^\theta\Big)^{2/\theta}\\&\le C\|u\|^{2/\theta}_{W^{1,\theta}(\mathbb{R}^N)}\|\phi\|^{2/\theta}_{W^{1,\theta}(\mathbb{R}^N)}<\infty.\end{aligned}$$

We can handle other terms in a similar way, so A, B, C are finite numbers. Now by letting $t\to 0$ in (A-1), we immediately get the Frèchet derivative as

$$DV(u)[\phi]=2\int_{\mathbb{R}^N}u\phi|\nabla u|^2+u^2\nabla u\cdot\nabla\phi.$$

Then in a similarly way to (A-2), one can prove that $DV(u)$ is continuous for $u\in\mathcal{X}$, so $V(u)\in\mathcal{C}^1(\mathcal{X})$ and $V'(u)=DV(u)$. □

Lemma A.2. *Assume that $I'_\mu(u_n)+\lambda u_n\to 0$ for some $\lambda\in\mathbb{R}$ with $u_n\in\mathcal{X}$, and that $u_n\rightharpoonup u$ in $\mathcal{X}$. Then up to a subsequence,*

(1) *$u_n\to u$ in $\mathcal{X}_{\text{loc}}:=W^{1,\theta}_{\text{loc}}(\mathbb{R}^N)\cap W^{1,2}_{\text{loc}}(\mathbb{R}^N)$,*

(2) *$u_n\nabla u_n\to u\nabla u$ in $(L^2_{\text{loc}}(\mathbb{R}^N))^N$,*

(3) *$I'_\mu(u)+\lambda u=0$.*

Proof. The proof is inspired by [29, Lemma 14.3]. Since $u_{\mu_n}\rightharpoonup u$ in $\mathcal{X}$, we have $\|u_n\|_{\mathcal{X}}\le C_0$ for any $n\ge 1$. For any $R>1$, we set $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ satisfying

$$0\le\phi\le 1,\quad \phi(x)=\begin{cases}1, & |x|\le R,\\ 0, & |x|\ge 2R,\end{cases}\quad\text{and}\quad |\nabla\phi|\le 2.$$

Then for any $n,m\in\mathbb{N}$,

$$\text{(A-3)}\quad \begin{aligned}o(1)_n+o(1)_m&=(I'_\mu(u_n)+\lambda u_n)[(u_n-u_m)\phi]-(I'_\mu(u_m)+\lambda u_m)[(u_n-u_m)\phi]\\&=\mu\int_{\mathbb{R}^N}(|\nabla u_n|^{\theta-2}\nabla u_n-|\nabla u_m|^{\theta-2}\nabla u_m)\cdot\nabla((u_n-u_m)\phi)\\&\qquad+\int_{\mathbb{R}^N}(\nabla u_n-\nabla u_m)\cdot\nabla((u_n-u_m)\phi)\\&\qquad+2\int_{\mathbb{R}^N}(u_n|\nabla u_n|^2-u_m|\nabla u_m|^2)(u_n-u_m)\phi\\&\qquad+2\int_{\mathbb{R}^N}(u_n^2\nabla u_n-u_m^2\nabla u_m)\cdot\nabla((u_n-u_m)\phi)\\&\qquad-\int_{\mathbb{R}^N}(|u_n|^{p-2}u_n-|u_m|^{p-2}u_m)(u_n-u_m)\phi\\&=:K_1+K_2+K_3+K_4+K_5.\end{aligned}$$

Next we estimate K_i for $i = 1, 2, 3, 4, 5$:

$$
\begin{aligned}
K_1 = \mu \int_{B_R} (|\nabla u_n|^{\theta-2}\nabla u_n - |\nabla u_m|^{\theta-2}\nabla u_m)\cdot\nabla(u_n-u_m) \\
+ \mu \int_{B_{2R}\setminus B_R} (|\nabla u_n|^{\theta-2}\nabla u_n - |\nabla u_m|^{\theta-2}\nabla u_m)\cdot\nabla(u_n-u_m)\phi \\
+ \mu \int_{B_{2R}\setminus B_R} (|\nabla u_n|^{\theta-2}\nabla u_n - |\nabla u_m|^{\theta-2}\nabla u_m)\cdot\nabla\phi(u_n-u_m) \\
\geq C\mu \int_{B_R} |\nabla u_n - \nabla u_m|^\theta + C\mu \int_{B_{2R}\setminus B_R} |\nabla u_n - \nabla u_m|^\theta \phi \\
- C(\|u_n\|_\theta^{\theta-1} + \|u_m\|_\theta^{\theta-1})\|u_n-u_m\|_{L^\theta(B_{2R})} \\
\geq C\mu \|\nabla u_n - \nabla u_m\|^\theta_{L^\theta(B_R)} - C\|u_n-u_m\|_{L^\theta(B_{2R})},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&K_2 \geq C\|\nabla u_n - \nabla u_m\|^2_{L^2(B_R)} - C\|u_n-u_m\|_{L^2(B_{2R})}, && K_3 \geq -C\|u_n-u_m\|_{L^\theta(B_{2R})}, \\
&K_4 \geq 2\|u_n\nabla u_n - u_m\nabla u_m\|^2_{L^2(B_R)} - C\|u_n-u_m\|_{L^\theta(B_{2R})}, && K_5 \geq -C\|u_n-u_m\|_{L^p(B_{2R})}.
\end{aligned}
$$

Substituting these estimates into (A-3), we obtain

$$
\begin{aligned}
&\mu\|\nabla u_n - \nabla u_m\|^\theta_{L^\theta(B_R)} + \|\nabla u_n - \nabla u_m\|^2_{L^2(B_R)} + \|u_n\nabla u_n - u_m\nabla u_m\|^2_{L^2(B_R)} \\
&\quad \leq C\|u_n-u_m\|_{L^\theta(B_{2R})} + C\|u_n-u_m\|_{L^2(B_{2R})} + C\|u_n-u_m\|_{L^\theta(B_{2R})} + o(1)_n + o(1)_m \\
&\quad \to 0, \quad \text{as } n\to\infty,\ m\to\infty,
\end{aligned}
$$

where in the last estimate we use the compact embedding theorem in bounded domains. Thus for any $R > 1$, u_n is a Cauchy sequence in $W^{1,\theta}(B_R)\cap W^{1,2}(B_R)$, and $u_n\nabla u_n$ is also a Cauchy sequence in $(L^2(B_R))^N$. So up to a subsequence $u_n \to u$ in $\mathcal{X}_{\text{loc}}$ and $u_n\nabla u_n \to u\nabla u$ in $(L^2_{\text{loc}}(\mathbb{R}^N))^N$. Finally, we need to prove that for any $\varphi \in \mathcal{X}$, there holds $(I'_\mu(u)+\lambda u)[\varphi] = 0$. Since $u_n\nabla u_n \to u\nabla u$ a.e. in $\mathbb{R}^N$ and u_n is bounded in $\mathcal{X}$, we obtain that

$$
\begin{aligned}
|\nabla u_n|^{\theta-2}\nabla u_n &\rightharpoonup |\nabla u|^{\theta-2}\nabla u && \text{in } L^{\frac{\theta}{\theta-1}}(\mathbb{R}^N), \\
u_n|\nabla u_n|^2 &\rightharpoonup u|\nabla u|^2 && \text{in } L^{\frac{4}{3}}(\mathbb{R}^N), \\
u_n^2\nabla u_n &\rightharpoonup u^2\nabla u && \text{in } (L^{\frac{4}{3}}(\mathbb{R}^N))^N,
\end{aligned}
$$

it follows that

$$
(I'_\mu(u)+\lambda u)[\varphi] = \lim_{n\to\infty}(I'_\mu(u_n)+\lambda u_n)[\varphi] = 0. \qquad \square
$$

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