Pacific
Journal of
Mathematics

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Volume 322 No. 1 January 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 322, No. 1, 2023
https://doi.org/10.2140/pjm.2023.322.139

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Let $\mathfrak{g}$ be a simple complex Lie algebra of finite dimension. This paper gives an inequality relating the order of an automorphism of $\mathfrak{g}$ to the dimension of its fixed-point subalgebra and characterizes those automorphisms of $\mathfrak{g}$ for which equality occurs. This amounts to an inequality/equality for Thomae's function on the automorphism group of $\mathfrak{g}$. The result has applications to characters of zero-weight spaces, graded Lie algebras, and inequalities for adjoint Swan conductors.

1. Introduction

Thomae's function $\tau : \mathbb{R} \to \mathbb{R}$ is discontinuous precisely on the rational numbers. It is traditionally defined as $\tau(x) = \frac{1}{m}$ if $x = \frac{n}{m}$ is rational in lowest terms with $m > 0$, and $\tau(x) = 0$ if x is irrational. So $\tau(n) = 1$ for every integer n, and on each open interval $(n, n+1)$ the maximum value of τ is $\frac{1}{2}$, taken just at the midpoint of the interval. More succinctly, $\tau(x)$ is the reciprocal of the order of x in the group $\mathbb{R}/\mathbb{Z}$, with the convention that $\frac{1}{\infty} = 0$.

Every group G has an analogous function $\tau_G : G \to \mathbb{R}$, whose value at $g \in G$ is equal to the reciprocal of the order of g.

Consider the group $G = \mathrm{SO}_3$ of rotations about a fixed point O in three-dimensional Euclidean space. Here, $\tau_G(g) = \frac{1}{m}$ if g rotates by a rational multiple $\frac{n}{m}$ (in lowest terms) of a full circle, and $\tau_G(g) = 0$ otherwise. So $\tau_G(g) = 1$ if g is the identity rotation, and elsewhere τ_G has maximum value $\frac{1}{2}$ taken just on the conjugacy class of half-turns. Since every element of G is conjugate to a rotation

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MSC2020: 22Exx.

Keywords: Lie groups, automorphisms, Thomae.

about a fixed axis through O, this example is essentially the same as Thomae's original one, but now we observe that $\frac{1}{2} = \frac{1}{h}$, where h is the Coxeter number of G.

Suppose G is either a compact Lie group or a complex algebraic group. For such groups the function τ_G is discontinuous precisely on the set of torsion elements in G. The proof is the same as for $\tau = \tau_{\mathbb{R}/\mathbb{Z}}$, using the facts: (1) torsion elements can be approximated by elements of infinite order, (2) for every $\epsilon > 0$, there are only finitely many conjugacy classes in G whose elements have order $\leq \frac{1}{\epsilon}$, and (3) the conjugacy class of any torsion element is closed in G.

If G is connected and simple as an abstract group, then on the regular elements of G we have $\tau_G(g) \leq \frac{1}{h}$, where h is the Coxeter number of G. Equality holds on just the conjugacy class of *principal elements*. These are the analogues of the half-turns in SO_3 and were studied be Kostant [1959].

The aim of this paper is to extend this inequality/equality for Thomae's function to singular elements in the group $G = \mathrm{Aut}(\mathfrak{g})$ of automorphisms of a simple complex Lie algebra $\mathfrak{g}$ of finite dimension. We also indicate some applications of the result.

We will measure the singularity of an element $\theta \in G$ by the dimension of the fixed-point subalgebra $\mathfrak{g}^\theta$. We will give an upper bound for $\tau_G(\theta)$ in terms of $\dim \mathfrak{g}^\theta$, along with precise conditions for equality.

To explain these conditions, we need some preparation. We say that an element $\theta \in G$ is *ell-reg* if θ normalizes a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that (i) $\mathfrak{t}^\theta = 0$ and (ii) the cyclic group generated by θ permutes the roots of $\mathfrak{t}$ in $\mathfrak{g}$ freely.

The set of ell-reg automorphisms in G is partitioned into finitely many conjugacy classes. Each ell-reg automorphism has finite order. In fact, for each integer $m > 1$, there is at most one ell-reg conjugacy class whose elements have order m. The classification of ell-reg automorphisms was given in [Reeder et al. 2012] and is recalled in the Appendix. A uniform set of representatives for each ell-reg class is given in [Reeder et al. 2012, Proposition 12], see Section 2.1 below for the inner case.[1]

For ell-reg automorphisms it is known that the automorphism of $\mathfrak{t}$ given by $\theta|_{\mathfrak{t}}$, as in (i) and (ii), has the same order as θ. It follows that if $\theta \in G$ is ell-reg, then

$$\tau_G(\theta) = \frac{\dim \mathfrak{g}^\theta}{\dim(\mathfrak{g}/\mathfrak{t})}, \tag{1}$$

where $\mathfrak{t}$ is any Cartan subalgebra of $\mathfrak{g}$.

Fix a connected component Γ of G, and let $e \in \{1, 2, 3\}$ be the order of Γ in the group $\mathrm{Out}(\mathfrak{g})$ of connected components of G. If $\theta \in \Gamma$, the rank of $\mathfrak{g}^\theta$ depends only on e; we write

$$n_e = \mathrm{rank}(\mathfrak{g}^\theta).$$

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[1] *Ell-reg* automorphisms are called *$\mathbb{Z}$-regular* in [Reeder et al. 2012], in deference to [Springer 1974]. Except for the classes P_Γ described below, ell-reg automorphisms of $\mathfrak{g}$ are not regular elements of G. The point of "ell-reg", besides brevity, is to avoid conflict between these two meanings of the word "regular".

In Γ there is a unique conjugacy class P_Γ of elements θ of minimal order for which $\mathfrak{g}^\theta$ is a Cartan subalgebra of $\mathfrak{g}^\theta$. This order, denoted h_e, is the *twisted Coxeter number* of the coset Γ [Reeder 2010]. The elements of P_Γ are ell-reg, and it is known that

$$\frac{1}{h_e} = \frac{n_e}{\dim(\mathfrak{g}/\mathfrak{t})}. \tag{2}$$

It follows that if $\theta \in \Gamma$ has order $m \geq h_e$, then

$$\tau_G(\theta) = \frac{1}{m} \leq \frac{\dim \mathfrak{g}^\theta}{\dim(\mathfrak{g}/\mathfrak{t})}, \tag{3}$$

with equality only if $\theta \in P_\Gamma$, where τ_G is Thomae's function for the group $G = \operatorname{Aut}(\mathfrak{g})$. In this paper, we extend (3) to all $\theta \in \operatorname{Aut}(\mathfrak{g})$ as follows:

Theorem 1. *Let $\mathfrak{g}$ be a simple complex Lie algebra of finite dimension, and let τ_G be Thomae's function for the group $G = \operatorname{Aut}(\mathfrak{g})$. Then for all $\theta \in G$, we have*

$$\tau_G(\theta) \leq \frac{\dim \mathfrak{g}^\theta}{\dim(\mathfrak{g}/\mathfrak{t})}. \tag{4}$$

Equality holds in (4) *if and only if θ is ell-reg.*

From (2), we have equality in (4) if $\theta \in P_\Gamma$. Also (4) holds trivially, and is a strict inequality, if the order of θ is larger than h_e, by (3). Equality in (4) holds for ell-reg elements, by (1). Therefore, the content of Theorem 1 is (i) the inequality (4) for all $\theta \in G$ whose order m lies in the range $1 < m < h_e$, and (ii) the assertion that only ell-reg automorphisms attain equality.

The proof of Theorem 1 consists of computations with Kac diagrams. It is given in Section 3.

It is a pleasure to thank the referee for carefully reading earlier versions of this paper and providing many helpful comments.

2. Applications

First we give some applications of Theorem 1 and connections to other results.

2.1. ***Characters of zero-weight spaces.*** The original motivation for Theorem 1 was to compute characters of zero weight spaces in [Reeder 2022].[2]

Let G be a connected and simply connected complex Lie group. Fix a maximal torus T in G, with Lie algebra $\mathfrak{t}$, normalizer N, and Weyl group $W = N/T$. In every finite-dimensional irreducible representation V of G, the zero-weight space V^T is a representation of W. The problem is to compute the W-character afforded by V^T, as a function of the highest weight of V.

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[2] The first version of this paper was an appendix to an earlier version of [Reeder 2022].

For example, Kostant [1976] used his results on principal elements to calculate the trace $\mathrm{tr}(\mathsf{cox}, V^T)$ of a Coxeter element $\mathsf{cox} \in W$. He showed that $\mathrm{tr}(\mathsf{cox}, V^T)$ is 0 or ± 1 and gave an explicit formula for this trace in terms of the highest weight of V.

In [Prasad 2016], Kostant's proof was reformulated in terms of the dual group $\hat{G}$ of G. Since G is simply connected, $\hat{G}$ is the group of inner automorphisms of the Lie algebra $\hat{\mathfrak{g}}$ whose root system is dual to that of $\mathfrak{g}$. In [Reeder 2022], Theorem 1 is applied to both $\mathrm{Ad}(G)$ and $\hat{G}$ to compute traces of other Weyl group elements on V^T. A brief description of this result, indicating the role of Theorem 1, is as follows:

We call an element $w \in W$ *ell-reg* if (i) $\mathfrak{t}^w = 0$ and (ii) the group $\langle w \rangle$ generated by w acts freely on the roots of $\mathfrak{t}$ in $\mathfrak{g}$. It is easy to see that w satisfies condition (i) if and only if all lifts of w in N are T-conjugate. By [Reeder et al. 2012, Proposition 1], condition (ii) is equivalent to Springer's notion of regularity of Weyl group elements in [Springer 1974]. Springer [1974, Theorem 4.2] showed that if two regular elements of W have the same order, then they are conjugate. Finally, if w is ell-reg, it follows from [Reeder et al. 2012, Proposition 12] that if n is a lift of w to N, then w and $\mathrm{Ad}(n)$ have the same order. From these facts it follows that the set $\mathcal{E}_m(N) = \{n \in N : nT \text{ is ell-reg in } W \text{ of order } m\}$, if nonempty, is a single conjugacy class in N whose elements have order m in $\mathrm{Ad}(N)$. Hence, there is an order-preserving bijection between the set of W-conjugacy classes of ell-reg elements in W and the set of G-conjugacy classes of ell-reg elements in $\mathrm{Ad}(G)$. The classification of these classes (in W and $\mathrm{Ad}(G)$) is given in the Appendix.

Let P and Q be the weight- and root-lattices of T. Let $R^+ \subset Q$ be a system of positive roots for T in G, and let $\rho \in P$ be the half-sum of the roots in R^+. We may regard P as the group of one-parameter subgroups of a dual maximal torus $\hat{T}$ of $\hat{G}$. Assuming $\mathcal{E}_m(N)$ is nonempty, we set $\zeta_m = e^{2\pi i/m}$. From [Reeder et al. 2012, Proposition 12], we have that $\rho(\zeta_m)$ has order m and is ell-reg in $\hat{G} \subset \mathrm{Aut}(\hat{\mathfrak{g}})$.

Now let $\lambda \in P$ be the highest weight of V (with respect to R^+), and let $\theta_\lambda \in \hat{T}$ be the value at ζ_m of the one-parameter subgroup $\lambda + \rho$. Let $n \in \mathcal{E}_m(N)$, and let $w = nT \in W$. Applying Theorem 1 to both $\mathrm{Ad}(n) \in \mathrm{Ad}(G)$ and $\theta_\lambda \in \hat{G}$, one obtains an inequality of centralizers

$$\dim C_G(n) \leq \dim C_{\hat{G}}(\theta_\lambda), \tag{5}$$

with equality if and only if $(\lambda + \rho) + mQ$ is conjugate to $\rho + mQ$ under the natural W-action on P/mQ, see [Reeder 2022, Section 3.1] for the proof. From the inequality (5) and the theory of W-harmonic polynomials, one can show that $\mathrm{tr}(w, V^T) = 0$ unless there exists $v \in W$ such that $v(\lambda + \rho) \in \rho + mQ$, in which case

$$\mathrm{tr}(w, V^T) = \mathrm{sgn}(v) \prod_{\check{\alpha} \in \check{R}_m^+} \frac{\langle v(\lambda + \rho), \check{\alpha} \rangle}{\langle \rho, \check{\alpha} \rangle},$$

where the product is over the positive coroots $\check{\alpha}$ of G for which $\langle \rho, \check{\alpha} \rangle \in m\mathbb{Z}$, see [Reeder 2022, Theorem 3.4]. If $m = h$ is the Coxeter number then $\check{R}^+_m$ is empty, the product is 1, and we recover Kostant's result for $\operatorname{tr}(\mathsf{cox}, V^T)$. If $m < h$, then R^+_m is nonempty.

2.2. ***Graded Lie algebras.*** Let $\theta \in \operatorname{Aut}(\mathfrak{g})$ have order m, and let $\zeta = e^{2\pi i/m}$. Then θ determines a $\mathbb{Z}/m\mathbb{Z}$ grading

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_k\,, \tag{6}$$

where $\mathfrak{g}_k = \{x \in \mathfrak{g} : \theta(x) = \zeta^k x\}$. Note that $\mathfrak{g}_0 = \mathfrak{g}^\theta$.

From [Reeder et al. 2012, Corollary 14], it is known that the following are equivalent:

(i) There exists a semisimple element $x \in \mathfrak{g}_1$ for which $\operatorname{ad}(x) : \mathfrak{g}_0 \to \mathfrak{g}_1$ is injective.

(ii) θ is ell-reg.

Therefore, we can also use (i) as the condition for equality in Theorem 1.

Theorem 1 makes no *a priori* assumptions on the kinds of elements contained in $\mathfrak{g}_1$. But let us now assume that $\mathfrak{g}_1$ contains nonzero semisimple elements. Such gradings are said to have *positive rank.* Their classification is contained in [Vinberg 1976; Levy 2009; Reeder et al. 2012].

In the case of positive rank gradings, Theorem 1 complements results of Panyushev. Assume $x \in \mathfrak{g}_1$ is semisimple. According to [Panyushev 2005, Proposition 2.1], we have

$$\dim[\mathfrak{g}_0, x] = \frac{\dim[\mathfrak{g}, x]}{m}. \tag{7}$$

Since $\dim[\mathfrak{g}_0, x] \leq \dim \mathfrak{g}_0$ with equality exactly when (i) holds for x, and since $\dim[\mathfrak{g}, x] \leq \dim(\mathfrak{g}/\mathfrak{t})$ with equality exactly when x is a regular element of $\mathfrak{g}$, Theorem 1 combines with (7) to interpose $\dim(\mathfrak{g}/\mathfrak{t})/m$ in $\dim[\mathfrak{g}_0, x] \leq \dim \mathfrak{g}_0$. That is, we have:

Corollary 2. *Assume $x \in \mathfrak{g}_1$ is semisimple. Then we have two inequalities*

$$\dim[\mathfrak{g}_0, x] \overset{(1)}{\leq} \frac{\dim(\mathfrak{g}/\mathfrak{t})}{m} \overset{(2)}{\leq} \dim \mathfrak{g}_0.$$

Here, *inequality* (1) *is equality if and only if x is regular* (*semisimple*), *and inequality* (2) *is equality if and only if θ is ell-reg.*

Under the additional assumption that $\mathfrak{g}_1$ contains a regular semisimple element, Panyushev [2005, Theorem 4.2] also showed that

$$\dim \mathfrak{g}_0 = \frac{\dim[\mathfrak{g}/\mathfrak{t}]}{m} + k_0,$$

where $k_0 \geq 0$ is an integer depending only on the orders m and e of θ in $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Out}(\mathfrak{g})$. For example, if $e = 1$, then k_0 is the number of exponents of $\mathfrak{g}$ divisible by m. This is a sharper form of Corollary 2 in the case that $\mathfrak{g}_1$ contains a regular semisimple element.

2.3. ***Adjoint Swan conductors.*** In the setting of Section 2.1, sending a representation V to its highest weight λ is a simple case of the much broader and still mostly conjectural local Langlands correspondence (LLC). In Section 2.1, we saw that the inequalities/equalities of Theorem 1 appear on the dual side of this LLC.

They also appear on the dual side of the LLC for reductive p-adic groups, now as measures of ramification.

We use notation parallel to that of Section 2.1. Let k be a p-adic field, and let G be the group of k-rational points in a connected and simply connected almost simple k-group $\boldsymbol{G}$.

Let $\hat{\mathfrak{g}}$ be a simple complex Lie algebra whose root system is dual to that of $\boldsymbol{G}$.

The LLC predicts the existence of a partition

$$\mathrm{Irr}^2(G) = \bigsqcup_{\varphi} \Pi_\varphi$$

of the set $\mathrm{Irr}^2(G)$ of irreducible discrete series representations of G (up to equivalence) into finite sets Π_φ, where φ ranges over certain representations

$$\varphi : \mathcal{W}_k \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\hat{\mathfrak{g}})$$

of the Weil group of k. For simplicity, we assume φ is trivial on $\mathrm{SL}_2(\mathbb{C})$. (See [Gross and Reeder 2010] for more background on the LLC.) It is of interest to find invariants relating the discrete series representation π of G to the parameter φ for which $\pi \in \Pi_\varphi$.

One invariant of φ is its *adjoint Swan conductor* $\mathsf{sw}(\varphi, \mathfrak{g})$. This is an integer depending only on the image $I = \varphi(\mathcal{I})$ of the inertia subgroup $\mathcal{I} \subset \mathcal{W}_k$. There is a factorization $I = S \ltimes P$, where P is a p-group and S is a cyclic group of order prime to p. We have $\mathsf{sw}(\varphi, \mathfrak{g}) \geq 0$, with equality if and only if P is trivial.

Expected properties of the LLC imply certain inequalities for $\mathsf{sw}(\varphi, \mathfrak{g})$ which have been found to hold unconditionally. For example, if φ is totally ramified (that is, if $\mathfrak{g}^I = 0$), then the LLC predicts that

$$\dim \mathfrak{g}^\theta \leq \mathsf{sw}(\varphi, \mathfrak{g}), \tag{8}$$

where θ is a generator of S. This inequality has been proved in [Reeder 2018] and [Bushnell and Henniart 2020].

Assume now that p does not divide the order of W. By a result of Borel and Serre [1953], this ensures that P is contained in a maximal torus of $\mathrm{Aut}(\hat{\mathfrak{g}})$, which we may choose to be normalized by θ.

Let m be the order of θ. Combining (8) with Theorem 1 gives the inequality

$$\frac{\dim(\mathfrak{g}/\mathfrak{t})}{m} \leq \mathsf{sw}(\varphi, \mathfrak{g}), \tag{9}$$

which is weaker than (8), but which depends only on the order m of S, not on S itself. Moreover, the two inequalities (8) and (9) coincide if and only if θ is ell-reg.

3. Proof of Theorem 1

The torsion automorphisms of $\mathfrak{g}$ are classified by Kac diagrams. We start with a summary of Kac diagrams so that the reader can follow the computations. For more background, see [Kac 1995; Reeder 2010].

3.1. ***Kac diagrams.*** Fix a divisor $e \in \{1, 2, 3\}$ of the order of the component group $\mathrm{Out}(\mathfrak{g})$ of $\mathrm{Aut}(\mathfrak{g})$. Let $\mathrm{Aut}(\mathfrak{g}, e)$ be the set of elements in $\mathrm{Aut}(\mathfrak{g})$ whose image in $\mathrm{Out}(\mathfrak{g})$ has order e. Then $\mathrm{Aut}(\mathfrak{g}, e)$ has one or two connected components, the latter only when $\mathfrak{g} = \mathfrak{so}_8$ and $e = 3$.

For any torsion automorphism $\theta \in \mathrm{Aut}(\mathfrak{g}, e)$, the rank of the fixed point subalgebra $\mathfrak{g}^\theta$ depends only on e; we denote this rank by n_e. If $e = 1$, then $G_1 := \mathrm{Aut}(\mathfrak{g}, 1)$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$ and n_1 is the rank of $\mathfrak{g}$.

To the pair $(\mathfrak{g}, e)$ one associates an affine Dynkin diagram $\mathcal{D}(\mathfrak{g}, e)$. As we vary over all pairs $(\mathfrak{g}, e)$, the diagrams $\mathcal{D}(\mathfrak{g}, e)$ range exactly over the affine Coxeter diagrams together with all possible orientations on the multiple edges. If $e = 1$, then $\mathcal{D}(\mathfrak{g}, 1)$ is the usual affine Dynkin diagram of $\mathfrak{g}$.

The vertices in $\mathcal{D}(\mathfrak{g}, e)$ are indexed by a set I whose cardinality is $n_e + 1$, and these vertices are labeled by certain positive integers $\{c_i : i \in I\}$, where $1 \leq c_i \leq 6$.

The automorphism group $\mathrm{Aut}(\mathcal{D}(\mathfrak{g}, e))$ of the oriented and labeled diagram $\mathcal{D}(\mathfrak{g}, e)$ contains a (very small) subgroup Ω with the following property: If $e > 1$, then $\Omega = \mathrm{Aut}(\mathcal{D}(\mathfrak{g}, e))$. If $e = 1$, then $\Omega \simeq \pi_1(G_1)$.

We fix a connected component Γ of $\mathrm{Aut}(\mathfrak{g}, e)$. For any positive integer m, let Γ_m be the set of elements of Γ having order m. Then Γ_m is nonempty only if e divides m. The G_1-conjugacy classes in Γ_m are parametrized as follows: Let S_m be the set of I-tuples $s = (s_i : i \in I)$ consisting of integers $s_i \geq 0$ such that $\gcd\{s_i : i \in I\} = 1$ and

$$m = e \cdot \sum_{i \in I} c_i s_i .$$

There is a surjective mapping from S_m to the set of G_1-conjugacy classes in Γ_m (Kac coordinates). The *Kac-diagram* of the conjugacy class corresponding to s consists of the diagram $\mathcal{D}(\mathfrak{g}, e)$ with each node i replaced by s_i. Two elements s and $s' \in S_m$ map to the same conjugacy class in Γ_m if and only if their Kac diagrams are conjugate under the group Ω.

For example, in Γ there is a unique conjugacy class of automorphisms of minimal order having abelian fixed-point subalgebras. Such automorphisms are called *principal*. They are ell-reg and have Kac coordinates $s = (s_i)$, where $s_i = 1$ for all i. The order of a principal automorphism in Γ, namely

$$h_e := e \cdot \sum_{i \in I} c_i,$$

is the Coxeter number of $\operatorname{Aut}(\mathfrak{g}, e)$. It is known from [Reeder 2010] that equality holds in Theorem 1 for principal elements, namely, we have

$$\frac{1}{h_e} = \frac{n_e}{[\mathfrak{g} : \mathfrak{t}]}. \tag{10}$$

The Kac diagrams of all ell-reg automorphisms of $\mathfrak{g}$ were tabulated in [Reeder et al. 2012, Section 7] and are recalled in the Appendix. These diagrams have all Kac-coordinates $s_i \in \{0, 1\}$ and are determined by the subset $J = \{j \in I : s_j = 0\} \subsetneq I$.

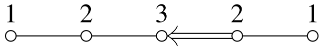
For any subset $J \subsetneq I$, we set

$$c_J = \sum_{j \in J} c_j \quad \text{and} \quad c^J = \sum_{i \notin J} c_i.$$

The subgraph of $\mathcal{D}(\mathfrak{g}, e)$ supported on J is the finite Dynkin graph of a reductive subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}$. Let $|R_J|$ be the number of roots of $\mathfrak{g}_J$.

Let $\theta \in \Gamma$ be a torsion automorphism with Kac-coordinates $s = (s_i)$, and let $J = \{j \in I : s_j = 0\}$. Then $J \neq I$, and we have $\mathfrak{g}^\theta \simeq \mathfrak{g}_J$.

Example. Consider $\mathfrak{g}$ of type E_6. The labeled diagram $\mathcal{D}(\mathfrak{g}, 2)$ for all outer automorphisms of $\mathfrak{g}$ is



The Kac diagram

$$1 \text{——} 1 \text{——} 0 \Leftarrow 0 \text{——} 1$$

represents the conjugacy class of an outer automorphism $\theta \in \operatorname{Aut}(\mathfrak{g})$ having order

$$m = 2 \cdot (1 \cdot 1 + 2 \cdot 1 + 3 \cdot 0 + 2 \cdot 0 + 1 \cdot 1) = 8.$$

We have $c_J = 3 + 2 = 5$, $c^J = 1 + 2 + 1 = 4$, and $\mathfrak{g}^\theta \simeq \mathfrak{so}_5$. This automorphism has minimal order among those with fixed-point subalgebra $\mathfrak{so}_5$.

Lemma 3. *The inequality in Theorem 1 for all torsion automorphisms in a component* $\Gamma \subset \operatorname{Aut}(\mathfrak{g}, e)$ *is equivalent to the inequality*

$$n_e \cdot c_J \leq c^J \cdot |R_J| \tag{11}$$

for every subset $J \subsetneq I$.

Proof. Let $\theta \in \Gamma_m$ have Kac coordinates (s_i), and let

$$J = \{j \in I : s_j = 0\}.$$

Then $m \geq e \cdot c^J$ with equality if and only if $s_i = 1$ for all $i \in I - J$. Since

$$\dim \mathfrak{g}^\theta = \dim \mathfrak{g}_J = n_e + |R_J| \quad \text{and} \quad \dim(\mathfrak{g}/\mathfrak{t}) = h_e n_e = e \cdot c_I \cdot n_e,$$

it follows that

$$\frac{1}{m} \leq \frac{1}{e \cdot c^J} \quad \text{and} \quad \frac{\dim \mathfrak{g}^\theta}{\dim(\mathfrak{g}/\mathfrak{t})} = \frac{n_e + |R_J|}{e \cdot c_I \cdot n_e}.$$

So, for every θ, the inequality in Theorem 1 is equivalent to having

$$e \cdot c_I \cdot n_e \leq (n_e + |R_J|) \cdot e \cdot c^J$$

for every J. Since $c_I = c^J + c_J$, the result follows. ☐

If J is empty then both sides of (11) are zero. We may assume from now on that J is nonempty and that $s_i = 1$ for all $i \in I - J$. Thus J is identified with a Kac diagram with labels in $\{0, 1\}$, where the nodes in J are labeled 0 and the nodes in $I - J$ are labeled 1.

We will show that the integer $f(\mathfrak{g}, e, J)$ defined by

$$f(\mathfrak{g}, e, J) = c^J |R_J| - n_e c_J$$

satisfies $f(\mathfrak{g}, e, J) \geq 0$. Our analysis will also find those J for which $f(\mathfrak{g}, e, J) = 0$. It turns out that the Kac diagrams of ell-reg automorphisms are exactly those for which $f(\mathfrak{g}, e, J) = 0$.

3.2. ***Type A_n.*** The case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $e = 1$ is very simple but different from the other cases, so we treat it separately here. Fix a nonempty subset $J \subsetneq I$. The root system R_J has type

$$\prod_{i=1}^{a} A_{q_i}$$

for some positive integers $q_1, \ldots, q_a$. Let $q = \sum q_i$. Since all $c_i = 1$, we have $c_J = q$ and $c^J = n + 1 - q \geq a$. Now,

$$\begin{aligned} f(\mathfrak{g}, 1, J) &= c^J \sum_{i=1}^{a} q_i(q_i + 1) - (c^J + q - 1)q \\ &= c^J \sum_{i=1}^{a} q_i^2 - q^2 + q \geq a \sum_{i=1}^{a} q_i^2 - q^2 + q \geq q, \end{aligned}$$

where the arithmetic-geometric inequality is used in the last step. Since $J \neq \varnothing$, we have $f(\mathfrak{g}, 1, J) \geq q > 0$.

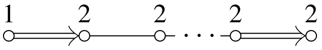
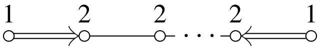
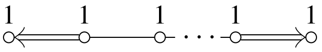
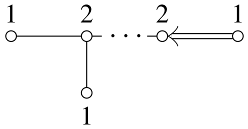
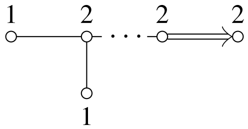
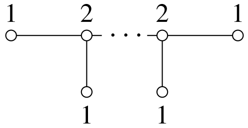
$(\mathfrak{g}, e)$	$\mathcal{D}(\mathfrak{g}, e)$	$h = e \cdot c_I$
${}^2A_{2n},\ n \geq 2$		$2(2n+1)$
$C_n,\ n \geq 2$		$2n$
${}^2D_{n+1},\ n \geq 2$		$2(n+1)$
${}^2A_{2n-1},\ n \geq 3$		$2(2n-1)$
$B_n,\ n \geq 3$		$2n$
$D_n,\ n \geq 4$		$2n-2$

Table 1. The relevant diagrams $\mathcal{D}(\mathfrak{g}, e)$ for $n \geq 2$.

3.3. ***The remaining classical Lie algebras.*** In this section, $(\mathfrak{g}, e)$ is of classical type not equal to $(\mathfrak{sl}_n, 1)$. We will write

$$n = n_e \quad \text{and} \quad h = h_e.$$

Since the criteria in Lemma 3 are easy to check for outer automorphisms of $\mathfrak{sl}_3$, we may assume $n \geq 2$.

The relevant diagrams $\mathcal{D}(\mathfrak{g}, e)$, for $n \geq 2$, are listed in Table 1. Each diagram has $n+1$ nodes. They are grouped according to their underlying Coxeter diagram. Note that ${}^2A_3 = {}^2D_3$ and $B_2 = C_2$.

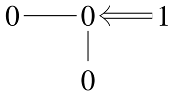
3.3.1. *Small rank.* For the reduction arguments to come, it is necessary to directly verify Theorem 1 for classical $\mathfrak{g}$ of minimal rank in Table 1. (One can shorten the task by using the first parts of Sections 3.4.1 and 3.4.2 below.) For $J \neq \varnothing$, we obtain the following:

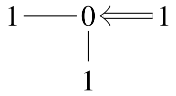
For $(\mathfrak{g}, e)$ of types ${}^2A_4$, C_2, and ${}^2D_3$, we have $f(\mathfrak{g}, e, J) \geq 0$ with equality just for the Kac diagrams:

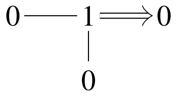
$$1 \Longrightarrow 0 \Longrightarrow 0 \qquad\qquad 1 \Longrightarrow 0 \Longleftarrow 1 \qquad\qquad 0 \Longleftarrow 1 \Longrightarrow 0$$

respectively. These diagrams represent the nonprincipal ell-reg automorphisms of $\mathfrak{sl}_5$, $\mathfrak{sp}_4$, and $\mathfrak{so}_6$; each is an involution. See Sections A.1, A.4, and A.5.

For $(\mathfrak{g}, e)$ of types ${}^2A_5$ and B_3, we have $f(\mathfrak{g}, e, J) \geq 0$, with equality just for the Kac diagrams:

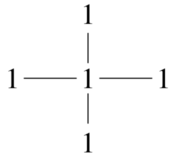


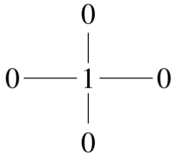


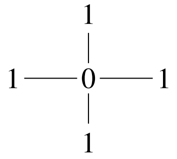


These are the nonprincipal ell-reg automorphisms of $\mathfrak{sl}_6$ and $\mathfrak{so}_7$; see Sections A.2 and A.3.

Finally consider $(\mathfrak{g}, e)$ of type D_4. We write $I = \{0, 1, 2, 3, 4\}$, where 0 is the degree-four vertex in $\mathcal{D}(\mathfrak{so}_8, 1)$. Let q be the number of degree-one vertices in J. One easily computes the following: If $s_0 = 1$, then $f(\mathfrak{so}_8, 1, J) = 2q(4-q)$. If $s_0 = 0$, then $f(\mathfrak{so}_8, 1, J) \geq 0$, with equality just for $q = 0$. Hence the inequality of Theorem 1 holds, with equality just for the Kac diagrams:







These are the Kac diagrams for the ell-reg inner automorphisms of $\mathfrak{so}_8$; see Section A.5.

3.4. ***Refinements.*** Let $\mathcal{X}$ be the set of all triples $(\mathfrak{g}, e, J)$, where $(\mathfrak{g}, e)$ is one of the above classical types for $n \geq 2$ and J is a nonempty proper subset of the set I of vertices of $\mathcal{D}(\mathfrak{g}, e)$. For any subset $\mathcal{Y} \subset \mathcal{X}$, let $\mathcal{Y}_0 = \{(\mathfrak{g}, e, J) \in \mathcal{Y} : f(\mathfrak{g}, e, J) = 0\}$. We must prove that $f \geq 0$ on $\mathcal{X}$ and that $\mathcal{X}_0$ consists precisely of the diagrams listed in the Appendix for classical $(\mathfrak{g}, e)$.

Definition. If $\mathcal{Y}' \subset \mathcal{Y}$ are subsets of $\mathcal{X}$, we say $\mathcal{Y}'$ is a *refinement* of $\mathcal{Y}$ if for every $(\mathfrak{g}, e, J) \in \mathcal{Y} - \mathcal{Y}'$, we have either:

(i) $f(\mathfrak{g}, e, J) > 0$ or

(ii) there exists $(\mathfrak{g}', e', J') \in \mathcal{Y}'$ and a positive integer c such that

$$c \cdot f(\mathfrak{g}, e, J) > f(\mathfrak{g}', e', J').$$

We note the following:

(i) Refinement is transitive: if $\mathcal{Y}''$ is a refinement of $\mathcal{Y}'$ and $\mathcal{Y}'$ is a refinement of $\mathcal{Y}$, then $\mathcal{Y}''$ is a refinement of $\mathcal{Y}$.

(ii) If $\mathcal{Y}$ is a refinement of $\mathcal{X}$ and $f \geq 0$ on $\mathcal{Y}$, then $f > 0$ on $\mathcal{X} - \mathcal{Y}$ and $\mathcal{X}_0 = \mathcal{Y}_0$.

From (ii), it suffices to find a refinement $\mathcal{Y}$ of $\mathcal{X}$ such that $f \geq 0$ on $\mathcal{Y}$ and $\mathcal{Y}_0$ consists precisely of the ell-reg triples listed in the Appendix.

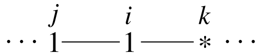
This classification guides our refinements. Ignoring the principal automorphisms as we may, we observe that in classical ell-reg Kac diagrams the vertices in $I-J$ are: (i) never adjacent and (ii) tend to be equally spaced from each other.

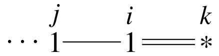
We say that a vertex $i \in I$ is *interior* if i is adjacent to at least two other vertices in $\mathcal{D}(\mathfrak{g}, e)$. If i is adjacent to just one other vertex in $\mathcal{D}(\mathfrak{g}, e)$, we say i is a *boundary vertex*. Since $n \geq 3$, every pair of adjacent vertices has at least one interior vertex. Table 1 shows that all interior i have the same value c of c_i ($c = 1$ in type ${}^2D_{n+1}$ and $c = 2$ in the other classical diagrams), and $c \geq c_i$ for all $i \in I$.

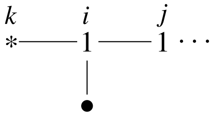
Lemma 4. *Let $\mathcal{Y}$ be the set of $(\mathfrak{g}, e, J) \in \mathcal{X}$ for which no two interior vertices of $I-J$ are adjacent in $\mathcal{D}(\mathfrak{g}, e)$. Then $\mathcal{Y}$ is a refinement of $\mathcal{X}$.*

Proof. Consider a triple $(\mathfrak{g}, e, J) \in \mathcal{X}$, and let $i, j \in I-J$ be adjacent interior vertices in $\mathcal{D}(\mathfrak{g}, e)$.

Let k be another vertex adjacent to i. The possible configurations of i, j, k in the Kac diagram are:







where the double bond has either orientation and $*, \bullet \in \{0, 1\}$ are arbitrary.

Removing i and joining j to k with a bond of the same type as the bond previously joining i to k, we obtain a diagram $\mathcal{D}(\mathfrak{g}', e)$ of the same type as $\mathcal{D}(\mathfrak{g}, e)$. The vertices of $\mathcal{D}(\mathfrak{g}', e)$ are indexed by $I' = I - \{i\}$, and we have $J \subset I'$. In this way, the diagram $\mathcal{D}(\mathfrak{g}, e, J)$ contracts by one vertex to the diagram $\mathcal{D}(\mathfrak{g}', e, J)$. The root system R'_J of $\mathfrak{g}'_J$ is isomorphic to R_J, we have $\sum_{i' \in I'-J} c_{i'} = c^J - c$, and c_J is unchanged. It follows that

$$f(\mathfrak{g}, e, J) - f(\mathfrak{g}', e, J) = c^J |R_J| - nc_J - (c^J - c)|R_J| + (n-1)c_J = c|R_J| - c_J.$$

Since $|R_J| \geq 2|J|$ and $c_J \leq c|J|$, we have

$$f(\mathfrak{g}, e, J) - f(\mathfrak{g}', e, J) \geq c|J| > 0. \tag{12}$$

Since $|I' - J| = |I - J| - 1$, repeating this procedure will eventually produce a diagram $\mathcal{D}(\mathfrak{g}'', e, J) \in \mathcal{Y}$, and we will have $f(\mathfrak{g}, e, J) > f(\mathfrak{g}'', e, J)$. $\square$

Our next refinement heads toward equilibrium for the interior components of R_J.

Given a diagram $\mathcal{D}(\mathfrak{g}, e, J) \in \mathcal{X}$, let J° be the set of interior vertices in J. We have a decomposition of root systems

$$R_J = R_J^\circ \sqcup R_{\partial J},$$

where R_J° (respectively, $R_{\partial J}$) is the union of those irreducible components of R_J whose bases are (respectively, are not) contained in J°. Let $R_1, R_2, \ldots, R_a$ be the components of R_J°. Each R_i has type A_{q_i} for some integer $q_i \geq 1$. Let

$$d(J) = \max\{|q_i - q_j| :\ 1 \leq i \leq j \leq a\}.$$

Lemma 5. *Let $\mathcal{Y}$ be as in Lemma 4, and let $\mathcal{Y}'$ be the set of $(\mathfrak{g}, e, J) \in \mathcal{Y}$ for which $d(J) \leq 1$. Then $\mathcal{Y}'$ is a refinement of $\mathcal{Y}$.*

Proof. The value of $f(\mathfrak{g}, e, J)$ is unchanged by permuting the components $R_1, \dots, R_a$. If $d(J) \geq 2$, then we may choose such a permutation to arrange that $q_1 - q_2 \geq 2$, and there are three interior vertices $\{i, j, k\}$ such that $j \in R_1$, $i \in I - J$, $k \in R_2$, as shown:

$$\cdots\ \overset{j}{0} \text{——} \overset{i}{1} \text{——} \overset{k}{0}\ \cdots$$

Now switch s_i and s_j to obtain a diagram

$$\mathcal{D}(\mathfrak{g}, e, J') \quad = \quad \cdots\ \overset{j}{1} \text{——} \overset{i}{0} \text{——} \overset{k}{0}\ \cdots$$

Note that $\mathcal{D}(\mathfrak{g}, e, J') \in \mathcal{Y}$, since $q_1 \geq 2$. The values n, c_J, and c^J are unchanged, and one checks that

$$f(\mathfrak{g}, e, J) - f(\mathfrak{g}, e, J') = 2c^J(q_1 - q_2 - 1) > 0.$$

Repeating this process, we eventually find a subset $J'' \subset I$ with $f(\mathfrak{g}, e, J) > f(\mathfrak{g}, e, J'')$ and $d(J'') \leq 1$. □

We next strengthen the refinement of Lemma 4 to include boundary vertices.

Lemma 6. *Let $\mathcal{Y}'$ be as in Lemma 5, and let $\mathcal{Z}$ be the set of $(\mathfrak{g}, e, J) \in \mathcal{Y}'$ for which no two vertices of $I - J$ are adjacent in $\mathcal{D}(\mathfrak{g}, e)$. Then $\mathcal{Z}$ is a refinement of $\mathcal{Y}'$.*

Proof. Assume $(\mathfrak{g}, e, J) \in \mathcal{Y}'$ and that i and j are adjacent vertices in $\mathcal{D}(\mathfrak{g}, e, J)$. Since $\mathcal{Y}' \subset \mathcal{Y}$, we may assume that i is an interior vertex and j is a boundary vertex. Lemma 6 has been proved for the minimal cases in Section 3.3.1, so we may also assume there is another interior vertex k adjacent to i. Near i, the possibilities for $\mathcal{D}(\mathfrak{g}, e, J)$ are as shown:

$$\text{(13)}\quad \text{(i)}\ \overset{j}{1} \Longrightarrow \overset{i}{1} \text{——} \overset{k}{0}\ \cdots \qquad \text{(ii)}\ \overset{j}{1} \Longleftarrow \overset{i}{1} \text{——} \overset{k}{0}\ \cdots \qquad \text{(iii)}\ \overset{j}{1} \text{——} \underset{\overset{|}{s}}{\overset{i}{1}} \text{——} \overset{k}{0}\ \cdots$$

where $s \in \{0, 1\}$.

In cases (i) and (ii), we proceed as in Lemma 4 by removing i and joining jk by the bond ji to obtain $\mathcal{D}(\mathfrak{g}', e, J)$. The same calculation as Lemma 4 shows that $f(\mathfrak{g}, e, J) > f(\mathfrak{g}', e, J)$.

Now for case (iii), let R_K be the component of R_J containing k, where $k \in K \subset J$, and let $q = |K| \geq 1$.

Suppose $R_K \subset R_{\partial J}$. Then R_K and the right-hand boundary of $\mathcal{D}(\mathfrak{g}, e, J)$ have one of these types (where $* \in \{0, 1\}$):

$$\begin{array}{c} \overset{k}{0} \cdots 0 \Longrightarrow 0 \end{array} \qquad \begin{array}{c} \overset{k}{0} \cdots 0 \Longleftarrow 0 \end{array} \qquad \begin{array}{ccccc} \overset{k}{0} & \cdots & 0 & \text{——} & 0 \\ & & | & & \\ & & * & & \end{array}$$

In view of (13), the diagram $\mathcal{D}(\mathfrak{g}, e, J)$ is specific enough to compute $f(\mathfrak{g}, e, J) > 0$ in each of these cases.

From now on, we may assume that R_K is an interior component of R_J, hence of type A_q, where $q \geq 1$. As in Lemma 5, after permuting components of R_J°, we may also assume that $R_J^\circ = xA_{q-1} + yA_q$ for integers x, y with $y > 0$. An expanded view of the neighborhood of i containing R_K, with single bonds omitted, is

$$\mathcal{D}(\mathfrak{g}, e, J) \quad = \quad \begin{array}{cccc} j & i & k & \\ 1 & 1 & 0 & \overbrace{0\ \ 0 \cdots 0}^{q-1 \text{ vertices}} \\ & s & & \end{array}$$

with $s \in \{0, 1\}$. Switch s_i and s_k to obtain

$$\mathcal{D}(\mathfrak{g}, e, J') \quad = \quad \begin{array}{ccccc} j & i & k & & \\ 1 & 0 & 1 & \overbrace{0\ \ 0 \cdots 0}^{q-1 \text{ vertices}} & \cdots \\ & s & & & \end{array} \tag{14}$$

Since $c^{J'} = c^J$, $n' = n$, and $c_{J'} = c_J$, we find that

$$f(\mathfrak{g}, e, J) - f(\mathfrak{g}, e, J') = 2(q + s - 2)c^J.$$

If $q + s > 2$, then $f(\mathfrak{g}, e, J) > f(\mathfrak{g}, e, J')$, so we may assume $q + s \leq 2$.

Assume that $q + s = 1$. Then $q = 1$ and $s = 0$, so $R_J^\circ = yA_1$. Since cases (i) and (ii) of (13) have been eliminated, we may assume $\mathcal{D}(\mathfrak{g}, e, J)$ has one of the forms below, where each diagram has y copies of 0 1 in the top row and single bonds are omitted:

$$\begin{array}{cccccccc} 1 & 1 & 0 & 1 & 0 & 1 \cdots 0 & 1 & \overbrace{0\ \ 0 \cdots \Rightarrow 0}^{B_r} \\ & 0 & & & & & & \end{array} \qquad \begin{array}{cccccccc} 1 & 1 & 0 & 1 & 0 & 1 \cdots 0 & 1 & \overbrace{0\ \ 0 \cdots \Leftarrow 0}^{C_r} \\ & 0 & & & & & & \end{array} \quad (r \geq 1)$$

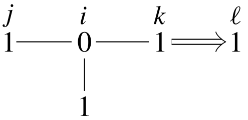
$$\begin{array}{cccccc} 1 & 1 & 0 & 1 \cdots 0 & 1 & 0 \Rightarrow 1 \\ & 0 & & & & \end{array} \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 \cdots 0 & 1 & 0 \Leftarrow 1 \\ & 0 & & & & \end{array}$$

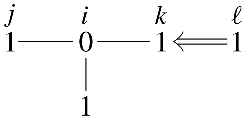
$$\begin{array}{cccccccc} 1 & 1 & 0 & 1 \cdots 0 & 1 & 0 \cdots & 0 & 0 \\ & 0 & & & & & 0 & \end{array}^{\quad D_r \text{ over } 0 \cdots 0\ 0} \qquad \begin{array}{cccccccc} 1 & 1 & 0 & 1 \cdots 0 & 1 & 0 \cdots & 0 & 0 \\ & 0 & & & & & 1 & \end{array}^{\quad A_r \text{ over } 0 \cdots 0\ 0} \quad (r \geq 2)$$

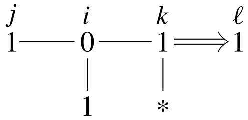
$$\begin{array}{ccccc} 1 & 1 & 0 & 1 \cdots 0 & 1 \\ & 0 & & 1 & \end{array} \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 \cdots 0 & 1 & 1 \\ & 0 & & & 1 & \end{array}$$

In each of the above cases, it is straightforward to calculate that $f(\mathfrak{g}, e, J) = y\beta(r) + \gamma(r)$, where β and γ are polynomials (of degree at most two) which are positive for all integer values of r.

Assume $q = s = 1$. Then we have $f(\mathfrak{g}, e, J) = f(\mathfrak{g}, e, J')$, with J' as in (14). Since k is interior, there is a boundary vertex ℓ adjacent to k, with $s_\ell = 1$. Then $\mathcal{D}(\mathfrak{g}, e, J')$ has one of the forms:







with $* \in \{0, 1\}$. Again, one easily checks that $f(\mathfrak{g}, e, J) > 0$.

For the remaining case $q = 2$ and $s = 0$, we have $f(\mathfrak{g}, e, J) = f(\mathfrak{g}, e, J')$ and

$$\mathcal{D}(\mathfrak{g}, e, J') \quad = \quad \begin{array}{ccccc} j & i & k & & \\ 1 & 0 & 1 & 0 & \cdots \\ & 0 & & & \end{array} \tag{15}$$

where single bonds have been omitted. Here, $R_{J'}$ has no adjacent vertices, except possibly at the other end of $\mathcal{D}(\mathfrak{g}, e, J')$, where one of the configurations of (13) could be mirrored. In that case, starting with (15), we repeat the above steps at the other end of $\mathcal{D}(\mathfrak{g}, e, J')$ to produce a triple $(\mathfrak{g}', e, J'') \in \mathcal{Z}$ such that $f(\mathfrak{g}, e, J) \geq f(\mathfrak{g}', e, J'')$. These steps only affect vertices to the right of k, so the A_2 boundary component of i in (15) persists in $R_{J''}$. In Sections 3.4.2 and 3.4.3, we will find by direct computation that $f > 0$ on every triple in $\mathcal{Z}$ having a boundary component of type A_n, for $n \geq 2$. This completes the proof of Lemma 6. □

To prove Theorem 1, it now suffices to calculate f on the set $\mathcal{Z}$ from Lemma 6. Recall that $\mathcal{Z}$ consists of those triples $(\mathfrak{g}, e, J)$ for which no two vertices in $I - J$ are adjacent and whose components of R_J° have at most two types A_{q-1} and A_q, occurring x and y times, respectively.

The refinement calculations made above were (mostly) local, using only data near the modification of the Kac diagram $\mathcal{D}(\mathfrak{g}, e, J)$ to estimate $f(\mathfrak{g}, e, J)$ from below. To actually calculate $f(\mathfrak{g}, e, J)$ requires the entire Kac diagram $\mathcal{D}(\mathfrak{g}, e, J)$, including the boundary. From here on we must proceed in cases, according to the various labeled boundaries of the graphs $\mathcal{D}(\mathfrak{g}, e)$.

Recall that $R_{\partial J}$ is the union of the components of R_J not in R_J°. Let ∂J be the subset of J supporting $R_{\partial J}$. Then $R_{\partial J}$ is a product of two classical root systems whose ranks (possibly zero) we will denote by p and r. We have

$$|R_J| = |R_{\partial J}| + q(q-1)x + q(q+1)y \quad \text{and} \quad c_J = c_{\partial J} + c(q-1)x + cqy,$$

where

$$c_{\partial J} = \sum_{j \in \partial J} c_j.$$

Define integers a and b by

$$c^J = a + cx + cy \quad \text{and} \quad n = b + qx + (q+1)y,$$

where c is the common value of c_i on the interior vertices of I. A straightforward computation gives the following:

Lemma 7. *For* $(\mathfrak{g}, e, J) \in \mathcal{Z}$, *the integer* $f(\mathfrak{g}, e, J) = |R_J|c^J - nc_J$ *has the form*

$$f(\mathfrak{g}, e, J) = cxy + \alpha x + \beta y + \gamma,$$

where α, β, *and* γ *are polynomial expressions in* p, q, *and* r *given by*:

$$\begin{aligned} \alpha &= \big(c|R_{\partial J}| + aq(q-1)\big) - \big(bc(q-1) + qc_{\partial J}\big), \\ \beta &= \big(c|R_{\partial J}| + aq(q+1)\big) - \big(bcq + (q+1)c_{\partial J}\big), \\ \gamma &= a|R_{\partial J}| - bc_{\partial J}. \end{aligned} \tag{16}$$

We will show that $\alpha, \gamma \geq 0$. Since β is obtained from α upon replacing q by $q+1$, then also $\beta \geq 0$, so this will imply that

$$f(\mathfrak{g}, e, J) \geq 0,$$

with equality if and only if $0 = xy = \alpha = \gamma$. Without loss of generality, we may then assume $y = 0$. Theorem 1 will then follow by comparison with the tables of ell-reg automorphisms in the Appendix.

3.4.1. *Types* ${}^2A_{2n}$, C_n, *and* ${}^2D_{n+1}$. The underlying Coxeter diagram with indexing set $I = \{0, 1, \dots, n\}$ is

$$0 \Longleftrightarrow 1 \text{---} 2 \text{---} \cdots \text{---} (n-1) \Longleftrightarrow n$$

The three types differ only in the labels c_i, which do not affect $|R_J|$. Let $(\mathfrak{g}, e)$ and $(\mathfrak{g}', e')$ be two of ${}^2A_{2n}$, C_n, and ${}^2D_{n+1}$, with corresponding labellings c_i and c_i'. For each subset $A \subset I$, we set

$$c_A = \sum_{i \in A} c_i \quad \text{and} \quad c_A' = \sum_{i \in A} c_i'.$$

We set $K = I - J$.

One more local calculation will reduce the number of cases further. Set:

$$f = f(\mathfrak{g}, e, J) = |R_J|c_K - nc_J \quad \text{and} \quad f' = f(\mathfrak{g}', e', J) = |R_J|c_K' - nc_J'.$$

Suppose $(\mathfrak{g}, e) = {}^2A_{2n}$ and $(\mathfrak{g}', e') = C_n$. If $n \in K$, then $c_K = c_K' + 1$ and $c_J = c_J'$, so $f > f'$. If $n \in J$, then $c_K = c_K'$ and $c_J = c_J' + 1$, so $f < f'$.

Suppose $(\mathfrak{g}, e) = {}^2A_{2n}$ and $(\mathfrak{g}', e') = {}^2D_{n+1}$. If $0 \in K$, then $1 + c_K = 2c_K'$ and $c_J = 2c_J'$, so $2f' > f$. If $0 \in J$, then $1 + c_J = 2c_J'$ and $c_K = 2c_K'$, so $f > 2f'$.

Suppose $(\mathfrak{g}, e) = C_n$ and $(\mathfrak{g}', e') = {}^2D_{n+1}$. If $\{0, n\} \in J$, then $2c'_K = c_K$ and $2c'_J = c_J + 2$, so $f = 2f' + 2n > 2f'$. If $0 \in J$ and $n \in K$, then $c_K + 1 = 2c'_K$ and $c_J + 1 = 2c'_J$, so $2f' = f + |R_J| - n$. Since no two vertices in K are adjacent, it follows that $|R_J| > n$, so $2f' > f$.

This discussion shows that we need only consider the following three cases:

(1) $(\mathfrak{g}, e) = {}^2A_{2n}$, with $0 \in K$ and $n \in J$,

(2) $(\mathfrak{g}, e) = C_n$, with $\{0, n\} \in K$,

(3) $(\mathfrak{g}, e) = {}^2D_{n+1}$, with $\{0, n\} \subset J$.

Indeed, if $f(\mathfrak{g}, e, J) \geq 0$ in Cases 1–3, then $f(\mathfrak{g}, e, J) \geq 0$ in all cases and $f(\mathfrak{g}, e, J) = 0$ can only occur in Cases 1–3.

Case 1. Assume $(\mathfrak{g}, e) = {}^2A_{2n}$ and $R_J = B_r + xA_{q-1} + yA_q$, with $r \geq 1$. Then:

$$\begin{aligned} |R_J| &= 2r^2 + q(q-1)x + q(q+1)y, & c_K &= 1 + 2x + 2y, \\ n &= r + xq + y(q+1), & c_J &= 2r + 2(q-1)x + 2qy, \\ \gamma &= 0, & \alpha &= (q-2r)(q-2r-1). \end{aligned}$$

Thus we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $q = 2r$ or $2r + 1$. These cases are the last two rows in the table in Section A.1 for $n \geq 2$.

Case 2. Assume $(\mathfrak{g}, e) = C_n$ and $R_J = xA_{q-1} + yA_q$. Then:

$$\begin{aligned} |R_J| &= q(q-1)x + q(q+1)y, & c_K &= 2x + 2y, \\ n &= qx + (q+1)y, & c_J &= 2(q-1)x + 2qy, \\ \gamma &= 0, & \alpha &= 0. \end{aligned}$$

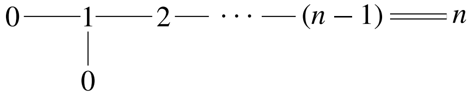
Thus we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $xy = 0$. These are the cases with $k = q$ in the table in Section A.4.

Case 3. Assume $(\mathfrak{g}, e) = {}^2D_{n+1}$ and $R_J = B_p + xA_{q-1} + yA_q + B_r$, with $p, r > 0$ and $q > 1$. Then:

$$\begin{aligned} |R_J| &= 2p^2+2r^2+q(q-1)x+q(q+1)y, & c_K &= 1+x+y, \\ n &= p+r+qx+(q+1)y, & c_J &= p+r+(q-1)x+qy, \\ \gamma &= (p-r)^2, & \alpha &= (p-r)^2+(p+r-q)(p+r-q+1). \end{aligned}$$

Thus we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $xy = 0$, $p = r$ and $q = 2p$ or $q = 2p + 1$. These are the cases in the last two rows of the table in Section A.6.

3.4.2. *Types* ${}^2A_{2n-1}$ *and* B_n. The underlying Coxeter diagram with indexing set $I = \{0, 1, \dots, n\}$ is



The two types differ only in the label $c_n = 1$ for ${}^2A_{2n-1}$ and $c_n = 2$ for B_n. Comparing, as in the previous section, we may assume $n \in K$ for ${}^2A_{2n-1}$ and $n \in J$ for B_n.

*Case A*1. Assume $n \in K$, $\{0, 1\} \subset J$, $R_J = D_p + xA_{q-1} + yA_q$, with $p \geq 2$. Then:

$$\begin{aligned} |R_J| &= 2p(p-1)+q(q-1)x+q(q+1)y, & c_K &= 1+2x+2y,\\ n &= p+qx+(q+1)y, & c_J &= 2(p-1)+2(q-1)x+2qy,\\ \gamma &= 0, & \alpha &= (2p-q)(2p-q-1). \end{aligned}$$

In this case, we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $xy = 0$ and $q = 2p$ or $q = 2p - 1$. These are the cases with $d = 1$ or $k = p$ in Section A.2.

*Case A*2. Assume $\{0, n\} \subset K$, $1 \in J$, and $R_J = A_p + xA_{q-1} + yA_q$. Then:

$$\begin{aligned} |R_J| &= p(p+1) + q(q-1)x + q(q+1)y, & c_K &= 2 + 2x + 2y,\\ n &= 1 + p + qx + (q+1)y, & c_J &= 2p - 1 + 2(q-1)x + 2qy,\\ \gamma &= p + 1, & \alpha &= 2(p - q + 1)^2 + q. \end{aligned}$$

In this case, we have $f(\mathfrak{g}, e, J) > 0$.

*Case A*3. Assume $\{0, 1, n\} \subset K$ and $R_J = xA_{q-1} + yA_q$, where $q \geq 2$. Then:

$$\begin{aligned} |R_J| &= q(q-1)x + q(q+1)y, & c_K &= 1 + 2x + 2y,\\ n &= 1 + qx + (q+1)y, & c_J &= 2(q-1)x + 2qy,\\ \gamma &= 0, & \alpha &= (q-1)(q-2). \end{aligned}$$

In this case, we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $q = 2$. This is the case $d = n$ in Section A.2.

*Case B*1. Assume $\{0, 1, n\} \subset J$ and $R_J = D_p + xA_{q-1} + yA_q + B_r$. Then:

$$\begin{aligned} |R_J| &= 2p(p-1)+2r^2+q(q-1)x+q(q+1)y, & c_K &= 2(1+x+y),\\ n &= p+r+qx+(q+1)y, & c_J &= 2(p+r-1)+2(q-1)x+2qy,\\ \gamma &= 2(p-r)(p-r-1), & \alpha &= 2(p-r)(p-r-1)+2(p+r-q)^2. \end{aligned}$$

In this case, we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $p = r$ and $q = 2r$, or $p = r + 1$ and $q = 2r + 1$. these are the cases in the last two rows of the table in Section A.3 with $k = q$.

*Case B*2. Assume $\{1, n\} \subset J$, $0 \in K$, and $R_J = A_p + xA_{q-1} + yA_q + B_r$, where $p, r \geq 1$. Then:

$$\begin{aligned}
|R_J| &= p(p+1)+2r^2+q(q-1)x+q(q+1)y, & c_K &= 3+2x+2y,\\
n &= p+r+1+qx+(q+1)y, & c_J &= 2p+2r-1+2(q-1)x+2qy,\\
\gamma &= (2r-p-1)^2+3r, & \alpha &= 2(p-q+1)^2+(q-2r)^2+2r.
\end{aligned}$$

In this case, we have $f(\mathfrak{g}, e, J) > 0$.

*Case B*3. Assume $n \in J$, $\{0, 1\} \subset K$, and $R_J = xA_{q-1}+yA_q+B_r$, where $r \geq 1$. Then:

$$\begin{aligned}
|R_J| &= 2r^2 + q(q-1)x + q(q+1)y, & c_K &= 2 + 2x + 2y,\\
n &= r + 1 + qx + (q+1)y, & c_J &= 2r + 2(q-1)x + 2qy,\\
\gamma &= 2r(r-1), & \alpha &= 2(q-r-1)^2 + 2r(r-1).
\end{aligned}$$

In this case, we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $r = 1$ and $q = 2$. This is the case $k = 2$ in Section A.3

3.4.3. *Type D_n.* Since the case $n = 4$ was covered in Section 3.3.1, we assume $n \geq 5$. Choose the indexing set $I = \{0, 1, \ldots, n\}$ as in [Bourbaki 2002], so that $\{i \in I : c_i = 1\} = \{0, 1, n-1, n\}$. Up to automorphisms of $\mathcal{D}(\mathfrak{so}_{2n}, 1)$, there are six cases for $J \cap \{0, 1, n-1, n\}$.

Case 1. Assume $\{0, 1, n-1, n\} \subset J$ and $R_J = D_p \times xA_{q-1} \times yA_q \times D_r$, where $p, q, r \geq 2$. Then:

$$\begin{aligned}
|R_J| &= 2p(p-1)+2r(r-1)+q(q-1)x & c_K &= 2+2x+2y,\\
&\qquad +q(q+1)y, & &\\
n &= p+r+qx+(q+1)y, & c_J &= 2\bigl(p+r-2+(q-1)x+qy\bigr),\\
\gamma &= 2(p-r)^2, & \alpha &= 2(p-r)^2+2(p-q+r)(p-q+r-1).
\end{aligned}$$

In this case, we have $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $p = r$ and $q = 2p$ or $q = 2p-1$. These are the cases $2 < k = q$ in Section A.5

Case 2. Assume $\{0, 1, n-1\} \subset J$, where $n \in K$, and $R_J = D_p \times xA_{q-1} \times yA_q \times A_r$, where $p, q, r \geq 2$. Then:

$$\begin{aligned}
|R_J| &= 2p(p-1)+r(r+1)+q(q-1)x & c_K &= 3+2x+2y,\\
&\qquad +q(q+1)y, & &\\
n &= 1+p+r+qx+(q+1)y, & c_J &= 2p+2r-3+2(q-1)x+2qy,\\
\gamma &= (2p-r-1)(2p-r-2)+p+r+1, & \alpha &= (2p-q-1)^2+2(q-r-1)^2+2p-1.
\end{aligned}$$

In this case, $f(\mathfrak{g}, e, J) > 0$.

Case 3. Assume $\{0, n\} \subset J$, $\{1, n-1\} \subset K$, and $R_J = A_{p-1} + xA_{q-1} + yA_q + A_{r-1}$, where $p, q, r \geq 2$. Then:

$$\begin{aligned} |R_J| &= p(p-1)+r(r-1)+q(q-1)x+q(q+1)y, & c_K &= 4+2x+2y, \\ n &= p+r+qx+(q+1)y, & c_J &= 2\big(p+r-3+(q-1)x+qy\big), \\ \gamma &= 2(p-r)^2+2(p+r), & \alpha &= 2(p-q)^2+2(q-r)^2+2q. \end{aligned}$$

In this case, $f(\mathfrak{g}, e, J) > 0$.

Case 4. Assume $\{0, 1\} \subset J$, $\{n-1, n\} \subset K$, and $R_J = D_p + xA_{q-1} + yA_q$, where $p \geq 2$. Then:

$$\begin{aligned} |R_J| &= 2p(p-1)+q(q-1)x+q(q+1)y, & c_K &= 2(1+x+y), \\ n &= 1+p+qx+(q+1)y, & c_J &= 2\big(p-1+(q-1)x+qy\big), \\ \gamma &= 2(p-1)^2, & \alpha &= 2(p-q+1)^2+2(p-2)(p-1) \\ & & &\qquad +2(q-2). \end{aligned}$$

In this case, $f(\mathfrak{g}, e, J) > 0$.

Case 5. Assume $0 \in J$, $\{1, n-1, n\} \subset K$, and $R_J = A_{p-1} + xA_{q-1} + yA_q$. Then:

$$\begin{aligned} |R_J| &= p(p-1) + q(q-1)x + q(q+1)y, & c_K &= 3 + 2x + 2y, \\ n &= 1 + p + qx + (q+1)y, & c_J &= 2p - 3 + 2(q-1)x + 2qy, \\ \gamma &= (p-1)^2 + 2, & \alpha &= 2(p-q)^2 + (q-1)^2 + 1. \end{aligned}$$

In this case, $f(\mathfrak{g}, e, J) > 0$.

Case 6. Assume $\{0, 1, n-1, n\} \subset K$ and $R_J = xA_{q-1} + yA_q$, where $q \geq 2$. Then:

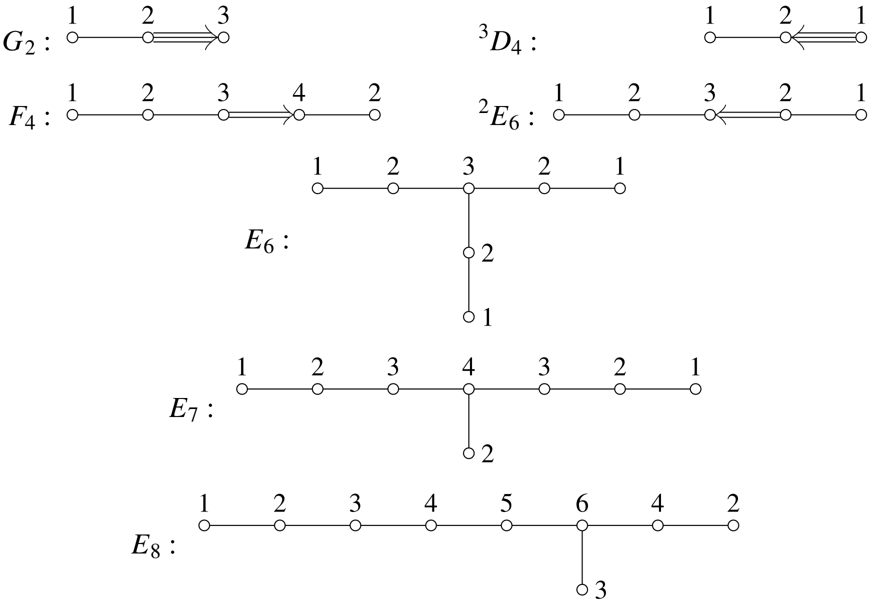
$$\begin{aligned} |R_J| &= q(q-1)x + q(q+1)y, & c_K &= 2 + 2x + 2y, \\ n &= 2 + qx + (q+1)y, & c_J &= 2(q-1)x + 2qy, \\ \gamma &= 0, & \alpha &= 2(q-1)(q-2). \end{aligned}$$

In this case, $f(\mathfrak{g}, e, J) \geq 0$, with equality if and only if $q = 2$. This is the case $k = 2$ in Section A.5.

4. Exceptional Lie algebras

On a computer one can verify Theorem 1 for the exceptional Lie algebras and ${}^3D_4$ by checking the theorem for each subset $J \subset I$. (See [Reeder 2010, (2.6)] for $\mathfrak{g} = E_8$.) The aim of this section is to make this verification somewhat more transparent.

Assume the diagram $\mathcal{D}(\mathfrak{g}, e)$, with labels c_i has one of the following types:

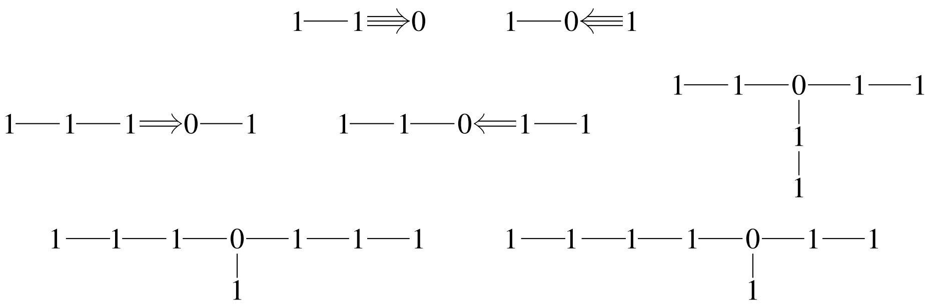


4.1. ***Small J.*** We begin with cases where $|R_J| \leq 8$.

When $R_J = A_1$, Theorem 1 follows from an observation which applies uniformly to all exceptional cases. Namely, each coefficient c_i is at most twice the average of the remaining coefficients, with equality just for the unique largest coefficient $c_{i_0} = c$; the vertex i_0 is the target of the arrow or is the branch node. Equivalently, we have

$$2c_I = (n+2)c. \tag{17}$$

On the other hand, the Kac diagrams:



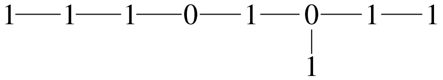
are those of the ell-reg automorphisms of order $h - ec$.

Now suppose $R_J = 2A_1$. Then $J = \{i, j\}$, where i, j are not adjacent in $\mathcal{D}(\mathfrak{g}, e)$. The maximum value of $c_i + c_j$ is $2c - 2$, with c as above. From (17), we obtain

$$|R_J|c^J - nc_J \geq 2(n - 2c + 4).$$

We check that the latter is ≥ 0, with equality only in G_2, F_4, and E_8. On the other hand, the Kac diagrams:

$$0 - 1 \Rrightarrow 0 \qquad 1 - 0 - 1 \Rightarrow 0 - 1$$



are those of ell-reg automorphisms of order $h - 2c + 2$.

If $R_J = A_2$, one finds similarly that

$$|R_J|c^J - nc_J = 6c_I - (n+6)(c_i + c_j) \geq 0,$$

with equality only in ${}^3D_4$. The Kac diagram

$$0 - 0 \Lleftarrow 1$$

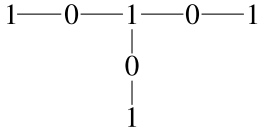
is the ell-reg outer automorphism $\mathfrak{so}_8$ of order $e = 3$.

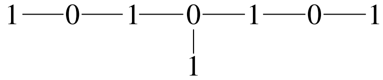
If $R_J = B_2$ or G_2, one finds that $|R_J|c^J - nc_J > 0$.

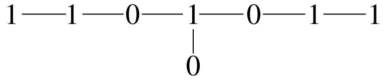
At this point, the theorem is proved for G_2 and ${}^3D_4$, and we may assume R_J has rank at least three in the remaining cases.

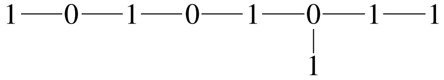
Assume that $R_J = 3A_1$. Then $f(\mathfrak{g}, e, J) = 6c_I - (n+6)c_J$. The Kac diagrams with maximal c_J are:

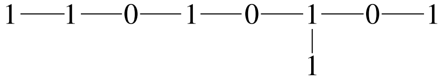
$$0 - 1 - 0 \Rightarrow 1 - 0 \qquad 0 - 1 - 0 \Leftarrow 1 - 0$$

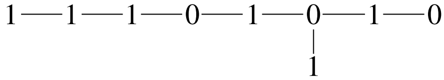


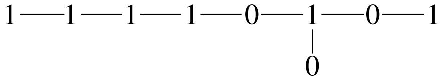












These all have $f(\mathfrak{g}, e, J) \geq 0$, with equality just in the E_6 case, where we find the Kac diagram of the ell-reg inner automorphism of $\mathfrak{g} = E_6$ of order six.

Assume that $R_J = A_1 + A_2$. In the same manner we find $f(\mathfrak{g}, e, J) \geq 0$, with equality only in the cases

$$1 - 0 - 1 \Rightarrow 0 - 0 \quad \text{and} \quad 1 - 0 - 0 \Leftarrow 1 - 0$$

which are the Kac diagrams for the ell-reg automorphisms of F_4 of order four and the outer ell-reg automorphism of E_6 of order six.

Assume that $R_J = 4A_1$. This only exists in type E. We find $f(\mathfrak{g}, e, J) \geq 0$, with equality only in the case

$$\begin{array}{ccccccccccccccc} 1 & \text{—} & 1 & \text{—} & 0 & \text{—} & 1 & \text{—} & 0 & \text{—} & 1 & \text{—} & 0 & \text{—} & 1 \\ &&&&&&&&&& | &&&& \\ &&&&&&&&&& 0 &&&& \end{array}$$

This is the ell-reg automorphism of E_8 of order 15.

4.2. ***Types F_4 and $^2E_6$.*** We now complete the proof of Theorem 1 for $(\mathfrak{g}, e)$ of types F_4 and $^2E_6$, for which $\mathcal{D}(\mathfrak{g}, e)$ has the same underlying Coxeter diagram. By the previous section, we may assume $|R_J| > 8$. Arguing as in Section 3.4.1, we need only consider cases of the form:

$$* \text{—} * \text{—} * \Longrightarrow 0 \text{—} 0 \qquad * \text{—} * \text{—} * \Longleftarrow 1 \text{—} 1$$

$$* \text{—} * \text{—} * \Longleftarrow 1 \text{—} 0 \qquad * \text{—} * \text{—} * \Longleftarrow 0 \text{—} 1$$

The Kac diagrams of these types, with $|R_J| > 8$ are tabulated as follows (the first four rows are for F_4 and the last six for $^2E_6$):

J	$R_J \cdot c^J$	$4 \cdot c_J$
$1 \text{—} 0 \text{—} 0 \Longrightarrow 0 \text{—} 0$	$48 \cdot 1$	$4 \cdot 11$
$0 \text{—} 1 \text{—} 0 \Longrightarrow 0 \text{—} 0$	$20 \cdot 2$	$4 \cdot 10$ ←
$0 \text{—} 0 \text{—} 1 \Longrightarrow 0 \text{—} 0$	$12 \cdot 3$	$4 \cdot 9$ ←
$1 \text{—} 1 \text{—} 0 \Longrightarrow 0 \text{—} 0$	$18 \cdot 3$	$4 \cdot 9$
$0 \text{—} 0 \text{—} 0 \Longleftarrow 1 \text{—} 1$	$12 \cdot 3$	$4 \cdot 6$
$0 \text{—} 0 \text{—} 0 \Longleftarrow 1 \text{—} 0$	$14 \cdot 2$	$4 \cdot 7$ ←
$0 \text{—} 0 \text{—} 0 \Longleftarrow 0 \text{—} 1$	$32 \cdot 1$	$4 \cdot 8$ ←
$1 \text{—} 0 \text{—} 0 \Longleftarrow 0 \text{—} 1$	$18 \cdot 2$	$4 \cdot 7$
$0 \text{—} 1 \text{—} 0 \Longleftarrow 0 \text{—} 1$	$10 \cdot 3$	$4 \cdot 6$

We have $f(\mathfrak{g}, e, J) \geq 0$ with equality in the cases marked by ←. These are the ell-reg automorphisms of orders 2 and 3 for F_4 and outer ell-reg automorphisms of E_6 of orders 4 and 2. This completes the proof of Theorem 1 in the cases F_4 and $^2E_6$.

4.3. ***Types E_6, E_7, and E_8.*** Here, $e = 1$. We consider the ends of the interval $1 < m < h$ in two steps:

Step 1. For each $1 < m < n$, we compute the minimum

$$r(m) = \min\{|R_J| : c^J = m\}.$$

In the tables below, we check that

$$r(m) \geq \frac{|R|}{m} - n \tag{18}$$

for each $m < n$, and we verify that equality holds in (18) for at most one J with $c^J = m$. This will prove Theorem 1 when $m < n$.

Next we will consider $|R_J|$, where $c^J \geq n$. If $|R_J| > h - n$, then

$$c^J|R_J| - nc_J > c^J(h-n) - nc_J = c^J h - n(c^J + c_J) = (c^J - n)h \geq 0,$$

so $f(\mathfrak{g}, 1, J) > 0$. Hence, we may also assume $|R_J| \leq h - n$. Since we have already proved Theorem 1 for $|R_J| \leq 8$, we may in fact assume that

$$10 \leq |R_J| \leq h - n.$$

Step 2. For each even integer $r \leq h - n$, we compute the minimum

$$m(r) = \min\{c^J : |R_J| = r\}.$$

In the tables below, we check that

$$r \geq \frac{|R|}{m(r)} - n, \tag{19}$$

and we verify that equality holds in (19) for at most one J with $|R_J| = r$. This will complete the proof of Theorem 1.

4.3.1. *Type* E_6. In Step 1 for E_6, we take $1 < m < 6$ and compute $r(m)$ in the following table. The types of R_J for which $c^J = m$ are shown; those for which $|R_J| = r(m)$ are in bold. We write the irreducible components of R_J multiplicatively. The rightmost column indicates the unique J for which $r(m) = (|R|/m) - n$, if it exists. The tabulations of Step 1 are as follows, with single bonds omitted:

m	types of R_J with $c^J = m$	$r(m)$	$(\lvert R\rvert/m) - 6$	J
2	$\boldsymbol{A_1A_5}$, D_5	32	30	none
3	$\boldsymbol{A_2^3}$, A_1A_4, D_4, A_5	18	18	$\begin{smallmatrix}0&0&1&0&0\\&&0&&\\&&0&&\end{smallmatrix}$
4	$\boldsymbol{A_1A_2^2}$, $\boldsymbol{A_1A_3}$, $A_1^2A_3$, A_4	14	12	none
5	$\boldsymbol{A_1^2A_2}$, $A_1A_2^2$, A_1A_3, A_3	10	$\frac{42}{5}$	none

Since $h - n = 12 - 6 < 8$, the proof of Theorem 1 for E_6 is completed by Step 1 alone.

4.3.2. *Type* E_7. In Step 1 for E_7, we take $1 < m < 7$ and compute $r(m)$ in the following table, using the same notational conventions as for E_6 above, with single bonds omitted:

m	types of R_J with $c^J = m$	$r(m)$	$(\lvert R\rvert/m)-7$	J
2	$\mathbf{A_7}$, A_1D_6, E_6	56	56	0 0 0 0 0 0 0 1
3	$\mathbf{A_2A_5}$, A_1D_5, A_6, D_6	36	35	none
4	$\mathbf{A_1A_3^2}$, $\mathbf{A_2A_4}$, $A_1^2D_4$, A_5, A_1A_5, D_5	26	$\frac{49}{2}$	none
5	$\mathbf{A_1A_2A_3}$, A_1A_4, A_2A_4, A_1D_4, A_5, A_1A_5	20	$\frac{91}{5}$	none
6	$\mathbf{A_1A_2^2}$, $A_1^2A_3$, A_2A_3, A_2^3, $A_1^3A_3$, A_4, A_1A_4, A_3^2, D_4, A_5	14	14	1 0 0 1 0 0 1 0

For Step 2, we need only consider $r = 10$. The only simply laced root systems with 10 roots are A_1^5 and $A_1^2A_2$. All occurrences of these as R_J in E_7 have $c^J \geq 8$. Since

$$\frac{|R|}{8} - 7 = \frac{35}{4} < 10,$$

Theorem 1 is now proved for E_7.

4.3.3. *Type* E_8. In Step 1 for E_8, we take $1 < m < 8$ and compute $r(m)$ in the following table, using the same notational conventions as for E_6 and E_7 above, with single bonds omitted:

m	types of R_J with $c^J = m$	$r(m)$	$(240/m) - 8$	J
2	$\mathbf{D_8}$, A_1E_7	112	112	0 0 0 0 0 0 0 1 0
3	$\mathbf{A_8}$, A_2E_6, D_7, E_7	72	72	0 0 0 0 0 0 0 0 1
4	$\mathbf{A_3D_5}$, A_7, A_1A_7, A_1D_6, A_1E_6	52	52	0 0 0 1 0 0 0 0 0
5	$\mathbf{A_4^2}$, A_1A_6, A_2D_5, A_7, D_6, A_1E_6	40	40	0 0 0 0 1 0 0 0 0
6	$\mathbf{A_3A_4}$, $A_1^2A_5$, A_3D_4, A_2A_5, $A_1A_2A_5$, A_1D_5, A_6, $A_1^2D_5$, A_7, E_6	32	32	1 0 0 0 1 0 0 0 0
7	$\mathbf{A_1A_2A_4}$, A_2D_4, A_3A_4, A_1A_5 A_1D_5, A_6, A_1A_6, A_2D_5	28	$\frac{184}{7}$	none

For Step 2, we take $r = 10, 12, \ldots, 22$ and compute $m(r)$ in the following table. The types of R_J for which $|R_J| = r$ are shown; those for which $c^J = m(r)$ are in bold; and that J for which $|R_J| = (240/c^J) - n$, if it exists, is shown in the right column (single bonds have been omitted).

r	types of R_J with $\lvert R_J\rvert = r$	$m(r)$	$[240/m(r)] - 8$	J
10	$\mathbf{A_1^5}$, $\mathbf{A_1^2A_2}$	14	$\frac{64}{7}$	none
12	$\mathbf{A_1^3A_2}$, A_2^2, A_3	12	12	1 0 1 0 0 1 0 1 0
14	$\mathbf{A_1^4A_2}$, $\mathbf{A_1A_2^2}$, $\mathbf{A_1A_3}$	12	12	none
16	$\mathbf{A_1^2A_2^2}$, $A_1^2A_3$	10	16	1 0 1 0 0 1 0 0 0
18	$\mathbf{A_2A_3}$, $\mathbf{A_1^3A_3}$, A_2^3	10	16	none
20	$\mathbf{A_1A_2A_3}$, $\mathbf{A_1A_2^3}$, A_4	9	$\frac{56}{3}$	none
22	$\mathbf{A_1^2A_2A_3}$, A_1A_4	8	22	0 1 0 0 0 1 0 0 0

In each case, we have

$$r \geq \left[\frac{240}{m(r)}\right] - 8,$$

and equality is achieved by at most one J, as indicated in the rightmost column.

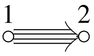
The proof of Theorem 1 for E_8 is now complete.

Appendix: The classification of ell-reg automorphisms

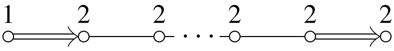
For reference in the proofs above, we recall the classification of ell-reg automorphisms given in [Reeder et al. 2012]. There is only one inner ell-reg automorphism of $\mathfrak{sl}_n$, namely the principal one, so we ignore this case. Recall that m denotes the order of an ell-reg automorphism of $\mathfrak{g}$.

A.1. ***Type*** $\mathbf{{}^2A_{2n}}$**.** The ell-reg outer automorphisms of $\mathfrak{sl}_{2n+1}$ correspond to odd quotients d of $2n$ and $2n+1$. The graphs $\mathcal{D}(\mathfrak{sl}_{2n+1}, 2)$ are as shown:

$n \geq 1$:



$n > 1$:



The ell-reg outer automorphisms of $\mathfrak{sl}_{2n+1}$ correspond to odd quotients d of $2n+1$ and $2n$. We write these quotients as

$$d = \frac{2n+1}{2k+1} \quad \text{and} \quad d = \frac{n}{k},$$

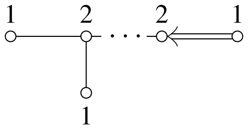
respectively. The cases overlap only when $d = 1$. The corresponding ell-reg automorphism has order $m = 2d$ in both cases:

$d = m/2$	s
3	$1 \Rrightarrow 1$
2	$1 \Rrightarrow 0$

$d = m/2$	s
$2n+1$	$1 \Rightarrow 1 - 1 - \cdots - 1 - 1 \Rightarrow 1$
1	$1 \Rightarrow 0 - 0 - \cdots - 0 - 0 \Rightarrow 0$
$\frac{2n+1}{2k+1}$	$1 \Rightarrow \overbrace{0 \cdots 0}^{A_{2k}} - 1 - \overbrace{0 \cdots 0}^{A_{2k}} - 1 - \cdots - 1 - \overbrace{0 \cdots 0 \Rightarrow 0}^{B_k}$
$\frac{n}{k}$	$1 \Rightarrow \overbrace{0 \cdots 0}^{A_{2k-1}} - 1 - \overbrace{0 \cdots 0}^{A_{2k-1}} - 1 - \cdots - 1 - \overbrace{0 \cdots 0 \Rightarrow 0}^{B_k}$

In the two last rows we have $0 < k < n$ such that d is odd and the number of type-A factors is $(d-1)/2$. The next-to-last row corrects an error in [Reeder et al. 2012].

A.2. ***Type*** $^2A_{2n-1}$**.** The graph $\mathcal{D}(\mathfrak{sl}_{2n}, 2)$, with $n \geq 3$ and labels $c_0, c_1, \ldots, c_n$, is shown here, with $c_0 = c_n = 1$:



The ell-reg outer automorphisms of $\mathfrak{sl}_{2n}$ correspond to odd quotients d of $2n-1$ and $2n$. We write these quotients as

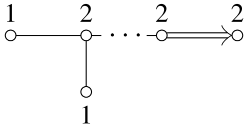
$$d = \frac{2n-1}{2k-1} \quad \text{and} \quad d = \frac{n}{k},$$

respectively. The cases overlap only when $d = 1$. The corresponding ell-reg automorphism has order $m = 2d$ in both cases.

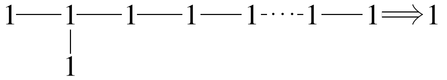
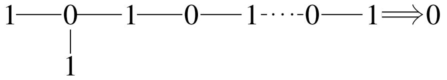
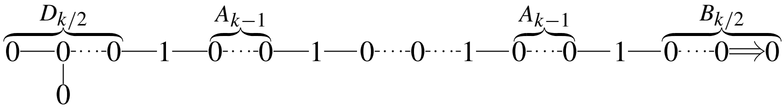
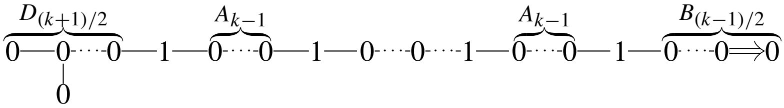
$d = m/2$	s
$2n-1$	$1 - \underset{\overset{\mid}{1}}{1} - 1 - 1 \cdots 1 - 1 \Leftarrow 1$
1	$0 - \underset{\overset{\mid}{0}}{0} - 0 - 0 \cdots 0 - 0 \Leftarrow 1$
$n,\ n$ odd	$1 - \underset{\overset{\mid}{1}}{0} - 1 - 0 - 1 \cdots 1 - 0 \Leftarrow 1$
$\frac{2n-1}{2k-1}$	$\overbrace{0 - \underset{\overset{\mid}{0}}{0} \cdots 0}^{D_k} - 1 - \overbrace{0 \cdots 0}^{A_{2k-2}} - 1 \cdots 1 - \overbrace{0 \cdots 0}^{A_{2k-2}} \Leftarrow 1$
$\frac{n}{k}$	$\overbrace{0 - \underset{\overset{\mid}{0}}{0} \cdots 0}^{D_k} - 1 - \overbrace{0 \cdots 0}^{A_{2k-1}} - 1 \cdots 1 - \overbrace{0 \cdots 0}^{A_{2k-1}} \Leftarrow 1$

In the last two rows we have $1 < k < n$ such that d is odd and there are $(d-1)/2$ components of type A.

A.3. ***Type B_n.*** The graph $\mathcal{D}(\mathfrak{so}_{2n+1}, 1)$ with labels $c_0, c_1, \ldots, c_n$ is shown here, with $c_0 = c_n = 1$:

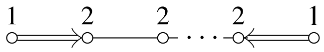


The ell-reg automorphisms of $\mathfrak{so}_{2n+1}$ are of the form π^k, where π is a principal automorphism and k is a divisor of n. The order m of π^k is $m = 2n/k$, and the Kac coordinates of π^k are given in the table below. We replace each node i by the Kac coordinate $s_i \in \{0, 1\}$, and also omit the single bonds in the graph. Recall that $J = \{i \in I : s_i = 0\}$.

$k \mid n$	m	$s = (s_0, s_1, \ldots, s_n)$
1	$2n$	
2	n	
$k > 2$, k even	$\frac{2n}{k}$	
$k > 1$, k odd	$\frac{2n}{k}$	

The second line, where $m = n$, only occurs if n is even. In the last two lines there are $(n/k) - 1$ factors of type A_{k-1}.

A.4. ***Type C_n.*** The graph $\mathcal{D}(\mathfrak{sp}_{2n}, 1)$ with labels $c_0, c_1, \ldots, c_n$ is shown here, with $c_0 = c_n = 1$:



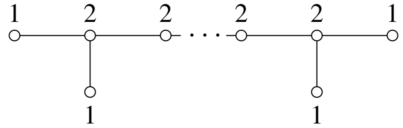
The Coxeter number is $2n$. As with $\mathfrak{so}_{2n+1}$, the ell-reg automorphisms of $\mathfrak{sp}_{2n}$ are powers π^k of a principal automorphism π, where k is a divisor of n. The order m

of π^k is $m = 2n/k$, and the Kac coordinates of π^k are these:

$k \mid n$	m	$s = (s_0, s_1, \ldots, s_n)$
1	$2n$	$1 \Rightarrow 1 - 1 - 1 - 1 \cdots 1 - 1 \Leftarrow 1$
$k > 1$	$\frac{2n}{k}$	$0 \Rightarrow \overbrace{0 \cdots 0}^{A_{k-1}} - 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 \cdots 1 - \overbrace{0 \cdots 0}^{A_{k-1}} \Leftarrow 1$

In the last line, for $k > 1$, there are n/k factors of type A_{k-1}.

A.5. ***Type D_n.*** The graph $\mathcal{D}(\mathfrak{so}_{2n}, 1)$ with labels $c_0,\ c_1,\ \ldots\ ,\ c_n$ is shown here, with $c_0 = c_1 = c_{n-1} = c_n = 1$:



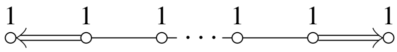
The ell-reg conjugacy classes in $\mathrm{Aut}(\mathfrak{so}_{2n}, 1)$ correspond to even divisors k of n, where $m = 2n/k$, and odd divisors k of $n - 1$, where $m = (2n - 2)/k$, as shown in the table below:

k	m	$s = (s_i)$
1	$2n-2$	$1 - \underset{\substack{\vert \\ 1}}{1} - 1 \cdots 1 - \underset{\substack{\vert \\ 1}}{1} - 1$
2	n, n even	$1 - \underset{\substack{\vert \\ 1}}{0} - 1 - 0 - 1 \cdots 1 - 0 - 1 - \underset{\substack{\vert \\ 1}}{0} - 1$
n, n even	2	$\overbrace{0 - \underset{\substack{\vert \\ 0}}{0} \cdots 0}^{D_{n/2}} - 1 - \overbrace{0 \cdots \underset{\substack{\vert \\ 0}}{0} - 0}^{D_{n/2}}$
k even, $k \mid n$, $2 < k < n$	$\frac{2n}{k}$	$\overbrace{0 - \underset{\substack{\vert \\ 0}}{0} \cdots 0}^{D_{k/2}} - 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 - 0 \cdots 0 - 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 - \overbrace{0 \cdots \underset{\substack{\vert \\ 0}}{0} - 0}^{D_{k/2}}$
k odd, $k \mid n-1$, $1 < k < n-1$	$\frac{2n-2}{k}$	$\overbrace{0 - \underset{\substack{\vert \\ 0}}{0} \cdots 0}^{D_{(k+1)/2}} - 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 - 0 \cdots 0 - 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 - \overbrace{0 \cdots \underset{\substack{\vert \\ 0}}{0} - 0}^{D_{(k+1)/2}}$

In the last two rows, the number of type-A factors is one less than n/k and $(n - 1)/k$, respectively.

A.6. ***Type $^2D_{n+1}$.*** The graph $\mathcal{D}(\mathfrak{so}_{2n+2}, 2)$, with $n \geq 2$ and $c_0 = c_1 = \cdots = c_n = 1$:

$^2D_{n+1}$:



The ell-reg classes in $\mathrm{Aut}(\mathfrak{so}_{2n+2}, 2)$ correspond to even divisors k of n with order $m = 2n/k$ and odd divisors k of $n+1$ with order $m = 2(n+1)/k$.

k	m	$s = (s_0, s_1, \dots, s_n)$
1	$2n+2$	$1 \Leftarrow 1 - 1 \cdots 1 - 1 \Rightarrow 1$
2	n, n even	$0 \Leftarrow 1 - 0 - 1 - 0 \cdots 0 - 1 - 0 - 1 \Rightarrow 0$
k even, $k \mid n$, $2 < k$	$\frac{2n}{k}$	$\overbrace{0 \Leftarrow 0 \cdots 0}^{B_{k/2}} - 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 \cdots 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 - \overbrace{0 \cdots 0 \Rightarrow 1}^{B_{k/2}}$
k odd, $k \mid n+1$, $1 < k$	$\frac{2n+2}{k}$	$\overbrace{0 \Leftarrow 0 \cdots 0}^{B_{(k-1)/2}} - 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 \cdots 1 - \overbrace{0 \cdots 0}^{A_{k-1}} - 1 - \overbrace{0 \cdots 0 \Rightarrow 0}^{B_{(k-1)/2}}$

In the last two rows, the number of type A factors is one less than n/k and $(n+1)/k$, respectively.

A.7. ***Exceptional Lie algebras.*** When only single bonds are present, they have been omitted.

E_6

m	s
12	$\begin{smallmatrix}1&1&1&1&1\\&&1&&\\&&1&&\end{smallmatrix}$
9	$\begin{smallmatrix}1&1&0&1&1\\&&1&&\\&&1&&\end{smallmatrix}$
6	$\begin{smallmatrix}1&0&1&0&1\\&&0&&\\&&1&&\end{smallmatrix}$
3	$\begin{smallmatrix}0&0&1&0&0\\&&0&&\\&&0&&\end{smallmatrix}$

$^2E_6$

m	s
18	$1 - 1 - 1 \Leftarrow 1 - 1$
12	$1 - 1 - 0 \Leftarrow 1 - 1$
6	$1 - 0 - 0 \Leftarrow 1 - 0$
4	$0 - 0 - 0 \Leftarrow 1 - 0$
2	$0 - 0 - 0 \Leftarrow 0 - 1$

E_7

m	s
18	$\begin{smallmatrix}1&1&1&1&1&1&1\\&&&1&&&\end{smallmatrix}$
14	$\begin{smallmatrix}1&1&1&0&1&1&1\\&&&1&&&\end{smallmatrix}$
6	$\begin{smallmatrix}1&0&0&1&0&0&1\\&&&0&&&\end{smallmatrix}$
2	$\begin{smallmatrix}0&0&0&0&0&0&0\\&&&1&&&\end{smallmatrix}$

E_8

m	s
30	$\begin{smallmatrix}1&1&1&1&1&1&1&1\\&&&&&1&&\end{smallmatrix}$
24	$\begin{smallmatrix}1&1&1&1&1&0&1&1\\&&&&&1&&\end{smallmatrix}$
20	$\begin{smallmatrix}1&1&1&0&1&0&1&1\\&&&&&1&&\end{smallmatrix}$
15	$\begin{smallmatrix}1&1&0&1&0&1&0&1\\&&&&&0&&\end{smallmatrix}$
12	$\begin{smallmatrix}1&0&1&0&0&1&0&1\\&&&&&0&&\end{smallmatrix}$
10	$\begin{smallmatrix}1&0&1&0&0&1&0&0\\&&&&&0&&\end{smallmatrix}$
8	$\begin{smallmatrix}0&1&0&0&0&1&0&0\\&&&&&0&&\end{smallmatrix}$
6	$\begin{smallmatrix}1&0&0&0&1&0&0&0\\&&&&&0&&\end{smallmatrix}$
5	$\begin{smallmatrix}0&0&0&0&1&0&0&0\\&&&&&0&&\end{smallmatrix}$
4	$\begin{smallmatrix}0&0&0&1&0&0&0&0\\&&&&&0&&\end{smallmatrix}$
3	$\begin{smallmatrix}0&0&0&0&0&0&0&0\\&&&&&1&&\end{smallmatrix}$
2	$\begin{smallmatrix}0&0&0&0&0&0&0&1\\&&&&&0&&\end{smallmatrix}$

G_2

m	s
6	$1 - 1 \Rrightarrow 1$
3	$1 - 1 \Rrightarrow 0$
2	$0 - 1 \Rrightarrow 0$

F_4

m	s
12	$1 - 1 - 1 \Rightarrow 1 - 1$
8	$1 - 1 - 1 \Rightarrow 0 - 1$
6	$1 - 0 - 1 \Rightarrow 0 - 0$
4	$1 - 0 - 1 \Rightarrow 0 - 0$
3	$0 - 0 - 1 \Rightarrow 0 - 0$
2	$0 - 1 - 0 \Rightarrow 0 - 0$

$^3D_4$

m	s
12	$1 - 1 \Lleftarrow 1$
6	$1 - 0 \Lleftarrow 1$
3	$0 - 0 \Lleftarrow 1$

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Received August 4, 2021. Revised September 12, 2022.

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PACIFIC JOURNAL OF MATHEMATICS

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 322 No. 1 January 2023