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**VANISHING THEOREMS AND ADJOINT LINEAR SYSTEMS
ON NORMAL SURFACES IN POSITIVE CHARACTERISTIC**

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We prove the Kawamata–Viehweg vanishing theorem for a large class of divisors on surfaces in positive characteristic. By using this vanishing theorem, Reider-type theorems and extension theorems of morphisms for normal surfaces are established. As an application of the extension theorems, we characterize nonsingular rational points on any plane curve over an arbitrary base field in terms of rational functions on the curve.

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1. Introduction

This paper is a continuation of [Enokizono 2020]. The purpose of this paper is to establish the vanishing theorem and the criterion for spannedness of adjoint linear systems $|K_X + D|$ for *positive* divisors D on normal surfaces X in positive characteristic.

Vanishing theorems. Kodaira-type vanishing theorems are fundamental tools in algebraic geometry. Unfortunately, Kodaira’s vanishing theorem fails in positive characteristic [Raynaud 1978]. In the case of surfaces, it is known that Kodaira’s vanishing, or more generally, the Kawamata–Viehweg vanishing for \mathbb{Z} -divisors, holds except for quasielliptic surfaces with Kodaira dimension 1 and surfaces of general type [Shepherd-Barron 1991; Terakawa 1999; Mukai 2013]. For these

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exceptional surfaces, Kodaira's vanishing also holds under some positive condition for divisors D (e.g., the self-intersection number D^2 is large to some extent) [Fujita 1983; Shepherd-Barron 1991; Terakawa 1999; Di Cerbo and Fanelli 2015; Zhang 2021]. However for the Kawamata–Viehweg vanishing for \mathbb{Q} -divisors, there exist counterexamples even for smooth rational surfaces [Cascini and Tanaka 2018; Bernasconi 2021]. Under some liftability conditions, it is known that the Kawamata–Viehweg vanishing holds in full generality (see [Hara 1998; Langer 2015]). But for an arbitrary surface, the only known results are the asymptotic versions of the Kawamata–Viehweg vanishing [Tanaka 2015]. One of the main result in this paper is the following vanishing theorem for surfaces in positive characteristic:

Theorem 1.1 (Theorem 4.17). *Let X be a normal proper surface over an algebraically closed field k of positive characteristic. Let D be a big \mathbb{Z} -positive divisor on X . If $\dim |D| \geq \dim H^1(\mathcal{O}_X)_n$, then $H^i(\mathcal{O}_X(K_X + D)) = 0$ holds for any $i > 0$.*

Here, $H^1(\mathcal{O}_X)_n$ denotes the nilpotent part of $H^1(\mathcal{O}_X)$ under the Frobenius action, and a divisor D on X is said to be \mathbb{Z} -positive if $B - D$ is not nef over B for any effective negative definite divisor $B > 0$ on X . Typical examples of \mathbb{Z} -positive divisors are the round-ups $D = \lceil M \rceil$ of nef \mathbb{Q} -divisors M and numerically connected divisors which are not negative definite. Theorem 1.1 is new even when X is smooth (and of general type) and D is ample.

A well-known proof of the Kawamata–Viehweg vanishing theorem is to use the covering method and reduce to the Kodaira vanishing theorem (see [Kawamata et al. 1987]). In positive characteristic, although the Kodaira vanishing theorem holds for almost all surfaces not of general type, it is difficult to apply the covering method. The reason is that the covering method reduces the vanishing of the cohomology on a given surface X to that on the total space Y of a covering $Y \rightarrow X$, but in many cases Y must be of general type, and so Kodaira's vanishing cannot be applied. For this reason, we use another method to prove Theorem 1.1. The key observation to prove Theorem 1.1 is to study the connectedness of effective divisors and to prove the following lemma:

Lemma 1.2 (Corollary 3.13). *Let $D > 0$ be an effective big \mathbb{Z} -positive divisor on a normal complete surface X over an algebraically closed field k . Then $H^0(\mathcal{O}_D) \cong k$.*

For the higher-dimensional case, we will prove the following vanishing theorem on H^1 :

Theorem 1.3 (Theorem 4.20). *Let X be a normal projective variety of dimension greater than 1 over an algebraically closed field k . Let D be a divisor on X such that $D = \lceil M \rceil$ for some nef \mathbb{R} -divisor M on X with $\kappa(D) \geq 2$ or $v(M) \geq 2$. If $\text{char } k > 0$, we further assume that $\dim |D| \geq \dim H^1(\mathcal{O}_X)_n$. Then $H^1(X, \mathcal{O}_X(-D)) = 0$.*

Adjoint linear systems. For adjoint linear systems $|K_X + D|$ on varieties X , it is expected that the “positivity” of the divisor D implies the “spannedness” of the linear system $|K_X + D|$ (Fujita’s conjecture is a typical statement). For smooth surfaces X in characteristic 0, Reider’s theorem [1988] roughly says that if the adjoint linear system $|K_X + D|$ for a nef and big divisor D has a base point, there exists a curve B on X obstructing the basepoint-freeness such that D and B satisfy some numerical conditions. Reider’s method enables us to give various applications of adjoint linear systems, especially, the affirmative answer to Fujita’s conjecture for surfaces in characteristic 0. On the other hand, although Shepherd-Barron and others [Shepherd-Barron 1991; Terakawa 1999; Moriwaki 1993; Di Cerbo and Fanelli 2015] studied adjoint linear systems on smooth surfaces in positive characteristic by using Reider’s method based on some Bogomolov-type inequalities, there exist counterexamples to Fujita’s conjecture for surfaces in positive characteristic [Gu et al. 2022]. In this paper, as applications of Theorem 1.1 and other vanishing results (Corollary 4.9, Propositions 4.11 and 4.12), we give some results for adjoint linear systems on not necessarily smooth surfaces in positive characteristic. Here, we state immediate corollaries of the main result, Theorem 5.2:

Corollary 1.4 (Corollary 5.8). *Let X be a normal complete surface over an algebraically closed field k . When $\text{char } k > 0$, we further assume that the Frobenius map on $H^1(\mathcal{O}_X)$ is injective. Let $x \in X$ be at most a rational singularity. Let L be a divisor on X which is Cartier at x . We assume that there exists an integral curve $D \in |L - K_X|$ passing through x such that (X, x) or (D, x) is singular and D is analytically irreducible at x when $\text{char } k > 0$. Then x is not a base point of $|L|$.*

Corollary 1.5 (Corollary 5.10). *Let X be a normal proper surface over an algebraically closed field k of positive characteristic. Assume that the geometric genus of any singular point of X is less than 4. Let D be a nef divisor on X such that $K_X + D$ is Cartier. Then $|K_X + D|$ is base point free if the following hold:*

- (i) $D^2 > 4$,
- (ii) $DB \geq 2$ for any curve B on X , and
- (iii) $\dim |D| \geq \dim H^1(\mathcal{O}_X)_n + 3$.

The following is a partial answer to Fujita’s conjecture for surfaces in positive characteristic:

Corollary 1.6 (Corollary 5.13). *Let X be a projective surface with at most rational double points over an algebraically closed field k of positive characteristic. Let H be an ample divisor on X . Then $|K_X + mH|$ is base point free for any $m \geq 3$ with $\dim |mH| \geq \dim H^1(\mathcal{O}_X)_n + 3$ and is very ample for any $m \geq 4$ with $\dim |mH| \geq \dim H^1(\mathcal{O}_X)_n + 6$.*

For other corollaries (e.g., for the pluri-(anti)canonical systems on normal surfaces), see Section 5.

Extension theorems. In this paper, *extension theorem* means the result of the extendability of morphisms defined on a divisor to the whole variety. For the surface case, the extension theorem goes back to the results of Saint-Donat [1974] and Reid [1976] for $K3$ surfaces. After that, Serrano [1987] and Paoletti [1995] proved extension theorems for integral curves on smooth surfaces. These results were generalized in [Enokizono 2020] for possibly reducible or nonreduced curves on normal surfaces in characteristic 0. In this paper, as an application of the Reider-type theorem (Theorem 5.5), we give a positive characteristic analog of this extension theorem.

Theorem 1.7 (Theorem 6.1). *Let $D > 0$ be an effective divisor on a normal complete surface X over an algebraically closed field k of positive characteristic, and assume that any prime component D_i of D has positive self-intersection number. Let $\varphi: D \rightarrow \mathbb{P}^1$ be a finite separable morphism of degree d . If $D^2 > \mu(q_X, d)$ and $\dim |D| \geq 3d + \dim H^1(\mathcal{O}_X)_n$, then there exists a morphism $\psi: X \rightarrow \mathbb{P}^1$ such that $\psi|_D = \varphi$.*

For the definition of $\mu(q_X, d)$, see Definition 5.4 (e.g., $\mu(q_X, d) \leq (d + 1)^2$ holds when X is smooth). Some variants of extension theorems are established in Section 6.

As an application of the extension theorems, we give a characterization of nonsingular k -rational points of plane curves $D \subseteq \mathbb{P}^2$ over any base field k in terms of rational functions on D , which is a natural generalization of the classical result in [Namba 1979] that the gonality of smooth complex plane curves of degree m is equal to $m - 1$.

Theorem 1.8 (Theorem 7.2). *Let $D \subseteq \mathbb{P}^2$ be a plane curve of degree $m \geq 3$ over an arbitrary base field k . Then there is a one-to-one correspondence between:*

- (i) *the set of nonsingular k -rational points of D which are not strange, and*
- (ii) *the set of finite separable morphisms $D \rightarrow \mathbb{P}^1$ of degree $m - 1$ up to automorphisms of \mathbb{P}^1 .*

Moreover, any finite separable morphism $D \rightarrow \mathbb{P}^1$ has degree greater than or equal to $m - 1$.

Structure of the paper. The present paper is organized as follows. In Section 2, we fix some notations and terminology used in this paper. In Section 3, we discuss chain-connected divisors, which play a central role in this paper. The key result in this section is the chain-connectedness of big \mathbb{Z} -positive divisors (Proposition 3.11). This is used to prove the main vanishing theorem (Theorem 4.17).

In the first half of Section 4, we study the kernel $\alpha(X, D)$ of the restriction map $H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_D)$ for divisors D on X following the arguments in [Mumford 1967], [Francia 1991] and [Barth et al. 2004]. The rest of Section 4 is devoted to the vanishing theorem on surfaces in positive characteristic and its generalization. The essential idea is the combination of Fujita's [1983, Theorem 7.4] and Mumford's [1967, p. 99] arguments and the chain-connectedness of big \mathbb{Z} -positive divisors. In Section 5, we study adjoint linear systems on normal surfaces in positive characteristic as an application of the vanishing theorems obtained in Section 4. The proof of the main result (Theorem 5.2) is almost similar to that of [Enokizono 2020, Theorem 5.2]. The only difference is the use of the chain-connected component decomposition (see [Konno 2010, Corollary 1.7]) instead of the integral Zariski decomposition [Enokizono 2020, Theorem 3.5]. In Section 6, we give extension theorems for normal surfaces in positive characteristic by using the Reider-type theorem (Theorem 5.5) obtained in Section 5. As an application of the extension theorems, nonsingular k -rational points of any plane curve $D \subseteq \mathbb{P}^2$ over an arbitrary base field k are characterized in terms of rational functions on D in Section 7. In the Appendix, Mumford's intersection form on a normal projective variety is formulated.

2. Notations and terminology

In this paper, we mainly work on the category of schemes over a field k . (Some results also hold on the category of complex analytic spaces).

- A *divisor* means a Weil divisor (not necessarily \mathbb{Q} -Cartier).
- For a divisor D on a normal variety X , we denote by $\mathcal{O}_X(D)$ the divisorial sheaf on X with respect to D . For an effective divisor D on X , we sometimes regard it as a subscheme of pure codimension 1 defined by the ideal sheaf $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$.
- For a \mathbb{Q} -divisor (respectively, nef \mathbb{R} -divisor) D on a normal proper variety X , we denote by $\kappa(D)$ (respectively, $\nu(D)$) the *Iitaka dimension* (respectively, *numerical dimension*) of D (for details, see [Kawamata et al. 1987, Chapter 6] or [Fujino 2017, Section 2.4]).
- For a normal projective variety X of dimension ≥ 2 over an infinite field k and a divisor D on X , we freely use the following Bertini-type result (for details, see [Huybrechts and Lehn 1997, Section 1.1]): Any general hyperplane Y on X is also a normal projective variety and satisfies $\mathcal{O}_Y(D|_Y) \cong \mathcal{O}_X(D)|_Y$.
- For a normal proper surface X , we freely use Mumford's intersection product

$$\mathrm{Cl}(X) \times \mathrm{Cl}(X) \rightarrow \mathbb{Q}, \quad (D, E) \mapsto DE,$$

extending the usual intersection product [Mumford 1961]. By using this intersection form, we can extend the numerical properties of Cartier divisors on X such as nef,

pseudoeffective, big and so on to that on Weil divisors naturally (for example, see [Enokizono 2020, Appendix A]). For a higher-dimensional analog of Mumford's intersection form, see the Appendix.

- For a scheme X over a field of characteristic $p > 0$, let us denote by F the absolute Frobenius morphism on X and the induced homomorphism on the cohomology $H^m(\mathcal{O}_X)$. Note that $F: H^m(\mathcal{O}_X) \rightarrow H^m(\mathcal{O}_X)$ is p -linear, that is, $F(av) = a^p F(v)$ for $a \in k$ and $v \in H^m(\mathcal{O}_X)$.
- For a p -linear transform $F: V \rightarrow V$ of a finite-dimensional vector space V over a field k of characteristic $p > 0$, we write the *semisimple part of V* (respectively, *nilpotent part of V*) by $V_s := \text{Im } F^l$ (respectively, $V_n := \text{Ker } F^l$), where $l \gg 0$. Then it is well known that $V = V_s \oplus V_n$, and there exists a k -basis $\{e_i\}$ of V_s such that $F(e_i) = e_i$ for each i when k is algebraically closed (for the proof, see [Chambert-Loir 1998, Exposé III, Lemma 3.3]).
- A finite surjective morphism $f: X \rightarrow Y$ from a proper scheme X to a variety Y is called *separable* if the restriction $f|_{X_i}$ induces a separable field extension $(f|_{X_{i,\text{red}}})^*: K(Y) \hookrightarrow K(X_{i,\text{red}})$ between function fields for each irreducible component X_i of X .

3. Chain-connected divisors

Connectedness of effective divisors. We introduce some notions about connectivity for effective divisors on normal surfaces, which are well known for smooth surfaces. In this section, X stands for a normal proper surface over a base field k (or a normal compact analytic surface) unless otherwise stated.

Definition 3.1 (connectedness for effective divisors). Let D be a nonzero effective divisor on X .

- (1) We say that D is *chain-connected* (respectively, *numerically connected*) if $-A$ is not nef over B , that is, $AC > 0$ for some curve C contained in the support of B (respectively, $AB > 0$) for any effective decomposition $D = A + B$ with $A, B > 0$.
- (2) Let $m \in \mathbb{Q}$. We say that D is *m -connected* (respectively, *strictly m -connected*) if $AB \geq m$ (respectively, $AB > m$) for any effective decomposition $D = A + B$ with $A, B > 0$. Clearly, numerical connectivity is equivalent to strict 0-connectivity and implies chain-connectivity.
- (3) For a subdivisor $0 < D_0 \leq D$, a *connecting chain from D_0 to D* is defined to be a sequence of subdivisors $D_0 < D_1 < \dots < D_m = D$ such that $C_i := D_i - D_{i-1}$ is prime and $D_{i-1}C_i > 0$ for each $i = 1, \dots, m$. We regard $D_0 = D$ as a connecting chain from D to D ($m = 0$ case).

The following lemma is easy and well known:

Lemma 3.2. *Let H be an ample Cartier divisor on X and $n > 1$. Then any effective divisor D which is numerically equivalent to nH is $(n - 1/H^2)$ -connected.*

Proof. Let $D = A + B$ be a nontrivial effective decomposition. By the Hodge index theorem, we can write $A \equiv aH + A'$ and $B \equiv bH + B'$, where $a := AH/H^2$, $b := BH/H^2$, $A'H = 0$ and $B'H = 0$, with $A'^2 \leq 0$ and $B'^2 \leq 0$. Note that both a and b are greater than or equal to $1/H^2$, since H is ample Cartier. Since $A + B \equiv nH$, it follows that $a + b = n$ and $A' + B' \equiv 0$. Thus,

$$AB = abH^2 + A'B' = a(n - a)H^2 - A'^2 \geq \frac{1}{H^2} \left(n - \frac{1}{H^2} \right) H^2 = n - \frac{1}{H^2}. \quad \square$$

Similarly, we can prove the following (for the proof, see [Ramanujam 1972, Lemma 2] or [Kawachi and Mařek 1998, p. 242]):

Lemma 3.3 (Ramanujam's connectedness lemma). *Let D be an effective, nef and big divisor on X . Then D is numerically connected.*

The following lemmas are due to [Konno 2010] (the proof also works on possibly singular normal surfaces):

Lemma 3.4 [Konno 2010, Proposition 1.2]. *Let $D > 0$ be an effective divisor on X . Then the following are equivalent:*

- (i) D is chain-connected.
- (ii) For any subdivisor $0 < D_0 \leq D$, there exists a connecting chain from D_0 to D .
- (iii) There exist a prime component D_0 of D and a connecting chain from D_0 to D .

Lemma 3.5 ([Konno 2010, Proposition 1.5 (3)]). *Let $D > 0$ be an effective divisor on X with connected support. Then there exists the greatest chain-connected subdivisor $0 < D_c \leq D$ such that $\text{Supp}(D_c) = \text{Supp}(D)$ and $-D_c$ is nef over $D - D_c$.*

Definition 3.6 (chain-connected component). Let $D > 0$ be an effective divisor on X with connected support. We say that the greatest chain-connected subdivisor D_c of D , as in Lemma 3.5, is called the *chain-connected component* of D . Similarly, for any effective divisor $D > 0$ on X , we can take the chain-connected component for each connected component of D .

The following proposition can be proved similarly to that of [Enokizono 2020, Proposition 3.19]:

Proposition 3.7. *Let $\pi : X' \rightarrow X$ be a proper birational morphism between normal complete surfaces. Then for any chain-connected divisor D on X , the round-up of the Mumford pullback $\lceil \pi^* D \rceil$ is chain-connected.*

Proof. We write

$$D' := \lceil \pi^* D \rceil = \pi^* D + D_\pi,$$

where D_π is a π -exceptional \mathbb{Q} -divisor on X' with $\lfloor D_\pi \rfloor = 0$. Note that D' has connected support since D does. Assume that D' is not chain-connected, that is, $D'_c < D'$. Let $B' := D' - D'_c$. It follows from Lemma 3.5 that $\text{Supp}(D'_c) = \text{Supp}(D')$ and $-D'_c$ is nef over B' . We write $B' = \pi^*B + B_\pi$, where $B := \pi_*B' \geq 0$ and B_π is a π -exceptional \mathbb{Q} -divisor. Let

$$B_\pi - D_\pi = G^+ - G^-$$

be the decomposition of effective π -exceptional \mathbb{Q} -divisors G^+ and G^- having no common components. Note that the support of G^+ is contained in that of B' . First we assume that $G^+ > 0$. Then there exists a prime component C of G^+ such that $G^+C < 0$ since G^+ is negative definite. It follows that $C \leq B'$ and

$$-D'_cC = (B' - D')C = (\pi^*(B - D) + G^+ - G^-)C = (G^+ - G^-)C < 0,$$

which contradicts the nefness of $-D'_c$ over B' . Hence, we have $G^+ = 0$. This and $\lfloor D_\pi \rfloor = 0$ imply $B > 0$. Indeed, if $B = 0$, then $B_\pi = B'$ is nonzero effective with integral coefficients, which contradicts $\lfloor B_\pi \rfloor = \lfloor D_\pi - G^- \rfloor \leq 0$. Since D is chain-connected and $D - B = \pi_*D'_c > 0$, there exists a prime component C of B such that $(B - D)C < 0$. Let \hat{C} be the proper transform of C on X' . Then it is a component of B' and

$$-D'_c\hat{C} = (B' - D')\hat{C} = (\pi^*(B - D) - G^-)\hat{C} = (B - D)C - G^-\hat{C} < 0,$$

which contradicts the nefness of $-D'_c$ over B' . Hence, we conclude that D' is chain-connected. \square

Example 3.8. The numerically connected version of Proposition 3.7 does not hold: Let $f: S \rightarrow \mathbb{P}^1$ be an elliptic surface having a singular fiber $F = f^{-1}(0)$ of type I_1 . Take three blow-ups

$$X' := S_3 \xrightarrow{\rho_3} S_2 \xrightarrow{\rho_2} S_1 \xrightarrow{\rho_1} S_0 := S$$

at single points $p_i \in S_{i-1}$, where p_1 and p_2 , respectively, are a general point in F and the node of F and p_3 is a point in the intersection of the ρ_2 -exceptional curve and the proper transform of F . Let C'_1 , C'_2 and C'_3 , respectively, denote the ρ_3 -exceptional curve, the proper transform of the ρ_2 -exceptional curve and the proper transform of F on X' . Then we have $(C'_i)^2 = -i$ and $C'_iC'_j = 1$ for $i \neq j$. Let $\pi: X' \rightarrow X$ be the contraction of C'_3 and put $C_i := \pi_*C'_i$ for $i = 1, 2$. Note that X has one cyclic quotient singularity at $\pi(C'_3)$ and $C'_i = \pi^*C_i - \frac{1}{3}C_3$ for $i = 1, 2$. Thus, we have $C_1^2 = -\frac{2}{3}$, $C_2^2 = -\frac{5}{3}$ and $C_1C_2 = \frac{4}{3}$. Then $D := 2C_1 + 2C_2$ is numerically connected but $\lceil \pi^*D \rceil = 2C'_1 + 2C'_2 + 2C'_3$ is not numerically connected since $(C'_1 + C'_2 + C'_3)^2 = 0$.

Lemma 3.9. *Let $D > 0$ be a chain-connected divisor on X . Then $H^0(\mathcal{O}_D)$ is a field. Moreover, if D contains a prime divisor C such that $H^0(\mathcal{O}_C) \cong H^0(\mathcal{O}_X)$, then we have $H^0(\mathcal{O}_D) \cong H^0(\mathcal{O}_X)$.*

Proof. First, we show the claim when X is regular. By Lemma 3.4, we can take a connecting chain $C =: D_0 < D_1 < \dots < D_m := D$ for any prime component C of D . Putting $C_i := D_i - D_{i-1}$, we have $D_{i-1}C_i > 0$ for each i . By the exact sequence

$$0 \rightarrow \mathcal{O}_{C_i}(-D_{i-1}) \rightarrow \mathcal{O}_{D_i} \rightarrow \mathcal{O}_{D_{i-1}} \rightarrow 0$$

and $H^0(\mathcal{O}_{C_i}(-D_{i-1})) = 0$ for each i , we have a chain of injections

$$H^0(\mathcal{O}_X) \hookrightarrow H^0(\mathcal{O}_D) \hookrightarrow H^0(\mathcal{O}_{D_{m-1}}) \hookrightarrow \dots \hookrightarrow H^0(\mathcal{O}_{D_0}).$$

Thus, $H^0(\mathcal{O}_D)$ is a subfield of $H^0(\mathcal{O}_C)$. If, moreover, we assume $H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_C)$, then all the injections above are isomorphisms.

For a general X , we take a resolution $\pi : X' \rightarrow X$ and put $D' := \lceil \pi^* D \rceil$. By Proposition 3.7, D' is also chain-connected. We note that there are natural injections

$$H^0(\mathcal{O}_X) \hookrightarrow H^0(\mathcal{O}_D) \hookrightarrow H^0(\mathcal{O}_{D'}).$$

Thus, $H^0(\mathcal{O}_D)$ is a subfield of $H^0(\mathcal{O}_{D'})$. If D contains a prime divisor C with $H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_C)$, then the proper transform \hat{C} of C satisfies $H^0(\mathcal{O}_{X'}) \cong H^0(\mathcal{O}_{\hat{C}})$ since $H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_{X'})$ and $H^0(\mathcal{O}_C) \cong H^0(\mathcal{O}_{\hat{C}})$. From the assertion for regular surfaces, we obtain $H^0(\mathcal{O}_{D'}) \cong H^0(\mathcal{O}_{X'})$. Hence, $H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_D)$. \square

Chain-connected vs. \mathbb{Z} -positive. Let us recall the notion of \mathbb{Z} -positive divisors on normal surfaces introduced in [Enokizono 2020].

Definition 3.10 (\mathbb{Z} -positive divisors). A divisor D on a normal complete surface X is called \mathbb{Z} -positive if $B - D$ is not nef over B (i.e., $BC < DC$ holds for some irreducible component C of B) for any effective negative definite divisor $B > 0$ on X .

Typical examples of \mathbb{Z} -positive divisors are chain-connected divisors which are not negative definite and the round-ups of nef \mathbb{R} -divisors (see [Enokizono 2020, Proposition 3.16]). In this subsection, we prove the following result, which is an analog of Lemma 3.3:

Proposition 3.11. *Any effective big \mathbb{Z} -positive divisor is chain-connected.*

Proof. Let $D > 0$ be an effective big \mathbb{Z} -positive divisor on X . First we note that the support of D is connected since D is obtained by a connecting chain from the round-up $\lceil P(D) \rceil$ of the positive part $P(D)$ in the Zariski decomposition of D (see [Enokizono 2020, Proposition 3.16]) and the support of $P(D)$ is connected by Lemma 3.3. Let D_c be the chain-connected component of D and suppose that $D - D_c \neq 0$. Let us take its Zariski decomposition

$$D - D_c = P + N.$$

If $P = 0$, then $D - D_c$ is negative definite, which contradicts the \mathbb{Z} -positivity of D . Thus, we have $P \neq 0$. Since $D_c P \leq 0$, $\text{Supp}(D_c) = \text{Supp}(D)$ and P is nef, it follows that P is numerically trivial over D . In particular, we have $P^2 = 0$. Since $\text{Supp}(D)$ is connected and $P \neq 0$, the support of P coincides with that of D . Thus, D is negative semidefinite. On the other hand, by the assumption that D is big, we have $P(D)^2 > 0$, which contradicts the negative semidefiniteness of D . Hence, we conclude that $D = D_c$. \square

Remark 3.12. (1) Conversely, any big chain-connected divisor is \mathbb{Z} -positive since for any effective divisor $D > 0$ with connected support, D is big if and only if D is not negative semidefinite (see [Enokizono 2020, Lemma A.12]). Thus, for any effective big divisor D with connected support, its \mathbb{Z} -positive part $P_{\mathbb{Z}}$ in the integral Zariski decomposition [Enokizono 2020, Theorem 3.5] coincides with the chain-connected component D_c .

(2) In general, effective \mathbb{Z} -positive divisors are not chain-connected even if D has connected support. For example, a multiple nF of a fiber $F = f^{-1}(t)$ of a fibration $f: X \rightarrow B$ over a curve B is \mathbb{Z} -positive, but not chain-connected for $n \geq 2$.

Combining Proposition 3.11 with Lemma 3.9, we obtain the following:

Corollary 3.13. *Let D be an effective big \mathbb{Z} -positive divisor on a normal complete surface. Then $H^0(\mathcal{O}_D)$ is a field.*

Base change property. In this subsection, let X be a normal proper geometrically connected surface over a field k and $k \subseteq k'$ a separable field extension. Then $X_{k'} := X \times_k k'$ is also a normal surface with $H^0(\mathcal{O}_{X_{k'}}) \cong k'$.

Lemma 3.14. *Let D be a pseudoeffective \mathbb{Z} -positive divisor on X . Then $D_{k'}$ is also a pseudoeffective \mathbb{Z} -positive divisor on $X_{k'}$.*

Proof. Let $D = P + N$ be the Zariski decomposition of D . Then there exists a connecting chain

$$D_0 := \lceil P \rceil < D_1 < \cdots < D_N := D$$

such that $C_i := D_i - D_{i-1}$ is prime and satisfies $D_{i-1} C_i > 0$ for each i from [Enokizono 2020, Proposition 3.16]. On the other hand, one can see that $D_{k'} = P_{k'} + N_{k'}$ is also the Zariski decomposition of $D_{k'}$. (Indeed, $P_{k'}$ is also nef and $P_{k'} N_{k'} = P N = 0$. Thus, the Hodge index theorem implies that $N_{k'}$ is negative definite if D is big. When D is not big, then by the pseudoeffectivity of D , this can be written by the limit of big divisors in the numerical class group of X . Thus, the claim holds by the continuity of the Zariski decomposition.) Since the extension k'/k is separable, it follows that

$$\lfloor N_{k'} \rfloor = \lfloor N \rfloor_{k'} = \sum_{i=1}^N C_{i,k'},$$

and $C_{i,k'}$ is reduced. Let $C_{i,k'} = \sum_{j=1}^{l(i)} C'_{i,j}$ be the irreducible decomposition. Since $D_{i-1,k'} C'_{i,j} > 0$ for any i and j , we can construct a connecting chain from $\lceil P_{k'} \rceil$ to $D_{k'}$, whence $D_{k'}$ is \mathbb{Z} -positive from [Enokizono 2020, Proposition 3.16]. \square

Combining Lemma 3.14 with Proposition 3.11, we obtain the following:

Corollary 3.15. *Let $D > 0$ be a chain-connected divisor on X which is not negative semidefinite. Then $D_{k'}$ is also chain-connected.*

Remark 3.16. In general, chain-connectivity is not preserved by a separable base change. Indeed, let X be a surface obtained by the blow-up of \mathbb{P}^2 at a closed point x such that the extension $k(x)/k$ is nontrivial and separable. Then the exceptional divisor E_x on X is chain-connected, but $E_{x,k(x)}$ is a disjoint union of the exceptional curves on $X_{k(x)}$.

Corollary 3.17. *Let D be an effective big \mathbb{Z} -positive divisor on X . Then $H^0(\mathcal{O}_D)/k$ is a purely inseparable field extension.*

Proof. Let k' be the separable closure of k in the field $H^0(\mathcal{O}_D)$ and assume $k \neq k'$. Then $H^0(\mathcal{O}_{D_{k'}}) \cong H^0(\mathcal{O}_D) \otimes_k k'$ is not a field, which contradicts Lemma 3.14 and Corollary 3.13. \square

Chain-connected divisors on projective varieties. We introduce chain-connected divisors on higher-dimensional varieties which are used later.

Definition 3.18 (chain-connected divisors). Let X be a normal projective variety of dimension $n \geq 3$ over an algebraically closed field k and H an ample divisor on X . A divisor D on X is called *H -nef* (respectively, *H -nef over an effective divisor B*) if $H^{n-2}DC \geq 0$ for any prime divisor C (respectively, any prime component $C \leq B$) on X , where we use Mumford's intersection form for $n - 2$ Cartier divisors and two Weil divisors on X (see the Appendix). An effective divisor $D > 0$ on X is said to be *chain-connected with respect to H* if $-A$ is not H -nef over B for any nontrivial effective decomposition $D = A + B$. We simply call D *chain-connected* if it is chain-connected with respect to some ample divisor H on X .

Example 3.19. The following effective divisors D are typical examples of chain-connected divisors:

- (i) D is reduced and connected.
- (ii) $D = \lceil M \rceil$, where M is an \mathbb{R} -divisor which is nef in codimension 1 and satisfies $v(M) \geq 2$ or $\kappa(D) \geq 2$.

Here, we say that an \mathbb{R} -Cartier \mathbb{R} -divisor D on a normal complete variety X is *nef in codimension 1* if there exists a closed subset Z of X with codimension ≥ 2 such that $DC \geq 0$ holds for any integral curve C on X not contained in Z (which is also called numerically semipositive in codimension one in [Fujita 1983]). A Weil \mathbb{R} -divisor D

on X is called *nef* (respectively, *nef in codimension 1*) if there exist an alteration $\pi : X' \rightarrow X$ and a nef (respectively, nef in codimension 1) \mathbb{R} -Cartier \mathbb{R} -divisor D' on X' such that $D = \pi_* D'$. By using Mumford's intersection form, the property $\nu(D) \geq 2$ makes sense as $H^{\dim X - 2} D^2 > 0$ for some ample divisor H on X . Note that for $\dim X = 2$, condition (ii) is equivalent to D being the round-up of a nef and big \mathbb{R} -divisor M . Therefore, the proof for the chain-connectivity of the divisor D in (ii) is reduced to the surface case by cutting with general hyperplanes in $|mH|$, with $m \gg 0$, and this is due to Proposition 3.11 and [Enokizono 2020, Corollary 3.18].

Proposition 3.20. *Let X be a normal projective variety over an algebraically closed field k . Let D be a chain-connected divisor on X . Then $H^0(\mathcal{O}_D) \cong k$ holds.*

Proof. We show the claim by induction on $n = \dim X$. The $n = 2$ case is due to Lemma 3.9. We take a general hyperplane $Y \in |mH|$ and consider the exact sequence

$$0 \rightarrow \mathcal{O}_D(-Y|_D) \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_{Y \cap D} \rightarrow 0.$$

Since $H^0(\mathcal{O}_D(-Y|_D)) = 0$, we have an injection $H^0(\mathcal{O}_D) \hookrightarrow H^0(\mathcal{O}_{Y \cap D})$. Since D is chain-connected with respect to H , the restriction $D|_Y$ on Y is also chain-connected with respect to $H|_Y$. Then we have $H^0(\mathcal{O}_{Y \cap D}) \cong k$ by the inductive hypothesis, whence $H^0(\mathcal{O}_D) \cong k$ holds. \square

Similarly, one can show the following:

Proposition 3.21. *Let X be a normal projective geometrically connected variety over a field k . Let $D = \lceil M \rceil$ be an effective divisor as in Example 3.19 (ii). Then $H^0(\mathcal{O}_D)/k$ is a purely inseparable field extension.*

Proof. Note that the condition of divisors in Example 3.19 (ii) is preserved by any separable base change, so we may assume that k is infinite. By the hyperplane cutting argument, as in the proof of Proposition 3.20, we conclude that $H^0(\mathcal{O}_D)$ is a field. The claim follows from the same argument in the proof of Corollary 3.17. \square

4. Vanishing theorem on H^1

In this section, we prove vanishing theorems of Ramanujam-, Kawamata–Viehweg- and Miyaoka-type for normal complete surfaces and normal projective varieties.

Picard schemes and $\alpha(X, D)$. We start with an elementary lemma, which is well known to experts.

Lemma 4.1. *Let X be a proper scheme over an algebraically closed field k of characteristic $p > 0$ with $H^0(\mathcal{O}_X) \cong k$. Then the following are equivalent:*

- (1) *The Frobenius map on $H^1(\mathcal{O}_X)$ is injective.*
- (2) *$H^1(X_{\text{fppf}}, \alpha_p, X) = 0$, that is, all α_p -torsors over X are trivial.*

- (3) *The Picard scheme $\text{Pic}_{X/k}$ does not contain α_p .*
 (4) *Any infinitesimal subgroup scheme of $\text{Pic}_{X/k}^\circ$ is linearly reductive, that is, of the form $\prod \mu_{p^{n_i}}$.*

Proof. The equivalence of (i) and (ii) follows from the fact that

$$H^1(X_{\text{fppf}}, \alpha_{p,X}) \cong \{\eta \in H^1(\mathcal{O}_X) \mid F(\eta) = 0\},$$

which is obtained by the exact sequence

$$0 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a^{(p)} \rightarrow 0$$

of commutative group schemes over k , where F is the relative Frobenius map. The equivalence of (ii) and (iii) follows from [Raynaud 1970, Proposition (6.2.1)] with $T = \text{Spec } k$ and $M = \alpha_p$. Indeed, there is a natural isomorphism

$$H^1(X_{\text{fppf}}, \alpha_{p,X}) \cong \text{Hom}_{\text{GSch}/k}(\alpha_p, \text{Pic}_{X/k})$$

as abelian groups. For the equivalence of (iii) and (iv), it suffices to show that for any $m \geq 1$, the m -th Frobenius kernel $\text{Pic}_{X/k}[F^m]$ does not contain α_p if and only if it is linearly reductive. This holds true for any infinitesimal group scheme by [Liedtke et al. 2021, Lemma 2.3]. \square

Definition 4.2. Let X and D be proper schemes over a field k and $\tau: D \rightarrow X$ a morphism. Then we denote by $\alpha(X, D)$ the kernel of $\tau^*: H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_D)$. It can be identified with the Lie algebra of the kernel of the homomorphism $\tau^*: \text{Pic}_{X/k} \rightarrow \text{Pic}_{D/k}$ of Picard schemes defined by the pullback of line bundles (see [Bosch et al. 1990, p. 231, Theorem 1]).

Let $\phi: E \rightarrow D$ and $\tau: D \rightarrow X$ be two morphisms of proper schemes over a field k . In this subsection, we are going to study the relations of $\alpha(X, D)$ and $\alpha(X, E)$, which will be used in Section 5. Let $\phi^*: \text{Pic}_{D/k}^\circ \rightarrow \text{Pic}_{E/k}^\circ$ and $\tau^*: \text{Pic}_{X/k}^\circ \rightarrow \text{Pic}_{D/k}^\circ$ be the corresponding homomorphisms of Picard schemes.

Lemma 4.3. *Let $\phi: E \rightarrow D$ and $\tau: D \rightarrow X$ be morphisms of proper schemes over a field k and assume one of the following:*

- (i) *$\text{char } k = 0$, X is normal and $\text{Ker}(\phi^*)$ is affine, or*
 (ii) *$\text{char } k > 0$, $H^0(\mathcal{O}_X) \cong k$, the Frobenius map on $H^1(\mathcal{O}_X)$ is injective and $\text{Ker}(\phi^*)$ is unipotent.*

Then $\alpha(X, D) = \alpha(X, E)$ holds.

Proof. It suffices to show that $\text{Ker}(\tau^*)^\circ = \text{Ker}(\phi^* \circ \tau^*)^\circ$. Taking the base change to an algebraic closure \bar{k} of k , we may assume that k is algebraically closed. Let us write $Q := \text{Im}(\tau^*|_{\text{Ker}(\phi^* \circ \tau^*)^\circ})$ and consider the exact sequence

$$1 \rightarrow \text{Ker}(\tau^*)^\circ \rightarrow \text{Ker}(\phi^* \circ \tau^*)^\circ \rightarrow Q \rightarrow 1.$$

First we assume condition (i). Then $\text{Pic}_{X/k}^\circ$ is proper over k since X is normal [Grothendieck 1962, Exposé 236, Théorème 2.1 (ii)]. Thus, $\text{Ker}(\phi^* \circ \tau^*)^\circ$ and $Q := \text{Im}(\tau^*|_{\text{Ker}(\phi^* \circ \tau^*)^\circ})$ are also proper over k . On the other hand, since $\text{Ker}(\phi^*)$ is affine, so is Q . Thus, we conclude that Q is infinitesimal, whence Q is trivial due to Cartier's theorem.

We assume condition (ii). It suffices to show that $Q[F]$ is trivial. By Lemma 4.1, the group scheme $\text{Pic}_{X/k}[F]$ is linearly reductive. Hence, $\text{Ker}(\phi^* \circ \tau^*)[F]$ and $Q[F]$ are also linearly reductive since the linear reductivity is preserved by taking subgroup schemes and quotient group schemes. On the other hand, $Q[F]$ is unipotent since $\text{Ker}(\phi^*)$ is. Thus, $Q[F]$ is trivial. \square

Let us recall the structure of the generalized Jacobian $\text{Pic}_{C/k}^\circ$ of a proper curve C . The following combinatorial data is useful in counting the rank of maximal tori in $\text{Pic}_{C/k}^\circ$:

Definition 4.4. Let C be a proper curve (i.e., purely 1-dimensional proper scheme) over an algebraically closed field k . Then the *extended dual graph* $\Gamma(C)$ of C is defined as follows: The vertex set of $\Gamma(C)$ is the union of the integral subcurves $\{C_i\}_i$ in C and the singularities $\{x_\lambda\}_\lambda$ of the reduced scheme C_{red} of C . For each singular point x_λ of C_{red} , we denote by B_1, \dots, B_m all the local analytic branches of C_{red} (that is, the minimal primes of the complete local ring $\hat{\mathcal{O}}_{C_{\text{red}}, x_\lambda}$). For each B_j , let $C_{i(j)}$ denote the corresponding integral curve in C . Then the edges of $\Gamma(C)$ are defined by connecting x_λ and $C_{i(j)}$ for each branch B_j .

For a proper curve C over an arbitrary field k , we define the extended dual graph of C by that of $C_{\bar{k}} = C \times_k \bar{k}$, where \bar{k} is an algebraic closure of k .

Proposition 4.5 [Bosch et al. 1990, Section 9.2, Proposition 10]. *Let C be a proper curve over an algebraically closed field k . Then the rank of maximal tori of $\text{Pic}_{C/k}^\circ$ is the first Betti number $b_1(\Gamma(C))$ of the extended dual graph of C .*

Proof. We may assume that C is connected. Let \tilde{C} denote the normalization of the reduced scheme C_{red} of C . Then the natural map $\pi : \tilde{C} \rightarrow C$ can be decomposed into

$$\tilde{C} \rightarrow C' \rightarrow C_{\text{red}} \rightarrow C,$$

as in the argument in [Bosch et al. 1990, Section 9.2], where the intermediate curve C' is canonically determined as the highest birational model of C_{red} which is homeomorphic to C_{red} . Since $\Gamma(C) = \Gamma(C_{\text{red}}) = \Gamma(C')$ and the kernel of $\text{Pic}_{C/k}^\circ \rightarrow \text{Pic}_{C'/k}^\circ$ is unipotent by Propositions 5 and 9 in [Bosch et al. 1990, Section 9.2], we may assume $C = C'$. Let x_λ , where $\lambda = 1, \dots, N$, denote the singular points of C , and for each λ , let $\tilde{x}_{\lambda, \mu}$, where $\mu = 1, \dots, n_\lambda$, denote the points of \tilde{C} lying over x_λ . Let C_i , where $i = 1, \dots, r$, denote the integral components of C .

Taking the cohomology of the exact sequence

$$1 \rightarrow \mathcal{O}_C^* \rightarrow \pi_* \mathcal{O}_{\tilde{C}}^* \rightarrow \mathcal{T} \rightarrow 1,$$

we obtain a long exact sequence

$$1 \rightarrow k^* \rightarrow \prod_{i=1}^r k^* \rightarrow \prod_{\lambda=1}^N \left(\prod_{\mu=1}^{n_\lambda} k(\tilde{x}_{\lambda,\mu})^* \right) / k(x_\lambda)^* \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(\tilde{C}) \rightarrow 1,$$

where the cokernel \mathcal{T} is a torsion sheaf supported at the singular points x_λ . Then the kernel of $\pi^*: \text{Pic}_{C/k} \rightarrow \text{Pic}_{\tilde{C}/k}$ is a torus of rank $\sum_{\lambda=1}^N (n_\lambda - 1) - r + 1$. On the other hand, the number of vertices and edges of $\Gamma(C)$ are $r + N$ and $\sum_{\lambda=1}^N n_\lambda$, respectively. Thus, the topological Euler number of the graph is

$$\chi_{\text{top}}(\Gamma(C)) = r + N - \sum_{\lambda=1}^N n_\lambda,$$

and then the first Betti number is

$$b_1(\Gamma(C)) = 1 - \chi_{\text{top}}(\Gamma(C)) = \sum_{\lambda=1}^N (n_\lambda - 1) - r + 1,$$

which completes the proof. \square

Lemma 4.6. *Let $\phi: E \rightarrow D$ be a morphism between proper schemes over a field k . Then the kernel of $\phi^*: \text{Pic}_{D/k}^\circ \rightarrow \text{Pic}_{E/k}^\circ$ is unipotent if one of the following holds:*

- (i) *The canonical immersion $D_{\text{red}} \hookrightarrow D$ factors through ϕ .*
- (ii) *D is a curve and there exists a birational morphism $\hat{D} \rightarrow D_{\text{red}}$ with $b_1(\Gamma(D)) = b_1(\Gamma(\hat{D}))$ such that the composition $\hat{D} \rightarrow D_{\text{red}} \hookrightarrow D$ factors through ϕ .*

Proof. In order to show the assertion for (i), we may assume that $E = D_{\text{red}}$. By taking a filtration of first-order thickenings $D_{\text{red}} \hookrightarrow D_1 \hookrightarrow \cdots \hookrightarrow D_N = D$, we further assume that $E \hookrightarrow D$ is a first-order thickening. Then one can show the assertion easily by taking the cohomology of the exact sequence

$$0 \rightarrow \mathcal{I}_{E/D} \rightarrow \mathcal{O}_D^* \rightarrow \mathcal{O}_E^* \rightarrow 1,$$

where the map on the left sends a local section a to $1 + a$. For case (ii), we may assume $E = \hat{D}$. By taking the base change to an algebraic closure \bar{k} of k , we may assume that k is algebraically closed. Note that the kernel of $\phi^*: \text{Pic}_{D/k}^\circ \rightarrow \text{Pic}_{\hat{D}/k}^\circ$ is affine, and it is unipotent if and only if it does not contain a torus (see Corollaries 11 and 12 in [Bosch et al. 1990, Section 9.2]). Then the claim follows from Proposition 4.5 and the surjectivity of $\phi^*: \text{Pic}_{D/k}^\circ \rightarrow \text{Pic}_{\hat{D}/k}^\circ$. \square

By combining Lemma 4.3 with Lemma 4.6, we obtain the following:

Proposition 4.7. *Let $\phi: E \rightarrow D$ and $\tau: D \rightarrow X$ be morphisms between proper schemes over a field k . If $\text{char } k > 0$ (respectively, $\text{char } k = 0$), we further assume that $H^0(\mathcal{O}_X) \cong k$ and the Frobenius map on $H^1(\mathcal{O}_X)$ is injective (respectively, X is normal). Then $\alpha(X, D) = \alpha(X, E)$ holds if one of the following holds:*

- (1) *The canonical immersion $D_{\text{red}} \hookrightarrow D$ factors through ϕ .*
- (2) *The scheme D is a curve and there exists a birational morphism $\hat{D} \rightarrow D_{\text{red}}$ such that the composition $\hat{D} \rightarrow D_{\text{red}} \hookrightarrow D$ factors through ϕ . Moreover, we further assume $b_1(\Gamma(D)) = b_1(\Gamma(\hat{D}))$ if $\text{char } k > 0$.*

Remark 4.8. (1) Proposition 4.7 also holds when ϕ and τ are morphisms between compact complex analytic spaces and X is a normal compact analytic variety in Fujiki's class \mathcal{C} , because the Picard variety $\text{Pic}^\circ(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})_{\text{free}}$ is a compact torus.

(2) If $\text{char } k > 0$ and Proposition 4.7 (1) holds, one can also show $\alpha(X, D) = \alpha(X, E)$ without using Picard schemes by the same proof of [Ramanujam 1972, Lemma 6].

The following is a generalization of [Francia 1991, Lemma 2.3]:

Corollary 4.9. *Let X be a normal proper variety over a field k , or normal compact analytic variety in Fujiki's class \mathcal{C} . Let D_1 and D_2 be closed subschemes of X . Let $\pi: X \rightarrow Y$ be a birational morphism to a normal variety Y . We assume the following three conditions:*

- (i) *The subschemes D_1 and D_2 are not contained in the exceptional locus of π and $H^1(\mathcal{O}_Y) \cong H^1(\mathcal{O}_X)$.*
- (ii) *The reduced image of D_1 by π coincides with that of D_2 and it is a curve, which is denoted by D .*
- (iii) *If $\text{char } k > 0$, the Frobenius map on $H^1(\mathcal{O}_X)$ is injective and $b_1(\Gamma(D)) = b_1(\Gamma(\hat{D}))$, where \hat{D} is the proper transform of D on X .*

Then we have $\alpha(X, D_1) = \alpha(X, D_2) \cong \alpha(Y, D)$.

Proof. We may assume that $H^0(\mathcal{O}_X) \cong k$ by replacing k with the field $H^0(\mathcal{O}_X)$. Since $\pi(\hat{D}) = D$, we may assume $D_1 = \hat{D}$. In particular, there is a closed immersion $D_1 \hookrightarrow D_2$. From Proposition 4.7, we have

$$\alpha(Y, D) = \alpha(Y, D_2) = \alpha(Y, D_1).$$

On the other hand, it follows from $H^1(\mathcal{O}_Y) \cong H^1(\mathcal{O}_X)$ that $\alpha(X, D_i) \cong \alpha(Y, D_i)$, where $i = 1, 2$. Hence, we conclude that

$$\alpha(X, D_1) = \alpha(X, D_2) \cong \alpha(Y, D). \quad \square$$

Vanishing of $\alpha(X, D)$. In this subsection, we are going to study the vanishing of $\alpha(X, D)$ when D is a divisor on a variety X .

Definition 4.10. Let X be a normal proper variety over a field k and D a divisor on X . The complete linear system $|D|$ on X defines a rational map $\phi_{|D|}: X \dashrightarrow \mathbb{P}^N$, where $N = \dim |D|$. Taking a resolution $X' \rightarrow X$ of the indeterminacy of $\phi_{|D|}$ and the Stein factorization, we obtain the morphisms

$$X \leftarrow X' \rightarrow B \rightarrow \mathbb{P}^N,$$

where the middle map is a fiber space. Then we say that $|D|$ is *composed with a (respectively, rational, irrational) pencil* if $\dim B = 1$ (respectively, and $H^1(\mathcal{O}_B) = 0$, $H^1(\mathcal{O}_B) \neq 0$).

Proposition 4.11. *Let X be a normal proper surface over a field k or analytic Moishezon surface. Let D be an effective and big divisor on X . Then $\alpha(X, D) = 0$ (respectively, $\alpha(X, D)_s = 0$) holds if $\text{char } k = 0$ (respectively, $\text{char } k > 0$ and either $k = \bar{k}$ or $H^1(\mathcal{O}_X)_n = 0$).*

Proof. We follow Mumford's argument [1967, p. 99]. Since $H^0(\mathcal{O}_X)$ is a field, we may assume $H^0(\mathcal{O}_X) = k$. By taking the base change to a separable closure of k , we may also assume that k is separably closed. Let $\tau: D \hookrightarrow X$ denote the natural immersion.

First we suppose that $\alpha(X, D) \neq 0$, i.e., $\text{Ker}(\tau^*)^\circ \neq 1$ and that $H^1(\mathcal{O}_X)_n = 0$ when $\text{char } k > 0$. Then we can take a subgroup scheme $\mu_p \subseteq \text{Ker}(\tau^*)^\circ$, where p is a prime number and $p = \text{char } k$ when $\text{char } k > 0$. Indeed, the characteristic 0 case is trivial, and so we may assume that $\text{char } k = p > 0$. Then by Lemma 4.1, $\text{Ker}(\tau^*)[F] \times_k \bar{k}$ is isomorphic to the product $\prod_i \mu_{p^{n_i}}$. Since k is separably closed, $\text{Ker}(\tau^*)[F]$ is also isomorphic to $\prod_i \mu_{p^{n_i}}$. In particular, $\text{Ker}(\tau^*)^\circ$ contains at least one μ_p . Thus, by the natural isomorphism

$$H^1(X_{\text{et}}, (\mathbb{Z}/p\mathbb{Z})_X) \cong \text{Hom}_{\text{GSch}/k}(\mu_p, \text{Pic}_{X/k}),$$

we can take a nontrivial étale cyclic covering $\pi: Y \rightarrow X$ of degree p with $\pi^*D = \sum_{i=1}^p D_i$, where all D_i are disjoint and $D_i \cong D$.

If $\alpha(X, D)_s \neq 0$ and the base field k is algebraically closed of characteristic $p > 0$, then there exists a nonzero element $\eta \in \alpha(X, D)$ such that $F(\eta) = \eta$. Thus, we can also take a nontrivial étale cyclic covering $\pi: Y \rightarrow X$ of degree p with $\pi^*D = \sum_{i=1}^p D_i$ corresponding to η by the isomorphism

$$H^1(X_{\text{et}}, (\mathbb{Z}/p\mathbb{Z})_X) \cong \{\eta \in H^1(\mathcal{O}_X) \mid F(\eta) = \eta\}.$$

Let $D = P + N$ and $D_i = P_i + N_i$, respectively, be the Zariski decompositions of D and D_i . Then $\pi^*P = \sum_{i=1}^p P_i$ holds and it is nef and big, which contradicts Lemma 3.3. \square

Proposition 4.12. *Let X be a normal proper surface over a field k of characteristic 0 or analytic in Fujiki's class \mathcal{C} . Let D be an effective divisor on X with $\kappa(D) = 1$. Then $\alpha(X, D) = 0$ holds if $|mD|$ is composed with a rational pencil for some $m > 0$.*

Proof. The proof is identical to that of [Barth et al. 2004, Lemma 12.8]. Since

$$\alpha(X, D) = \alpha(X, mD) \subseteq \alpha(mD - \text{Fix } |mD|)$$

holds for any $m \geq 1$ from Proposition 4.7, we may assume that $|D|$ has no fixed parts. Then it is base point free since $\kappa(D) = 1$. Let $f: X \rightarrow B$ be the fibration with connected fibers induced by $|D|$. Then we can write $D = \sum_{i=1}^l F_{t_i}$, where $F_{t_i} = f^{-1}(t_i)$ are the fibers of f at some closed points $t_i \in B$. By the Leray spectral sequence

$$H^p(R^q f_* \mathcal{O}_X) \Rightarrow H^{p+q}(\mathcal{O}_X)$$

and the assumption $H^1(\mathcal{O}_B) = 0$, we have $H^1(\mathcal{O}_X) \cong H^0(R^1 f_* \mathcal{O}_X)$. Let η be an element of $\alpha(X, D)$. Then $\eta|_{F_t} = 0$ for some $t \in B$ (for example, take $t = t_1$), that is, $\eta \in \alpha(X, F_t)$. By Proposition 4.7, we have $\eta \in \alpha(X, nF_t)$ for any $n \geq 1$. Thus, the formal function theorem implies that η maps to 0 by the composition

$$H^1(\mathcal{O}_X) \cong H^0(R^1 f_* \mathcal{O}_X) \rightarrow (R^1 f_* \mathcal{O}_X)_t.$$

Hence, there exists an open neighborhood $U \subseteq B$ of t such that $\eta|_U = 0$, which implies $\eta = 0$ since $R^1 f_* \mathcal{O}_X$ is locally free (note that f contains no wild fibers by the assumption $\text{char } k = 0$). \square

Example 4.13. When $\text{char } k = p > 0$, there exist counterexamples to Proposition 4.12 as follows: Let $G = \mathbb{Z}/p\mathbb{Z}$ be a constant group scheme over an algebraically closed field k with $\text{char } k = p > 0$ and $g \in G$ a generator. Then G acts on \mathbb{A}^1 as the translation $g: t \mapsto t + 1$. This extends to an action on \mathbb{P}^1 . Let E be an ordinary elliptic curve and take a p -torsion point $a \in E(k)$. Then G acts freely on E as $g: x \mapsto x + a$. Thus, the diagonal action of G to $E \times \mathbb{P}^1$ is free and the quotient $X := (E \times \mathbb{P}^1)/G$ admits a structure of elliptic surfaces

$$f: X \rightarrow \mathbb{P}^1/G \cong \mathbb{P}^1$$

via the second projection. This admits one wild fiber $f^{-1}(\infty) = pE_\infty$ at the infinity point $\infty \in \mathbb{P}^1$. Then a simple calculation shows that

$$R^1 f_* \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{T},$$

where \mathcal{T} is a torsion sheaf supported at ∞ with length 1 (see [Katsura and Ueno 1985, p. 313, Section 8]). Now we consider a fiber $D := f^{-1}(t)$ at a point $t \neq \infty$. Since

$$H^1(\mathcal{O}_X) \cong H^0(R^1 f_* \mathcal{O}_X) \cong H^0(\mathcal{T}),$$

we have $\alpha(X, D) = H^1(\mathcal{O}_X) \cong k$. We note that the Frobenius map on $H^1(\mathcal{O}_X)$ is injective since $H^1(\mathcal{O}_X) \cong H^1(\mathcal{O}_{E_\infty})$ and E_∞ is ordinary.

Next we consider the higher-dimensional cases. The following is a generalization of [Alzati and Tortora 2002, Theorem 2.1]:

Proposition 4.14. *Let X be a normal projective variety of dimension $n \geq 2$ over an infinite field k . Let D be an effective divisor on X such that $D|_S$ is big on a complete intersection surface $S := H_1 \cap \cdots \cap H_{n-2}$ for general hyperplanes H_1, \dots, H_{n-2} on X . Then $\alpha(X, D) = 0$ (respectively, $\alpha(X, D)_s = 0$) holds if $\text{char } k = 0$ (respectively, $\text{char } k > 0$ and either $k = \bar{k}$ or $H^1(\mathcal{O}_X)_n = 0$).*

Proof. The proof is similar to that of Proposition 4.11. Suppose that $\alpha(X, D) \neq 0$ (respectively, $\alpha(X, D)_s \neq 0$). Then we can also take a nontrivial étale cyclic covering $\pi : Y \rightarrow X$ of prime degree p with $\pi^*D = \sum_{i=1}^p D_i$, where all D_i are disjoint and $D_i \cong D$. Let H_1, \dots, H_{n-2} be general hyperplanes on X such that

$$S := H_1 \cap \cdots \cap H_{n-2}$$

is a normal surface. Then $S' := \pi^{-1}(S)$ is normal and

$$(\pi|_{S'})^*(D|_S) = D_1|_{S'} + \cdots + D_p|_{S'}$$

is big, which is a contradiction. \square

Proposition 4.15. *Let X be a normal projective variety of dimension $n \geq 2$ over a field k of characteristic 0. Let D be an effective divisor on X such that $|mD|$ is composed with a rational pencil for some $m > 0$. Then $\alpha(X, D) = 0$ holds.*

Proof. We use induction on the dimension $n = \dim X$. The $n = 2$ case is due to Propositions 4.11 and 4.12. Assume $n \geq 3$. Let us take a general hyperplane Y on X . Then $D|_Y$ satisfies $\kappa(D|_Y) \geq 2$ or $|mD|$ is composed with a rational pencil. Hence, the claim holds by the natural inclusion $\alpha(X, D) \hookrightarrow \alpha(Y, D|_Y)$ due to Enriques–Severi–Zariski’s lemma, Proposition 4.14 and the inductive assumption. \square

Vanishing on H^1 . Combining Propositions 4.11 and 4.12 with Lemma 3.9, we obtain the following vanishing theorem on normal surfaces:

Theorem 4.16. *Let X be a normal proper surface over a field k or analytic in Fujiki’s class \mathcal{C} . Let D be a chain-connected divisor on X having a prime component C with $H^0(\mathcal{O}_X) \cong H^0(\mathcal{O}_C)$. Assume that $|mD|$ is not composed with an irrational pencil and has positive dimension for some $m > 0$. If $\text{char } k > 0$, we further assume that D is big and the Frobenius map on $H^1(\mathcal{O}_X)$ is injective. Then we have $H^1(\mathcal{O}_X(-D)) = 0$, or equivalently, $H^1(\mathcal{O}_X(K_X + D)) = 0$.*

The following is a positive characteristic analog of [Enokizono 2020, Theorem 4.1] and includes a Kawamata–Viehweg type vanishing theorem for surfaces in positive characteristic:

Theorem 4.17. *Let X be a normal proper geometrically connected surface over a perfect field k of positive characteristic. Let D be a big divisor on X and $D = P_{\mathbb{Z}} + N_{\mathbb{Z}}$ be the integral Zariski decomposition as in [Enokizono 2020, Theorem 3.5]. Let \mathcal{L}_D and \mathcal{L}'_D , respectively, be the rank 1 sheaves on $N_{\mathbb{Z}}$ defined by the cokernel of the homomorphisms $\mathcal{O}_X(K_X + P_{\mathbb{Z}}) \rightarrow \mathcal{O}_X(K_X + D)$ and $\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X(-P_{\mathbb{Z}})$ induced by multiplying a defining section of $N_{\mathbb{Z}}$. If $\dim |D| \geq \dim H^1(\mathcal{O}_X)_n$, then we have*

$$H^1(X, \mathcal{O}_X(K_X + D)) \cong H^1(N_{\mathbb{Z}}, \mathcal{L}_D) \quad \text{and} \quad H^1(X, \mathcal{O}_X(-D)) \cong H^0(N_{\mathbb{Z}}, \mathcal{L}'_D).$$

Proof. Since $\dim |D| = \dim |P_{\mathbb{Z}}|$, it suffices to show the vanishing of $H^1(X, \mathcal{O}_X(-D))$ under the additional assumption that D is \mathbb{Z} -positive (see the proof of [Enokizono 2020, Theorem 4.1]). We may assume that k is algebraically closed by Lemma 3.14. Now, we use a slight modification of Fujita's argument (see [Fujita 1983, Theorem 7.4]). First note that $H^0(\mathcal{O}_{D_s}) \cong k$ holds for any member $D_s \in |D|$ from Corollary 3.13. Thus, each nonzero section $s \in H^0(\mathcal{O}_X(D))$ defines an injection

$$\times s : H^1(\mathcal{O}_X(-D)) \hookrightarrow H^1(\mathcal{O}_X).$$

Moreover, the image of $\times s$ is contained in $H^1(\mathcal{O}_X)_n$ by Proposition 4.11. Let

$$U := H^0(\mathcal{O}_X(D)), \quad V := H^1(\mathcal{O}_X(-D)), \quad W := H^1(\mathcal{O}_X)_n, \quad M := \text{Hom}_k(V, W)$$

for simplicity, and consider these as affine varieties over k . Then the correspondence $s \mapsto \times s$, as above, defines a k -morphism $\Phi : U \rightarrow M$. Suppose that $V \neq 0$. Note that

$$1 \leq \dim V \leq \dim W < \dim U$$

holds by assumption and the above argument. Then Φ induces a morphism

$$\bar{\Phi} : \mathbb{P}(U^*) = |D| \rightarrow \text{Gr}(r, W),$$

where $\text{Gr}(r, W)$ is the Grassmann variety parametrizing all $r := \dim V$ -dimensional k -linear subspaces of W . Since $\dim W < \dim U$, it follows from [Tango 1974, Corollary 3.2] that the morphism $\bar{\Phi}$ is constant. We denote by $(I \subseteq W)$ the image of $\bar{\Phi}$. Thus, Φ induces the morphism

$$\det \Phi : U \xrightarrow{\Phi} \text{Hom}_k(V, I) \xrightarrow{\det} \text{Hom}_k(\wedge^r V, \wedge^r I) \cong \mathbb{A}_k^1,$$

the restriction to $U \setminus \{0\}$ of which is nonzero everywhere. This contradicts $\dim U \geq 2$, since $(\det \Phi)^{-1}(0)$ must be a divisor. \square

Corollary 4.18 (Kawamata–Viehweg type vanishing theorem). *Let X be a normal proper surface over a perfect field k of positive characteristic with $H^0(\mathcal{O}_X) \cong k$. Let M be a nef and big \mathbb{R} -divisor on X . If $\dim \lceil M \rceil \geq \dim H^1(\mathcal{O}_X)_n$, then*

$$H^i(X, \mathcal{O}_X(K_X + \lceil M \rceil)) = 0$$

for any $i > 0$.

For the higher-dimensional cases, by combining Propositions 4.14 and 4.15 with Proposition 3.20, we obtain the following:

Theorem 4.19 (generalized Ramanujam vanishing theorem). *Let X be a normal projective variety of dimension $n \geq 2$ over an algebraically closed field k . If $\text{char } k > 0$, we assume that the Frobenius map on $H^1(\mathcal{O}_X)$ is injective. Let D be a chain-connected divisor on X which satisfies one of the following conditions:*

- (1) $D|_S$ is big on a complete intersection surface $S := H_1 \cap \cdots \cap H_{n-2}$ for general hyperplanes H_1, \dots, H_{n-2} .
- (2) $|mD|$ is composed with a rational pencil for some $m > 0$ and $\text{char } k = 0$.

Then $H^1(X, \mathcal{O}_X(-D)) = 0$.

The following theorem can be seen as a higher-dimensional generalization of Corollary 4.18 and [Miyaoka 1980, Theorem 2.7]:

Theorem 4.20 (generalized Miyaoka vanishing theorem). *Let X be a normal projective geometrically connected variety of dimension $n \geq 2$ over an infinite perfect field k . Let D be a divisor on X . We assume the following three conditions:*

- (i) $D = \lceil M \rceil + E$ for some \mathbb{R} -divisor M and the sum of prime divisors $E = \sum_{i=1}^m E_i$ (possibly $m = 0$).
- (ii) There exist $n - 2$ hyperplanes H_1, \dots, H_{n-2} on X with $S := H_1 \cap \cdots \cap H_{n-2}$ a normal surface such that $M|_S$ is nef, $D|_S$ is big and for each j ,

$$H_1 \cdots H_{n-2} \left(\lceil M \rceil + \sum_{i=1}^{j-1} E_i \right) E_j > 0.$$

- (iii) $\text{char } k = 0$ or $\dim |D| \geq \dim H^1(\mathcal{O}_X)_n$.

Then $H^1(X, \mathcal{O}_X(-D)) = 0$.

Proof. We may assume that k is algebraically closed since conditions (i), (ii) and (iii) are preserved by any separable base change. We note that for any effective divisor D satisfying conditions (i) and (ii), $D|_S$ is big \mathbb{Z} -positive on a general complete intersection surface $S = H_1 \cap \cdots \cap H_{n-2}$. Indeed, this can be checked from the fact that $D + C$ is \mathbb{Z} -positive for any \mathbb{Z} -positive divisor D and any prime divisor C on S with $DC > 0$. Thus, $H^0(\mathcal{O}_D) \cong k$ from Corollary 3.17.

First assume that $\text{char } k = 0$. We use induction on $n = \dim X$. The $n = 2$ case is due to [Enokizono 2020, Theorem 4.1 (1)]. Assume that $n \geq 3$. Taking a general hyperplane Y on X , the restriction of D to Y

$$D|_Y = \lceil M|_Y \rceil + E|_Y$$

also satisfies conditions (i), (ii) and (iii) in Theorem 4.20. Then the claim follows from the injection

$$H^1(\mathcal{O}_X(-D)) \hookrightarrow H^1(\mathcal{O}_Y(-D|_Y))$$

due to Enriques–Severi–Zariski’s lemma and the inductive assumption.

We assume that $\text{char } k > 0$. Note that $\alpha(X, D)$ is contained in $H^1(\mathcal{O}_X)_n$ by Proposition 4.14. Hence, the proof is identical to that of Theorem 4.17. \square

Remark 4.21. (1) Theorems 4.16 and 4.19 can be seen as generalizations of Ramanujam’s 1-connected vanishing for smooth surfaces (see [Barth et al. 2004, Chapter IV, Theorem 12.5]). Theorem 4.19 is also a generalization of [Ramanujam 1972, Theorem 2].

(2) Theorem 4.20 recovers [Fujino 2017, Theorem 3.5.3]. Indeed, for a smooth complete variety X of dimension ≥ 2 and a nef \mathbb{R} -divisor M with $\nu(M) \geq 2$, we reduce the vanishing $H^1(X, \mathcal{O}_X(-\lceil M \rceil)) = 0$ to Theorem 4.20 as follows: Take a birational morphism $\pi : X' \rightarrow X$ from a smooth projective variety by using Chow’s lemma and apply the Leray spectral sequence $H^p(R^q \pi_* \mathcal{O}_{X'}(-\lceil \pi^* M \rceil)) \Rightarrow H^{p+q}(\mathcal{O}_X(-\lceil M \rceil))$.

(3) Conditions (i) and (ii) in Theorem 4.20 are satisfied for any divisor D of the form $D = \lceil M \rceil$, where M is an \mathbb{R} -divisor which is nef in codimension 1 and satisfies $\nu(M) \geq 2$ or $\kappa(D) \geq 2$.

Example 4.22. (1) Raynaud surfaces [Raynaud 1978; Mukai 2013] are smooth projective surfaces X having positive characteristic with ample divisors D with $H^1(X, \mathcal{O}_X(-D)) \neq 0$. By construction, we can take the divisor D effective. Thus, these examples show that Theorem 4.17 does not hold if we only assume the weaker condition that $|D| \neq \emptyset$.

(2) The examples constructed in [Cascini and Tanaka 2018] (respectively, in [Bernasconi 2021]) are smooth (respectively, klt) rational surfaces X with a divisor D of the form $D = \lceil M \rceil$, a nef and big \mathbb{Q} -divisor M (respectively, an ample divisor D) on X such that $H^1(X, \mathcal{O}_X(-D)) \neq 0$. Here, by Theorem 4.17, we cannot take the divisor D effective because $H^1(\mathcal{O}_X) = 0$ in this case.

(3) If the base field k is not perfect, Theorem 4.17 does not hold. Indeed, Maddock [2016] constructed a regular del Pezzo surface X_2 over an imperfect field k of characteristic 2 with $\dim H^1(\mathcal{O}_{X_2}) = 1$ and $K_{X_2}^2 = 2$. Then, by the Riemann–Roch theorem, $\dim | -K_{X_2} | \geq \dim H^1(\mathcal{O}_{X_2}) = 1$.

(4) For any $i \geq 2$, there exists a normal projective variety X of dimension ≥ 3 and an ample Cartier divisor D on X such that $H^i(X, \mathcal{O}_X(-D)) \neq 0$ even for characteristic 0 [Sommese 1986]. Thus, the similar results of Theorem 4.20 for the vanishing on H^i , where $i \geq 2$, cannot be expected.

5. Adjoint linear systems for effective divisors

We are now going to apply our vanishing theorems to the study of adjoint linear systems on normal surfaces. In this section, let X be a normal proper surface over a field k or a normal compact analytic surface in Fujiki's class \mathcal{C} . First, let us recall the invariant $\delta_\zeta(\pi, Z)$ for the germ of a cluster (X, ζ) .

Definition 5.1 (see [Enokizono 2020, Definition 5.1]). Let ζ be a *cluster* on X , that is, a 0-dimensional subscheme (or analytic subset) of X . Let $\pi: X' \rightarrow X$ be a resolution of singularities of X contained in ζ and $Z > 0$ be an effective π -exceptional divisor on X' with $\pi_*\mathcal{I}_Z \subseteq \mathcal{I}_\zeta$. Let Δ be the *anticanonical cycle* of π , namely the π -exceptional \mathbb{Q} -divisor defined by $\Delta = \pi^*K_X - K_{X'}$, where π^* is the Mumford pullback of π . Then we define the number $\delta_\zeta(\pi, Z)$ to be 0 if $\Delta - Z$ is effective, and $-(\Delta - Z)^2$ otherwise.

For a cluster ζ and an effective divisor $D > 0$ on X , we say that the above pair (π, Z) *satisfies the condition $(E)_{D, \zeta}$* if $\pi_*\mathcal{I}_Z \subseteq \mathcal{I}_\zeta$ and $\pi^*D + \Delta - Z$ is effective.

The first main theorem in this section is as follows:

Theorem 5.2 (Reider-type theorem I). *Let X be a normal proper surface over a field k or analytic in Fujiki's class \mathcal{C} . Let $D > 0$ be an effective divisor on X , and assume there is a chain-connected component D_c of D containing a prime divisor C with $H^0(\mathcal{O}_C) \cong k$. Let ζ be a cluster on X along which $K_X + D$ is Cartier. Let (π, Z) be a pair satisfying condition $(E)_{D_c, \zeta}$ in Definition 5.1 and $D' := \pi^*D_c + \Delta - Z$. Assume that*

$$H^0(\mathcal{O}_X(K_X + D)) \rightarrow H^0(\mathcal{O}_X(K_X + D)|_\zeta)$$

is not surjective. Then there exists an effective decomposition $D = A + B$ such that both A and B intersect ζ and $AB \leq \frac{1}{4}\delta_\zeta(\pi, Z)$ holds if one of the following holds:

- (i0) $\text{char } k = 0$ and $H^1(\mathcal{O}_X) \cong H^1(\mathcal{O}_{X'})$.
- (ip) $\text{char } k > 0$ and either $H^1(\mathcal{O}_{X'}) = 0$ or $H^1(\mathcal{O}_{X'})_n = 0$ and $b_1(\Gamma(\hat{D}_c)) = b_1(\Gamma(D_c))$, where \hat{D}_c is the proper transform of D_c on X' .
- (ii0) $\text{char } k = 0$, $\kappa(D') \geq 1$ and $|mD'|$ is not composed with irrational pencils for $m \gg 0$.
- (iip) $\text{char } k > 0$, $H^1(\mathcal{O}_{X'})_n = 0$ and D' is big.
- (iiip) k is perfect with $\text{char } k > 0$, $\dim |D'| \geq \dim H^1(\mathcal{O}_{X'})_n$ and D' is big.

Proof. We first show the claim when D is chain-connected. Note that the restriction map

$$H^0(\mathcal{O}_X(K_X + D)) \rightarrow H^0(\mathcal{O}_X(K_X + D)|_\zeta)$$

is surjective if and only if the natural homomorphism

$$H^1(\mathcal{I}_\zeta \mathcal{O}_X(K_X + D)) \rightarrow H^1(\mathcal{O}_X(K_X + D))$$

is injective by the long exact sequence. By the Leray spectral sequence

$$E_2^{p,q} = H^p(R^q \pi_* \mathcal{O}_{X'}(K_{X'} + D')) \Rightarrow E^{p+q} = H^{p+q}(\mathcal{O}_{X'}(K_{X'} + D'))$$

and the assumption $\pi_* \mathcal{I}_Z \subseteq \mathcal{I}_\zeta$, the injectivity of

$$H^1(\mathcal{I}_\zeta \mathcal{O}_X(K_X + D)) \rightarrow H^1(\mathcal{O}_X(K_X + D))$$

follows from that of the natural homomorphism

$$H^1(\mathcal{O}_{X'}(K_{X'} + D')) \rightarrow H^1(\mathcal{O}_{X'}(K_{X'} + D' + Z)).$$

By the Serre duality, the injectivity of

$$H^1(\mathcal{O}_{X'}(K_{X'} + D')) \rightarrow H^1(\mathcal{O}_{X'}(K_{X'} + D' + Z))$$

is equivalent to the surjectivity of

$$H^1(\mathcal{O}_{X'}(-D' - Z)) \rightarrow H^1(\mathcal{O}_{X'}(-D')).$$

If $\alpha(X', D' + Z) = \alpha(X', D')$, then $H^1(\mathcal{O}_{X'}(-D' - Z)) \rightarrow H^1(\mathcal{O}_{X'}(-D'))$ is surjective if and only if the restriction map

$$H^0(\mathcal{O}_{D'+Z}) \rightarrow H^0(\mathcal{O}_{D'})$$

is surjective because of the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_{X'}) & \longrightarrow & H^0(\mathcal{O}_{D'+Z}) & \longrightarrow & H^1(\mathcal{O}_{X'}(-D' - Z)) & \longrightarrow & \alpha(X', D' + Z) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{O}_{X'}) & \longrightarrow & H^0(\mathcal{O}_{D'}) & \longrightarrow & H^1(\mathcal{O}_{X'}(-D')) & \longrightarrow & \alpha(X', D') & \longrightarrow & 0. \end{array}$$

Now we assume that $H^0(\mathcal{O}_X(K_X + D)) \rightarrow H^0(\mathcal{O}_X(K_X + D)|_\zeta)$ is not surjective. Note that any one of the conditions (i0), (ip), (ii0) and (iip) implies

$$\alpha(X', D' + Z) = \alpha(X', D')$$

by Propositions 4.11 and 4.12 and Corollary 4.9. Thus, the above argument implies that $H^0(\mathcal{O}_{D'+Z}) \rightarrow H^0(\mathcal{O}_{D'})$ is not surjective. Hence, D' is not chain-connected by Lemma 3.9. When condition (iip) holds, Theorem 4.17 implies that D' is also not chain-connected. Therefore, D' decomposes into $D' = A' + B'$ such that A' is chain-connected and contains the proper transform \hat{C} of C , B' is nonzero effective and $-A'$ is nef over B' by Lemma 3.5. Let us define $A := \pi_* A'$ and $B := \pi_* B'$. Now we show that B is nonzero and intersects ζ . If $B = 0$, then B' is π -exceptional. Replacing Z by $Z + B'$, it follows from the same argument

as above that $H^0(\mathcal{O}_{D'+Z}) \rightarrow H^0(\mathcal{O}_{D'-B'})$ is not surjective, which contradicts the chain-connectivity of $A' = D' - B'$. If $B \cap \zeta = \emptyset$, then for any prime component E of B , the proper transform \hat{E} coincides with the Mumford pullback π^*E . Since $-A'$ is nef over B' ,

$$-AE = -A'\pi^*E = -A'\hat{E} \geq 0,$$

which contradicts D being chain-connected. Similarly, one can see that A intersects ζ . Suppose that $\delta_\zeta(\pi, Z) = 0$, that is, $\Delta - Z$ is effective. Then we may assume that $D' = \lceil \pi^*D \rceil$ by replacing Z with the effective divisor $\Delta - \{-\pi^*D\}$. It follows from Proposition 3.7 that D' is chain-connected, which is a contradiction. Hence, we have $\delta_\zeta(\pi, Z) > 0$. Let us write $B' = \pi^*B + B_\pi$ for some π -exceptional \mathbb{Q} -divisor B_π on X' . Since $-A'$ is nef over B' ,

$$0 \leq -A'B' = -AB + (B_\pi - \Delta + Z)B_\pi.$$

Thus, we obtain

$$AB \leq \left(B_\pi - \frac{\Delta - Z}{2} \right)^2 + \frac{\delta_\zeta(\pi, Z)}{4} \leq \frac{\delta_\zeta(\pi, Z)}{4},$$

which completes the proof for the case that D is chain-connected.

For a general D , we consider the effective decomposition $D = D_1 + D_2$, where $D_1 := D_c$ is the chain-connected component of D containing C and $D_2 := D - D_1$. Then $-D_1$ is nef over D_2 . If D_2 intersects ζ , then $A := D_1$ and $B := D_2$ satisfy the assertion of the theorem. Thus, we may assume that D_2 and ζ are disjoint. Then $H^0(\mathcal{O}_X(K_X + D_1)) \rightarrow H^0(\mathcal{O}_X(K_X + D_1)|_\zeta)$ is also not surjective. As shown in the first half of the proof, we can take an effective decomposition $D_1 = A_1 + B_1$ such that both A_1 and B_1 intersect ζ and $A_1B_1 \leq \delta_\zeta(\pi, Z)/4$. If $D_2B_1 \leq 0$, then

$$(D - B_1)B_1 \leq A_1B_1 \leq \frac{\delta_\zeta(\pi, Z)}{4}.$$

Thus, $A := A_1 + D_2$ and $B := B_1$ satisfy the claim. If $D_2B_1 > 0$, then one can see that $A := A_1$ and $B := B_1 + D_2$ satisfy the claim. \square

Remark 5.3. (1) The condition $H^0(\mathcal{O}_C) \cong k$ in Theorem 5.2 is used in the proof only to ensure that the chain-connectivity of D' implies $H^0(\mathcal{O}_{D'}) \cong k$. Hence, if we assume that D' is big and X is geometrically connected over a perfect field k , then the assumption $H^0(\mathcal{O}_C) \cong k$ is not needed by Corollary 3.17.

(2) In the situation of Theorem 5.2, we further assume that $\pi_*\mathcal{I}_Z = \mathcal{I}_\zeta$ and $R^1\pi_*\mathcal{I}_Z = 0$ (e.g., $\Delta - Z$ is π - \mathbb{Z} -positive). Then the above proof of the theorem says that $H^0(\mathcal{O}_X(K_X + D)) \rightarrow H^0(\mathcal{O}_X(K_X + D)|_\zeta)$ is surjective if and only if $H^0(\mathcal{O}_{D'+Z}) \rightarrow H^0(\mathcal{O}_{D'})$ is surjective, where $D' := \pi^*D + \Delta - Z$. This is a generalization of [Francia 1991, Theorem 2].

(3) If we further assume that $D_\zeta^2 > \delta_\zeta(\pi, Z)$ in Theorem 5.2, then $2A - D_c$ is automatically big by the construction of A and B (see [Enokizono 2020, Theorem 5.2]).

Next we give a variant of Reider-type theorems which is used in Section 6.

Definition 5.4. (1) Let X be a normal complete surface and ζ be a cluster on X . We define the invariants q_X and $q_{X,\zeta}$ as

$$q_X := \min\{E^2 \mid E \text{ is an effective divisor on } X \text{ with } E^2 > 0\},$$

$$q_{X,\zeta} := \min\{E^2 \mid E \text{ is an effective divisor on } X \text{ with } E \cap \zeta \neq \emptyset \text{ and } E^2 > 0\}.$$

(2) Let us define a function $\mu: \mathbb{R}_{>0}^2 \rightarrow \mathbb{R}$ as

$$\mu(x, d) := \min\{x, d\} \left(\frac{d}{\min\{x, d\}} + 1 \right)^2.$$

Note that $\mu(-, d)$ is a nonincreasing function which takes the minimum value $4d$ and $\mu(x, -)$ is monotonically increasing for any fixed numbers x and d . Let ζ and (π, Z) be as in Definition 5.1. We define the number $\delta'_\zeta(\pi, Z)$ as

$$\delta'_\zeta(\pi, Z) := \mu(q_{X,\zeta}, \frac{1}{4}\delta_\zeta(\pi, Z)).$$

Note that $\delta'_\zeta(\pi, Z) \geq \delta_\zeta(\pi, Z)$ with the equality holding if and only if $q_{X,\zeta} \geq \frac{1}{4}\delta_\zeta(\pi, Z)$.

The second main theorem in this section is a positive characteristic analog of [Enokizono 2020, Theorem 5.4].

Theorem 5.5 (Reider-type theorem II). *Let X be a normal geometrically connected proper surface over a perfect field k of positive characteristic. Let D be an effective and nef divisor on X . Let ζ be a cluster on X along which $K_X + D$ is Cartier. Let (π, Z) be a pair satisfying condition (E) $_{D,\zeta}$ in Definition 5.1. We assume that $D^2 > \delta'_\zeta(\pi, Z)$ (respectively, $D^2 = \delta'_\zeta(\pi, Z) > \delta_\zeta(\pi, Z)$) and*

$$\dim |D'| \geq \dim H^1(\mathcal{O}_{X'})_n,$$

where $D' := \pi^*D + \Delta - Z$. If the restriction map

$$H^0(\mathcal{O}_X(K_X + D)) \rightarrow H^0(\mathcal{O}_X(K_X + D)|_\zeta)$$

is not surjective, then there exists an effective decomposition $D = A + B$ with $A, B > 0$ intersecting ζ such that $A - B$ is big, B is negative semidefinite and $AB \leq \frac{1}{4}\delta_\zeta(\pi, Z)$ (respectively, or $B^2 = q_{X,\zeta}$ and $D \equiv (\delta_\zeta(\pi, Z)/(4q_{X,\zeta}) + 1)B$).

Proof. Note that $D' := \pi^*D + \Delta - Z$ is automatically big since $D^2 > \delta'_\zeta(\pi, Z)$. Thus, the assumption of the theorem implies condition (iiip) in Theorem 5.2. Hence, there exists an effective decomposition $D = A + B$ such that both A and B intersect ζ , $A - B$ is big and $AB \leq \delta_\zeta(\pi, Z)/4$, where we note that D is chain-connected from Lemma 3.3. The rest of the proof is similar to that of [Enokizono 2020, Theorem 5.4]. \square

Corollaries of Reider-type theorems. In this subsection, we collect corollaries of Theorems 5.2 and 5.5. For simplicity, the base field k is assumed to be algebraically closed. First we consider the criterion of the basepoint-freeness.

Definition 5.6. Let X be a normal proper surface and $x \in X$ be a closed point. Let $\pi : X' \rightarrow X$ be the blow-up at x if $x \in X$ is smooth, or the minimal resolution at x otherwise. Let $Z > 0$ denote the exceptional (-1) -curve (respectively, the fundamental cycle of π , the round-up $\lceil \Delta \rceil$, the round-down $\lfloor \Delta \rfloor$) if $x \in X$ is smooth (respectively, Du Val; Kawamata log terminal but not Du Val; not Kawamata log terminal). We simply write by δ_x the number $\delta_x(\pi, Z)$ in Definition 5.1. Then we define the number τ_x to be 3 (respectively, 1, $\dim V_n$) if $x \in X$ is smooth (respectively, Du Val, otherwise), where V_n is the nilpotent part of the k -vector space $V := (R^1\pi_*\mathcal{O}_{X'})_x$ under the Frobenius action.

The following lemma is easy:

Lemma 5.7. *Let the situation be as in Definition 5.6 and D be an effective divisor on X passing through x such that $K_X + D$ is Cartier at x . Then the following hold:*

- (1) $\delta_x = 4$ (respectively, $\delta_x = 2$, $0 < \delta_x < 2$, $\delta_x = 0$) if $x \in X$ is smooth (respectively, Du Val; Kawamata log terminal, but not Du Val; not Kawamata log terminal).
- (2) $\dim |D'| - \dim H^1(\mathcal{O}_{X'})_n + \tau_x \geq \dim |D| - \dim H^1(\mathcal{O}_X)_n$, with $D' := \pi^*D + \Delta - Z$.

Proof. In order to prove (1), we may assume that $x \in X$ is a Kawamata log terminal singularity. Let $\Delta = \sum_i a_i E_i$ and $Z = \sum_i b_i E_i$ denote the irreducible decompositions. Then

$$\begin{aligned} (\Delta - Z)^2 &= (\Delta - Z)\Delta - (\Delta - Z)Z = \sum_i (a_i - b_i) E_i (-K_{X'}) + (K_{X'} + Z)Z \\ &= \sum_i (b_i - a_i) K_{X'} E_i + 2p_a(Z) - 2 \geq -2, \end{aligned}$$

and it is easy to see that equality holds if and only if $x \in X$ is Du Val. Claim (2) follows from the exact sequence

$$0 \rightarrow H^1(\mathcal{O}_X)_n \rightarrow H^1(\mathcal{O}_{X'})_n \rightarrow V_n$$

induced by the Leray spectral sequence with the Frobenius action. □

Theorem 5.2 (i0) and (ip) and Lemma 5.7 imply the following criterion of basepoint-freeness:

Corollary 5.8. *Let X be a normal proper surface. Let $x \in X$ be at most a rational singularity. Let L be a divisor on X which is Cartier at x . We assume that there exists a chain-connected member $D \in |L - K_X|$ passing through x satisfying:*

- (i) (X, x) or (D, x) is singular.
- (ii) D is strictly $(\delta_x/4)$ -connected if $x \in X$ is Kawamata log terminal.

- (iii) *The Frobenius map on $H^1(\mathcal{O}_X)$ is injective and $b_1(\Gamma(D)) = b_1(\Gamma(\hat{D}))$ when $\text{char } k > 0$, where \hat{D} is the proper transform of D by the minimal resolution of (X, x) when (X, x) is singular or by the blow-up at x when (X, x) is smooth.*

Then x is not a base point of $|L|$.

Remark 5.9. All of the conditions of D in Corollary 5.8 are satisfied if D is an integral curve passing through x , (X, x) or (D, x) is singular, and D is analytically irreducible at x when $\text{char } k > 0$.

Theorem 5.2 (iiip) and Lemma 5.7 imply the following corollary:

Corollary 5.10. *Let X be a normal proper surface. Let $x \in X$ be a closed point. Let D be a nef divisor on X such that $K_X + D$ is Cartier at x . Then x is not a base point of $|K_X + D|$ if the following conditions hold:*

- (i) *There exist rational numbers α and β with $\alpha \geq \delta_x$ and $4\beta(1 - \beta/\alpha) \geq \delta_x$ such that $D^2 > \alpha$ and $DB \geq \beta$ for any curve B on X passing through x .*
- (ii) *$\dim |D| \geq \dim H^1(\mathcal{O}_X)_n + \tau_x$ when $\text{char } k > 0$.*

Proof. Assume to the contrary that x is a base point of $|K_X + D|$. By Theorem 5.2 and Remark 5.3 (3) (or [Enokizono 2020, Theorem 5.2] when $\text{char } k = 0$), there exists a curve B on X passing through x such that $(D - B)B \leq \delta_x/4$ and $D - 2B$ is big. It follows from the Hodge index theorem that

$$DB \leq \frac{1}{4}\delta_x + B^2 \leq \frac{1}{4}\delta_x + \frac{(DB)^2}{D^2},$$

that is, $(DB)^2 - D^2(DB) + D^2\delta_x/4 \geq 0$. Since $(D - 2B)D > 0$,

$$DB \leq \frac{D^2 - \sqrt{D^2(D^2 - \delta_x)}}{2}.$$

It follows from $D^2 > \alpha$ and $DB \geq \beta$ that

$$\beta \leq DB \leq \frac{D^2 - \sqrt{D^2(D^2 - \delta_x)}}{2} < \frac{\alpha - \sqrt{\alpha(\alpha - \delta_x)}}{2}.$$

Thus, we have $4\beta(1 - \beta/\alpha) < \delta_x$, which contradicts assumption (i). \square

The very ample cases can be obtained similarly.

Corollary 5.11. *Let X be a normal proper surface with at most Du Val singularities. Let D be a Cartier divisor on X . Then $|K_X + D|$ is very ample if the following conditions hold:*

- (i) *There exist rational numbers α and β with $\alpha \geq 8$ and $\beta(1 - \beta/\alpha) \geq 2$ such that $D^2 > \alpha$ and $DB \geq \beta$ for any curve B on X .*
- (ii) *$\dim |D| \geq \dim H^1(\mathcal{O}_X)_n + 6$ when $\text{char } k > 0$.*

Proof. It suffices to show that $|K_X + D|$ separates any cluster ζ on X such that it is of length 2 and its support contains at least one smooth point of X or the defining ideal is $\mathfrak{m}_x^2 \subseteq \mathcal{O}_X$ for some Du Val singularity $x \in X$. Now we show this when the support of ζ has a single point $x \in X$ (the case $\zeta = x + y$, with $x \neq y$, is similar).

First, we assume that $x \in X$ is smooth. By assumption, ζ is a tangent vector at x . Let $\pi : X' \rightarrow X$ be the blow-up along ζ , that is, the composition of the blow-ups at x and at the point infinitely near x corresponding to the tangent vector. Let Z be the sum of the total transforms of the two exceptional (-1) -curves. Then one can see that $\delta_\zeta(\pi, Z) = 8$ and $\dim |D'| \geq \dim |D| - 6$, where $D' := \pi^*D + \Delta - Z$.

We assume that $x \in X$ is Du Val. Let $\pi : X' \rightarrow X$ be the minimal resolution of x and E be its fundamental cycle. Putting $Z := 2E$, one can see that $\pi_*\mathcal{O}_{X'}(-Z) = \mathfrak{m}_x^2$, $\delta_\zeta(\pi, Z) = 8$ and $\dim |D'| \geq \dim |D| - 4$.

The rest of the proof is similar to that of Corollary 5.10. \square

Remark 5.12. One can obtain a similar result to Corollary 5.11 when X is not necessarily canonical (but need the estimation of $\delta_\zeta(\pi, Z)$). For the direction, see [Sakai 1990; Kawachi and Maşek 1998; Langer 2000].

The following is a partial answer to the Fujita conjecture for surfaces in positive characteristic (although there are counterexamples to the Fujita conjecture [Gu et al. 2022]).

Corollary 5.13. *Let X be a projective surface with at most Du Val singularities in positive characteristic. Let H be an ample Cartier divisor on X . Then $|K_X + mH|$ is base point free for any $m \geq 3$ (or $m = 2$ and $H^2 > 1$) with $\dim |mH| \geq 3 + \dim H^1(\mathcal{O}_X)_n$, and is very ample for any $m \geq 4$ (or $m = 3$ and $H^2 > 1$) with $\dim |mH| \geq 6 + \dim H^1(\mathcal{O}_X)_n$.*

Proof. This follows from Corollary 5.10 with $(\alpha, \beta) = (4, 2)$ and Corollary 5.11 with $(\alpha, \beta) = (9, 3)$ (or directly from Lemma 3.2 and Theorem 5.2). \square

For pluri-(anti)canonical maps, the following hold:

Corollary 5.14. *Let X be a normal projective surface with at most singularities of geometric genera $p_g \leq 3$ in positive characteristic. Then the following hold:*

- (1) *If K_X is ample Cartier, then $|mK_X|$ is base point free for $m \geq 4$ (or $m = 3$ and $K_X^2 > 1$) with $\dim |(m-1)K_X| \geq \dim H^1(\mathcal{O}_X)_n + 3$.*
- (2) *If X is canonical and K_X is ample Cartier, then $|mK_X|$ is very ample for any $m \geq 5$ (or $m = 4$ and $K_X^2 > 1$) with $\dim |(m-1)K_X| \geq \dim H^1(\mathcal{O}_X)_n + 6$.*
- (3) *If $-K_X$ is ample Cartier (that is, X is canonical del Pezzo), then $|-mK_X|$ is base point free for $m \geq 2$ (or $m = 1$ and $K_X^2 > 1$), and is very ample for any $m \geq 3$ (or $m = 2$ and $K_X^2 > 1$).*

- (4) If K_X is ample with Cartier index $r \geq 2$, then $|mrK_X|$ is base point free for $m \geq 3$ with $\dim |(mr-1)K_X| \geq \dim H^1(\mathcal{O}_X)_n + 3$.
- (5) If $-K_X$ is ample with Cartier index $r \geq 2$ (that is, X is klt del Pezzo), then $| -mrK_X|$ is base point free for $m \geq 2$ with $\dim | -(mr+1)K_X| \geq 3$.

Proof. This follows from Corollaries 5.10 and 5.13. Note that $\dim |-(m+1)K_X| \geq 6$ automatically holds in case (3) by the Riemann–Roch theorem and that any klt del Pezzo surface is rational. Hence, $H^1(\mathcal{O}_X) = 0$ by [Tanaka 2015, Theorem 3.5]. \square

Remark 5.15. (1) When X is canonical, Corollary 5.14 (1) and (2) were obtained by [Ekedahl 1988, Main theorem, p. 97] without the condition for $\dim |(m-1)K_X|$. (2) Corollary 5.14 (3) was shown in [Bernasconi and Tanaka 2022, Proposition 2.14].

For bicanonical maps on smooth surfaces of general type, the following can be shown (compare to [Shepherd-Barron 1991, Theorems 26 and 27]):

Corollary 5.16. *Let X be a smooth minimal projective surface of general type in positive characteristic. Then $|2K_X|$ is base point free if $K_X^2 > 4$ and $\chi(\mathcal{O}_X) \geq 5 - h_s^{0,1}$, and $|2K_X|$ defines a birational morphism if $K_X^2 > 9$, $\chi(\mathcal{O}_X) \geq 8 - h_s^{0,1}$ and X does not admit genus 2 fibrations, where $h_s^{0,1}$ is the dimension of the semisimple part $H^1(\mathcal{O}_X)_s$.*

Proof. The base point free case follows from Theorem 5.2 and the 2-connectedness of K_X [Bombieri 1973, Lemma 1]. Note that the condition

$$\dim |K_X| \geq \dim H^1(\mathcal{O}_X)_n + 3$$

is equivalent to $\chi(\mathcal{O}_X) \geq 5 - h_s^{0,1}$. Next we consider the birational case. If the bicanonical map is not birational (hence, generically finite of degree ≥ 2), there exist infinitely many clusters $\zeta = x + y$ of degree 2 with $x \neq y$ such that $|2K_X|$ does not separate ζ . One can see easily that $\delta_\zeta(\pi, Z) = 8$ and $\delta'_\zeta(\pi, Z) = 8$ or 9 , where $\pi : X' \rightarrow X$ is the blow-up along $\zeta = x + y$ and $Z = E_x + E_y$ is the sum of two exceptional (-1) -curves. It follows from Theorem 5.5 that there exists a negative semidefinite curve B_ζ intersecting ζ such that $(K_X - B_\zeta)B_\zeta = 2$, where the equality is due to the 2-connectivity of K_X . Thus, $(K_X B_\zeta, B_\zeta^2) = (2, 0)$ or $(1, -1)$. Since the number of curves B_ζ satisfying the later case is finite, there exist infinitely many curves (but belong to finitely many numerical classes) B_ζ satisfying the former case. By applying [Enokizono 2020, Proposition 6.7], see Lemma 6.8, these B_ζ define a genus 2 fibration on X . \square

6. Extension theorems in positive characteristic

In this section, let X be a normal proper geometrically connected surface over an infinite perfect field k of positive characteristic. We will prove the following

extension theorem, which is a positive characteristic analog of [Enokizono 2020, Theorem 6.1], by using Theorem 5.5 instead of [Enokizono 2020, Theorem 5.4]:

Theorem 6.1 (extension theorem). *Let $D > 0$ be an effective divisor on X and assume that any prime component D_i of D has positive self-intersection number. Let $\varphi: D \rightarrow \mathbb{P}^1$ be a finite separable morphism of degree d . If $D^2 > \mu(q_X, d)$ and $\dim |D| \geq 3d + \dim H^1(\mathcal{O}_X)_n$, then there exists a morphism $\psi: X \rightarrow \mathbb{P}^1$ such that $\psi|_D = \varphi$.*

Remark 6.2. Theorem 6.1 generalizes a result of Serrano [1987, Remark 3.12]. Paoletti proved another variant of extension theorems in positive characteristic by using Bogomolov-type inequalities [Paoletti 1995, Theorem 3.1].

The following two theorems are positive characteristic analogs of [Enokizono 2020, Theorems 6.10 and 6.11] (see [Enokizono 2020, Section 6] for notation and discussions):

Theorem 6.3 (extension theorem with base points). *Let $D > 0$ be an effective divisor on X and assume that any prime component D_i of D has positive self-intersection number. Let $\varphi: D \rightarrow \mathbb{P}^1$ be a finite separable morphism of degree d which cannot be extended to a morphism on X . We assume that $D^2 = \mu(q_X, d)$, $q_X < d$ and $\dim |D| \geq 3d + \dim H^1(\mathcal{O}_X)_n$. Then there exists a linear pencil $\{F_\lambda\}_\lambda$ with $F_\lambda^2 = q_X$ and no fixed parts such that the induced rational map $\psi: X \dashrightarrow \mathbb{P}^1$ satisfies $\psi|_D = \varphi$.*

Theorem 6.4 (extension theorem on movable divisors). *Let $D > 0$ be an effective divisor on X and assume that all prime components D_i of D have nontrivial numerical linear systems and positive self-intersection numbers. Let $\varphi: D \rightarrow \mathbb{P}^1$ be a finite separable morphism of degree d on X . If $D^2 > \mu(q_{X,\infty}, d)$ (respectively, $D^2 = \mu(q_{X,\infty}, d)$, $q_{X,\infty} < d$) and $\dim |D| \geq 3d + \dim H^1(\mathcal{O}_X)_n$, then there exists a morphism $\psi: X \rightarrow \mathbb{P}^1$ (respectively, or a rational map $\psi: X \dashrightarrow \mathbb{P}^1$ induced by a linear pencil $\{F_\lambda\}_\lambda$ with $F_\lambda^2 = q_{X,\infty}$ and no fixed parts) such that $\psi|_D = \varphi$.*

Proof of the extension theorem. The proofs of Theorems 6.1, 6.3 and 6.4 are almost identical to those of [Enokizono 2020, Theorems 6.1, 6.10 and 6.11]. We only sketch here the proof of Theorem 6.1 (the remaining cases are left to the reader).

Let Λ be the set of closed points of \mathbb{P}^1 such that $(\varphi|_{D_{\text{red}}})^{-1}(\lambda)$ is reduced and contained in the smooth loci of X and D_{red} . It is a dense subset of \mathbb{P}^1 since X is normal and $\varphi|_D$ is separable. For a closed point $\lambda \in \mathbb{P}^1$, we put $\alpha_\lambda := \varphi^{-1}(\lambda)$.

Lemma 6.5 [Enokizono 2020, Lemma 6.4]. *For any k -rational point $\lambda \in \Lambda$, the restriction $H^0(\mathcal{O}_X(K_X + D)) \rightarrow H^0(\mathcal{O}_X(K_X + D)|_{\alpha_\lambda})$ is not surjective.*

Lemma 6.6 (see [Enokizono 2020, Lemma 6.5]). *For any k -rational point $\lambda \in \Lambda$, there exist a member $D_\lambda \in |D|$ and a pair (π, Z) satisfying condition $(E)_{D_\lambda, \alpha_\lambda}$ in Definition 5.1 such that $\delta_{\alpha_\lambda}(\pi, Z) = 4d$.*

Proof. As in the proof of [Enokizono 2020, Lemma 6.5], we can take a blow-up $\pi : X' \rightarrow X$ along α_λ and a π -exceptional divisor $Z > 0$ such that $\pi_*\mathcal{I}_Z = \mathcal{I}_{\alpha_\lambda}$ and $\delta_{\alpha_\lambda}(\pi, Z) = 4d$. Since $\dim |D| \geq 3d$, we can take a member $D'_\lambda \in |\pi^*D + \Delta - Z|$. Then $D_\lambda := \pi_*D'_\lambda$ is a desired one. \square

We fix a k -rational point $\lambda \in \Lambda$ arbitrarily. Then D_λ , α_λ and the pair (π, Z) obtained by Lemma 6.6 satisfy condition $(E)_{D_\lambda, \alpha_\lambda}$. Thus, we can apply Theorem 5.5 to this situation since

$$D_\lambda^2 = D^2 > \mu(q_X, d) \geq \delta'_{\alpha_\lambda}(\pi, Z)$$

and $\dim |D'_\lambda| \geq \dim H^1(\mathcal{O}_{X'})_n$. Hence, there exists an effective decomposition $D_\lambda = A_\lambda + B_\lambda$ with A_λ and B_λ intersecting α_λ such that $A_\lambda - B_\lambda$ is big, B_λ is negative semidefinite and $A_\lambda B_\lambda \leq d$. Moreover, the following lemma can be shown similarly to [Enokizono 2020, Lemma 6.6]:

Lemma 6.7 [Enokizono 2020, Lemma 6.6]. *In the above situation, we have $B_\lambda^2 = 0$ and $D \cap B_\lambda \subseteq \alpha_\lambda$ scheme-theoretically.*

Let \mathcal{B} be the set of all prime divisors C such that $C \leq B_\lambda$ for some k -rational point $\lambda \in \Lambda$ and $DC > 0$. This is an infinite set, because $D \cap B_\lambda \subseteq \alpha_\lambda$ by Lemma 6.7 and $\alpha_\lambda \cap \alpha_{\lambda'} = \emptyset$ for $\lambda \neq \lambda'$. On the other hand, the set \mathcal{B} consists of finitely many numerical equivalence classes, say $B_{(1)}, \dots, B_{(m)}$, since $0 < DC \leq DB_\lambda \leq d$ for any $C \in \mathcal{B}$. We put

$$\mathcal{B}_{(i)} := \{C \in \mathcal{B} \mid C \equiv B_{(i)}\}.$$

Then there is at least one $\mathcal{B}_{(i)}$ which has infinite elements. We choose such a $\mathcal{B}_{(i)}$ and put $\mathcal{B} := \mathcal{B}_{(i)}$ again.

Lemma 6.8 [Enokizono 2020, Proposition 6.7]. *Let X be a normal proper surface over an infinite perfect field k . Let \mathcal{B} be an infinite family of prime divisors on X , any member of which has the same numerical equivalence class B with $B^2 = 0$. Then there exists a fibration $f : X \rightarrow Y$ onto a smooth curve Y such that any member of \mathcal{B} is a fiber of f .*

By using Lemma 6.8 in this situation, there exists a fibration $f : X \rightarrow Y$ onto a smooth curve Y such that any member of \mathcal{B} is a fiber of f . Let \bar{D} denote the scheme-theoretic image of the morphism

$$(f|_D, \varphi) : D \rightarrow Y \times \mathbb{P}^1$$

and $h : \bar{D} \rightarrow Y$ denote the restriction of the first projection $Y \times \mathbb{P}^1 \rightarrow Y$ to \bar{D} .

Lemma 6.9 [Enokizono 2020, Lemma 6.8]. $h: \bar{D} \rightarrow Y$ is an isomorphism.

Let $\gamma: Y \rightarrow \mathbb{P}^1$ be the composition of h^{-1} and the second projection $\bar{D} \rightarrow \mathbb{P}^1$. Then $\varphi: D \rightarrow \mathbb{P}^1$ decomposes into $\varphi = \gamma \circ f|_D: X \rightarrow Y \rightarrow \mathbb{P}^1$. Hence, $\psi := \gamma \circ f$ is the desired one.

7. Applications to plane curves

In this section, we are going to apply our extension theorems obtained in Section 6 to the geometry of plane curves.

Definition 7.1 (strange points for plane curves). Let $D \subseteq \mathbb{P}^2$ be a plane curve over a field k . For a k -rational point $x \in \mathbb{P}^2$, we define the open subset $U_{D,x}$ of D to be the set of points y of D such that the reduced plane curve $(D_{k(y)})_{\text{red}} \subseteq \mathbb{P}^2_{k(y)}$ is nonsingular at y and its tangent line $L_y \subseteq \mathbb{P}^2_{k(y)}$ at y does not pass through x . Then we say that x is *strange* with respect to D if $U_{D,x}$ is not dense in D . If $\text{char } k = 0$, all the strange points are k -rational points on lines contained in D . If D is smooth and has strange points, then D is a line or a conic with $\text{char } k = 2$ (see [Hartshorne 1977, Chapter IV, Theorem 3.9]). For a nonsingular k -rational point x of D , we can see that x is not strange if and only if the inner projection $D \rightarrow \mathbb{P}^1$ from x is finite and separable.

Theorem 7.2. Let $D \subseteq \mathbb{P}^2$ be a plane curve of degree $m \geq 3$ over an arbitrary base field k . Then there is a one-to-one correspondence between:

- (i) the set of nonsingular k -rational points of D which is not strange, and
- (ii) the set of finite separable morphisms $D \rightarrow \mathbb{P}^1$ of degree $m - 1$ up to automorphisms of \mathbb{P}^1 .

Moreover, any finite separable morphism $D \rightarrow \mathbb{P}^1$ has degree greater than or equal to $m - 1$.

Proof. Let x be a nonsingular, nonstrange k -rational point of D . Then the inner projection from x defines a finite separable morphism $\text{pr}_x: D \rightarrow \mathbb{P}^1$ of degree $m - 1$. This correspondence $x \mapsto \text{pr}_x$ defines a map from the set of (i) to that of (ii), which is injective since $m \geq 3$. Thus, in order to prove the first claim, it suffices to show that any finite separable morphism $D \rightarrow \mathbb{P}^1$ of degree $m - 1$ is obtained by the inner projection from some k -rational point of D . Let $\varphi: D \rightarrow \mathbb{P}^1$ be such a morphism. If the base field k is algebraically closed (or infinite and perfect), then by Theorem 6.3 (or [Enokizono 2020, Theorem 6.10] when $\text{char } k = 0$), there exists a rational map $\psi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ induced by a linear pencil of lines such that $\psi|_D = \varphi$. Since φ is a morphism, ψ is nothing but an inner projection from a k -rational point. Suppose that k is not algebraically closed. Taking the base change to an algebraic closure \bar{k} of k , we obtain a finite separable morphism $\varphi_{\bar{k}}: D_{\bar{k}} \rightarrow \mathbb{P}^1_{\bar{k}}$ of degree $m - 1$.

From the above argument, this comes from an inner projection from a \bar{k} -rational point x of $D_{\bar{k}}$. Now, we take a k -rational point $\lambda \in \mathbb{P}^1$ and write

$$\varphi_{\bar{k}}^{-1}(\lambda) = \{x_1, \dots, x_l\}$$

as sets. Then these points x, x_1, \dots, x_l lie on the same line $L \subseteq \mathbb{P}_{\bar{k}}^2$. Moreover, the set $\{x_1, \dots, x_l\}$ is G -invariant under the Galois action $G := \text{Gal}(\bar{k}/k)$ on $\mathbb{P}_{\bar{k}}^2$. On the other hand, each $\sigma \in G$ sends the line L to another line $\sigma(L)$, which also contains $\{x_1, \dots, x_l\}$. Now we show $\sigma(L) = L$, that is, L is G -invariant. If $l \geq 2$, then this is clear since the line passing through fixed two points is unique. Thus, we may assume $l = 1$. Then both L and $\sigma(L)$ are tangent lines at x_1 with multiplicity $m - 1$, which implies $L = \sigma(L)$. By taking another k -rational point λ' of \mathbb{P}^1 and using the same argument as above, we can take another G -invariant line L' in $\mathbb{P}_{\bar{k}}^2$ such that $L \cap L' = \{x\}$. Thus x is G -invariant and descends to a k -rational point x^G of D . Since x is a smooth point of $D_{\bar{k}}$, so is x^G . Since the lines L and L' descend to lines L^G and L'^G which intersect at x^G , we conclude that $\varphi: D \rightarrow \mathbb{P}^1$ is the inner projection from x^G . The last claim is due to Theorem 6.1 (or [Enokizono 2020, Theorem 6.1] when $\text{char } k = 0$) since \mathbb{P}^2 does not admit any nonconstant morphism to \mathbb{P}^1 . \square

Appendix: Mumford's intersection form on a normal projective variety

In this appendix, we extend Mumford's intersection form on a normal surface [Mumford 1961] to a higher-dimensional variety over a field k .

Theorem A.1. *Let X be a normal projective variety of dimension $n \geq 2$ over a field k . Then there exists a multilinear form*

$$Q: \underbrace{\text{Pic}(X) \times \cdots \times \text{Pic}(X)}_{n-2} \times \text{Cl}(X) \times \text{Cl}(X) \rightarrow \mathbb{Q},$$

which we call Mumford's intersection form, such that the following conditions hold:

(i) Q is an extension of the usual intersection form

$$\text{Pic}(X) \times \cdots \times \text{Pic}(X) \times \text{Cl}(X) \rightarrow \mathbb{Z}.$$

(ii) Q is symmetric with respect to the first $n - 2$ terms and the last two terms.

(iii) Q is compatible with the base change to any separable field extension k' of k .

(iv) If k is an infinite field and $S := H_1 \cap \cdots \cap H_{n-2}$ is a normal surface obtained by the intersection of $n - 2$ general hyperplanes, then $Q(H_1, \dots, H_{n-2}, D_1, D_2)$ coincides with Mumford's intersection number of $D_1|_S$ and $D_2|_S$ on S .

Definition A.2 (Mumford pullback). Let X be a normal projective variety over an infinite field k . Let $\pi: X' \rightarrow X$ be a resolution of X . Let $\{E_i\}_i$ denote the

set of π -exceptional prime divisors on X' such that the center $C_i := \pi(E_i)$ is of codimension 2. For each i , let F_i denote the numerical equivalence class of the 1-cycle

$$\frac{1}{[k(x) : k]}(\pi|_{E_i})^{-1}(x)$$

(a fiber of $E_i \rightarrow C_i$ “at a rational point”), which is independent of the choice of a general closed point $x \in C_i$. For a Weil divisor D on X , we define the *Mumford pullback* of D by π , which is denoted by π^*D , as $\hat{D} + \sum_i d_i E_i$, where \hat{D} is the proper transform of D on X' and the coefficients $d_i \in \mathbb{Q}$ are determined by the equation

$$\left(\hat{D} + \sum_i d_i E_i\right)F_j = 0$$

for each j . Note that this condition is equivalent to

$$\left(\hat{D} + \sum_i d_i E_i\right)\pi^*H_1 \cdots \pi^*H_{n-2}E_j = 0$$

for some ample divisors H_1, \dots, H_{n-2} on X since

$$(H_1 \cdots H_{n-2}C_j)F_j \equiv \pi^*H_1 \cdots \pi^*H_{n-2}E_j.$$

Remark A.3. (1) The definition of the Mumford pullback makes sense if the intersection matrix $(E_i F_j)_{i,j}$ is invertible. The invertibility can be checked as follows: Let H_1, \dots, H_{n-2} be general hyperplanes on X such that $S := H_1 \cap \cdots \cap H_{n-2}$ is a normal surface. Let $\rho: S' \rightarrow \pi^{-1}(S)$ denote the normalization and E'_i denote the pullback of E_i under ρ . Then E'_i is a nonzero effective $(\pi \circ \rho)$ -exceptional divisor on S' , and thus $(E'_i E'_j)_{i,j}$ is negative definite. Since

$$E'_i E'_j = \pi^*H_1 \cdots \pi^*H_{n-2}E_i E_j = (H_1 \cdots H_{n-2}C_j)E_i F_j,$$

the matrix $(E_i F_j)_{i,j}$ is invertible.

(2) The definition of the Mumford pullback seems to be unnatural because all the coefficients of π -exceptional divisors contracting to codimension ≥ 3 centers are zero. It seems to be natural to consider that the Mumford pullback is determined modulo π -exceptional divisors contracting to codimension ≥ 3 centers. Indeed, the terms of such π -exceptional divisors do not affect the intersection numbers of $n - 2$ Cartier divisors and two Weil divisors defined later. For more general treatment of Mumford pullbacks, see [Boucksom et al. 2012].

Definition A.4. Let X be a normal projective variety of dimension $n \geq 2$ over an infinite field k . Let L_1, \dots, L_{n-2} be Cartier divisors on X . Let D_1 and D_2 be Weil divisors on X .

(1) For a resolution $\pi : X' \rightarrow X$ of X , we define $(L_1 \cdots L_{n-2} D_1 D_2)_\pi$ to be the rational number $\pi^* L_1 \cdots \pi^* L_{n-2} \pi^* D_1 \pi^* D_2$.

(2) Let $\pi : Y' \rightarrow X$ be an alteration from a regular projective variety Y' . Let $Y' \xrightarrow{\psi} Y \xrightarrow{\varphi} X$ denote the Stein factorization of π , where ψ is a resolution of a normal projective variety Y and φ is finite. Then, we define

$$(L_1 \cdots L_{n-2} D_1 D_2)_\pi := \frac{1}{\deg \varphi} (\varphi^* L_1 \cdots \varphi^* L_{n-2} \varphi^* D_1 \varphi^* D_2)_\psi,$$

where $\varphi^* D$ is the pullback of a Weil divisor D by the finite morphism φ .

Lemma A.5. *Let $X, L_1, \dots, L_{n-2}, D_1, D_2$ be as in Definition A.4. Then the numbers $(L_1 \cdots L_{n-2} D_1 D_2)_\pi$ are independent of the choice of an alteration π .*

Proof. We show $(L_1 \cdots L_{n-2} D_1 D_2)_{\pi_1} = (L_1 \cdots L_{n-2} D_1 D_2)_{\pi_2}$ for two alterations $\pi_i : Y'_i \rightarrow X$ with Y'_i regular, where $i = 1, 2$. Taking an alteration from a regular variety Y'_3 to the normalization of the main component of $Y'_1 \times_X Y'_2$ (for existence, see [de Jong 1996]) and replacing Y'_2 by Y'_3 , we may assume that there exists a generically finite morphism $\rho : Y'_2 \rightarrow Y'_1$ such that $\pi_1 \circ \rho = \pi_2$. Let $Y'_i \xrightarrow{\psi_i} Y_i \xrightarrow{\varphi_i} X_i$ denote the Stein factorization of π_i . Then there exists a finite morphism $\tau : Y_2 \rightarrow Y_1$ such that $\varphi_2 = \varphi_1 \circ \tau$. Now we have

$$\begin{aligned} (L_1 \cdots L_{n-2} D_1 D_2)_{\pi_1} &= \frac{1}{\deg \varphi_1} (\psi_1^* \varphi_1^* L_1 \cdots \psi_1^* \varphi_1^* L_{n-2} \psi_1^* \varphi_1^* D_1 \psi_1^* \varphi_1^* D_2) \\ &= \frac{1}{\deg \varphi_2} (\psi_2^* \varphi_2^* L_1 \cdots \psi_2^* \varphi_2^* L_{n-2} \rho^* \psi_1^* \varphi_1^* D_1 \rho^* \psi_1^* \varphi_1^* D_2) \end{aligned}$$

and

$$(L_1 \cdots L_{n-2} D_1 D_2)_{\pi_2} = \frac{1}{\deg \varphi_2} (\psi_2^* \varphi_2^* L_1 \cdots \psi_2^* \varphi_2^* L_{n-2} \psi_2^* \varphi_2^* D_1 \psi_2^* \varphi_2^* D_2).$$

Thus, it suffices to show that for any Weil divisor D on Y_1 , $\rho^* \psi_1^* D$ equals $\psi_2^* \tau^* D$ modulo ψ_2 -exceptional divisors contracting to codimension ≥ 3 centers. To prove this, it is enough to show that there exist ample divisors H_1, \dots, H_{n-2} on Y_2 such that

$$\rho^* \psi_1^* D \psi_2^* H_1 \cdots \psi_2^* H_{n-2} E_j = 0$$

for any ψ_2 -exceptional prime divisor E_j whose center has codimension 2. Now, we take ample divisors A_1, \dots, A_{n-2} on Y_1 and put $H_i := \tau^* A_i$, which are ample since τ is finite. Then, we have

$$\rho^* \psi_1^* D \psi_2^* H_1 \cdots \psi_2^* H_{n-2} E_j = \psi_1^* D \psi_1^* A_1 \cdots \psi_1^* A_{n-2} \rho_* E_j = 0,$$

since $\rho_* E_j$ is ψ_1 -exceptional or 0. \square

Definition A.6 (intersection numbers). Let X be a normal projective variety of dimension $n \geq 2$ over a field k . Let L_1, \dots, L_{n-2} be Cartier divisors on X .

Let D_1 and D_2 be Weil divisors on X . Then we define the *intersection number* of L_1, \dots, L_{n-2}, D_1 and D_2 , which is denoted by $L_1 \cdots L_{n-2} D_1 D_2$, as follows:

(1) If the base field k is infinite, then we define

$$L_1 \cdots L_{n-2} D_1 D_2 := (L_1 \cdots L_{n-2} D_1 D_2)_\pi,$$

where $\pi : Y' \rightarrow X$ is an alteration with Y' regular [de Jong 1996].

(2) If k is finite and $H^0(\mathcal{O}_X) = k$, then we take an algebraic closure \bar{k} of k and define

$$L_1 \cdots L_{n-2} D_1 D_2 := L_{1, \bar{k}} \cdots L_{n-2, \bar{k}} D_{1, \bar{k}} D_{2, \bar{k}},$$

where we put $X_{\bar{k}} := X \times_k \bar{k}$ and the divisors $L_{i, \bar{k}}$ and $D_{i, \bar{k}}$ are, respectively, the pullbacks of L_i and D_i via the projection $X_{\bar{k}} \rightarrow X$. Note that $X_{\bar{k}}$ is normal since k is perfect.

(3) If k is finite and $k_X := H^0(\mathcal{O}_X) \neq k$, then X is geometrically integral and geometrically normal over k_X . Then, we define

$$L_1 \cdots L_{n-2} D_1 D_2 := [k_X : k](L_1 \cdots L_{n-2} D_1 D_2)_X,$$

where $(L_1 \cdots L_{n-2} D_1 D_2)_X$ is the intersection number on X over k_X defined in (2).

Proof of Theorem A.1. We define the multilinear form Q as

$$Q(L_1, \dots, L_{n-2}, D_1, D_2) := L_1 \cdots L_{n-2} D_1 D_2.$$

One can see easily that this is well defined and satisfies the conditions (i), (ii), (iii) and (iv). \square

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
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