Pacific Journal of Mathematics

Volume 325 No. 1 July 2023

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 325, No. 1, 2023
https://doi.org/10.2140/pjm.2023.325.1

ESTIMATE FOR THE FIRST FOURTH STEKLOV EIGENVALUE OF A MINIMAL HYPERSURFACE WITH FREE BOUNDARY

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We dedicate this paper to João Xavier da Cruz Neto on the occasion of his sixtieth birthday.

We explore the fourth-order Steklov problem of a compact embedded hypersurface Σ^n with free boundary in a $(n+1)$-dimensional compact manifold M^{n+1} which has nonnegative Ricci curvature and strictly convex boundary. If Σ is minimal we establish a lower bound for the first eigenvalue of this problem. When $M = B^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$, if Σ has constant mean curvature H^Σ we prove that the first eigenvalue satisfies $\sigma_1 \le n + |H^\Sigma|$. In the minimal case ($H^\Sigma = 0$), we prove that $\sigma_1 = n$.

1. Introduction

Let Σ^n be an n-dimensional compact Riemannian manifold with nonempty boundary $\partial\Sigma \neq \varnothing$. Consider the fourth-order Steklov eigenvalue problem

$$\begin{cases} \Delta^2\xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta\xi = \sigma \frac{\partial\xi}{\partial\nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{1}$$

where σ is a real number, Δ is the Laplacian operator on Σ and ν_Σ denotes the outward unit normal on $\partial\Sigma$. The first nonzero eigenvalue of the above problem will be denoted by $\sigma_1 = \sigma(\Sigma)$. The first eigenvalue of (1) has the following variational characterization:

$$\sigma_1 = \inf_{w_{|\partial\Sigma}=0} \frac{\int_\Sigma (\Delta w)^2}{\int_{\partial\Sigma} \left(\frac{\partial w}{\partial\nu_\Sigma}\right)^2}. \tag{2}$$

Wang and Xia [2009] proved that if Σ has nonnegative Ricci curvature and the mean curvature of $\partial\Sigma$ is bounded below by a positive constant c then $\sigma_1 \ge c \cdot n$. Furthermore, equality occurs if and only if Σ is isometric to an n-dimensional Euclidean ball of radius $\frac{1}{c}$.

MSC2020: primary 53C20, 53C42; secondary 35J40, 35P15.
Keywords: fourth Steklov eigenvalue, hypersurface with free boundary.

Since their first appearance in [Stekloff 1902], elliptic problems with parameters in the boundary conditions are called Steklov problems. Kuttler [1972] and Payne [1970] studied the isoperimetric properties of the first eigenvalue σ_1 of the fourth-order Steklov problem (1). Moreover, as already noticed in [Kuttler 1972; 1979; Kuttler and Sigillito 1985], σ_1 is the sharp constant for L^2 a priori estimates for solutions of the (second-order) Laplace equation under nonhomogeneous Dirichlet boundary conditions. In [Ferrero et al. 2005] the authors studied the spectrum of the biharmonic Steklov problem (1) and obtained a characterization of it, and presented a physical interpretation of σ_1. For comprehensive references on such Steklov problems, see [Berchio et al. 2006; Bucur et al. 2009; Wang and Xia 2009].

It should be pointed out that the problem

$$\begin{cases} \Delta^2\xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \frac{\partial^2\xi}{\partial\nu_\Sigma^2} = \lambda\frac{\partial\xi}{\partial\nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{3}$$

is a natural Steklov problem and one can check that when the mean curvature of $\partial\Sigma$ is constant, it is equivalent to (1).

Let M be a compact Riemannian manifold with nonempty boundary ∂M and $\Sigma \subset M$ a compact hypersurface (with boundary $\partial\Sigma$) properly embedded into M, that is, $\Sigma \cap \partial M = \partial\Sigma$. We say that Σ is a minimal hypersurface with free boundary if Σ is a minimal hypersurface and Σ meets ∂M orthogonally along $\partial\Sigma$. In this setting, Fraser and Li [2014] obtained a lower bound for the first eigenvalue of the second-order Steklov problem.

If $M = B^n$ is the unit ball in $\mathbb{R}^n$, it is known [Fraser and Schoen 2013] that the coordinate functions are eigenfunctions of the second-order Steklov problem with eigenvalue 1. Taking that into consideration, Fraser and Li [2014] conjectured that the first eigenvalue of the second-order Steklov problem of a compact properly embedded minimal hypersurface in B^n is 1 and proved that this is limited from below by $\frac{1}{2}$.

On the one hand, we did not find in the literature an extrinsic approach to the fourth-order Steklov eigenvalue problem. Motivated by the work of Fraser and Li, in this paper we consider the fourth-order Steklov problem of a compact properly embedded minimal hypersurface Σ with free boundary in a compact manifold M.

On the other hand, Ferrero, Gazzola and Weth [Ferrero et al. 2005] explored the fourth-order Steklov eigenvalue problem in a bounded domain Ω of $\mathbb{R}^n$ and proved that the first eigenvalue of this problem is equal to n when $\Omega = B^n$. It is known that the unit ball B^n is a minimal hypersurface with free boundary in B^{n+1}. In this setting, we have established an upper estimate for the first eigenvalue of the fourth-order Steklov problem of a compact properly embedded CMC hypersurface in B^{n+1} with free boundary on ∂B^{n+1}:

Proposition 1. *Let Σ^n be a compact properly embedded hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$. Assume that Σ has constant normalized mean curvature H^Σ. Then*

$$\sigma_1 \leq n + |H^\Sigma|.$$

It follows from Proposition 1 that if Σ is minimal ($H^\Sigma = 0$), then $\sigma_1 \leq n$. This, together with the result of Ferrero, Gazzola and Weth [Ferrero et al. 2005], naturally led us to formulate and prove the main result of this paper:

Theorem 2. *Let Σ^n be a compact properly embedded minimal hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$. Then the first eigenvalue of the fourth-order Steklov problem of Σ is equal to n.*

Wang and Xia [2009] proved that any compact connected Riemannian manifold Σ with boundary $\partial\Sigma$ satisfies

$$|\Sigma|\sigma_1 \leq |\partial\Sigma|, \tag{4}$$

where $|\partial\Sigma|$ and $|\Sigma|$ denote the area of $\partial\Sigma$ and the volume of Σ, respectively. If in addition the Ricci curvature of Σ is nonnegative and the equality holds, then Σ is isometric to an n-dimensional Euclidean ball. In our context, the equality always holds even for codimension greater than 1 (see Proposition 2.4 in [Li 2020]), i.e.,

$$k|\Sigma| = |\partial\Sigma|$$

for every k-dimensional immersed free boundary minimal submanifold Σ^k in the unit ball B^{n+1}. As a consequence of this equality and from (4) we get that $\sigma_1 \leq k$ for free boundary minimal submanifolds $\Sigma^k \subset B^{n+1}$.

Taking that into consideration, it is natural to consider the following question.

Problem 3. *Under what additional assumption is it possible to ensure that a compact properly embedded minimal hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$, such that $\sigma_1 = n$ is the unit ball B^n?*

In our next result, we prove a lower estimate for σ_1 when Σ^n is a compact properly embedded minimal hypersurface with free boundary in a compact manifold which has nonnegative Ricci curvature and strictly convex boundary. More precisely, we prove the following theorem.

Theorem 4. *Let M^{n+1} be an $(n+1)$-dimensional compact orientable Riemannian manifold with nonnegative Ricci curvature and nonempty boundary ∂M. Assume the second fundamental form of ∂M satisfies $A^{\partial M}(v, v) \geq k > 0$, for any unit vector v tangent to ∂M.*

Let Σ^n be a properly embedded minimal hypersurface in M with free boundary on ∂M. Assume $\partial\Sigma$ has constant mean curvature $H^{\partial\Sigma}$. If

(i) Σ *is orientable*, *or*

(ii) $\pi_1(M)$ *is finite*,

then we have the eigenvalue estimate $\sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}$, *where* σ_1 *is the first eigenvalue of the fourth-order Steklov problem on* Σ.

This estimate for σ_1 is analogous to the estimates of Fraser and Li [2014] for the first nonzero Steklov eigenvalue of the Dirichlet-to-Neumann map on Σ.

Remark 5. If $M = B^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$ and $\Sigma = B^n \subset B^{n+1}$ is the unit ball in $\mathbb{R}^n$ ("equatorial disk"), then $H^{\partial\Sigma} = n-1$ and $k = 1$, and we get that $\sigma_1 = H^{\partial\Sigma} + k$. For this reason, we believe that $\sigma_1 \geq H^{\partial\Sigma} + k$ is the sharp estimate. Consequently, the hypothesis in Theorem 4 that $\partial\Sigma$ has constant mean curvature becomes natural to assume.

Combining the inequality (4) with our Theorem 4 we deduce the following corollary.

Corollary 6. *Let* M^{n+1} *be an* $(n+1)$*-dimensional compact orientable Riemannian manifold with nonnegative Ricci curvature and nonempty boundary* ∂M. *Assume the second fundamental form of* ∂M *satisfies* $A^{\partial M}(v, v) \geq k > 0$, *for any unit vector* v *tangent to* ∂M.

Let Σ *be a properly embedded minimal hypersurface in* M *with free boundary on* ∂M. *Assume* $\partial\Sigma$ *has constant mean curvature* $H^{\partial\Sigma}$. *Then*

$$|\partial\Sigma| \geq \left(H^{\partial\Sigma} + \frac{k}{2}\right)|\Sigma|.$$

2. Preliminaries

In this section we will collect some basic results that are essential to deduce Theorem 4. Let M^{n+1} be a $(n+1)$-dimensional compact Riemannian manifold with nonempty boundary ∂M. Denote by $\langle \cdot , \cdot \rangle$ the metric on M and D the Riemannian connection on M. We define the second fundamental form of the boundary ∂M with respect to the outward unit normal μ by $A^{\partial M}(u, v) = \langle D_u \mu, v\rangle$, where u, v are tangent to ∂M. The mean curvature $H^{\partial M}$ of ∂M is then defined as the trace of $A^{\partial M}$, i.e.,

$$H^{\partial M} = \sum_{j=1}^{n} A^{\partial M}(e_j, e_j),$$

where $e_1, \ldots, e_n$ is any orthonormal basis for $T\partial M$.

The following, known as Reilly's formula, was settled in [Fraser and Li 2014, Lemma 2.6]; see also [Choi and Wang 1983].

Proposition 7 [Fraser and Li 2014]. *Let* Ω *be a compact* $(n+1)$*-manifold with piecewise smooth boundary* $\partial\Omega = \bigcup \sum_{i=1}^{k} \Sigma_i$. *Suppose* f *is a continuous function* *on* Ω *where* $f \in C^\infty(\Omega \setminus S)$, $S = \bigcup \sum_{i=1}^{k} \partial \Sigma_i$, *and there exists some* $C > 0$ *such that* $\|f\|_{C^3(\Omega')} \leq C$ *for all* $\Omega' \subset \Omega \setminus S$. *Then, Reilly's formula holds*:

$$
(5) \quad 0 = \int_\Omega \mathrm{Ric}^\Omega(Df, Df) - (\Delta_\Omega f)^2 + \|\mathrm{Hess}_\Omega f\|^2 \\
+ \sum_{i=1}^{k} \int_{\Sigma_i} \Big[\Big(\Delta_{\Sigma_i} f + H^{\Sigma_i} \frac{\partial f}{\partial \eta_i} \Big) \frac{\partial f}{\partial \eta_i} - \Big\langle \nabla^{\Sigma_i} f, \nabla^{\Sigma_i} \frac{\partial f}{\partial \eta_i} \Big\rangle + h^{\Sigma_i}(\nabla^{\Sigma_i} f, \nabla^{\Sigma_i} f) \Big].
$$

Here, Ric^Ω *is the Ricci tensor of* Ω; Δ_Ω, Hess_Ω *and* ∇_Ω *are the Laplacian*, *Hessian and gradient operators on* Ω, *respectively*; Δ_{Σ_i} *and* ∇^{Σ_i} *are the Laplacian and gradient operators on each* Σ_i, *respectively*; η_i *is the outward unit normal of* Σ_i; H^{Σ_i} *and* h^{Σ_i} *are the mean curvature and second fundamental form of* Σ_i *in* Ω *with respect to the outward unit normal*, *respectively.*

To prove our main result we need a few considerations. Let $\varphi : \Sigma \to M$ be a properly embedded minimal hypersurface with free boundary in a compact orientable manifold M. Assume that ∂M is strictly convex and M has nonnegative Ricci curvature. Under these assumptions, ∂M is connected [Fraser and Li 2014, Proposition 2.8], and any properly embedded minimal hypersurface in M with free boundary is connected [Fraser and Li 2014, Lemma 2.5]. Furthermore, if both Σ and M are orientable then $M \setminus \varphi(\Sigma)$ consists of two components Ω_1 and Ω_2 (see [Fraser and Li 2014, Corollary 2.10]). Take $\Omega = \Omega_1$. Let $\partial\Omega = \Sigma \cup \Gamma$ where $\Gamma \subset \partial M$. Thus, $\partial\Sigma = \partial\Gamma$. Note that Γ is not necessarily connected, but each component of Γ must intersect Σ along some component of $\partial\Sigma$. Otherwise, ∂M would have more than one component, a contradiction.

Remark 8. From a result due to M. C. Li [2011, Theorem 1.1.8], any compact Riemannian 3-manifold M with nonempty boundary ∂M admits a nontrivial compact embedded minimal surface Σ with free boundary. Some examples of free boundary submanifolds in the unit ball are given in [Fraser and Schoen 2013].

3. Proof of the results

3.1. *Proof of Proposition 1.*

Proof. Let $\xi : B^{n+1} \to \mathbb{R}$ be defined by $\xi(x) = 1 - \|x\|^2$. As can be easily seen

$$
\xi_{|\partial\Sigma} = 0 \quad \text{and} \quad \Delta_\Sigma \xi(x) = -2\big(n + H^\Sigma \langle x, N(x)\rangle\big),
$$

where N is a unit vector field normal to Σ^n in B^{n+1}. Thus,

$$
(\Delta_\Sigma \xi)^2 \leq 4n^2 \bigg(1 + \frac{|H^\Sigma|}{n}\bigg)^2.
$$

On the other hand, if ν_Σ is the outward unit conormal along $\partial\Sigma$ and x_i are the coordinate functions, the condition

$$\frac{\partial x_i}{\partial \nu_\Sigma} = x_i$$

is equivalent to $\nu_\Sigma = x$, which is equivalent to the condition that Σ meets ∂B^n orthogonally. Then, Σ meets ∂B^n orthogonally if and only if

$$\frac{\partial \xi}{\partial \nu_\Sigma} = -2.$$

Now, using the variational characterization of σ_1 we get

$$\sigma_1 \cdot |\partial\Sigma| \leq n^2\left(1+\frac{|H^\Sigma|}{n}\right)^2 \cdot |\Sigma|,$$

and applying inequality (4) we conclude that

$$\sigma_1 \leq n + |H^\Sigma|. \qquad \square$$

3.2. *Proof of Theorem 2.*

Proof. Again let us consider the function $\xi : B^{n+1} \to \mathbb{R}$ defined by $\xi(x) = 1 - \|x\|^2$. Since Σ is minimal, it follows from the proof of Proposition 1 that $\Delta_\Sigma \xi = -2n$. Thus

$$\begin{cases} \Delta^2_\Sigma \xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta_\Sigma \xi = n \frac{\partial \xi}{\partial \nu_\Sigma} & \text{on } \partial\Sigma, \end{cases}$$

which implies that n is an eigenvalue. Now we will show that $\sigma_1 = n$.

It is known (see Theorem 1 in [Berchio et al. 2006]) that the infimum in (2) is achieved and that, up to a multiplicative constant, the minimizer is unique, smooth up to the boundary, positive in Σ, and the normal derivative relative to the outward unit normal is negative on $\partial\Sigma$. Arguing as in the proof of Lemma 2.2 in [Ferrero et al. 2005] we conclude that $\sigma_1 = n$. $\square$

3.3. *Proof of Theorem 4.*

Proof. Firstly suppose that Σ is orientable. Since M is orientable we have Σ is connected and $M \setminus \varphi(\Sigma)$ consists of two components Ω_1 and Ω_2 (see [Fraser and Li 2014, Corollaries 2.5 and 2.10]). Let $\Omega = \Omega_1$ and $\partial\Omega = \Sigma \cup \Gamma$, where $\Gamma \subset \partial M$, so that $\partial\Sigma = \partial\Gamma$.

Let $\xi \in C^\infty(\Sigma)$ be an eigenfunction corresponding to the first eigenvalue σ_1 of the fourth-order Steklov problem, that is,

$$\begin{cases} \Delta^2_\Sigma \xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta_\Sigma \xi = \sigma_1 \frac{\partial \xi}{\partial \nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{6}$$

where ν_Σ is the outward conormal vector of $\partial\Sigma$ with respect to Σ. Next, we consider the Dirichlet–Neumann boundary value problem on the compact $(n+1)$-manifold Ω with piecewise smooth boundary $\partial\Omega = \Sigma \cup \Gamma$

$$\begin{cases} \Delta_\Omega f = 0 & \text{in } \Omega, \\ f = \Delta_\Sigma \xi & \text{on } \Sigma, \\ \frac{\partial f}{\partial \eta_\Gamma} = (\sigma_1 - H^{\partial\Sigma}) f & \text{on } \Gamma. \end{cases} \tag{7}$$

Analyzing the relationship between the first eigenvalues of problems (1) and (3) it is possible to conclude that $\sigma_1 > H^{\partial\Sigma}$. To ensure the existence of a solution for problem (7), we will consider the homogeneous problem

$$\begin{cases} \Delta_\Omega f = 0 & \text{in } \Omega, \\ f = 0 & \text{on } \Sigma, \\ \frac{\partial f}{\partial \eta_\Gamma} = \mu f & \text{on } \Gamma. \end{cases} \tag{8}$$

This mixed Steklov–Dirichlet problem has a discrete spectrum $\{\mu_i\}$ (see [Guo and Xia 2019, Section 2]) where

$$0 < \mu_1 \le \mu_2 \le \cdots \to +\infty.$$

Next, we will establish a lower bound for μ_1. Consider f_1 an eigenfunction associated with μ_1 and assume without loss of generality that $\int_\Sigma h^\Sigma(\nabla^\Sigma f_1, \nabla^\Sigma f_1) \ge 0$ (otherwise, we choose $\Omega = \Omega_2$ instead). We get by Reilly's formula (5) applied to f_1

$$0 \ge nk \int_\Gamma \left(\frac{\partial f_1}{\partial \eta_\Gamma}\right)^2 + \int_\Gamma \Delta_\Gamma f \frac{\partial f_1}{\partial \eta_\Gamma} - \int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle + k \int_\Gamma |\nabla^\Gamma f_1|^2,$$

where η_Σ and η_Γ are the outward unit normals of Σ and Γ, respectively, with respect to Ω. Integrating by parts we get

$$\int_\Gamma \Delta_\Gamma f_1 \frac{\partial f_1}{\partial \eta_\Gamma} = -\int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle + \int_{\partial\Gamma} \frac{\partial f_1}{\partial \nu_\Gamma} \frac{\partial f_1}{\partial \eta_\Gamma},$$

where ν_Σ and ν_Γ are the outward conormal vectors of $\partial\Sigma = \partial\Gamma$ with respect to Σ and Γ, respectively. Since Σ meets Γ orthogonally along $\partial\Sigma = \partial\Gamma$, we have $\nu_\Sigma = \eta_\Gamma$ and $\eta_\Sigma = \nu_\Gamma$ along the common boundary $\partial\Sigma$. Thereby

$$0 = \int_{\partial\Sigma} \frac{\partial f_1}{\partial \nu_\Sigma} \frac{\partial f_1}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f_1}{\partial \nu_\Sigma} \frac{\partial f_1}{\partial \nu_\Gamma} = \int_{\partial\Gamma} \frac{\partial f_1}{\partial \nu_\Gamma} \frac{\partial f_1}{\partial \eta_\Gamma},$$

which implies

$$2 \int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle \ge nk \int_\Gamma \left(\frac{\partial f_1}{\partial \eta_\Gamma}\right)^2 + k \int_\Gamma |\nabla^\Gamma f_1|^2.$$

We conclude that $\mu_1 \ge \frac{k}{2}$. Having proved this fact, we will make an analysis divided into two cases. Namely, if there is $i \in \mathbb{N}$ such that $\sigma_1 - H^{\partial\Sigma} = \mu_i \ge \mu_1$ we get $\sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}$. Otherwise, $\sigma_1 - H^{\partial\Sigma} \neq \mu_i$ for all $i \in \mathbb{N}$. So, the homogeneous problem (8) has only the trivial solution, and it follows from standard elliptic PDE theory, more specifically from the Fredholm alternative, that the problem (7) has a unique solution f. Note that $\Delta_\Sigma(f|_\Sigma) = \Delta_\Sigma^2 \xi = 0$ in Σ, and assuming without loss of generality that $\int_\Sigma h^\Sigma(\nabla^\Sigma f, \nabla^\Sigma f) \geq 0$, by substituting this function f in formula (5) we obtain

$$0 \geq -\int_\Sigma \left\langle \nabla^\Sigma f, \nabla^\Sigma \frac{\partial f}{\partial \eta_\Sigma} \right\rangle + nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + \int_\Gamma \Delta_\Gamma f \frac{\partial f}{\partial \eta_\Gamma} - \int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + k \int_\Gamma |\nabla^\Gamma f|^2.$$

Now, using that

$$\int_\Sigma \left\langle \nabla^\Sigma f, \nabla^\Sigma \frac{\partial f}{\partial \eta_\Sigma} \right\rangle = -\int_\Sigma \frac{\partial f}{\partial \eta_\Sigma} \Delta_\Sigma f + \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma}$$

and

$$\int_\Gamma \Delta_\Gamma f \frac{\partial f}{\partial \eta_\Gamma} = -\int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma},$$

we have

$$0 \geq -\int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} + \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma} - 2\int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + k \int_\Gamma |\nabla^\Gamma f|^2.$$

As we saw previously,

$$\int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \nu_\Gamma} = \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma}.$$

Therefore,

$$2\int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle \geq nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + k \int_\Gamma |\nabla^\Gamma f|^2.$$

Now, using the last equality in (7) we get

$$2(\sigma_1 - H^{\partial\Sigma}) \geq k \implies \sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}.$$

This proves the theorem when Σ is orientable. In the case when Σ nonorientable and $\pi_1(M)$ finite, we can argue as in [Fraser and Li 2014, Theorem 3.1]. □

3.4. *Proof of Corollary 6.*

Proof. The proof of Corollary 6 follows directly from (4) and Theorem 4. □

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Received October 25, 2022. Revised March 15, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 325, No. 1, 2023
https://doi.org/10.2140/pjm.2023.325.11

CATENOID LIMITS OF SINGLY PERIODIC MINIMAL SURFACES WITH SCHERK-TYPE ENDS

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We construct families of embedded, singly periodic minimal surfaces of any genus g in the quotient with any even number $2n > 2$ of almost parallel Scherk ends. A surface in such a family looks like n parallel planes connected by $n-1+g$ small catenoid necks. In the limit, the family converges to an n-sheeted vertical plane with $n-1+g$ singular points, termed nodes, in the quotient. For the nodes to open up into catenoid necks, their locations must satisfy a set of balance equations whose solutions are given by the roots of Stieltjes polynomials.

Introduction

The goal of this paper is to construct families of singly periodic minimal surfaces (SPMSs) of any genus in the quotient with any even number $2n > 2$ of Scherk ends (asymptotic to vertical planes). Each family is parameterized by a small positive real number $\tau > 0$. In the limit $\tau \to 0$, the Scherk ends tend to be parallel, and the surface converges to an n-sheeted vertical plane with singular points termed nodes. As τ increases, the nodes open up into catenoid necks, and the surface looks like parallel planes connected by these catenoid necks.

There are many previously known examples of such SPMSs. Scherk [1835] discovered examples with genus zero and four Scherk ends. Karcher [1988] generalized Scherk's surface with any even number $2n > 2$ of Scherk ends. In this paper, examples of genus zero will be called "Karcher–Scherk saddle towers" or simply "saddle towers", and saddle towers with four Scherk ends will be called "Scherk saddle towers". Karcher also added handles between adjacent pairs of ends, producing SPMSs of genus n with $2n$ Scherk ends. Traizet glued Scherk saddle towers into SPMSs of genus $(n^2-3n+2)/2$ with $2n > 2$ Scherk ends because he was desingularizing simple arrangements of $n > 1$ vertical planes. Martín and Ramos Batista [2006] replaced the ends of Costa's surface by Scherk ends, thereby constructing an embedded SPMS of genus one with six Scherk ends and, for the first

MSC2020: 53A10.
Keywords: minimal surfaces, saddle towers, node opening.

time, without any horizontal symmetry plane. Hauswirth, Morabito, and Rodríguez [Hauswirth et al. 2009] generalized this result, using an end-to-end gluing method to replace the ends of Costa–Hoffman–Meeks surfaces by Scherk ends, thereby constructing SPMSs of higher genus with six Scherk ends. Da Silva and Ramos Batista [2010] constructed an SPMS of genus two with eight Scherk ends based on Costa's surface. Also, Yucra Hancco, Lobos, and Ramos Batista [Yucra Hancco et al. 2014] constructed SPMSs with genus $2n$ and $2n$ Scherk ends.

The examples of da Silva and Ramos Batista as well as all examples of Traizet admit catenoid limits that can be constructed using techniques in the present paper.

One motivation of this work is an ongoing project to address various technical details in the gluing constructions.

Roughly speaking, given any "graph" G that embeds in the plane and minimizes the length functional, one could desingularize $G \times \mathbb{R}$ into an SPMS by placing a saddle tower at each vertex. Previously, this was only proved for simple graphs under the assumption of a horizontal reflection plane [Traizet 1996; 2001]. Recently, we managed to allow the graph to have parallel edges, to remove the horizontal reflection plane by Dehn twist [Chen and Traizet 2021], and to prove embeddedness by analyzing the bendings of Scherk ends [Chen 2021].

However, we still require that the vertices of G are neither "degenerate" nor "special". Here, a vertex of degree $2k$ is said to be *degenerate* (resp. *special*) if k (resp. $k-1$) of its adjacent edges extend in the same direction while the other k (resp. $k-1$) edges extend in the opposite direction. This limitation is due to the fact that a saddle tower with $2k$ Scherk ends cannot have $k-1$ ends extending in the same direction while the other $k-1$ ends extend in the opposite direction. Therefore, it is not possible to place a saddle tower at a degenerate or special vertex.

Nevertheless, we do know SPMSs that desingularize $G \times \mathbb{R}$ where G is a graph with a degenerate vertex. To include these in the gluing construction, we need to place catenoid limits of saddle towers, as those constructed in this paper, at degenerate vertices. From this point of view, the present paper can be seen as preparatory: the insight gained here will help us to glue saddle towers with catenoid limits of saddle towers in a future project.

This paper reproduces the main result of the thesis of Li [2012]. Technically, the construction implemented in [Li 2012] was in the spirit of [Traizet 2002b], which defines the Gauss map and the Riemann surface at the same time, and the period of the surface was assumed horizontal. Here, for the convenience of future applications, we present a construction in the spirit of [Traizet 2008; Chen and Traizet 2021; Chen 2021], which defines all three Weierstrass integrands by prescribing their periods, and the period of the surface is assumed vertical. In particular, we will reveal that a balance condition in [Li 2012] is actually a disguise of the balance of Scherk ends: the unit vectors in the directions of the ends sum up to zero.

1. Main result

1.1. ***Configuration.*** We consider $L+1$ vertical planes, $L \geq 1$, labeled by integers $l \in [1, L+1]$. Up to horizontal rotations, we assume that these planes are all parallel to the xz-plane, which we identify as the complex plane $\mathbb{C}$, with the x-axis (resp. z-axis) corresponding to the real (resp. imaginary) axis. We use the term “layer” for the space between two adjacent parallel planes. So there are L layers.

We want $n_l \geq 1$ catenoid necks on layer l, i.e., between the planes l and $l+1$, $1 \leq l \leq L$. For convenience, we adopt the convention that $n_l = 0$ if $l < 1$ or $l > L$, and write $N = \sum n_l$ for the total number of necks. Each neck is labeled by a pair (l, k), where $1 \leq l \leq L$ and $1 \leq k \leq n_l$.

To each neck is associated a complex number $q_{l,k} \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $1 \leq l \leq L$, $1 \leq k \leq n_l$. Then the positions of the necks are prescribed at $\ln q_{l,k} + 2m\pi\mathrm{i}$, $m \in \mathbb{Z}$. Recall that the z-axis is identified as the imaginary axis of the complex plane $\mathbb{C}$, so the necks are periodic with period vector $(0, 0, 2\pi)$. Note that, if we multiply $q_{l,k}$'s by the same complex factor c, then the necks are all translated by $\ln c \pmod{2\pi\mathrm{i}}$. So we may quotient out translations by fixing $q_{1,1} = 1$.

Also, each plane has two ends asymptotic to vertical planes. We label the end of plane l that expands in the $-x$ (resp. x) direction by 0_l (resp. ∞_l). To be compatible with the language of graph theory that were used for gluing saddle towers [Chen and Traizet 2021], we use

$$\mathsf{H} = \{\eta_l : 1 \leq l \leq L+1,\ \eta \in \{0, \infty\}\}$$

to denote the set of ends. When 0_l is used as subscript for parameter x, we write $x_{l,0}$ instead of x_{0_l} to ease the notation; the same applies to ∞_l.

To each end is associated a real number $\dot\theta_h$, $h \in \mathsf{H}$. They prescribe infinitesimal changes of the directions of the ends. More precisely, for small τ, we want the unit vector in the direction of the end h to have a y-component of order $\tau\dot\theta_h + \mathcal{O}(\tau^2)$.

Remark 1. Multiplying $\dot\theta$ by a common real constant leads to a reparameterization of the family. Adding a common real constant to $\dot\theta_{l,0}$ and subtracting the same constant from $\dot\theta_{l,\infty}$ leads to horizontal rotations of the surface.

In the following, a *configuration* refers to the pair $(q, \dot\theta)$, where

$$q = (q_{l,k})_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant n_l}} \quad \text{and} \quad \dot\theta = (\dot\theta_h)_{h \in \mathsf{H}}.$$

1.2. ***Force.*** Given a configuration $(q, \dot\theta)$, let c_l be the real numbers that solve

$$-n_l c_l + n_{l-1} c_{l-1} + \dot\theta_{l,0} + \dot\theta_{l,\infty} = 0, \quad 1 \leq l \leq L+1. \tag{1}$$

Recall the convention $n_l = 0$ if $l < 1$ or $l > L$, so we also adopt the convention $c_l = 0$ if $l < 1$ or $l > L$. A summation over l yields

$$\Theta_1 = \sum_{h \in \mathsf{H}} \dot\theta_h = 0. \tag{2}$$

If (2) is satisfied, the real numbers c_l are determined by (1) as functions of $\dot\theta$.

For $1 \le l \le L+1$, let ψ_l be the meromorphic 1-form on the Riemann sphere $\hat{\mathbb{C}}$ with simple poles at $q_{l,k}$ with residue $-c_l$ for each $1 \le k \le n_l$, at $q_{l-1,k}$ with residue c_{l-1} for each $1 \le k \le n_{l-1}$, at 0 with residue $\dot\theta_{l,0}$, and at ∞ with residue $\dot\theta_{l,\infty}$. More explicitly,

$$\psi_l = \left(\sum_{k=1}^{n_l} \frac{-c_l}{z - q_{l,k}} + \sum_{k=1}^{n_{l-1}} \frac{c_{l-1}}{z - q_{l-1,k}} + \frac{\dot\theta_{l,0}}{z} \right) dz.$$

We then see that (1) arises from the residue theorem.

Remark 2. In the definition of configuration, we may replace $\dot\theta$ by the parameters $(c_l, \dot\theta_{l+1,0} - \dot\theta_{l,0})_{1 \le l \le L}$. Then $\dot\theta_{l,0}$'s are defined up to an additive constant (corresponding to a rotation), $\dot\theta_{l,\infty}$'s are determined by (1), and (2) is automatically satisfied. To quotient out reparameterizations of the family, we may assume that $c_l = 1$ for some $1 \le l \le L$.

We define the force $F_{l,k}$ by

$$F_{l,k} = \operatorname{Res}\left(\frac{\psi_l^2 + \psi_{l+1}^2}{2} \frac{z}{dz}, q_{l,k} \right). \tag{3}$$

Or, more explicitly,

$$\begin{aligned} F_{l,k} = \sum_{1 \le k \ne j \le n_l} \frac{2c_l^2 q_{l,k}}{q_{l,k} - q_{l,j}} &- \sum_{1 \le j \le n_{l+1}} \frac{c_l c_{l+1} q_{l,k}}{q_{l,k} - q_{l+1,j}} \\ &- \sum_{1 \le j \le n_{l-1}} \frac{c_l c_{l-1} q_{l,k}}{q_{l,k} - q_{l-1,j}} + c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0}). \end{aligned} \tag{4}$$

In [Li 2012], the force had two different formulas depending on the parity of l. One verifies that both are equivalent to (4).

Remark 3 (electrostatic interpretation). The force equation (4) can be expressed as

$$\begin{aligned} F_{l,k} = \sum_{1 \le k \ne j \le n_l} \frac{c_l^2 (q_{l,k} + q_{l,j})}{q_{l,k} - q_{l,j}} &- \sum_{1 \le j \le n_{l+1}} \frac{c_l c_{l+1} (q_{l,k} + q_{l+1,j})}{2(q_{l,k} - q_{l+1,j})} \\ - \sum_{1 \le j \le n_{l-1}} \frac{c_l c_{l-1} (q_{l,k} + q_{l-1,j})}{2(q_{l,k} - q_{l-1,j})} &+ \frac{c_l}{2} (\dot\theta_{l,\infty} - \dot\theta_{l,0} - \dot\theta_{l+1,\infty} + \dot\theta_{l+1,0}). \end{aligned}$$

Note that

$$\frac{a+b}{a-b} = \coth\frac{\ln a - \ln b}{2} = \frac{2}{\ln a - \ln b} + \sum_{m=1}^{\infty}\left(\frac{2}{\ln a - \ln b - 2m\pi \mathrm{i}} + \frac{2}{\ln a - \ln b + 2m\pi \mathrm{i}}\right).$$

Disregarding absolute convergence, we write this formally as

$$\frac{a+b}{a-b} = \sum_{m\in\mathbb{Z}} \frac{2}{\ln a - \ln b - 2m\pi \mathrm{i}}.$$

Then the force is given, formally, by

$$\begin{aligned} F_{l,k} = \sum_{0\neq m\in\mathbb{Z}} \frac{2c_l^2}{2m\pi \mathrm{i}} + \sum_{\substack{m\in\mathbb{Z}\\ 1\le k\neq j\le n_l}} \frac{2c_l^2}{\ln q_{l,k} - \ln q_{l,j} - 2m\pi \mathrm{i}} \\ - \sum_{\substack{m\in\mathbb{Z}\\ 1\le j\le n_{l+1}}} \frac{c_l c_{l+1}}{\ln q_{l,k} - \ln q_{l+1.j} - 2m\pi \mathrm{i}} - \sum_{\substack{m\in\mathbb{Z}\\ 1\le j\le n_{l-1}}} \frac{c_l c_{l-1}}{\ln q_{l,k} - \ln q_{l-1.j} - 2m\pi \mathrm{i}} \\ + \frac{c_l}{2}(\dot\theta_{l,\infty} - \dot\theta_{l,0} - \dot\theta_{l+1,\infty} + \dot\theta_{l+1,0}). \end{aligned}$$

Recall that $\ln q_{l,k} + 2m\pi \mathrm{i}$ are the real positions of the necks. So this formal expression has an electrostatic interpretation similar to those in [Traizet 2002b; 2008]. Here, each neck interacts not only with all other necks in the same or adjacent layers, but also with background constant fields given by $\dot\theta$.

Remark 4 (another electrostatic interpretation). In fact, $(4)/q_{l,k}$ has a similar electrostatic interpretation. But this time, the necks are seen as placed at $q_{l,k}$. Each neck interacts with all other necks in the same and adjacent layers, as well as a virtual neck at 0 with "charge" $c_l + \dot\theta_{l+1,0} - \dot\theta_{l,0}$. This is no surprise, as electrostatic laws are known to be preserved under conformal mappings (such as $\ln z$).

1.3. *Main result.* In the following, we write $F = (F_{l,k})_{1\le l\le L, 1\le k\le n_l}$.

Definition 5. The configuration is *balanced* if $F = 0$ and $\Theta_1 = 0$.

Summing up all forces yields a necessary condition for the configuration to be balanced, namely

$$\begin{aligned} \Theta_2 = \sum_{\substack{1\le l\le L\\ 1\le k\le n_l}} F_{l,k} &= \sum_{1\le l\le L+1} -\frac{1}{2}\left(\mathrm{Res}\left(\frac{z\psi_l^2}{dz}, 0\right) + \mathrm{Res}\left(\frac{z\psi_l^2}{dz}, \infty\right)\right) \\ &= \sum_{1\le l\le L+1} \frac{\dot\theta_{l,\infty}^2 - \dot\theta_{l,0}^2}{2} = 0. \end{aligned}$$

Lemma 6. *The Jacobian matrix $\partial(\Theta_1, \Theta_2)/\partial\dot\theta$ has real rank 2 as long as $c_l \neq 0$ for some $1 \le l \le L$.*

The assumption of the lemma simply says that the surface does not remain a degenerate plane to the first order.

Proof. The proposition says that the matrix has an invertible minor of size 2×2. Explicitly, we have

$$\frac{\partial(\Theta_1,\Theta_2)}{\partial(\dot\theta_{l,0},\dot\theta_{l,\infty})}=\begin{pmatrix}1 & 1\\ -\dot\theta_{l,0} & \dot\theta_{l,\infty}\end{pmatrix}.$$

This minor is invertible if and only if $\dot\theta_{l,0}+\dot\theta_{l,\infty}$ does not equal 0. This must be the case for at least one $1\le l\le L$ because, otherwise, we have $c_l=0$ for all $1\le l\le L$. □

Definition 7. The configuration is *rigid* if the complex rank of $\partial F/\partial q$ is $N-1$.

Remark 8. In fact, the complex rank of $\partial F/\partial q$ is at most $N-1$. We have seen that a complex scaling of q corresponds to a translation of $\ln q_{l,k}+2m\pi\mathrm{i}$, $m\in\mathbb{Z}$, which does not change the force. It then makes sense to normalize q by fixing $q_{1,1}=1$.

Theorem 9. *Let $(q,\dot\theta)$ be a balanced and rigid configuration such that $c_l\ne 0$ for $1\le l\le L$. Then for $\tau>0$ sufficiently small, there exists a smooth family M_τ of complete singly periodic minimal surfaces of genus $g=N-L$, period $(0,0,2\pi)$, and $2(L+1)$ Scherk ends such that, as $\tau\to 0$:*

- *M_τ converges to an $(L+1)$-sheeted xz-plane with singular points at*

$$\ln q_{l,k}+2m\pi\mathrm{i},\quad m\in\mathbb{Z}.$$

  Here, the xz-plane is identified as the complex plane $\mathbb{C}$, with the x-axis (resp. z-axis) identified as the real (resp. imaginary) axis.
- *After suitable scaling and translation, each singular point opens up into a neck that converges to a catenoid.*
- *The unit vector in the direction of each Scherk end h has the y-component $\tau\dot\theta_h+\mathcal{O}(\tau^2)$.*

Also, M_τ is embedded if

$$\dot\theta_{1,0}>\cdots>\dot\theta_{L+1,0}\quad\text{and}\quad\dot\theta_{1,\infty}>\cdots>\dot\theta_{L+1,\infty}.\tag{5}$$

Remark 10. The family M_τ also depends smoothly on $\dot\theta$ belonging to the local smooth manifold defined by $\Theta_1=0$ and $\Theta_2=0$. Up to reparameterizations of the family and horizontal rotations, we obtain families parameterized by $2L-1$ parameters. Since we have $2(L+1)$ Scherk ends, this parameter count is compatible with the fact that Karcher–Scherk saddle towers with $2k$ ends form a family parameterized by $2k-3$ parameters.

Remark 11. If the embeddedness condition (5) is satisfied and $\Theta_1 = 0$, the sequence $\dot\theta_{l,0} + \dot\theta_{l,\infty}$ is strictly monotonically decreasing, and changes sign once and only once. Then the sequence $n_l c_l$ is strictly concave (that is, $n_{l-1}c_{l-1} + n_{l+1}c_{l+1} < 2n_l c_l$ for $1 \le l \le L$). Hence c_l, $1 \le l \le L$, are strictly positive, and the condition of Lemma 6 is satisfied.

Remark 12. We could allow some c_l to be negative, with the price of losing embeddedness. Even worse, with negative c_l, the vertical planes in the limit will not be geometrically ordered as they are labeled. For instance, if $L = 2$, $c_1 > 0$, but $c_2 < 0$, then the catenoid necks, as well as the first and third "planes", will all lie on the same side of the second "plane".

Remark 13. We did not allow any c_l to be 0 in Theorem 9. Otherwise, the surface might still have nodes. In that case, the claimed family might not be smooth, and the claimed genus would be incorrect.

2. Examples

2.1. ***Surfaces of genus zero.*** When the genus satisfies $g = N - L = 0$, we have $n_l = 1$ for all $1 \le l \le L$, i.e., there is only one neck on every layer. It then makes sense to drop the subscript k. For instance, the position and the force for the neck on layer l are simply denoted by q_l and F_l, respectively. We assume $L > 1$ in this part.

In this case, if $\Theta_1 = 0$, (1) can be explicitly solved by

$$c_l = \sum_{i=1}^{l} (\dot\theta_{i,0} + \dot\theta_{i,\infty}), \quad 1 \le l \le L,$$

and the force can be written in the form

$$F_l = -\widetilde{Q}_l + \widetilde{Q}_{l-1} + c_l(\dot\theta_{l,\infty} + \dot\theta_{l+1,0}), \quad 1 \le l \le L,$$

where we changed to the parameters

$$\widetilde{Q}_l = \frac{c_{l+1}c_l}{1 - q_{l+1}/q_l}, \quad 1 \le l < L,$$

with the convention that $\widetilde{Q}_0 = \widetilde{Q}_L = 0$. Then the forces are linear in $\widetilde{Q}$ and, if $\Theta_2 = 0$, the balance condition $F = 0$ is uniquely solved by

$$\widetilde{Q}_l = \sum_{i=1}^{l} c_i(\dot\theta_{i+1,0} + \dot\theta_{i,\infty}) = -\sum_{i=l+1}^{L} c_i(\dot\theta_{i+1,0} + \dot\theta_{i,\infty}), \quad 1 \le l < L. \tag{6}$$

Therefore, if we fix $q_1 = 1$, all other q_l, $1 < l \le L$, are uniquely determined.

Recall from Remark 11 that, under the embeddedness condition (5), the numbers c_l, $1 \le l \le L$, are positive. Furthermore, the summands in (6) change sign at most once, so the sequence $\widetilde{Q}$ is unimodal, i.e., there exists $1 \le l' < L$ such that

$$0 = \widetilde{Q}_0 \le \widetilde{Q}_1 \le \cdots \le \widetilde{Q}_{l'} \ge \cdots \ge \widetilde{Q}_{L-1} \ge \widetilde{Q}_L = 0.$$

Hence $\widetilde{Q}_l$, $1 \le l \le L$, are nonnegative. Lastly,

$$\begin{aligned}
\widetilde{Q}_l &< \sum_{i=1}^{l} c_i(\dot{\theta}_{i,0} + \dot{\theta}_{i,\infty}) = \sum_{i=1}^{l}(c_i^2 - c_{i-1}c_i) \le c_l^2 \le c_{l+1}c_l && \text{if } l < l',\\
\widetilde{Q}_l &< -\sum_{i=l+1}^{L} c_i(\dot{\theta}_{i+1,0} + \dot{\theta}_{i+1,\infty}) = \sum_{i=l+1}^{L}(c_i^2 - c_{i+1}c_i) \le c_{l+1}^2 \le c_{l+1}c_l && \text{if } l \ge l'.
\end{aligned}$$

So q consists of real numbers and $q_{l+1}/q_l < 0$ for all $1 \le l < L$.

We have proved the following:

Proposition 14. *If the genus satisfies $g = N - L = 0$, and $\dot{\theta}$ satisfies the balancing condition $\Theta_1 = \Theta_2 = 0$ as well as the embeddedness condition* (5), *then up to complex scalings, there exist unique values for the parameters q, depending analytically on $\dot{\theta}$, such that the configuration $(q, \dot{\theta})$ is balanced. All such configurations are rigid. If we fix $q_1 = 1$, then q consist of real numbers, and we have $q_l > 0$* (*resp.* < 0) *if l is odd* (*resp. even*).

2.2. ***Surfaces with four ends.*** When $L = 1$, $\Theta_1 = \Theta_2 = 0$ implies that

$$\dot{\theta}_{1,0} + \dot{\theta}_{2,\infty} = \dot{\theta}_{2,0} + \dot{\theta}_{1,\infty} = 0.$$

Up to reparameterizations of the family, we may assume that $c_1 = 1$. It makes sense to drop the subscript l, and write F_k for $F_{1,k}$, q_k for $q_{1,k}$, and n for n_1. The goal of this part is to prove the following classification result.

Proposition 15. *Up to a complex scaling, a configuration with $L = 1$ and n nodes must be given by $q_k = \exp(2\pi \mathrm{i} k/n)$, and such a configuration is rigid.*

Such a configuration is an n-covering of the configuration for Scherk saddle towers. As a consequence, the arising minimal surfaces are n-coverings of Scherk saddle towers. This is compatible with the result of [Meeks and Wolf 2007] that the Scherk saddle towers are the only connected SPMSs with four Scherk ends.

Proof. To find the positions q_k such that

$$F_k = \sum_{1 \le k \ne j \le n} \frac{2q_k}{q_k - q_j} - (n-1) = 0, \quad 1 \le k \le n, \tag{7}$$

we use the polynomial method. Consider the polynomial

$$P(z) = \prod_{k=1}^{n}(z - q_k).$$

Then we have

$$\begin{aligned}
P' &= P\sum_{k=1}^{n}\frac{1}{z-q_k},\\
P'' &= P\sum_{k=1}^{n}\sum_{1\le k\ne j\le n}\frac{1}{z-q_j}\frac{1}{z-q_k} = 2P\sum_{k=1}^{n}\frac{1}{z-q_k}\sum_{1\le k\ne j\le n}\frac{1}{q_k-q_j}\\
&= P\sum_{k=1}^{n}\frac{n-1}{q_k(z-q_k)} = (n-1)P\sum_{k=1}^{n}\frac{1}{z}\left(\frac{1}{q_k}+\frac{1}{z-q_k}\right) \quad \text{(by (7))}\\
&= \frac{n-1}{z}\left(P' - \frac{P'(0)}{P(0)}P\right).
\end{aligned}$$

For the last equation to have a polynomial solution, we must have $P'(0) = 0$. Otherwise, the left-hand side would be a polynomial of degree $n-2$, but the right-hand side would be a polynomial of degree $n-1$.

Consequently, $F_k = 0$ if and only if

$$zP''(z) - (n-1)P'(z) = 0,$$

which, up to a complex scaling, is uniquely solved by

$$P(z) = z^n - 1.$$

So a balanced 4-end configuration must be given by the roots of unity $q_k = \exp(2\pi\mathrm{i}k/n)$, $0 \le k \le n-1$.

We now verify that the configuration is rigid. For this purpose, we compute

$$\frac{\partial F_k}{\partial q_j} = \begin{cases} 2\frac{q_k}{(q_k-q_j)^2}, & j \ne k,\\ 2\sum_{1\le k\ne i\le n}\frac{-q_i}{(q_k-q_i)^2}, & j = k. \end{cases}$$

Note that $\sum_{j=1}^{n} q_j \partial F_k/\partial q_j = 0$ while

$$q_j\frac{\partial F_k}{\partial q_j} = 2\frac{q_j q_k}{(q_k-q_j)^2} = 2\frac{e^{2\pi\mathrm{i}\frac{j+k}{n}}}{(e^{2\pi\mathrm{i}\frac{j}{n}} - e^{2\pi\mathrm{i}\frac{k}{n}})^2} \in \mathbb{R}_{<0}$$

when $j \ne k$, so the matrix

$$\frac{\partial F}{\partial q}\,\mathrm{diag}(q_1, \dots, q_n)$$

has real entries, has a kernel of complex dimension 1 (spanned by the all-one vector), and any of its principal submatrices are diagonally dominant. We then conclude that the matrix, as well as the Jacobian $\partial F/\partial q$, has a complex rank $n-1$. This finishes the proof of rigidity. □

Remark 16. The perturbation argument as in the proof of [Traizet 2002b, Proposition 1] also applies here, word by word, to prove the rigidity.

2.3. ***Gluing two saddle towers of different periods.*** We want to construct a smooth family of configurations depending on a positive real number λ such that, for small λ, the configuration looks like two columns of nodes far away from each other, one with period $2\pi/n_1$, and the other with period $2\pi/n_2$. If balanced and rigid, these configurations would give rise to minimal surfaces that look like two Scherk saddle towers with different periods that are glued along a pair of ends. The construction is in the same spirit as [Traizet 2002b, §2.5; 2008, §4.3.4].

Proposition 17. *For a real number $\lambda > 0$ sufficiently small, there are balanced and rigid configurations $(q(\lambda), \dot\theta(\lambda))$ with $L = 2$ depending smoothly on λ such that, at $\lambda = 0$,*

$$\frac{q_{2,j}}{q_{1,k}} = 0, \quad 1 \le k \le n_1, 1 \le j \le n_2.$$

Up to a complex scaling and reparameterization, we may fix $q_{1,1} = 1$, and write $q_{2,1} = \lambda \exp(\mathrm{i}\phi)$. Then, at $\lambda = 0$, we have

$$\dot\theta_{1,0} + \dot\theta_{2,\infty} = \dot\theta_{2,0} + \dot\theta_{3,\infty} = \dot\theta_{3,0} + \dot\theta_{1,\infty} = 0 \tag{8}$$

and,

$$q_{1,k} = \exp\left(\frac{k-1}{n_1} 2\pi \mathrm{i}\right), \quad 1 \le k \le n_1,$$
$$\widetilde{q}_{2,k} := q_{2,k}/q_{2,1} = \exp\left(\frac{k-1}{n_2} 2\pi \mathrm{i}\right), \quad 1 \le k \le n_2,$$

where $\phi\, \mathrm{lcm}(n_1, n_2)$ is necessarily a multiple of π.

In other words, the construction only works if the configuration admits a reflection symmetry.

Remark 18. H. Chen was shown a video suggesting that, when two Scherk saddle towers are glued into a minimal surface, one can slide one saddle tower with respect to the other while the surface remains minimal. The proposition above suggests that this is not possible.

In fact, the family of configurations also depends on $\dot\theta$ belonging to the local manifold defined by $\Theta_1 = \Theta_2 = 0$ and (one equation from) (8). Up to rotations of the configuration and reparameterizations of the family of minimal surfaces, the family of configurations is parameterized, as expected, by two parameters.

Proof. Let us first study the situation at $\lambda = 0$. We compute, at $\lambda = 0$,

$$\begin{aligned}\frac{F_{1,k}}{c_1^2} &= \sum_{1\le k\ne j\le n_1} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - \sum_{1\le j\le n_2} \frac{c_2}{c_1}\frac{q_{1,k}}{q_{1,k}-q_{2,j}} + 1 + \frac{\dot\theta_{2,0}-\dot\theta_{1,0}}{c_1}\\ &= \sum_{1\le k\ne j\le n_1} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - n_2\frac{c_2}{c_1} + 1 + \frac{\dot\theta_{2,0}-\dot\theta_{1,0}}{c_1},\\ \frac{F_{2,k}}{c_2^2} &= \sum_{1\le k\ne j\le n_2} \frac{2q_{2,k}}{q_{2,k}-q_{2,j}} - \sum_{1\le j\le n_1} \frac{c_1}{c_2}\frac{q_{2,k}}{q_{2,k}-q_{1,j}} + 1 + \frac{\dot\theta_{3,0}-\dot\theta_{2,0}}{c_2}\\ &= \sum_{1\le k\ne j\le n_2} \frac{2q_{2,k}}{q_{2,k}-q_{2,j}} + 1 + \frac{\dot\theta_{3,0}-\dot\theta_{2,0}}{c_2}.\end{aligned}$$

Write $G_l = \sum_k F_{l,k}$. Summing the above over k gives, at $\lambda = 0$,

$$\frac{1}{n_1}\frac{G_1}{c_1^2} = n_1 - n_2\frac{c_2}{c_1} + \frac{\dot\theta_{2,0}-\dot\theta_{1,0}}{c_1}, \qquad \frac{1}{n_2}\frac{G_2}{c_2^2} = n_2 + \frac{\dot\theta_{3,0}-\dot\theta_{2,0}}{c_2}.$$

So $G_1 = G_2 = 0$, at $\lambda = 0$, only if

$$\begin{aligned}0 &= -(\dot\theta_{2,0}+\dot\theta_{3,\infty}) = \dot\theta_{3,0}-\dot\theta_{2,0}+n_2c_2\\ &= -(\dot\theta_{1,0}+\dot\theta_{2,\infty}) = n_1c_1 - n_2c_2 + \dot\theta_{2,0}-\dot\theta_{1,0}.\end{aligned}$$

This together with $\Theta_1 = 0$ proves (8).

Now assume that (8) is satisfied. Then we have, at $\lambda = 0$,

$$\begin{aligned}\frac{F_{1,k}}{c_1^2} &= \sum_{1\le k\ne j\le n_1} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - (n_1-1),\\ \frac{F_{2,k}}{c_2^2} &= \sum_{1\le k\ne j\le n_2} \frac{2q_{2,k}}{q_{2,k}-q_{2,j}} - (n_2-1).\end{aligned}$$

These expressions are identical to the force (7) for single layer configurations. So we know for $l = 1, 2$ that, at $\lambda = 0$, the configuration is balanced only if

$$\widetilde{q}_{l,k} := \frac{q_{l,k}}{q_{l,1}} = \exp\left(\frac{k-1}{n_l}2\pi \mathrm{i}\right).$$

Up to complex scaling, we may fix $q_{1,1} = 1$ so $\widetilde{q}_{1,k} = q_{1,k}$. And up to reparameterization of the family (of configurations), we write $q_{2,1} = \lambda\exp(\mathrm{i}\phi)$.

Now assume these initial values for $\widetilde{q}_{l,k}$. Then we have, at $\lambda = 0$,

$$\frac{G_2}{c_1 c_2} = -\sum_{k=1}^{n_2}\sum_{j=1}^{n_1} \frac{q_{2,k}}{q_{2,k} - q_{1,j}} = \sum_{k=1}^{n_2}\sum_{j=1}^{n_1}\sum_{m=1}^{\infty} \left(\frac{q_{2,k}}{q_{1,j}}\right)^m$$
$$= \sum_{k=1}^{n_2}\sum_{j=1}^{n_1}\sum_{m=1}^{\infty} q_{2,1}^m \exp\left(2mi\pi\left(\frac{k-1}{n_2} - \frac{j-1}{n_1}\right)\right).$$

Seen as a power series of $q_{2,1}$, the coefficient for $q_{2,1}^m$ is

$$\sum_{k=1}^{n_2}\sum_{j=1}^{n_1} \exp\left(2mi\pi\left(\frac{k-1}{n_2} - \frac{j-1}{n_1}\right)\right).$$

It is nonzero only if m is a common multiple of n_1 and n_2, in which case the coefficient of $q_{2,1}^m$ equals $n_1 n_2$. In particular, let $\mu = \mathrm{lcm}(n_1, n_2)$; then, at $\lambda = 0$,

$$\mathrm{Im}\, \frac{G_2}{\lambda^\mu} = c_1 c_2 n_1 n_2 \sin(\mu\phi) \tag{9}$$

vanishes if and only if $\mu\phi$ is a multiple of π.

Now we use the implicit function theorem to find balanced configurations with $\lambda > 0$. From the proof for Proposition 15, we know that $(\partial F_{l,k}/\partial \widetilde{q}_{l,j})_{2\le j,k\le n_l}$, $l = 1, 2$, are invertible. Hence for λ sufficiently small, there exist unique values for $(\widetilde{q}_{l,k})_{l=1,2;2\le k\le n_l}$, depending smoothly on λ, $\dot\theta$, and ϕ, where $(F_{l,k})_{l=1,2;2\le k\le n_l} = 0$. By (9), there exists a unique value for ϕ, depending smoothly on λ and $\dot\theta$, such that $\mathrm{Im}\, G_2/\lambda^\mu = 0$. Note also that $\mathrm{Re}\, G_2$ is linear in $\dot\theta$. By Lemma 6, the solutions $(\lambda, \dot\theta)$ to $\mathrm{Re}\, G_2 = 0$ and $\Theta_1 = \Theta_2 = 0$ form a manifold of dimension 4 (including multiplication by common real factor on $\dot\theta$ and rotation of the configuration). Finally, we have $G_1 = 0$ by the residue theorem, and the balance is proved.

For the rigidity of the configurations with sufficiently small λ, we need to prove that the matrix

$$\begin{pmatrix} \left(\frac{\partial F_{1,k}}{\partial q_{1,j}}\right)_{2\le j,k\le n_1} & & \\ & \left(\frac{\partial F_{2,k}}{\partial \widetilde{q}_{2,j}}\right)_{2\le j,k\le n_2} & \\ & & \frac{\partial G_2}{\partial q_{2,1}} \end{pmatrix}$$

is invertible. We know that the first two blocks are invertible at $\lambda = 0$. By continuity, they remain invertible for λ sufficiently small. The last block is clearly nonzero for $\lambda \ne 0$ sufficiently small. □

2.4. *Surfaces with six ends of type (n, 1).* In this section, we investigate examples with $L = 2$ (hence six ends), $n_1 = n$, $n_2 = 1$. Up to a reparameterization of the family, we may assume that $c_1 = 1$. Up to a complex scaling, we may assume that $q_{2,1} = 1$.

We will prove that the $q_{1,k}$'s are given by the roots of hypergeometric polynomials. Let us first recall their definitions. A *hypergeometric function* is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} $$

with $a, b, c \in \mathbb{C}$, c is not a nonpositive integer,

$$ (a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, $$

and $(a)_0 = 1$. The hypergeometric function $w = {}_2F_1(a,b;c;z)$ solves the *hypergeometric differential equation*

$$ z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0. \tag{10} $$

If $a = -n$ is a negative integer,

$$ {}_2F_1(-n,b;c;z) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(b)_k}{(c)_k} z^k $$

is a polynomial of degree n, and is referred to as a *hypergeometric polynomial.*

Proposition 19. *Let $(q,\dot\theta)$ be a balanced configuration with $L=2$, $c_1=1$, $n_1=n$, $n_2=1$. Then, up to a complex scaling, we have $q_{2,1}=1$ and $(q_{1,k})_{1\le k\le n}$ are the roots of the hypergeometric polynomial ${}_2F_1(-n,b;c;z)$ with*

$$ b := n - c_2 + \dot\theta_{2,0} - \dot\theta_{1,0}, \quad c := 1 + \dot\theta_{2,0} - \dot\theta_{1,0}. $$

As long as b and c are not nonpositive integers, and $c-b$ is not a nonpositive integer bigger than $-n$, the configuration is rigid.

Proof. The force equations are

$$ F_{1,k} = \sum_{1\le k\ne j\le n} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - \frac{q_{1,k}c_2}{q_{1,k}-1} + c, \quad 1 \le k \le n, $$

$$ F_{2,1} = -\sum_{j=1}^{n} \frac{c_2}{1-q_{1,j}} + c_2^2 + c_2(\dot\theta_{3,0} - \dot\theta_{2,0}), $$

where $c := 1 + \dot\theta_{2,0} - \dot\theta_{1,0}$. To solve $F_{1,k} = 0$ for $k = 1, 2, \ldots, n$, we use again the polynomial method. Let

$$ P(z) = \prod_{k=1}^{n} (z - q_{1,k}). $$

Then we have

$$\begin{aligned} P' &= P\sum_{k=1}^{n}\frac{1}{z-q_{1,k}}; \\ P'' &= 2P\sum_{k=1}^{n}\frac{1}{z-q_{1,k}}\sum_{1\le k\neq j\le n}\frac{1}{q_{1,k}-q_{1,j}} \\ &= P\sum_{k=1}^{n}\frac{1}{(z-q_{1,k})}\left(\frac{c_2}{q_{1,k}-1}-\frac{c}{q_{1,k}}\right) \quad (\text{by } F_{1,k}=0) \\ &= P\sum_{k=1}^{n}\left(\frac{c_2}{(z-1)(z-q_{1,k})}+\frac{c_2}{(z-1)(q_{1,k}-1)}-\frac{c}{z(z-q_{1,k})}-\frac{c}{zq_{1,k}}\right). \end{aligned}$$

So the configuration is balanced if and only if

$$P''+\left(\frac{-c_2}{z-1}+\frac{c}{z}\right)P'+\left(\frac{c_2}{z-1}\frac{P'(1)}{P(1)}-\frac{c}{z}\frac{P'(0)}{P(0)}\right)P=0. \tag{11}$$

Define

$$b := n-1-c_2+c.$$

For (11) to have a polynomial solution of degree n, we must have

$$c_2\frac{P'(1)}{P(1)} = c\frac{P'(0)}{P(0)} = -nb,$$

so that the leading coefficients cancel. Then (11) becomes the hypergeometric differential equation

$$z(1-z)P''+[c-(-n+b+1)z]P'+nbP=0$$

to which the only polynomial solution (up to a multiplicative constant) is given by the hypergeometric polynomial $P(z) = {}_2F_1(-n,b;c;z)$ of degree n.

Furthermore, in order for $F_{2,1}=0$, we must have

$$\dot\theta_{3,0}-\dot\theta_{2,0} = \sum_{j=1}^{n}\frac{1}{1-q_{1,j}}-c_2 = \frac{P'(1)}{P(1)}-c_2 = -\frac{nb}{c_2}-c_2. \tag{12}$$

Note that b and c are real. If b is not a nonpositive integer, and $c-b$ is not a nonpositive integer bigger than $-n$, then all the n roots of $P(z) = {}_2F_1(-n,b;c;z)$ are simple. Indeed, under these assumptions, we have $P(0)=1$ and $P(1) = (c-b)_n/(c)_n \neq 0$ by the Chu–Vandermonde identity. If z_0 is a root of $P(z)$, then $z_0 \neq 0, 1$. In view of the hypergeometric differential equation, if z_0 is not simple, we have $P(z_0)=P'(z_0)=0$; hence $P(z)\equiv 0$ by the uniqueness theorem.

The rigidity means that no perturbation of $q_{1,k}$ preserve the balance to the first order. To prove this fact, we use a perturbation argument similar to that in the proof of [Traizet 2002b, Proposition 1].

Let $(q_{1,k}(t))_{1\le k\le n}$ be a deformation of the configuration such that $q_{1,k}(0)=q_{1,k}$ and $(\dot{F}_{1,k}(0))_{1\le k\le n}=0$, where dot denotes derivative with respect to t. Define

$$P_t(z)=\sum_{j=0}^{n} a_j(t)z^j := \prod_{k=1}^{n}(z-q_{1,k}(t)).$$

Then we have

$$z(1-z)P_t''+[c-(-n+b+1)z]P_t'+nbP_t=o(t),$$

meaning that the coefficients from the left side are all $o(t)$. So the coefficients of P_t must satisfy

$$(b+j)(n-j)a_j(t)+(j^2+j+cj)a_{j+1}(t)=o(t),\quad 0\le j\le n. \tag{13}$$

Note that $P_t(z)$ is monic by definition, meaning that $a_n(t)\equiv 1$. Since b and c are not nonpositive integers, we conclude that $a_j(t)=o(t)$ for all $0\le j\le n$. The simple roots depend analytically on the coefficients, so $q_{1,k}(t)=q_{1,k}+o(t)$. □

The simple roots of ${}_2F_1(-n,b;c;z)$ are either real or form conjugate pairs. As a consequence, if rigid, the configurations in the proposition above will give rise to minimal surfaces with horizontal symmetry planes.

Example 20. For each integer $n\ge 2$, Dominici, Johnston, and Jordaan [Dominici et al. 2013] enumerated the real parameters (b, c) for which ${}_2F_1(-n,b;c;z)$ has only real simple roots. The results are plotted in blue in Figure 1. The embeddedness conditions (5) are

$$\begin{aligned}
\dot\theta_{1,0}>\dot\theta_{2,0} &\implies c<1,\\
\dot\theta_{1,\infty}>\dot\theta_{2,\infty} &\implies b>-n,\\
\dot\theta_{2,0}>\dot\theta_{3,0} &\implies c_2^2>-nb,\\
\dot\theta_{2,\infty}>\dot\theta_{3,\infty} &\implies c_2^2>n(c_2+b),
\end{aligned}$$

where $c_2=n-1-b+c$. The region defined by these is plotted in red in Figure 1. Then noninteger parameters (b, c) in the intersection of red and blue regions give rise to balanced and rigid configurations with real $q_{1,k}$.

Figure 2 shows the configurations of three examples with $n=5$. □

Remark 21. As $c\to 0$, ${}_2F_1(-n,b;c;z)/\Gamma(c)$ converges to a polynomial with a root at 0. One may interpret that, as c increases across 0, a root moves from the interval $(-\infty, 0)$ to the interval $(0, 1)$ through 0.

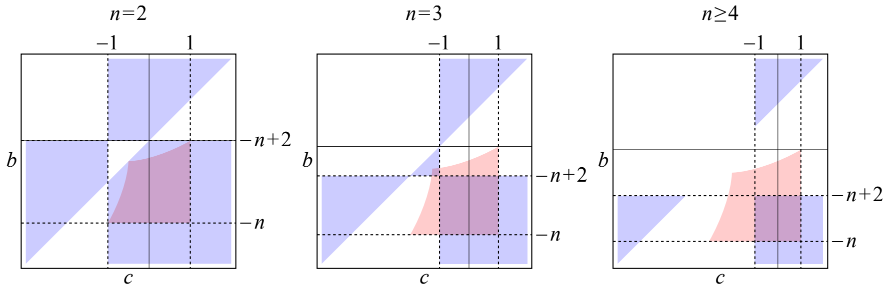

Figure 1. The set of b and c for which ${}_2F_1(-n, b; c; z)$ has only real simple roots (blue), and for which the embeddedness conditions are satisfied (red).

When $b = 1 - n$, ${}_2F_1(-n, b; c; z)$ becomes a polynomial of degree $n-1$. One may interpret that, as b increases across $1 - n$, a root moves from the interval $(-\infty, 0)$ to the interval $(1, \infty)$ through the infinity.

Example 22. Assume that $b + c = 1 - n$ (hence $c_2 = -2b$). Then by the identity

$$ {}_2F_1(-n, b; c; z) = \frac{(b)_n}{(c)_n}(-z)^n {}_2F_1\left(-n, 1 - c - n; 1 - b - n; \frac{1}{z}\right), $$

the simple roots must be symmetrically placed. That is, if z_0 is a root, so is $1/z_0$. This symmetry appears in the resulting minimal surfaces as a rotational symmetry. If the simple roots are real, the rotation reduces to a vertical reflectional. In view of Figure 1, we obtain the following concrete examples.

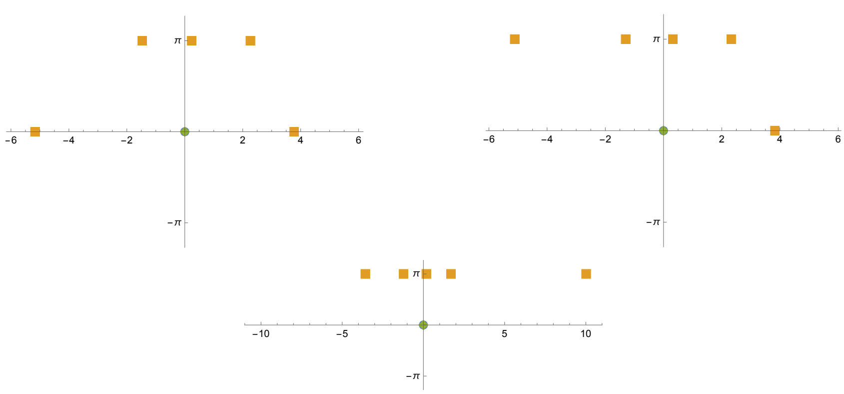

Figure 2. $(5, 1)$ balanced configurations with $b = -3.4$, $c = -0.1$ (top left), $b = -3.4$, $c = 0.1$ (top right), and $b = -4.001$, $c = 0.5$ (bottom). The circles and squares represent the necks at levels one and two, respectively.

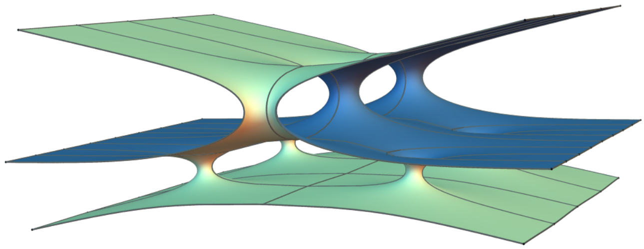

Figure 3. Genus one example with $n = 2$ and $0 < c < 1$.

- $n \geq 2$ and $0 < c < 1$. In this case ${}_2F_1(-n, b; c; z)$ has n simple negative roots. See Figure 3 for an example of this type with $n = 2$. Figure 4 shows the configurations of two examples with $n = 5$.
- $n \geq 3$ and $-1 < c < 0$, or $n = 3$ and $-\frac{5}{4} < c < -1$, or $n = 2$ and $-\frac{1}{2} < c < 0$. In these cases, ${}_2F_1(-n, b; c; z)$ has $n - 2$ simple negative roots, one root $0 < z_0 < 1$, and another root $1/z_0 > 1$. Figure 5 shows the configurations of two examples with $n = 5$. □

Remark 23. Examples with six Scherk ends are parameterized by three real parameters, here by b, c, and the family parameter τ. We see that the relation $b+c = 1-n$ imposes a rotational symmetry. It can be imagined that removing the relation would break this symmetry.

Remark 24. The polynomial method is often used to find balanced configurations of interacting points in the plane. In minimal surface theory, it has been employed in many implementations of Traizet's node-opening technique [Traizet 2002a; 2002b; Traizet and Weber 2005; Li 2012; Connor and Weber 2012; Connor 2017a; 2017b; Chen and Freese 2022].

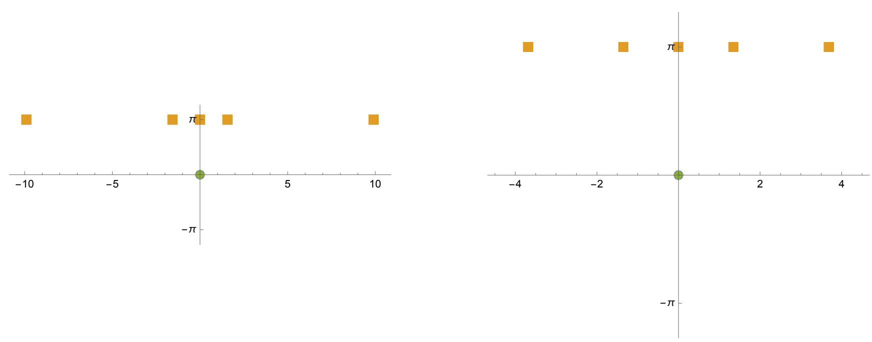

Figure 4. (5, 1) balanced configurations with $c = 0.001$ (left) and $c = 0.5$ (right). The circles and squares represent the necks at levels one and two, respectively.

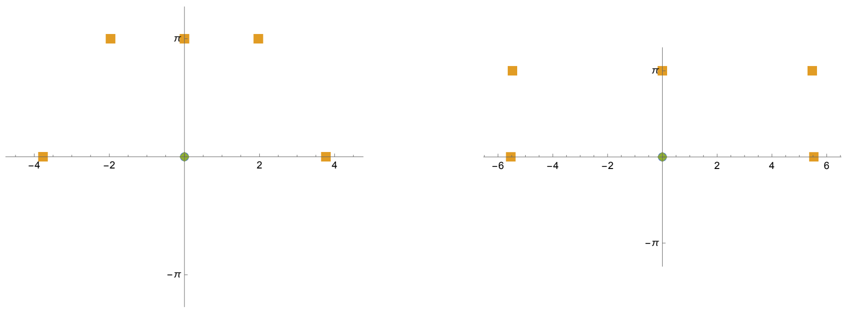

Figure 5. (5, 1) balanced configurations with $c = -0.5$ (left) and $c = -0.999$ (right). The circles and squares represent the necks at levels one and two, respectively.

2.5. ***Surfaces with eight ends of type (1, n, 1).*** Proposition 19 generalizes to the following lemma with similar proof:

Lemma 25. *We fix $q_{l\pm1,k}$'s and assume that $c_l = 1$. Then $q_{l,k}$'s in a balanced configuration are given by the roots of a Stieltjes polynomial $P(z)$ of degree n_l that solves the generalized Lamé equation (a.k.a. second-order Fuchsian equation)* [Marden 1966]

$$P'' + \left(\frac{c}{z} + \sum_{k=1}^{n_{l-1}} \frac{-c_{l-1}}{z - q_{l-1,k}} + \sum_{k=1}^{n_{l+1}} \frac{-c_{l+1}}{z - q_{l+1,k}}\right) P' + \left(\frac{\gamma_0}{z} + \sum_{k=1}^{n_{l-1}} \frac{\gamma_{l-1,k}}{z - q_{l-1,k}} + \sum_{k=1}^{n_{l+1}} \frac{\gamma_{l+1,k}}{z - q_{l+1,k}}\right) P = 0, \tag{14}$$

where $c = 1 + \dot\theta_{l+1,0} - \dot\theta_{l,0}$, subject to conditions

$$\gamma_0 + \sum_{k=1}^{n_{l-1}} \gamma_{l-1,k} + \sum_{k=1}^{n_{l+1}} \gamma_{l+1,k} = 0,$$

$$\sum_{k=1}^{n_{l-1}} \gamma_{l-1,k} q_{l-1,k} + \sum_{k=1}^{n_{l+1}} \gamma_{l+1,k} q_{l+1,k} =: -n_l b,$$

and

$$c - n_{l-1}c_{l-1} - n_{l+1}c_{l+1} = 1 - n_l + b.$$

Also, the matrix $(\partial F_{l,k}/\partial q_{l,j})_{1\le j,k\le n_l}$ is nonsingular as long as b is not a nonpositive integer bigger than n_l.

A root of $P(z)$ is simple if and only if it does not coincide with 0 or any $q_{l\pm1,k}$. If the roots $(q_{l,k})$ of $P(z)$ are all simple, then they solve the equations [Marden 1966]

$$\sum_{1\le k\ne j\le n_l} \frac{2}{q_{l,k} - q_{l,j}} + \sum_{1\le j\le n_{l+1}} \frac{-c_{l+1}}{q_{l,k} - q_{l+1,j}} + \sum_{1\le j\le n_{l-1}} \frac{-c_{l-1}}{q_{l,k} - q_{l-1,j}} + \frac{c}{q_{l,k}} = \frac{F_{l,k}}{q_{l,k}} = 0,$$

which is exactly our balance condition; see Remark 4. In addition, an equation system generalizing (13) has been obtained in [Heine 1878, §136], from which we may conclude the nonsingularity of the Jacobian. In fact, there are

$$\binom{n_{l-1}+n_l+n_{l+1}-1}{n_{l-1}+n_{l+1}-1}$$

choices of γ for which (14) has a polynomial solution of degree n_l [Heine 1878, §135].

This observation allows us to easily construct balanced and rigid configurations of type $(1, n, 1)$. Up to reparametrizations and complex scalings, we may assume that $c_2 = 1$ and $q_{1,1} = 1$. Then $q_{3,1}$ must be real, and $(q_{2,k})_{1\le k\le n}$ are given by roots of a Heun polynomial. Such a configuration depends locally on four real parameters, namely $q_{3,1}$, c_1, c_3 and c (or b). When these are given, we have $n+1$ Heun polynomials, each of which gives balanced positions of $q_{2,k}$'s. For each of the Heun polynomials P, we have

$$\begin{aligned}
\dot\theta_{2,0}-\dot\theta_{1,0} &= \frac{P'(1)}{P(1)}-c_1,\\
\dot\theta_{3,0}-\dot\theta_{2,0} &= c-1 = b+c_1+c_3-n,\\
\dot\theta_{4,0}-\dot\theta_{3,0} &= \frac{P'(q_{3,1})}{P(q_{3,1})}-c_3.
\end{aligned}$$

Together with the family parameter τ, the surface depends locally on five parameters, which is expected because there are eight ends.

Example 26 (symmetric examples). When $q_{3,1}=q_{1,1}=1$, the Heun polynomial reduces to a hypergeometric polynomial ${}_2F_1(-n, b; c; z)$, where $c_1+c_3=n-1-b+c$. Assume further that $b+c=1-n$, so $c_1+c_3=-2b$. This imposes a symmetry in the configuration. Because $(c_1+c_3)P'(1)/P(1)=-nb$, the embeddedness conditions simplify to

$$c_1 > \tfrac12 n,\quad c_3 > \tfrac12 n,\quad 1-\tfrac12 n < c < 1,\quad -n < b < -\tfrac12 n.$$

As explained in Example 22, the hypergeometric polynomial has real roots if b and c lie in the blue regions of Figure 1. More specifically:

- When $n\ge 2$ and $0<c<1$, ${}_2F_1(-n, b; c; z)$ has n simple negative roots. See Figure 6 for an example with $n=5$.
- When $n\ge 4$ and $-1<c<0$, or $n=3$ and $-\frac12<c<0$, ${}_2F_1(-n, b; c; z)$ has $n-2$ simple negative roots, one root $0<z_0<1$, and another root $1/z_0>1$. □

Example 27 (offset handles). There are embedded examples in which the handles are not symmetrically placed. For instance, one balanced configuration of type

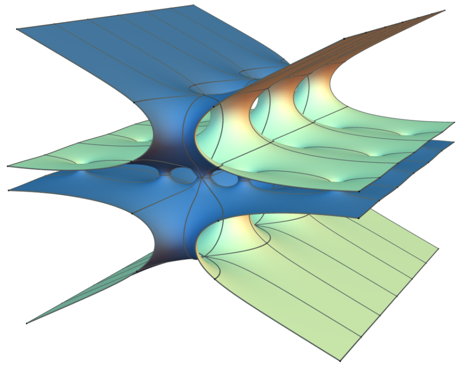

Figure 6. Genus four example with $n = 5$ and $0 < c < 1$.

(1, 2, 1) is given by

$$q_{3,1} = \tfrac{2}{3}, \quad q_{2,1} = -\tfrac{1}{3}, \quad q_{2,2} = -23,$$
$$c_1 = \tfrac{8}{5}, \quad c_3 = \tfrac{4189}{2890}, \quad b = -\tfrac{9857}{8670},$$

so $c = \frac{3956}{4335}$. □

2.6. ***Concatenating surfaces of type (1, n, 1).*** We describe a family of examples in the same spirit as [Traizet 2002a, Proposition 2.3]. Assume that we are in possession of R configurations of type $(1, n^{(r)}, 1)$, $n^{(r)} > 1$, $1 \le r \le R$. In the following, we use superscript (r) to denote the parameters of the r-th configuration. Up to reparameterizations and complex scalings, we may assume that $c_2^{(r)} = 1$ and $q_{1,1}^{(r)} = 1$. Then we may concatenate these configurations into one of type

$$(1, n_2, 1, n_4, 1, \ldots, 1, n_{2R}, 1)$$

such that $q_{1,1} = 1$, $c_1 = 1$, and for $1 \le r \le R$, we have $n_{2r} = n^{(r)}$,

$$q_{2r,k} = q_{2r-1,1} q_{2,k}^{(r)}, \quad q_{2r+1,1} = q_{2r-1,1} q_{3,1}^{(r)}, \quad c_{2r} = \frac{c_{2r-1}}{c_1^{(r)}}, \quad c_{2r+1} = c_{2r-1} \frac{c_3^{(r)}}{c_1^{(r)}},$$

and

$$\dot\theta_{2r+1,0} - \dot\theta_{2r,0} = c_{2r}(c^{(r)} - 1).$$

The balance of even layers then follows from the balance of each subconfiguration. The balance of odd layers leads to

$$\dot\theta_{2r,0} - \dot\theta_{2r-1,0} = c_{2r}(\dot\theta_{2,0}^{(r)} - \dot\theta_{1,0}^{(r)} + c_1^{(r)}) + c_{2r-2}(\dot\theta_{4,0}^{(r-1)} - \dot\theta_{3,0}^{(r-1)} + c_3^{(r-1)}) - c_{2r-1}$$

for $1 \le r \le R + 1$. As expected, such a configuration depends locally on $4R$ real parameters, namely $q_{3,1}^{(r)}$, $c_1^{(r)}$, $c_3^{(r)}$, and $c^{(r)}$, $1 \le r \le R$.

We may impose symmetry by assuming that $q^{(r)}_{3,1} = 1$, so $q_{2r+1,1} = 1$ for all $0 \le r \le R$, and that $b^{(r)}+c^{(r)} = 1-n^{(r)}$, so $c^{(r)}_1+c^{(r)}_3 = n^{(r)}-1-b^{(r)}+c^{(r)} = -2b^{(r)}$. Then $q_{2r,k} = q^{(r)}_{2,k}$, $1 \le k \le n^{(r)}$, are given by the roots of ${}_2F_1(-n^{(r)}, b^{(r)}; c^{(r)}; z)$, $1 \le r \le R$. Recall from Remark 11 that the embeddedness conditions simplifies to the concavity of the sequence $(n_l c_l)_{1\le l\le L}$. For even l, the concavity implies that $b^{(r)} > -n$; hence $c^{(r)} < 1$ for all $1 \le r \le R$. We may choose, for instance, $n_l c_l = \ln(1 + l)$ or $n_l c_l = (\exp l - 1)/\exp(l - 1)$ to obtain embedded minimal surfaces.

Remark 28. We can also append a configuration of type $(1, n^{(r)})$ to the sequence of $(1, n^{(r)}, 1)$-configurations to obtain a configuration of type

$$(1, n_2, 1, n_4, 1, \dots, 1, n_{2R-2}, 1, n_{2R}),$$

where the $q_{l,k}$, c_l, $\dot\theta_{l,0}$ terms are defined as above. Therefore, an embedded example of any genus with any even number (> 2) of ends can be constructed.

2.7. ***Numerical examples.*** The balance equations can be combined into one differential equation that is much easier to solve. A solution to this differential equation corresponds to several balance configurations that are equivalent by permuting the locations of the nodes.

Lemma 29. *Let L be a positive integer, $n_1, n_2, \dots, n_L \in \mathbb{N}$, and suppose $\{q_{l,k}\}$ is a configuration such that the $q_{l,k}$ are distinct. Let*

$$P_l(z) = \prod_{k=1}^{n_l}(z - q_{l,k}), \qquad P(z) = \prod_{l=1}^{L} P_l(z), \qquad P_0(z) = P_{L+1}(z) = 1,$$

and

$$\mathcal{F}P(z) = \sum_{l=1}^{L}\biggl(\frac{c_l^2 z P_l''(z)P(z)}{P_l(z)} - \frac{c_l c_{l+1} z P_l'(z)P_{l+1}'(z)P(z)}{P_l(z)P_{l+1}(z)} + (c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0}))\frac{P_l'(z)P(z)}{P_l(z)}\biggr).$$

Then the configuration $\{q_{l,k}\}$ is balanced if and only if $\mathcal{F}P(z) \equiv 0$.

Proof. We have seen that

$$\frac{P_l''(q_{l,k})}{P_l'(q_{l,k})} = \sum_{1\le k\ne j\le n_l}\frac{2}{q_{l,k} - q_{l,j}}, \qquad \frac{P_{l\pm1}'(q_{l,k})}{P_{l\pm1}(q_{l,k})} = \sum_{j=1}^{n_{l\pm1}}\frac{1}{q_{l,k} - q_{l\pm1,j}}.$$

Define

$$F_l(z) = \frac{c_l^2 z P_l''(z)}{P_l'(z)} - \frac{c_l c_{l+1} z P_{l+1}'(z)}{P_{l+1}(z)} - \frac{c_l c_{l-1} z P_{l-1}'(z)}{P_{l-1}(z)} + c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0}).$$

Then $F_{l,k} = F_l(q_{l,k})$. Set

$$\begin{aligned} Q_l(z) &= \frac{P_l'(z)P(z)}{P_l(z)} F_l(z) \\ &= \frac{c_l^2 z P_l''(z)P(z)}{P_l(z)} - \frac{c_l c_{l+1} z P_l'(z)P_{l+1}'(z)P(z)}{P_l(z)P_{l+1}(z)} - \frac{c_l c_{l-1} z P_{l-1}'(z)P_l'(z)P(z)}{P_{l-1}(z)P_l(z)} \\ &\qquad + (c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0})) \frac{P_l'(z)P(z)}{P_l(z)}. \end{aligned}$$

Then $F_{l,k} = 0$ if and only if $Q_l(q_{l,k}) = 0$.

Now observe that $Q_l(z)$ and $Q(z) = \mathcal{F}P(z)$ are polynomials with degree strictly less than

$$\deg P = N = \sum_{l=1}^{L} n_l,$$

and $Q(q_{l,k}) = Q_l(q_{l,k})$ for $1 \le k \le n_l$ and $1 \le l \le L$. If $Q \equiv 0$ then $Q_l(q_{l,k}) = 0$ and so $\{q_{l,k}\}$ is a balanced configuration. If $\{q_{l,k}\}$ is a balanced configuration then $Q(q_{l,k}) = Q_l(q_{l,k}) = F_{l,k} = 0$. Hence, Q has at least N distinct roots. Since the degree of Q is strictly less than N, we must have $Q \equiv 0$. $\square$

It is relatively easy to numerically solve $\mathcal{F}P(z) \equiv 0$ as long as we don't have too many levels and necks. So we use this lemma to find balanced configurations. Since all previous examples admit a horizontal reflection symmetry, we are most interested in examples without this symmetry, or with no nontrivial symmetry at all.

Figure 7 shows an example with $L = 3$,

$$\begin{gathered} n_1 = 1, \quad n_2 = 3, \quad n_3 = 2, \\ c_1 = 2, \quad c_2 = 1, \quad c_3 = \tfrac{13}{16}, \\ \theta_{1,0} = 0, \quad \theta_{2,0} = -\tfrac{1}{2}, \quad \theta_{3,0} = -\tfrac{27}{16}, \quad \theta_{4,0} = -\tfrac{29}{16}. \end{gathered}$$

This configuration corresponds to an embedded minimal surface with eight ends and genus three in the quotient. It has no horizontal reflectional symmetry, but does have a rotational symmetry.

Figure 8 shows two examples with $L = 3$,

$$\begin{gathered} n_1 = 1, \quad n_2 = 4, \quad n_3 = 3, \\ c_1 = \tfrac{7}{2}, \quad c_2 = 1, \quad c_3 = \tfrac{3}{4}, \\ \theta_{1,0} = 0, \quad \theta_{2,0} = -2, \quad \theta_{3,0} = -\tfrac{13}{5}, \quad \theta_{4,0} = -\tfrac{541}{180}. \end{gathered}$$

These configurations correspond to embedded minimal surfaces with eight ends and genus five in the quotient, with no nontrivial symmetry.

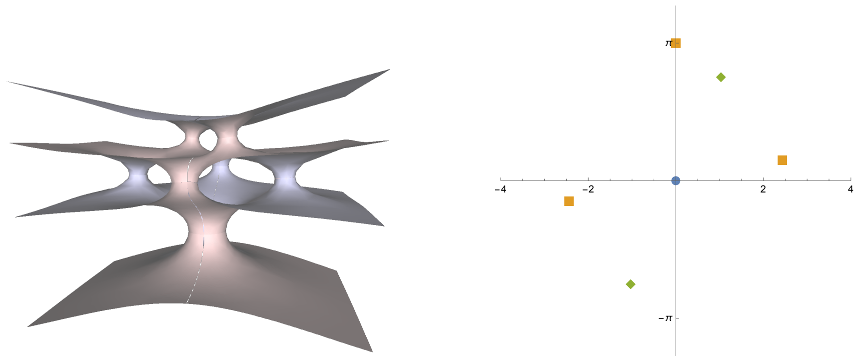

Figure 7. A $(1,3,2)$ balanced configuration with no horizontal reflectional symmetry. The circles, squares, and diamonds represent the necks at levels one, two, and three, respectively.

Figure 9 shows two examples with $L=3$,

$$\begin{gathered} n_1 = 1, \quad n_2 = 7, \quad n_3 = 3, \\ c_1 = \tfrac{17}{7}, \quad c_2 = 1, \quad c_3 = \tfrac{3}{2}, \\ \theta_{1,0} = 0, \quad \theta_{2,0} = -\tfrac{1}{2}, \quad \theta_{3,0} = -\tfrac{3}{2}, \quad \theta_{4,0} = -\tfrac{2468}{441}. \end{gathered}$$

These configurations correspond to embedded minimal surfaces with eight ends and genus eight in the quotient, with no nontrivial symmetry.

3. Construction

3.1. ***Opening nodes.*** To each vertical plane is associated a punctured complex plane $\mathbb{C}_l^\times \simeq \mathbb{C}\setminus\{0\}$, $1 \le l \le L+1$. They can be seen as Riemann spheres $\hat{\mathbb{C}}_l \simeq \mathbb{C}\cup\{\infty\}$ with two fixed punctures at $p_{l,0} = 0$ and $p_{l,\infty} = \infty$, corresponding to the two ends.

To each neck is associated a puncture $p^\circ_{l,k} \in \mathbb{C}_l^\times$ and a puncture $p'^\circ_{l,k} \in \mathbb{C}_{l+1}^\times$. Our initial surface at $\tau = 0$ is the noded Riemann surface Σ_0 obtained by identifying $p^\circ_{l,k}$ and $p'^\circ_{l,k}$ for $1 \le l \le L$ and $1 \le k \le n_l$.

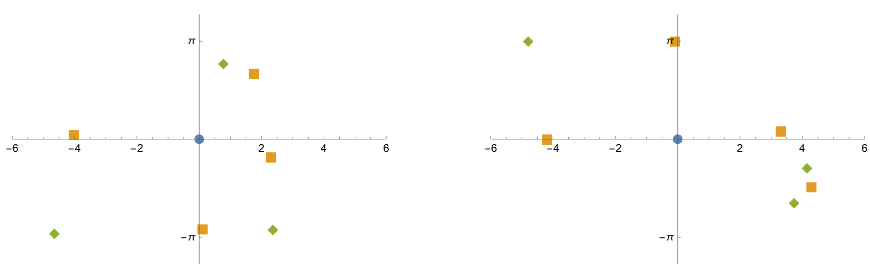

Figure 8. $(1,4,3)$ balanced configurations with no symmetries. The circles, squares, and diamonds represent the necks at levels one, two, and three, respectively.

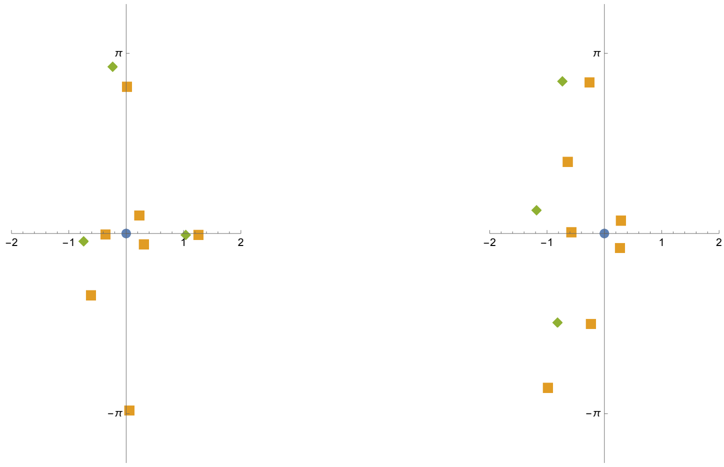

Figure 9. $(1, 7, 3)$ balanced configurations with no symmetries. The circles, squares, and diamonds represent the necks at levels one, two, and three, respectively.

As τ increases, we open the nodes into necks as follows. Fix local coordinates $w_{l,0} = z$ in the neighborhood of $0 \in \hat{\mathbb{C}}_l$ and $w_{l,\infty} = 1/z$ in the neighborhood of $\infty \in \hat{\mathbb{C}}_l$. For each neck, we consider parameters $(p_{l,k}, p'_{l,k})$ in the neighborhoods of $(p^\circ_{l,k}, p'^\circ_{l,k})$ and local coordinates

$$w_{l,k} = \ln \frac{z}{p_{l,k}} \quad \text{and} \quad w'_{l,k} = \ln \frac{z}{p'_{l,k}}$$

in a neighborhood of $p_{l,k}$ and $p'_{l,k}$, respectively. In this paper, the branch cut of $\ln z$ is along the negative real axis, and we use the principal value of $\ln z$ with imaginary part in the interval $(-\pi, \pi]$.

As we only open finitely many necks, we may choose $\delta > 0$ independent of k and l such that the disks

$$|w_h| < 2\delta, \quad h \in \mathsf{H}\ (= [1, L+1] \times \{0, \infty\}),$$
$$|w_{l,k}| < 2\delta \quad \text{and} \quad |w'_{l,k}| < 2\delta, \qquad 1 \le l \le L,\ 1 \le k \le n_k$$

are all disjoint. For parameters $t = (t_{l,k})_{1 \le l \le L, 1 \le k \le n_l}$ in a neighborhood of 0 with $|t_{l,k}| < \delta^2$, we remove the disks

$$|w_{l,k}| < \frac{|t_{l,k}|}{\delta} \quad \text{and} \quad |w'_{l,k}| < \frac{|t_{l,k}|}{\delta}$$

and identify the annuli

$$\frac{|t_{l,k}|}{\delta} \le |w_{l,k}| \le \delta \quad \text{and} \quad \frac{|t_{l,k}|}{\delta} \le |w'_{l,k}| \le \delta$$

by

$$w_{l,k}w'_{l,k} = t_{l,k}.$$

If $t_{l,k} \neq 0$ for all $1 \leq l \leq L$ and $1 \leq k \leq n_l$, we obtain a Riemann surface denoted by Σ_t.

3.2. ***Weierstrass data.*** We construct a conformal minimal immersion using the Weierstrass parameterization in the form

$$z \mapsto \operatorname{Re}\int^z (\Phi_1, \Phi_2, \Phi_3),$$

where Φ_i are meromorphic 1-forms on Σ_t satisfying the conformality equation

$$Q := \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0. \tag{15}$$

3.2.1. *A-periods.* We consider the following fixed domains in all Σ_t:

$$\begin{aligned} U_{l,\delta} = \big\{ z \in \hat{\mathbb{C}}_v : |w^\circ_{l,k}(z)| > \delta/2 \ \ \forall 1 \leq k \leq n_l \quad & \text{if } 1 \leq l \leq L \\ \text{and} \quad |w'^\circ_{l,k}(z)| > \delta/2 \ \ \forall 1 \leq k \leq n_{l-1} \quad & \text{if } 2 \leq l \leq L+1 \big\} \end{aligned}$$

and $U_\delta = \bigsqcup_{1 \leq l \leq L} U_{l,\delta}$.

Let $A_{l,k}$ denote a small counterclockwise circle in $U_{l,\delta}$ around $p_{l,k}$; it is then homologous in Σ_t to a clockwise circle in $U_{l+1,\delta}$ around $p'_{l,k}$. Moreover, let $A_{l,0}$ (resp. $A_{l,\infty}$) denote a small counterclockwise circle in $U_{l,\delta}$ around 0 (resp. ∞).

Recall that the vertical period vector is assumed to be $(0, 0, 2\pi)$, so we need to solve the A-period problems

$$\operatorname{Re}\int_{A_h} (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 2\pi\sigma_h) \quad \text{and} \quad \operatorname{Re}\int_{A_{l,k}} (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 0)$$

for $h \in \mathsf{H}$, $1 \leq l \leq L$, and $1 \leq k \leq n_l$. Here, the orientation $\sigma_h = \pm 1$ satisfies

$$\sigma_h = -\sigma_{\varsigma(h)},$$

where the "counterclockwise rotation" ς on H is defined by

$$\begin{cases} \varsigma(0_l) = 0_{l-1}, & 2 \leq l \leq L+1, \\ \varsigma(0_1) = \infty_1, & \\ \varsigma(\infty_l) = \infty_{l+1}, & 1 \leq l \leq L, \\ \varsigma(\infty_{L+1}) = 0_{L+1}. & \end{cases} \tag{16}$$

In particular, we have $\sigma_{l,0} = -\sigma_{l,\infty}$ for all $1 \leq l \leq L+1$.

Recall that the surface tends to an $(L+1)$-sheeted xz-plane in the limit $\tau \to 0$. So we define the meromorphic functions Φ_1, Φ_2 and Φ_3 as the unique regular 1-forms on Σ_t (see [Traizet 2013, §8]) with simple poles at p_h, $h \in \mathsf{H}$, and the A-periods

$$\int_{A_h} (\Phi_1, \widetilde{\Phi}_2, \Phi_3) = 2\pi \mathrm{i}(\alpha_h, \beta_h, \gamma_h - \mathrm{i}\sigma_h), \quad h \in \mathsf{H},$$
$$\int_{A_{l,k}} (\Phi_1, \widetilde{\Phi}_2, \Phi_3) = 2\pi \mathrm{i}(\alpha_{l,k}, \beta_{l,k}, \gamma_{l,k}), \quad 1 \le l \le L, 1 \le k \le n_l,$$

where $\Phi_2 = \tau \widetilde{\Phi}_2$ and, by the residue theorem, it is necessary that

$$\alpha_{l,0} + \alpha_{l,\infty} + \sum_{1 \le k \le n_l} \alpha_{l,k} - \sum_{1 \le k \le n_{l-1}} \alpha_{l-1,k} = 0, \tag{17}$$

$$\beta_{l,0} + \beta_{l,\infty} + \sum_{1 \le k \le n_l} \beta_{l,k} - \sum_{1 \le k \le n_{l-1}} \beta_{l-1,k} = 0, \tag{18}$$

$$\gamma_{l,0} + \gamma_{l,\infty} + \sum_{1 \le k \le n_l} \gamma_{l,k} - \sum_{1 \le k \le n_{l-1}} \gamma_{l-1,k} = 0, \tag{19}$$

for $1 \le l \le L+1$. Then the A-period problems are solved by definition.

3.2.2. *Balance of ends.* Summing up (18) over l gives

$$\sum_{h \in \mathsf{H}} \beta_h = 0, \tag{20}$$

which we use to replace (18) with $l = L+1$.

In this paper, the punctures $p_{l,0}$ and $p_{l,\infty}$ correspond to Scherk-type ends. Hence we fix

$$\alpha_h^2 + \tau^2 \beta_h^2 \equiv 1 \quad \text{and} \quad \gamma_h \equiv 0 \tag{21}$$

for all $h \in \mathsf{H}$, so that (the stereographic projection of) the Gauss map

$$G = -\frac{\Phi_1 + \mathrm{i}\Phi_2}{\Phi_3}$$

extends holomorphically to the punctures p_h with unitary values. Then (19) is not independent: if it is solved for $1 \le l \le L$, it is automatically solved for $l = L+1$.

In particular, at $\tau = 0$, we have $\alpha_h^2 = 1$. In view of the orientation of the ends, we choose $\alpha_{l,0} = 1$ and $\alpha_{l,\infty} = -1$ so that $G(p_{l,\infty}) = G(p_{l,0}) = \mathrm{i}\sigma_{l,0}$.

Summing up (17) over l gives

$$\sum_{1 \le l \le L+1} \left(\sqrt{1 - \tau^2 \beta_{l,\infty}^2} - \sqrt{1 - \tau^2 \beta_{l,0}^2} \right) = 0, \tag{22}$$

which we use to replace (17) with $l = L+1$.

Remark 30. The conditions (20) and (22) are disguises of the balance condition of Scherk ends, namely that the unit vectors in their directions should sum up to 0.

3.2.3. *B-periods.* For $1 \le l \le L+1$, we fix a point $O_l \in U_{l,\delta}$. For every $1 \le l \le L$ and $1 \le k \le n_l$ and $t_{l,k} \neq 0$, let $B_{l,k}$ be the concatenation of

(1) a path in $U_{l,\delta}$ from O_l to $w_{l,k} = \delta$,
(2) the path parameterized by $w_{l,k} = \delta^{1-2s}\, t_{l,k}^s$ for $s \in [0, 1]$, from $w_{l,k} = \delta$ to $w_{l,k} = t_h/\delta$, which is identified with $w'_{l,k} = \delta$, and
(3) a path in $U_{l+1,\delta}$ from $w'_{l,k} = \delta$ to O_{l+1}.

We need to solve the B-period problem, namely that

$$\operatorname{Re}\int_{B_{l,k}} (\Phi_1, \Phi_2, \Phi_3) = \operatorname{Re}\int_{B_{l,1}} (\Phi_1, \Phi_2, \Phi_3). \tag{23}$$

3.2.4. *Conformality.*

Lemma 31. *For t sufficiently close to* 0, *the conformality condition* (15) *is equivalent to*

$$\mathfrak{G}_{l,k} := \int_{A_{l,k}} \frac{w_{l,k} Q}{dw_{l,k}} = 0, \quad 1 \le l \le L,\ 1 \le k \le n_l, \tag{24}$$

$$\mathfrak{F}_{l,k} := \int_{A_{l,k}} \frac{Q}{dw_{l,k}} = 0, \quad 1 \le l \le L,\ 2 \le k \le n_l, \tag{25}$$

$$\mathfrak{F}'_{l,k} := \int_{A'_{l,k}} \frac{Q}{dw'_{l,k}} = 0, \quad 1 \le l \le L,\ 1+\delta_{l,L} \le k \le n_l, \tag{26}$$

where $A'_{l,k}$ in (26) *denotes a small counterclockwise circle in $U_{l+1,\delta}$ around $p'_{l,k}$* (*hence homologous to $-A_{l,k}$*), *and $\delta_{l,L} = 1$ if $l = L$ and* 0 *otherwise.*

Proof. By our choice of α_h and γ_h, the quadratic differential Q has at most simple poles at the $2L+2$ punctures p_h, $h \in \mathsf{H}$. The space of such quadratic differentials is of complex dimension $3(N-L)-3+(2L+2) = 3N-L-1$. We will prove that

$$Q \mapsto (\mathfrak{G}, \mathfrak{F}, \mathfrak{F}')$$

is an isomorphism. We prove the claim at $t = 0$; then the claim follows by continuity.

Consider Q in the kernel. Recall from [Traizet 2008] that a regular quadratic differential on Σ_0 has at most double poles at the nodes $p_{l,k}$ and $p'_{l,k}$. Then (24) guarantees that Q has at most simple poles at the nodes. By (25) and (26), Q may only have simple poles at $p_{l,1} \in \mathbb{C}_l^\times$, $1 \le l \le L$, and $p'_{L,1} \in \mathbb{C}_{L+1}^\times$. So, on each Riemann sphere $\hat{\mathbb{C}}_l$, Q is a quadratic differential with at most simple poles at three punctures; the other two being $0, \infty$. But such a quadratic differential must be 0. □

3.3. ***Using the implicit function theorem.*** All parameters vary in a neighborhood of their central values, denoted by a superscript $\circ$. We will see that

$$\beta_h^\circ = \dot{\theta}_h, \quad \alpha_{l,k}^\circ = \gamma_{l,k}^\circ = 0, \quad \beta_{l,k}^\circ = -c_l, \quad p'^\circ_{l,k} = \overline{p_{l,k}}.$$

Let us first solve (20) and (22).

Proposition 32. *Suppose we are given a configuration $(q, \dot\theta)$ such that $\Theta_1 = \Theta_2 = 0$. For τ sufficiently small and β_h close to $\beta_h^\circ = \dot\theta_h$, the solutions (τ, β) to* (20) *and* (22) *form a smooth manifold of dimension* $2L+1$.

Proof. At $\tau = 0$, (20) is solved by $\beta_h^\circ = \dot\theta_h$ if $\Theta_1 = 0$. Taking the derivative of (22) with respect to τ^2 gives

$$\sum_{1\le l\le L+1} \frac{\beta_{l,\infty}^2 - \beta_{l,0}^2}{2} = 0, \tag{27}$$

which is solved by $\beta_h^\circ = \dot\theta_h$ if $\Theta_2 = 0$. The proposition then follows from Lemma 6 and the implicit function theorem. □

From now on, we assume that the parameters $(\tau, (\beta_h)_{h\in\mathsf{H}})$ are solutions to (20) and (22) in a neighborhood of $(0, \dot\theta)$.

3.3.1. *Solving conformality problems.*

Proposition 33. *For τ sufficiently small and $\beta_{l,k}$, $p_{l,k}$, and $p'_{l,k}$ in a neighborhood of their central values, there exist unique values of $t_{l,k}$, $\alpha_{l,k}$, and $\gamma_{l,k}$, depending real-analytically on (τ^2, β, p, p'), such that the balance equations* (17) *and* (19) *with $1 \le l \le L$ and the conformality equations* (24) *and* (25) *are solved. Also, at $\tau = 0$, we have $t_{l,k} = 0$, $\alpha_{l,k} = \gamma_{l,k} = 0$,*

$$\frac{\partial t_{l,k}}{\partial(\tau^2)} = \frac{1}{4}\beta_{l,k}^2,$$

and, for $2 \le k \le n_l$,

$$\frac{\partial}{\partial(\tau^2)}(\alpha_{l,k} - \mathrm{i}\sigma_{l,0}\gamma_{l,k}) = -\frac{1}{2}\operatorname{Res}\left(\frac{\widetilde\Phi_2^2}{dw_{l,k}}, p_{l,k}\right) = -\frac{1}{2}\operatorname{Res}\left(\frac{z\widetilde\Phi_2^2}{dz}, p_{l,k}\right). \tag{28}$$

Note that, according to this proposition, if $\beta_{l,k}^\circ \ne 0$, then $t_{l,k} > 0$ for sufficiently small τ.

Proof. At $\tau = 0$, for $2 \le k \le n_l$ we have

$$\mathfrak{G}_{l,k} = \int_{A_{l,k}} \frac{w_{l,k}Q}{dw_{l,k}} = 2\pi\mathrm{i}(\alpha_{l,k}^2 + \gamma_{l,k}^2) = 0,$$

which vanishes when

$$\alpha_{l,k} = \gamma_{l,k} = 0.$$

Recall that $\alpha_h = \pm1$ at $\tau = 0$ and that $\gamma_h \equiv 0$. Then by the residue theorem, we have

$$\alpha_{l,1} = \gamma_{l,1} = 0.$$

As a consequence, we have at $\tau = 0$

$$\Phi_1^\circ = \frac{dz}{z}, \quad \Phi_2^\circ = 0, \quad \text{and} \quad \Phi_3^\circ = -\mathrm{i}\sigma_{l,0}\frac{dz}{z};$$

so $Q = 0$ as we expect.

We then compute the partial derivatives at $\tau = 0$:

$$\begin{aligned}
\frac{\partial}{\partial \alpha_{l,k}}\mathfrak{F}_{l,k} &= \int_{A_{l,k}} 2\frac{\Phi_1^\circ}{dw_{l,k}}\frac{\partial \Phi_1}{\partial \alpha_{l,k}}\bigg|_{\tau=0} = \int_{A_{l,k}} 2\frac{dz/z}{dz/p}\frac{dz}{z-p_{l,k}} = 4\pi\mathrm{i},\\
\frac{\partial}{\partial \gamma_{l,k}}\mathfrak{F}_{l,k} &= \int_{A_{l,k}} 2\frac{\Phi_3^\circ}{dw_{l,k}}\frac{\partial \Phi_3}{\partial \gamma_{l,k}}\bigg|_{\tau=0} = \int_{A_{l,k}} -2i\sigma_{l,0}\frac{dz/z}{dz/p}\frac{dz}{z-p_{l,k}} = 4\pi\sigma_{l,0},\\
\frac{\partial}{\partial t_{l,k}}\mathfrak{G}_{l,k} &= \int_{A_{l,k}} \frac{2w_{l,k}}{dw_{l,k}}\left(\Phi_1^\circ\frac{\partial \Phi_1^\circ}{\partial t_{l,k}} + \Phi_3^\circ\frac{\partial \Phi_3^\circ}{\partial t_{l,k}}\right)\\
&= \frac{-1}{\pi\mathrm{i}}\left(\int_{A_{l,k}}\frac{\Phi_1^\circ}{w_{l,k}}\int_{A'_{l,k}}\frac{\Phi_1^\circ}{w'_{l,k}} + \int_{A_{l,k}}\frac{\Phi_3^\circ}{w_{l,k}}\int_{A'_{l,k}}\frac{\Phi_3^\circ}{w'_{l,k}}\right)\\
&= -8\pi\mathrm{i},
\end{aligned}$$

where the second to last line is true by [Traizet 2008, Lemma 3]. All other partial derivatives vanish. Therefore, by the implicit function theorem, there exist unique values of $\alpha_{l,k}$, $\gamma_{l,k}$ (with $2 \le k \le n_l$), and $t_{l,k}$ (with $1 \le k \le n_l$) that solve the conformality equations (24) and (25). Recall that α_h are determined by (21). Then $\alpha_{l,1}$ and $\gamma_{l,1}$ are uniquely determined by the linear balance equations (17) and (19).

Moreover,

$$\frac{\partial}{\partial(\tau^2)}\mathfrak{F}_{l,k} = \int_{A_{l,k}}\frac{\widetilde{\Phi}_2^2}{dw_{l,k}}, \qquad \frac{\partial}{\partial(\tau^2)}\mathfrak{G}_{l,k} = 2\pi\mathrm{i}\beta_{l,k}^2.$$

Hence the total derivatives satisfy

$$\frac{d}{d(\tau^2)}\mathfrak{F}_{l,k} = 4\pi\mathrm{i}\frac{\partial \alpha_{l,k}}{\partial(\tau^2)} + 4\pi\sigma_{l,0}\frac{\partial \gamma_{l,k}}{\partial(\tau^2)} + 2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{\widetilde{\Phi}_2^2}{dw_{l,k}}, p_{l,k}\right) = 0$$

and

$$\frac{d}{d(\tau^2)}\int_{A_{l,k}}\mathfrak{G}_{l,k} = -8\pi\mathrm{i}\frac{\partial t_{l,k}}{\partial(\tau^2)} + 2\pi\mathrm{i}\beta_{l,k}^2 = 0.$$

This proves the claimed partial derivatives with respect to τ^2. □

Remark 34. We see from the computations that our local coordinates w and w' are chosen for convenience. Had we used other coordinates, the computations would be very different, but $\partial(\alpha_{l,k} - \mathrm{i}\sigma_{l,0}\gamma_{l,k})/\partial(\tau^2)$ would be invariant, and $\partial t_{l,k}/\partial(\tau^2)$ would be rescaled to keep the conformal type of Σ_t (to the first order). So the choice of local coordinates has no substantial impact on our construction.

3.3.2. *Solving B-period problems.* In the following, we make a change of variable $\tau = \exp(-1/\xi^2)$.

Proposition 35. *Let the parameters $t_{l,k}$, $\alpha_{l,k}$, and $\gamma_{l,k}$ be given by Proposition 33. For ξ sufficiently small and $p_{l,k}$ and $p'_{l,k}$ in a neighborhood of their central values, there exist unique values of $\beta_{l,k}$, depending smoothly on (ξ, p, p') and $(\beta_h)_{h\in\mathsf{H}}$, such that the balance equation* (18) *with $1 \le l \le L$ and the y-component of the B-period problem* (23) *are solved. In addition, at $\xi = 0$ and $\beta_h = \beta_h^\circ = \dot{\theta}_h$, we have $\beta_{l,k} = \beta_{l,1} = -c_l$ where c_l is given by* (1).

Proof. By Lemma 8.3 of [Chen and Traizet 2021],

$$\Big(\int_{B_{l,k}} \widetilde{\Phi}_2\Big) - \beta_{l,k} \ln t_{l,k}$$

extends holomorphically to $t = 0$ as bounded analytic functions of other parameters. We have seen that $t_{l,k} \sim \tau^2 \beta_{l,k}^2/4$. So

$$\mathfrak{H} := -\frac{\xi^2}{2}\operatorname{Re}\Big(\int_{B_{l,k}} \widetilde{\Phi}_2 - \int_{B_{l,1}} \widetilde{\Phi}_2\Big) = \beta_{l,k} - \beta_{l,1}$$

at $\xi = 0$. Therefore, $\mathfrak{H} = 0$ is solved at $\xi = 0$ by $\beta_{l,k} = \beta_{l,1}$ for all $2 \le k \le n_l$, and $\beta_{l,1} = -c_l$ follows as (1) is just a reformulation of (18). The proposition then follows by the implicit function theorem. □

Proposition 36. *Assume that the parameters $t_{l,k}$, $\alpha_{l,k}$, $\beta_{l,k}$ and $\gamma_{l,k}$ are given by Propositions 33 and 35. For ξ sufficiently small and $p_{l,k}$ in a neighborhood of their central values, there exist unique values of $p'_{l,k}$, depending smoothly on ξ, p, and $(\beta_h)_{h\in\mathsf{H}}$, such that the x- and z-components of the B-period problem* (23) *are solved. In addition, up to complex scalings on $\mathbb{C}^\times_{l+1}$, $1 \le l \le L$, we have $p'_{l,k} = \overline{p_{l,k}}$ at $\xi = 0$ for any $1 < k \le n_l$.*

Proof. At $\xi = 0$, recall that $\Phi_1 = dz/z$ and $\Phi_3 = -\mathrm{i}\sigma_{l,0}\, dz/z$. So

$$\operatorname{Re}\int_{B_{l,k}} \Phi_1 - \operatorname{Re}\int_{B_{l,1}} \Phi_1 = \operatorname{Re}\ln\frac{p_{l,k}}{p_{l,1}} - \operatorname{Re}\ln\frac{p'_{l,k}}{p'_{l,1}};$$

$$\operatorname{Re}\int_{B_{l,k}} \Phi_3 - \operatorname{Re}\int_{B_{l,1}} \Phi_3 = \sigma_{l,0}\Big(\operatorname{Im}\ln\frac{p_{l,k}}{p_{l,1}} + \operatorname{Im}\ln\frac{p'_{l,k}}{p'_{l,1}}\Big).$$

They vanish if and only if $\ln(p_{l,k}/p_{l,1}) = \overline{\ln(p'_{l,k}/p'_{l,1})}$. We normalize the complex scaling on $\mathbb{C}^\times_{l+1}$, $1 \le l \le L$, by fixing $p'_{l,1} = \overline{p_{l,1}}$. Then the B-period problem is solved at $\xi = 0$ with $p'_{l,k} = \overline{p_{l,k}}$. By the same argument as in [Traizet 2008], the integrals are smooth functions of ξ and other parameters, so the proposition follows by the implicit function theorem. □

3.3.3. *Balancing conditions.* Define

$$\mathfrak{R}_{l,k} = \operatorname{Res}\Big(\frac{z\widetilde{\Phi}_2^2}{dz}, p_{l,k}\Big) \quad \text{and} \quad \mathfrak{R}'_{l,k} = \operatorname{Res}\Big(\frac{z\widetilde{\Phi}_2^2}{dz}, p'_{l,k}\Big).$$

Let the central values $p^\circ_{l,k}$ equal $\operatorname{conj}^l q_{l,k}$, where q is from a balanced configuration. So the central values $p'^\circ_{l,k}$ equal $\operatorname{conj}^{l+1} q_{l,k}$ and

$$\widetilde{\Phi}_2^\circ = \begin{cases} \overline{\operatorname{conj}^* \psi_l} & \text{on } \mathbb{C}_l^\times \text{ for } l \text{ odd,} \\ \psi_l & \text{on } \mathbb{C}_l^\times \text{ for } l \text{ even.} \end{cases}$$

Then we have

$$\overline{\mathfrak{R}_{l,k}} + \mathfrak{R}'_{l,k} = 2 \operatorname{conj}^{l+1} F_{l,k}$$

at the central values, where $F_{l,k}$ is the force given by (4). Also, by the residue theorem on $\mathbb{C}_l^\times$,

$$\sum_{k=1}^{n_{l-1}} \mathfrak{R}'_{l-1,k} + \sum_{k=1}^{n_l} \mathfrak{R}_{l,k} + \beta_{l,0}^2 - \beta_{l,\infty}^2 = 0. \tag{29}$$

Proposition 37. *Assume that the parameters $t_{l,k}$, $\alpha_{l,k}$, and $\gamma_{l,k}$ are given as analytic functions of τ^2 by Proposition 33. Then $\widetilde{\mathfrak{F}}'_{l,k} := \tau^{-2}\mathfrak{F}'_{l,k}$ extends analytically to $\tau = 0$ with the value*

$$\begin{cases} 4\pi \mathrm{i} \operatorname{conj}^{l+1} F_{l,k}, & 2 \le k \le n_l, \\ 4\pi \mathrm{i} \operatorname{conj}^{l+1}\big(F_{l,1} + \sum_{j=1}^{l-1}\sum_{k=1}^{n_j} F_{j,k}\big), & k = 1. \end{cases}$$

Proof. If $f(z)$ is an analytic function in z and $f(0) = 0$, then $f(z)/z$ extends analytically to $z = 0$ with the value $df/dz\,|_{z=0}$. We compute at $\tau = 0$ that

$$\frac{\partial}{\partial\alpha}\mathfrak{F}'_{l,k} = -4\pi\mathrm{i} \quad \text{and} \quad \frac{\partial}{\partial\gamma}\mathfrak{F}'_{l,k} = 4\pi\sigma_{l,0}.$$

Then

$$\frac{d}{d(\tau^2)}\mathfrak{F}'_{l,k} = -4\pi\mathrm{i}\frac{\partial\alpha_{l,k}}{\partial(\tau^2)} + 4\pi\sigma_{l,0}\frac{\partial\gamma_{l,k}}{\partial(\tau^2)} + 2\pi\mathrm{i}\mathfrak{R}'_{l,k}.$$

For $2 \le k \le n_l$, by (28), $\widetilde{\mathfrak{F}}'_{l,k} := \tau^{-2}\mathfrak{F}'_{l,k}$ extends to $\tau = 0$ with the value

$$\frac{d}{d(\tau^2)}\mathfrak{F}'_{l,k} = 2\pi\mathrm{i}(\overline{\mathfrak{R}_{l,k}} + \mathfrak{R}'_{l,k}) = 4\pi\mathrm{i} \operatorname{conj}^{l+1} F_{l,k}.$$

As for $k=1$ and $l<L$, we compute at $\tau=0$

$$\begin{aligned}
\sum_{k=1}^{n_l}\frac{d\mathfrak{F}'_{l,k}}{d(\tau^2)}+\sum_{k=1}^{n_{l-1}}\operatorname{conj}\Bigl(\frac{d\mathfrak{F}'_{l-1,k}}{d(\tau^2)}\Bigr)
&= -4\pi\mathrm{i}\frac{\partial}{\partial\tau^2}\Bigl(\sum_{k=1}^{n_l}\alpha_{l,k}-\sum_{k=1}^{n_{l-1}}\alpha_{l-1,k}\Bigr)\\
&\qquad +4\pi\sigma_{l,0}\frac{\partial}{\partial\tau^2}\Bigl(\sum_{k=1}^{n_l}\gamma_{l,k}-\sum_{k=1}^{n_{l-1}}\gamma_{l-1,k}\Bigr) \qquad \text{(because } \sigma_{l-1,0}=-\sigma_{l,0}\text{)}\\
&\qquad\qquad +2\pi\mathrm{i}\Bigl(\sum_{k=1}^{n_l}\mathfrak{R}'_{l,k}-\sum_{k=1}^{n_{l-1}}\overline{\mathfrak{R}'_{l-1,k}}\Bigr)\\
&= 4\pi\mathrm{i}\frac{\partial}{\partial\tau^2}(\alpha_{l,0}+\alpha_{l,\infty})-4\pi\sigma_{l,0}\frac{\partial}{\partial\tau^2}(\gamma_{l,0}+\gamma_{l,\infty}) \qquad \text{(by (17) and (19))}\\
&\qquad +2\pi\mathrm{i}\Bigl(\sum_{k=1}^{n_l}\mathfrak{R}'_{l,k}+\sum_{k=1}^{n_l}\overline{\mathfrak{R}_{l,k}}+\beta_{l,0}^2-\beta_{l,\infty}^2\Bigr) \qquad \text{(by (29))}\\
&= 2\pi\mathrm{i}\Bigl(\beta_{l,\infty}^2-\beta_{l,0}^2+\sum_{k=1}^{n_l}(\overline{\mathfrak{R}_{l,k}}+\mathfrak{R}'_{l,k})+\beta_{l,0}^2-\beta_{l,\infty}^2\Bigr) \qquad \text{(by (21))}\\
&= 4\pi\mathrm{i}\sum_{k=1}^{n_l}\operatorname{conj}^{l+1}F_{l,k},
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=1}^{n_l}\frac{d\mathfrak{F}'_{l,k}}{d(\tau^2)} &= \frac{d\mathfrak{F}'_{l,1}}{d(\tau^2)}+4\pi\mathrm{i}\operatorname{conj}^{l+1}\sum_{k=2}^{n_l}F_{l,k}\\
&= (-\operatorname{conj})^l\sum_{m=1}^{l}(-\operatorname{conj})^m\Bigl(\sum_{k=1}^{n_m}\frac{d\mathfrak{F}'_{m,k}}{d(\tau^2)}+\sum_{k=1}^{n_{m-1}}\operatorname{conj}\Bigl(\frac{d\mathfrak{F}'_{m-1,k}}{d(\tau^2)}\Bigr)\Bigr)\\
&= (-\operatorname{conj})^l\sum_{m=1}^{l}(-\operatorname{conj})^m\Bigl(4\pi\mathrm{i}\sum_{k=1}^{n_m}\operatorname{conj}^{m+1}F_{m,k}\Bigr)\\
&= 4\pi\mathrm{i}\operatorname{conj}^{l+1}\sum_{m=1}^{l}\sum_{k=1}^{n_m}F_{m,k},
\end{aligned}$$

so $\widetilde{\mathfrak{F}}'_{l,1}:=\tau^{-2}\mathfrak{F}'_{l,1}$ extends to $\tau=0$ with the value

$$\frac{d\mathfrak{F}'_{l,1}}{d(\tau^2)}=4\pi\mathrm{i}\operatorname{conj}^{l+1}\Bigl(F_{l,1}+\sum_{m=1}^{l-1}\sum_{k=1}^{n_m}F_{m,k}\Bigr). \qquad \square$$

Therefore, if $(q, \dot\theta)$ is balanced, $\widetilde{\mathfrak{F}}' = 0$ is solved at $\tau = 0$. Recall that we normalize the complex scaling on $\mathbb{C}_1^\times$ by fixing $p_{1,1}$. If $(q, \dot\theta)$ is rigid, because $\Theta_2 = \sum F_{l,k} = 0$ independent of p, the partial derivative of $(\widetilde{\mathfrak{F}}')_{(l,k)} \neq (L, 1)$ with respect to $(p_{l,k})_{(l,k)\neq(1,1)}$ is an isomorphism from $\mathbb{C}^{N-1}$ to $\mathbb{C}^{N-1}$. The following proposition then follows by the implicit function theorem.

Proposition 38. *Assume that the parameters $t_{l,k}$, $\alpha_{l,k}$, $\beta_{l,k}$, $\gamma_{l,k}$, $p'_{l,k}$ are given by Propositions 33, 35, and 36. Assume further that the central values $q_{l,k} = \operatorname{conj}^l p^\circ_{l,k}$ and $\dot\theta_h = \beta^\circ_h$ form a balanced and rigid configuration $(q, \dot\theta)$. Then for (τ, β) in a neighborhood of $(0, \dot\theta)$ that solves* (20) *and* (22), *there exists values for $p_{l,k}$, unique up to a complex scaling, depending smoothly on τ and $(\beta_h)_{h\in\mathsf{H}}$, such that $p_{l,k}(0, \dot\theta) = p^\circ_{l,k}$ and the conformality condition* (26) *is solved.*

3.4. ***Embeddedness.*** It remains to prove that:

Proposition 39. *The minimal immersion given by the Weierstrass parameterization is regular and embedded.*

Proof. The immersion is regular if $|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 > 0$. This is easily verified on U_δ. On the necks and the ends, the regularity follows if we prove that $\widetilde\Phi_2$ has no zeros outside U_δ. At $\tau = 0$, $\widetilde\Phi_2$ has $n_l + n_{l-1} + 2$ poles on $\hat{\mathbb{C}}_l$, hence $n_l + n_{l-1}$ zeros. By taking δ sufficiently small, we may assume that all these zeros lie in $U_{l,\delta}$. By continuity, $\widetilde\Phi_2$ has $n_l + n_{l-2}$ zeros in $U_{l,\delta}$ also for τ sufficiently small. But for $\tau \neq 0$, $\widetilde\Phi_2$ is meromorphic on a Riemann surface Σ_τ of genus $g = N - L$ and has $2L + 2$ simple poles, hence has $2(N - L) - 2 + 2L + 2 = 2N$ zeros. So $\widetilde\Phi_2$ has no further zeros in Σ_t, and, in particular, not outside U_δ.

We now prove that the immersion

$$z \mapsto \operatorname{Re} \int^z (\Phi_1, \widetilde\Phi_2, \Phi_3)$$

is an embedding, and the limit positions of the necks are as prescribed.

On $U_{l,\delta}$, the Gauss map $G = -(\Phi_1 + \mathrm{i}\Phi_2)/\Phi_3$ converges to $\mathrm{i}\sigma_{l,0}$, so the immersion is locally a graph over the xz-plane. Fix an orientation $\sigma_{1,0} = -1$; then up to translations, we have

$$\lim_{\tau\to 0} \Big(\operatorname{Re} \int^z \Phi_1 + \mathrm{i} \operatorname{Re} \int^z \Phi_3 \Big) = \operatorname{conj}^l (\ln z) + 2m\pi\mathrm{i},$$

where m depends on the integral path, and

$$\lim_{\tau\to 0} \operatorname{Re} \int^z \widetilde\Phi_2 = \operatorname{Re} \int^z (\operatorname{conj}^*)^l \psi_l =: \Psi_l(\operatorname{conj}^l z),$$

which is well defined for $z \in U_{l,\delta}$ because the residues of ψ_l are all real.

With a change of variable $z \mapsto \ln z$, we see that the immersion restricted to $U_{l,\delta}$ converges to a periodic graph over the xz-planes, defined within bounded

x-coordinate and away from the points $\ln q_{l,k} + 2m\pi\mathrm{i}$, and the period is $2\pi\mathrm{i}$. Here, again, we identified the xz-plane with the complex plane.

This graph must be included in a slab parallel to the xz-plane with bounded thickness. We have seen from the integration along B_k that the distance between adjacent slabs is of the order $\mathcal{O}(\ln\tau)$. So the slabs are disjoint for τ sufficiently small.

As for the necks and ends, note that there exists $Y > 0$ such that $\Psi_l^{-1}([-Y, Y])$ is bounded by $n_l + n_{l-1} + 2$ convex curves. After the change of variable $z \mapsto \ln z$, all but two of these curves remain convex; those around 0 and ∞ become periodic infinite curves. If Y is chosen sufficiently large, there exists $X > 0$ independent of l such that the curves $|z| = \exp(\pm X)$ are included in $\Psi_l^{-1}([-Y, Y])$ for every $1 \le l \le L+1$. After the change of variable $z \mapsto \ln z$, these curves become curves with $\operatorname{Re} z = \pm X$.

Hence for τ sufficiently small, we may find Y_l^+ and Y_l^-, with $Y_l^- < Y_l^+ < Y_{l+1}^-$, and $X > 0$, such that:

- The immersion with $Y_l^- < y < Y_l^+$ and $-X < x < X$ is a graph bounded by $n_l + n_{l-1}$ planar convex curves parallel to the xz-plane and two periodic planar infinite curves parallel to the yz-plane.
- The immersion with $Y_l^+ < y < Y_{l+1}^-$ and $-X < x < X$ consists of annuli, each bounded by two planar convex curves parallel to the xz-plane. These annuli are disjoint and, by a theorem of Schiffman [1956], all embedded.
- The immersion with $|x| > X$ are ends, i.e., graphs over vertical half-planes, extending in the direction $(-1, -\dot\theta_{l,0})$ and $(+1, -\dot\theta_{l,\infty})$, $1 \le l \le L+1$. If the inequality (5) is satisfied, these graphs are disjoint.

This finishes the proof of embeddedness. $\square$

Acknowledgements

H. Chen is partially supported by an Individual Research Grant from Deutsche Forschungsgemeinschaft within the project "Defects in Triply Periodic Minimal Surfaces", Projektnummer 398759432. We would like to thank the referee for many helpful comments.

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Received July 12, 2022. Revised May 17, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 325, No. 1, 2023
https://doi.org/10.2140/pjm.2023.325.47

THE STRONG HOMOTOPY STRUCTURE OF BRST REDUCTION

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We propose a reduction scheme for polydifferential operators phrased in terms of L_∞-morphisms. The desired reduction L_∞-morphism has been obtained by applying an explicit version of the homotopy transfer theorem. Finally, we prove that the reduced star product induced by this reduction L_∞-morphism and the reduced star product obtained via the formal Koszul complex are equivalent.

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1. Introduction

This paper aims to propose a reduction scheme for equivariant polydifferential operators that is phrased in terms of L_∞-morphisms, generalizing the results from [Esposito et al. 2022b], obtained for polyvector fields. Our main motivation comes from formal deformation quantization: deformation quantization has been introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [Bayen et al. 1978a; 1978b] and it relies on the idea that the quantization of a phase space described by a Poisson manifold M is described by a formal deformation, so-called *star product*, of the commutative algebra of smooth complex-valued functions $\mathscr{C}^\infty(M)$ in a formal parameter $\hbar$. The existence and classification of star products on Poisson manifolds has been provided by Kontsevich's formality theorem [2003], whereas the invariant setting of Lie group actions has been treated by Dolgushev

MSC2020: 53D20, 53D55.
Keywords: reduction, quantization, BRST, formality.

[2005a; 2005b]. More explicitly, the formality theorem provides an L_∞-quasi-isomorphism between the differential graded Lie algebra (DGLA) of polyvector fields $T_{\mathrm{poly}}(M)$ and polydifferential operators $D_{\mathrm{poly}}(M)$ as well as between their invariant versions. As such, it maps Maurer–Cartan elements in the DGLA of polyvector fields, i.e., (formal) Poisson structures, to Maurer–Cartan elements in the DGLA of polydifferential operators, which correspond to star products.

One open question and our main motivation is to investigate the compatibility of deformation quantization and phase space reduction in the Poisson setting, and in this present paper we propose a way to describe the reduction on the quantum side by an L_∞-morphism. Given a Lie group G acting on a manifold M, we aim to reduce *equivariant star products* $(\star, H)$, that is, pairs consisting of an invariant star product $\star$ and a quantum momentum map $H = \sum_{r=0}^{\infty} \hbar^r J_r : \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)[[\hbar]]$, where $\mathfrak{g}$ is the Lie algebra of G. In this case, J_0 is a classical momentum map for the Poisson structure induced by $\star$. Interpreting it as smooth map $J_0 : M \longrightarrow \mathfrak{g}^*$ and assuming that $0 \in \mathfrak{g}^*$ is a value and regular value, it follows that $C = J^{-1}(\{0\})$ is a closed embedded submanifold of M and by the Poisson version of the Marsden–Weinstein reduction [1974] we know that under suitable assumptions the reduced manifold $M_{\mathrm{red}} = C/\,\mathrm{G}$ is again a Poisson manifold if the action on C is proper and free. In this setting, there is a well-known BRST-like reduction procedure [Bordemann et al. 2000; Gutt and Waldmann 2010] of equivariant star products on M to star products on M_{red}.

In order to describe this reduction by an L_∞-morphism, we have to fix at first the DGLA controlling Hamiltonian actions in the quantum setting, i.e., a DGLA whose Maurer–Cartan elements correspond to equivariant star products. We denote it by

$$(D_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial^{\mathfrak{g}} - [J_0, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}),$$

where $\lambda = \sum_i e^i \otimes (e_i)_M$ is given by the fundamental vector fields of the G-action in terms of a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ with dual basis $e^1, \dots, e^n$ of $\mathfrak{g}^*$. It is called the DGLA of *equivariant polydifferential operators*.

The construction of the desired L_∞-morphism to $(D_{\mathrm{poly}}(M_{\mathrm{red}}), \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}})$ is then based on the following steps:

- Assuming for simplicity $M = C \times \mathfrak{g}^*$, which always holds locally in suitable situations, we can perform a Taylor expansion around C and end up with a DGLA $D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$. Using a 'partial homotopy', we find a deformation retract to a DGLA structure on the space $\big(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}$, that is, we get rid of differentiations in the $\mathfrak{g}^*$-direction.

- For the polyvector fields in [Esposito et al. 2022b] we used the canonical linear Poisson structure π_{KKS} on the dual of the action Lie algebroid $C \times \mathfrak{g}$ for the reduction. The analogue structure in our quantum setting is the product on the quantized universal enveloping algebra $U_\hbar(C \times \mathfrak{g})$ of the action Lie algebroid. We use this product to perturb the deformation retract from the last point. This is more complicated than the polyvector field case since we have to use now the homological perturbation lemma to perturb the involved chain maps, and the deformed maps are no longer compatible with the Lie brackets.

- We use the homotopy transfer theorem to construct the L_∞-projection from the Taylor expansion to $\big(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}$ with transferred L_∞-structure. Notice that in the polyvector field case it was not necessary to transfer the DGLA structure.

- We check in Proposition 3.10 that the transferred L_∞-structure is just a DGLA structure, and in Proposition 3.11 that the transferred Lie bracket is compatible with the projection to $D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!]$. Thus we get the reduction L_∞-morphism from the Taylor expansion to the polydifferential operators on M_{red}. Twisting it by the product on the universal enveloping algebra ensures that we start in the right curved DGLA structure.

Finally, the morphism can be globalized to general smooth manifolds M with sufficiently nice Lie group actions and we get the following result (Theorem 3.15):

Theorem. *There exists an L_∞-morphism*

$$
\begin{aligned}
\text{(1-1)}\quad D_{\mathrm{red}}:(D_{\mathfrak{g}}(M)[\![\hbar]\!],\hbar\lambda,\partial^{\mathfrak{g}}-[J_0,\cdot\,]_{\mathfrak{g}},[\,\cdot\,,\cdot\,]_{\mathfrak{g}})&\\
\longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!],0,\partial,[\,\cdot\,,\cdot\,]_{\mathrm{G}})&,
\end{aligned}
$$

called the ***reduction L_∞-morphism****.*

Finally, we compare the reduction of equivariant star products via D_{red} to a slightly modified version of the BRST reduction from [Bordemann et al. 2000; Gutt and Waldmann 2010]; see Theorem 4.4:

Theorem. *Let $(\star, H)$ be an equivariant star product on M. Then the reduced star product induced by D_{red} from* (1-1) *and the reduced star product via the formal Koszul complex are equivalent.*

Together with [Esposito et al. 2022b, Theorem 5.1] we have now the diagram

$$
\begin{array}{ccc}
(T^\bullet_{\mathfrak{g}}(M)[\![\hbar]\!],\hbar\lambda,[-J_0,\cdot\,]_{\mathfrak{g}},[\,\cdot\,,\cdot\,]_{\mathfrak{g}}) & & (D^\bullet_{\mathfrak{g}}(M)[\![\hbar]\!],\hbar\lambda,\partial^{\mathfrak{g}}-[J_0,\cdot\,]_{\mathfrak{g}},[\,\cdot\,,\cdot\,]_{\mathfrak{g}})\\
\Big\downarrow{\scriptstyle \mathrm{T}_{\mathrm{red}}} & & \Big\downarrow{\scriptstyle D_{\mathrm{red}}}\\
(T^\bullet_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!],0,0,[\,\cdot\,,\cdot\,]_S) & \xrightarrow{\;F_{\mathrm{red}}\;} & (D^\bullet_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!],0,\partial,[\,\cdot\,,\cdot\,]_{\mathrm{G}})
\end{array}
$$

where F_{red} is the standard Dolgushev formality with respect to a torsion-free covariant derivative on M_{red}. Also, in [Esposito et al. 2022a] we show that the Dolgushev formality is compatible with λ under suitable flatness assumptions. In these flat cases it induces an L_∞-morphism

$$F^{\mathfrak{g}} : (T^\bullet_{\mathfrak{g}}(M)[\![\hbar]\!], \hbar\lambda, [-J_0, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \longrightarrow (D^\bullet_{\mathfrak{g}}(M)[\![\hbar]\!], \hbar\lambda, \partial^{\mathfrak{g}} - [J_0, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}),$$

which gives the fourth arrow in the above diagram, and we plan to investigate its commutativity (up to homotopy) in future work.

The results of this paper are partially based on [Kraft 2021] and the paper is organized as follows. In Section 2 we recall the basic notions of (curved) L_∞-algebras, L_∞-morphisms and twists and fix the notation. Then we introduce in Section 2B the curved DGLA of equivariant polydifferential operators and show that they indeed control Hamiltonian actions. In Section 3 we construct the global reduction L_∞-morphism to the polydifferential operators on the reduced manifold. Finally, we compare in Section 4 the reduction via this reduction morphism D_{red} with a slightly modified BRST reduction of equivariant star products as explained in Appendix A, where we also recall the homological perturbation lemma. In Appendix B we give explicit formulas for the transferred L_∞-structure and the L_∞-projection induced by the homotopy transfer theorem.

2. Preliminaries

2A. ***L_∞-algebras, Maurer–Cartan elements and twisting.*** In this section we recall the notions of (curved) L_∞-algebras, L_∞-morphisms and their twists by Maurer–Cartan elements to fix the notation. Proofs and further details can be found in [Dolgushev 2005a; 2005b; Esposito and de Kleijn 2021].

We denote by $V^\bullet$ a graded vector space over a field $\mathbb{K}$ of characteristic 0 and define the *shifted* vector space $V[k]^\bullet$ by

$$V[k]^\ell = V^{\ell+k}.$$

A degree $+1$ coderivation Q on the coaugmented counital conilpotent cocommutative coalgebra $S^c(\mathfrak{L})$ cofreely cogenerated by the graded vector space $\mathfrak{L}[1]^\bullet$ over $\mathbb{K}$ is called an L_∞*-structure* on the graded vector space $\mathfrak{L}$ if $Q^2 = 0$. The (universal) coalgebra $S^c(\mathfrak{L})$ can be realized as the symmetrized deconcatenation coproduct on the space $\bigoplus_{n\geq 0} \mathrm{S}^n\mathfrak{L}[1]$ where $\mathrm{S}^n\mathfrak{L}[1]$ is the space of coinvariants for the usual (graded) action of S_n (the symmetric group in n letters) on $\otimes^n(\mathfrak{L}[1])$; see, for example, [Esposito and de Kleijn 2021]. Any degree $+1$ coderivation Q on $S^c(\mathfrak{L})$ is uniquely determined by the components

$$Q_n : \mathrm{S}^n(\mathfrak{L}[1]) \longrightarrow \mathfrak{L}[2] \tag{2-1}$$

through the formula

$$(2\text{-}2)\qquad Q(\gamma_1\vee\cdots\vee\gamma_n)=\sum_{k=0}^{n}\sum_{\sigma\in \mathrm{Sh}(k,n-k)}\epsilon(\sigma)Q_k(\gamma_{\sigma(1)}\vee\cdots\vee\gamma_{\sigma(k)})\vee\gamma_{\sigma(k+1)}\vee\cdots\vee\gamma_{\sigma(n)}.$$

Here $\mathrm{Sh}(k,n-k)$ denotes the set of $(k,n-k)$ shuffles in S_n, $\epsilon(\sigma)=\epsilon(\sigma,\gamma_1,\dots,\gamma_n)$ is a sign given by the rule $\gamma_{\sigma(1)}\vee\cdots\vee\gamma_{\sigma(n)}=\epsilon(\sigma)\gamma_1\vee\cdots\vee\gamma_n$ and we use the conventions that $\mathrm{Sh}(n,0)=\mathrm{Sh}(0,n)=\{\mathrm{id}\}$ and that the empty product equals the unit. Note in particular that we also consider a term Q_0 and thus we are actually considering curved L_∞-algebras. Sometimes we also write $Q_k=Q_k^1$ and, following [Canonaco 1999], we denote by Q_n^i the component of $Q_n^i:\mathrm{S}^n\mathfrak{L}[1]\to\mathrm{S}^i\mathfrak{L}[2]$ of Q. It is given by

$$(2\text{-}3)\qquad Q_n^i(x_1\vee\cdots\vee x_n)=\sum_{\sigma\in\mathrm{Sh}(n+1-i,i-1)}\epsilon(\sigma)Q^1_{n+1-i}(x_{\sigma(1)}\vee\cdots\vee x_{\sigma(n+1-i)})\vee x_{\sigma(n+2-i)}\vee\cdots\vee x_{\sigma(n)},$$

where Q^1_{n+1-i} are the usual structure maps.

Example 2.1 (curved DGLA). A basic example of an L_∞-algebra is that of a (curved) differential graded Lie algebra $(\mathfrak{g},R,\mathrm{d},[\,\cdot\,,\,\cdot\,])$ obtained by setting $Q_0(1)=-R$, $Q_1=-\mathrm{d}$, $Q_2(\gamma\vee\mu)=-(-1)^{|\gamma|}[\gamma,\mu]$ and $Q_i=0$ for all $i\geq 3$. Note that we denoted by $|\cdot|$ the degree in $\mathfrak{g}[1]$.

Let us consider two L_∞-algebras $(\mathfrak{L},Q)$ and $(\widetilde{\mathfrak{L}},\widetilde{Q})$. A degree-0 counital coalgebra morphism

$$F:S^c(\mathfrak{L})\longrightarrow S^c(\widetilde{\mathfrak{L}})$$

such that $FQ=\widetilde{Q}F$ is said to be an *L_∞-morphism*. A coalgebra morphism F from $S^c(\mathfrak{L})$ to $S^c(\widetilde{\mathfrak{L}})$ such that $F(1)=1$ is uniquely determined by its components (also called *Taylor coefficients*)

$$F_n:\mathrm{S}^n(\mathfrak{L}[1])\longrightarrow\widetilde{\mathfrak{L}}[1],$$

where $n\geq 1$. Namely, we set $F(1)=1$ and use the formula

$$F(\gamma_1\vee\cdots\vee\gamma_n)=\sum_{p\geq1}\sum_{\substack{k_1,\dots,k_p\geq1\\ k_1+\cdots+k_p=n}}\sum_{\sigma\in\mathrm{Sh}(k_1,\dots,k_p)}\frac{\epsilon(\sigma)}{p!}F_{k_1}(\gamma_{\sigma(1)}\vee\cdots\vee\gamma_{\sigma(k_1)})\vee\cdots\vee F_{k_p}(\gamma_{\sigma(n-k_p+1)}\vee\cdots\vee\gamma_{\sigma(n)}),$$

where $\mathrm{Sh}(k_1,\dots,k_p)$ denotes the set of $(k_1,\dots,k_p)$-shuffles in S_n (again we set $\mathrm{Sh}(n)=\{\mathrm{id}\}$). We also write $F_k=F_k^1$ and similarly to (2-3) we get coefficients $F_n^j:\mathrm{S}^n\mathfrak{L}[1]\to\mathrm{S}^j\widetilde{\mathfrak{L}}[1]$ of F by taking the corresponding terms in [Dolgushev 2006, Equation (2.15)]. Note that F_n^j only depends on $F_k^1 = F_k$ for $k \leq n - j + 1$. Given an L_∞-morphism F of (noncurved) L_∞-algebras $(\mathfrak{L}, Q)$ and $(\widetilde{\mathfrak{L}}, \widetilde{Q})$, we obtain the map of complexes

$$F_1 : (\mathfrak{L}, Q_1) \longrightarrow (\widetilde{\mathfrak{L}}, \widetilde{Q}_1).$$

In this case the L_∞-morphism F is called an *L_∞-quasi-isomorphism* if F_1 is a quasi-isomorphism of complexes. Given a DGLA $(\mathfrak{g}, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ and an element $\pi \in \mathfrak{g}[1]^0$ we can obtain a curved Lie algebra by defining a new differential $\mathrm{d}+[\pi, \cdot\,]$ and considering the curvature $R^\pi = \mathrm{d}\pi + \frac{1}{2}[\pi, \pi]$. In fact the same procedure can be applied to a curved Lie algebra $(\mathfrak{g}, R, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ to obtain the *twisted* curved Lie algebra $(\mathfrak{L}, R^\pi, \mathrm{d} + [\pi, \cdot\,], [\,\cdot\,,\,\cdot\,])$, where

$$R^\pi := R + \mathrm{d}\pi + \tfrac{1}{2}[\pi, \pi]. \tag{2-4}$$

The element π is called a *Maurer–Cartan element* if it satisfies the equation

$$R + \mathrm{d}\pi + \tfrac{1}{2}[\pi, \pi] = 0. \tag{2-5}$$

Finally, it is important to recall that given a DGLA morphism, or more generally an L_∞-morphism, $F : \mathfrak{g} \to \mathfrak{g}'$ between two DGLAs, one may associate to any (curved) Maurer–Cartan element $\pi \in \mathfrak{g}[1]^0$ a (curved) Maurer–Cartan element

$$\pi_F := \sum_{n \geq 1} \frac{1}{n!} F_n(\pi \vee \cdots \vee \pi) \in \mathfrak{g}'[1]^0. \tag{2-6}$$

In order to make sense of these infinite sums we consider DGLAs with complete descending filtrations

$$\cdots \supseteq \mathcal{F}^{-2}\mathfrak{g} \supseteq \mathcal{F}^{-1}\mathfrak{g} \supseteq \mathcal{F}^{0}\mathfrak{g} \supseteq \mathcal{F}^{1}\mathfrak{g} \supseteq \cdots, \quad \mathfrak{g} \cong \varprojlim \mathfrak{g}/\mathcal{F}^n\mathfrak{g} \tag{2-7}$$

and

$$\mathrm{d}(\mathcal{F}^k\mathfrak{g}) \subseteq \mathcal{F}^k\mathfrak{g} \quad \text{and} \quad [\mathcal{F}^k\mathfrak{g}, \mathcal{F}^\ell\mathfrak{g}] \subseteq \mathcal{F}^{k+\ell}\mathfrak{g}. \tag{2-8}$$

In particular, $\mathcal{F}^1\mathfrak{g}$ is a projective limit of nilpotent DGLAs. In most cases the filtration is bounded below, i.e., bounded from the left with $\mathfrak{g} = \mathcal{F}^k\mathfrak{g}$ for some $k \in \mathbb{Z}$. If the filtration is unbounded, then we assume always that it is exhaustive, i.e., that

$$\mathfrak{g} = \bigcup_n \mathcal{F}^n\mathfrak{g}, \tag{2-9}$$

even if we do not mention it explicitly. Also, we assume that the DGLA morphisms are compatible with the filtrations. Considering only Maurer–Cartan elements in $\mathcal{F}^1\mathfrak{g}^1$ ensures the well-definedness of (2-6). Mainly, the filtration is induced by formal power series in a formal parameter $\hbar$. Starting with a DGLA $(\mathfrak{g}, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$, its $\hbar$-linear extension to formal power series $\mathfrak{G} = \mathfrak{g}[[\hbar]]$ of a DGLA $\mathfrak{g}$ has the complete descending filtration $\mathcal{F}^k\mathfrak{G} = \hbar^k\mathfrak{G}$.

One cannot only twist the DGLAs and L_∞-algebras, but also the L_∞-morphisms between them. Below we need the following result; see [Dolgushev 2006, Proposition 2; 2005b, Proposition 1].

Proposition 2.2. *Let* $F : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ *be an* L_∞*-morphism of DGLAs,* $\pi \in \mathcal{F}^1\mathfrak{g}^1$ *a Maurer–Cartan element and* $S = F^1(\overline{\exp}(\pi)) \in \mathcal{F}^1\mathfrak{g}'^1$.

(i) *The map*

$$F^\pi = \exp(-S\vee) F \exp(\pi\vee) : \bar{\mathrm{S}}(\mathfrak{g}[1]) \longrightarrow \bar{\mathrm{S}}(\mathfrak{g}'[1])$$

defines an L_∞*-morphism between the DGLAs* $(\mathfrak{g}, \mathrm{d}+[\pi, \cdot\,])$ *and* $(\mathfrak{g}', \mathrm{d}+[S, \cdot\,])$.

(ii) *The structure maps of* F^π *are given by*

$$F_n^\pi(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F_{n+k}(\pi, \dots, \pi, x_1, \dots, x_n). \tag{2-10}$$

(iii) *Let* F *be an* L_∞*-quasi-isomorphism where* F_1^1 *is not only a quasi-isomorphism of filtered complexes* $L \to L'$ *but even induces a quasi-isomorphism*

$$F_1^1 : \mathcal{F}^k L \longrightarrow \mathcal{F}^k L'$$

for each k. *Then* F^π *is an* L_∞*-quasi-isomorphism.*

2B. ***Equivariant polydifferential operators.*** In the following we present some basic results concerning equivariant polydifferential operators, which are basically folklore knowledge and are based on [Tsygan 2010].

Let us consider the DGLA of *polydifferential operators* on a smooth manifold M

$$(D^\bullet_{\mathrm{poly}}(M), \partial = [\mu, \cdot\,]_{\mathrm{G}}, [\,\cdot\,,\cdot\,]_{\mathrm{G}}) \tag{2-11}$$

Here

$$D^\bullet_{\mathrm{poly}}(M) = \bigoplus_{n=-1}^{\infty} D^n_{\mathrm{poly}}(M),$$

where $D^n_{\mathrm{poly}}(M) = \mathrm{Hom}_{\mathrm{diff}}(\mathscr{C}^\infty(M)^{\otimes n+1}, \mathscr{C}^\infty(M))$ are the differentiable Hochschild cochains vanishing on constants. We use the sign convention from [Bursztyn et al. 2012] for the Gerstenhaber bracket $[\,\cdot\,,\cdot\,]$, not the original one from [Gerstenhaber 1963]. Explicitly

$$[D, E]_{\mathrm{G}} = (-1)^{|E||D|}(D \circ E - (-1)^{|D||E|} E \circ D) \tag{2-12}$$

with

$$D \circ E(a_0, \dots, a_{d+e}) = \sum_{i=0}^{|D|} (-1)^{i|E|} D(a_0, \dots, a_{i-1}, E(a_i, \dots, a_{i+e}), a_{i+e+1}, \dots, a_{d+e}) \tag{2-13}$$

for homogeneous $D, E \in D^{\bullet}_{\mathrm{poly}}(M)$ and $a_0, \dots, a_{d+e} \in \mathscr{C}^\infty(M)$. Also, μ denotes the commutative pointwise product on $\mathscr{C}^\infty(M)[[\hbar]]$ and ∂ is the usual Hochschild differential.

We are interested in the case of group actions where we always consider a (left) action $\Phi : \mathrm{G} \times M \to M$ of a connected Lie group G. Let M be now equipped with a G-invariant star product $\star$, that is, an associative product $\star = \mu + \sum_{r=1}^{\infty} \hbar^r C_r = \mu_0 + \hbar m_\star \in (D^1_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]]$. Recall that a linear map $H : \mathfrak{g} \to \mathscr{C}^\infty(M)[[\hbar]]$ is called a *quantum momentum map* if

$$\mathscr{L}_{\xi_M} = -\frac{1}{\hbar}[H(\xi), \cdot\,]_\star \quad \text{and} \quad \frac{1}{\hbar}[H(\xi), H(\eta)]_\star = H([\xi, \eta]),$$

where ξ_M denotes the fundamental vector field corresponding to the action Φ.

A pair $(\star, H)$ consisting of an invariant star product $\star = \mu + \hbar m_\star$ and a quantum momentum map H is also called *equivariant star product*. They are useful since they allow for a BRST like reduction scheme; see Appendix A. We introduce now the DGLA that contains the data of Hamiltonian actions, i.e., of equivariant star products. Here we follow [Tsygan 2010].

Definition 2.3 (equivariant polydifferential operators). The DGLA of *equivariant polydifferential operators* $(D^{\bullet}_{\mathfrak{g}}(M), \partial^{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}})$ is defined by

$$D^k_{\mathfrak{g}}(M) = \bigoplus_{2i+j=k} (\mathrm{S}^i \mathfrak{g}^* \otimes D^j_{\mathrm{poly}}(M))^{\mathrm{G}} \tag{2-14}$$

with bracket

$$[\alpha \otimes D_1, \beta \otimes D_2]_{\mathfrak{g}} = \alpha \vee \beta \otimes [D_1, D_2]_{\mathrm{G}} \tag{2-15}$$

and differential

$$\partial^{\mathfrak{g}}(\alpha \otimes D_1) = \alpha \otimes \partial D_1 = \alpha \otimes [\mu, D_1]_{\mathrm{G}} \tag{2-16}$$

for $\alpha \otimes D_1, \beta \otimes D_2 \in D^{\bullet}_{\mathfrak{g}}(M)$. Here we denote by ∂ and $[\,\cdot\,,\cdot\,]_{\mathrm{G}}$ the usual Hochschild differential and Gerstenhaber bracket on the polydifferential operators, respectively, and by μ the pointwise multiplication of $\mathscr{C}^\infty(M)$.

Notice that invariance with respect to the group action means invariance under the transformations $\mathrm{Ad}^*_g \otimes \Phi^*_g$ for all $g \in G$, and that the equivariant polydifferential operators can be interpreted as equivariant polynomial maps $\mathfrak{g} \to D_{\mathrm{poly}}(M)$. We introduce the canonical linear map

$$\lambda : \mathfrak{g} \ni \xi \longmapsto \mathscr{L}_{\xi_M} \in D^0_{\mathrm{poly}}(M),$$

and see that $\lambda \in D^2_{\mathfrak{g}}(M)$ is central and moreover $\partial^{\mathfrak{g}}\lambda = 0$. This implies that we can see $D^\bullet_{\mathfrak{g}}(M)$ either as a flat DGLA with the above structures or as a curved DGLA with the above structures and curvature λ. In the case of formal power series we rescale the curvature again by $\hbar^2$ and obtain the following characterization of Maurer–Cartan elements:

Lemma 2.4. *A curved formal Maurer–Cartan element* $\Pi \in \hbar D^1_{\mathfrak{g}}(M)[[\hbar]]$, *that is, an element* Π *satisfying*

$$\hbar^2\lambda + \partial^{\mathfrak{g}}\Pi + \tfrac{1}{2}[\Pi, \Pi]_{\mathfrak{g}} = 0, \tag{2-17}$$

is equivalent to a pair $(m_\star, H)$, *where* $m_\star \in D^1_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]]$ *defines a* G*-invariant star product via* $\star = \mu + \hbar m_\star$ *with quantum momentum map* $H : \mathfrak{g} \to \mathscr{C}^\infty(M)[[\hbar]]$. *In other words*, $(\star, H)$ *is an equivariant star product.*

Proof. We have the decomposition

$$\Pi = \hbar m_\star - \hbar H \in \hbar(D^1_{\mathrm{poly}}(M))^{\mathrm{G}} \oplus (\mathfrak{g}^* \otimes D^{-1}_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]].$$

Then the curved Maurer–Cartan equation applied to an element $\xi \in \mathfrak{g}$ reads

$$\begin{aligned} -\hbar^2\mathscr{L}_{\xi_M} = -\hbar^2\lambda(\xi) &= \partial^{\mathfrak{g}}\Pi(\xi) + \tfrac{1}{2}[\Pi, \Pi]_{\mathfrak{g}}(\xi) \\ &= \hbar[\mu, m_\star]_{\mathrm{G}} + \tfrac{1}{2}\hbar^2[m_\star, m_\star]_{\mathrm{G}} - \hbar^2[m_\star, H(\xi)]_{\mathrm{G}}. \end{aligned}$$

This is equivalent to the fact that $\hbar m_\star$ is Maurer–Cartan in the flat setting and that $\mathscr{L}_{\xi_M} = -\frac{1}{\hbar}[H(\xi), -]_\star$, since $\hbar[m_\star, H(\xi)]_{\mathrm{G}}(f) = -[H(\xi), f]_\star$ for $f \in \mathscr{C}^\infty(M)$. Then the invariance of both elements implies that $\star = \mu + \hbar m_\star$ is a G-invariant star product with quantum momentum map H. □

Two equivariant star products $\hbar(m_\star - H)$ and $\hbar(m'_\star - H')$ are called *equivariantly equivalent* if they are gauge equivalent, i.e., if there exists an $\hbar T \in \hbar D^0_{\mathrm{poly}}(M)^{\mathrm{G}}[[\hbar]] \subset D^0_{\mathfrak{g}}(M)$ such that

$$\hbar(m'_\star - H') = \exp(\hbar[T, \cdot\,]_{\mathfrak{g}}) \rhd \hbar(m_\star - H) = \exp(\hbar[T, \cdot\,]_{\mathfrak{g}})(\mu + \hbar(m_\star - H)) - \mu.$$

This means that $S = \exp(\hbar T)$ satisfies for all $f, g \in \mathscr{C}^\infty(M)[[\hbar]]$

$$S(f \star g) = Sf \star' Sg \quad \text{and} \quad SH = H'.$$

3. Reduction of the equivariant polydifferential operators

Now we aim to describe a reduction scheme for general equivariant polydifferential operators via an L_∞-morphism denoted by D_{red}, generalizing the results for the polyvector fields from [Esposito et al. 2022b].

Let M be a smooth manifold with action $\Phi : \mathrm{G} \times M \to M$ of a connected Lie group and let $(\star, H = J + \hbar J')$ be an equivariant star product, that is, a curved formal Maurer–Cartan element in the equivariant polydifferential operators; see Lemma 2.4. Here the component $J : M \to \mathfrak{g}^*$ of the quantum momentum map H in $\hbar$-order zero is a classical momentum map with respect to the Poisson structure induced by the skew-symmetrization of the $\hbar^1$-part of $\star$. We assume from now on that $0 \in \mathfrak{g}^*$ is a value and a regular value of J and set $C = J^{-1}(\{0\})$. In addition, we require the action to be proper around C and free on C. Then $M_{\mathrm{red}} = C/\mathrm{G}$ is a smooth manifold and we denote by $\iota : C \to M$ the inclusion and by $\mathrm{pr} : C \to M_{\mathrm{red}}$ the projection on the quotient. Moreover, the properness around C implies that there exists an G-invariant open neighborhood $M_{\mathrm{nice}} \subseteq M$ of C and a G-equivariant diffeomorphism $\Psi : M_{\mathrm{nice}} \to U_{\mathrm{nice}} \subseteq C \times \mathfrak{g}^*$, where U_{nice} is an open neighborhood of $C \times \{0\}$ in $C \times \mathfrak{g}^*$. Here the Lie group G acts on $C \times \mathfrak{g}^*$ as $\Phi_g = \Phi^C_g \times \mathrm{Ad}^*_{g^{-1}}$, where Φ^C is the induced action on C, and the momentum map on U_{nice} is the projection to $\mathfrak{g}^*$ (see [Bordemann et al. 2000, Lemma 3; Gutt and Waldmann 2010]).

From now on we assume $M = M_{\mathrm{nice}}$. Then we can define an equivariant *prolongation map* by

$$\mathrm{prol} : \mathscr{C}^\infty(C) \ni \phi \longmapsto (\mathrm{pr}_1 \circ \Psi)^* \phi \in \mathscr{C}^\infty(M_{\mathrm{nice}})$$

and we directly get $\iota^* \mathrm{prol} = \mathrm{id}_{\mathscr{C}^\infty(C)}$.

Consider the Taylor expansion around C in the $\mathfrak{g}^*$-direction as in [Esposito et al. 2022b, Section 4.1], which is a map

$$D_{\mathfrak{g}^*} : D^k_{\mathrm{poly}}(C \times \mathfrak{g}^*) \longmapsto \prod_{i=0}^\infty (\mathrm{S}^i \mathfrak{g} \otimes T^{k+1}(\mathrm{S}\mathfrak{g}^*) \otimes D^k_{\mathrm{poly}}(C)),$$

where $T^\bullet(\mathrm{S}\mathfrak{g}^*)$ denotes the tensor algebra of $\mathrm{S}\mathfrak{g}^*$. Note that we are only interested in a subspace since we consider polydifferential operators vanishing on constants. Slightly abusing the notation, the Taylor expansion of the equivariant polydifferential operators takes then the following form:

$$D_{\mathrm{Tay}}(C \times \mathfrak{g}^*) = \left(\mathrm{S}\mathfrak{g}^* \otimes \prod_{i=0}^\infty (\mathrm{S}^i \mathfrak{g} \otimes T(\mathrm{S}\mathfrak{g}^*) \otimes D_{\mathrm{poly}}(C)) \right)^{\mathrm{G}} \tag{3-1}$$

and one easily checks that this yields an equivariant DGLA morphism

$$D_{\mathfrak{g}^*} : (D_{\mathfrak{g}}(M), \lambda, \partial^{\mathfrak{g}}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}}) \longrightarrow (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, \partial, [\,\cdot\,,\,\cdot\,]). \tag{3-2}$$

Our goal consists in finding a reduction morphism from

$$D_{\mathrm{red}} : (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]], \hbar\lambda, \partial+[-J,\cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}).$$

Following a similar strategy as in [Esposito et al. 2022b], we construct L_∞-morphisms

$$D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]] \longrightarrow \left(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[[\hbar]] \longrightarrow D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]] \tag{3-3}$$

with suitable L_∞-structures on the three spaces, where $\left(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[[\hbar]]$ is a candidate for a Cartan model.

3A. ***A 'partial' homotopy for the Hochschild differential.*** In order to find a suitable analogue of the Cartan model for the polydifferential operators, we need to understand the cohomology of

$$(D_{\mathfrak{g}}(M), \partial^{\mathfrak{g}}-[J,\cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}})$$

and in particular the role of the differential $[-J,\cdot\,]_{\mathfrak{g}}$. To this end we construct a 'partial' homotopy for $\partial^{\mathfrak{g}}-[J,\cdot\,]_{\mathfrak{g}}$. Here we use the results concerning the homotopy for the Hochschild differential from [De Wilde and Lecomte 1995]. In particular, we restrict ourselves to the subspace of normalized differential Hochschild cochains, i.e., polydifferential operators vanishing on constants. One can show that they are quasi-isomorphic to the differential ones. Recall the maps

$$\Phi : D^a_{\mathrm{poly}}(M) \longrightarrow D^{a-1}_{\mathrm{poly}}(M),$$
$$\Phi(A)(f_0,\dots,f_{a-1}) = \sum_{t=1}^{n}\sum_{i}\sum_{j=i}^{a-1}(-1)^i A\Big(f_0,\dots,f_{i-1},x^t,\dots,\frac{\partial}{\partial x^t}f_j,\dots,f_{a-1}\Big),$$

for $f_1,\dots,f_{a-1}\in\mathscr{C}^\infty(M)$, and

$$\Psi : D^a_{\mathrm{poly}}(M)\ni A\longmapsto(-1)^a[x^i,A]_{\mathrm{G}}\cup\frac{\partial}{\partial x^i}=(-1)^{a+1}\sum_{i=1}^{n}(A\circ x^i)\cup\frac{\partial}{\partial x^i}\in D^a_{\mathrm{poly}}(M),$$

for local coordinates $(x^1,\dots,x^n)$ of M. They satisfy, by [De Wilde and Lecomte 1995, Proposition 4.1], the condition

$$\Phi\circ\partial+\partial\circ\Phi = -(\deg_D\cdot\mathrm{id}+\Psi), \tag{3-4}$$

where $\deg_D$ is the order of the differential operator.

We assume from now on for simplicity $M = C\times\mathfrak{g}^*$ and $J=\mathrm{pr}_{\mathfrak{g}^*}$ and we want to find a suitable Cartan model for the polydifferential operators. Similarly to

[Esposito et al. 2022b, Definition 4.14] for the polyvector field case, we want to obtain a DGLA structure on

$$\left(\prod_{i=0}^{\infty}(\mathrm{S}^i\,\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}. \tag{3-5}$$

Hence we adapt the maps Φ and Ψ in such a way that they only include coordinates $J_i=\alpha_i=e_i$ on $\mathfrak{g}^*$ with $i=1,\dots,n$:

$$\Phi(A)(f_0,\dots,f_{a-1})=\sum_{t=1}^{n}\sum_{i\le j<a}(-1)^i A\Big(f_0,\dots,f_{i-1},e_t,\dots,\frac{\partial}{\partial e_t}f_j,\dots,f_{a-1}\Big),$$

$$\Psi(A)=(-1)^{a+1}\sum_{i=1}^{n}(A\circ e_i)\cup\frac{\partial}{\partial e_i},$$

where $A\in D^a_{\mathrm{poly}}(C\times\mathfrak{g}^*)$ and $f_0,\dots,f_{a-i}\in\mathscr{C}^\infty(C\times\mathfrak{g}^*)$.

Proposition 3.1. *One has on* $D_{\mathrm{poly}}(C\times\mathfrak{g}^*)$

$$\Phi\circ\partial+\partial\circ\Phi=-(\deg_{\mathfrak{g}}\cdot\mathrm{id}+\Psi), \tag{3-6}$$

where $\deg_{\mathfrak{g}}$ *is the order of differentiations in the direction of* $\mathfrak{g}^*$*-coordinates.*

Proof. The proof follows the same lines as in [De Wilde and Lecomte 1995, Proposition 4.1]. It is proven by induction on the degree of a of $A\in D^a_{\mathrm{poly}}(C\times\mathfrak{g}^*)$. For $a=0$ and $A\in D^0_{\mathrm{poly}}(C\times\mathfrak{g}^*)$ as well as $f\in\mathscr{C}^\infty(C\times\mathfrak{g}^*)$ we get

$$\begin{aligned}((\Phi\circ\partial+\partial\circ\Phi)(A))(f)&=(\partial A)\Big(e_i,\frac{\partial}{\partial e_i}f\Big)\\&=e_iA\Big(\frac{\partial}{\partial e_i}f\Big)-A\Big(e_i\frac{\partial}{\partial e_i}f\Big)+A(e_i)\frac{\partial}{\partial e_i}f\\&=(-\deg_{\mathfrak{g}}(A)\,A-\Psi(A))(f).\end{aligned}$$

Note that Ψ has the following compatibility with the $\cup$-product:

$$\Psi(A\cup B)=(\Psi A)\cup B+A\cup(\Psi B)+(-1)^a(A\circ e_i)\cup\Big(\frac{\partial}{\partial e_i}\cup B+(-1)^bB\cup\frac{\partial}{\partial e_i}\Big).$$

Writing $\mathrm{i}(A)(\,\cdot\,)=(\,\cdot\,)\circ A$ one computes

$$\begin{aligned}(\Phi\circ\partial+\partial\circ\Phi)(A\cup B)&=((\Phi\circ\partial+\partial\circ\Phi)A)\cup B+A\cup((\Phi\circ\partial+\partial\circ\Phi)B)\\&\quad+((\mathrm{i}(e_i)\circ\partial+\partial\circ\mathrm{i}(e_i))A)\cup\mathrm{i}\Big(\frac{\partial}{\partial e_i}\Big)B\\&\quad+(-1)^a(\mathrm{i}(e_i)A)\cup\Big(\partial\circ\mathrm{i}\Big(\frac{\partial}{\partial e_i}\Big)-\mathrm{i}\Big(\frac{\partial}{\partial e_i}\Big)\circ\partial\Big)B.\end{aligned} \tag{3-7}$$

The operators $(\mathrm{i}(e_i)\circ\partial+\partial\circ\mathrm{i}(e_i))$ and $(\partial\circ\mathrm{i}(\partial/\partial e_i)-\mathrm{i}(\partial/\partial e_i)\circ\partial)$ are graded commutators of derivations of the $\cup$-product and are therefore graded derivations.

Thus they are determined by their action on $D^{-1}_{\mathrm{poly}}(C \times \mathfrak{g}^*)$ and $D^0_{\mathrm{poly}}(C \times \mathfrak{g}^*)$. The first one obviously vanishes. The second coincides on these generators with

$$A \longmapsto -\Big(\frac{\partial}{\partial e_i} \cup A + (-1)^a A \cup \frac{\partial}{\partial e_i}\Big)$$

and the proposition is shown. □

As in [Esposito et al. 2022b], we define a homotopy on the equivariant polydifferential operators

$$\begin{aligned}\hat{h} : (\mathrm{S}\mathfrak{g}^* \otimes D^d_{\mathrm{poly}}(C \times \mathfrak{g}^*))^{\mathrm{G}} \ni P \otimes D \longmapsto& \\ (-1)^{d+1}\, \mathrm{i_s}(e_i) P \otimes D \cup \frac{\partial}{\partial e_i} &\in (\mathrm{S}\mathfrak{g}^* \otimes D^{d+1}_{\mathrm{poly}}(C \times \mathfrak{g}^*))^{\mathrm{G}}.\end{aligned}$$

The fact that $\hat{h}$ maps invariant elements to invariant ones follows as in the case of polyvector fields. Finally, note that Φ and Ψ are equivariant, whence they can be extended to the equivariant polydifferential operators, where we can show:

Proposition 3.2. *One has on* $(\mathrm{S}\mathfrak{g}^* \otimes D_{\mathrm{poly}}(C \times \mathfrak{g}^*))^{\mathrm{G}}$

$$[\hat{h} - \Phi,\, \partial^{\mathfrak{g}} + [-J, \cdot\,]_{\mathfrak{g}}] = (\deg_{\mathrm{S}\mathfrak{g}^*} + \deg_{\mathfrak{g}})\,\mathrm{id}, \tag{3-8}$$

where $\deg_{\mathfrak{g}}$ *is again the order of differentiations in the direction of* $\mathfrak{g}^*$*-coordinates.*

Proof. From (3-6) we know $[\Phi, \partial^{\mathfrak{g}}] = -(\deg_{\mathfrak{g}} \cdot \mathrm{id} + \Psi)$. In addition, one has for homogeneous $P \otimes D$

$$\begin{aligned}\hat{h} \circ \partial^{\mathfrak{g}}(P \otimes D) &= (-1)^{d+2}\, \mathrm{i_s}(e_i) P \otimes (\partial D) \cup \frac{\partial}{\partial e_i} = -(-1)^{d+1}\, \mathrm{i_s}(e_i) P \otimes \partial\Big(D \cup \frac{\partial}{\partial e_i}\Big) \\ &= -\partial^{\mathfrak{g}} \circ \hat{h}(P \otimes D).\end{aligned}$$

Since we consider only differential operators vanishing on constants, one checks easily that also $[\Phi, [-J, \cdot\,]_{\mathfrak{g}}] = 0$. Finally,

$$\begin{aligned}[\hat{h}, [-J, \cdot\,]_{\mathfrak{g}}](P \otimes D) &= (-1)^d\, \mathrm{i_s}(e_i)(e^j \vee P) \otimes [-J_j, D] \cup \frac{\partial}{\partial e_i} \\ &\qquad + (-1)^{d+1} e^j \vee \mathrm{i_s}(e_i) P \otimes \Big[-J_j, D \cup \frac{\partial}{\partial e_i}\Big] \\ &= -\Psi(P \otimes D) + (-1)^d e^j \vee \mathrm{i_s}(e_i) P \otimes [-J_j, D] \cup \frac{\partial}{\partial e_i} \\ &\qquad + (-1)^{d+1} e^j \vee \mathrm{i_s}(e_i) P \otimes [-J_j, D] \cup \frac{\partial}{\partial e_i} + \deg_{\mathrm{S}\mathfrak{g}^*}(P)\, P \otimes D \\ &= (\deg_{\mathrm{S}\mathfrak{g}^*} \cdot \mathrm{id} - \Psi) P \otimes D.\end{aligned}$$

Thus the proposition is shown. □

The above constructions work also for the Taylor series expansion of the equivariant polydifferential operators, where we restrict ourselves again to polydifferential operators vanishing on constants. We slightly abuse the notation and denote them again by $D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$; see (3-1). Writing

$$h = \begin{cases} \frac{1}{\deg_{\mathrm{S}\mathfrak{g}^*} + \deg_{\mathfrak{g}}}(\hat{h} - \Phi) & \text{if } \deg_{\mathrm{S}\mathfrak{g}^*} + \deg_{\mathfrak{g}} \neq 0, \\ 0 & \text{else,} \end{cases} \tag{3-9}$$

we get the following result:

Proposition 3.3. *One has a deformation retract*

$$\Big(\big(\textstyle\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}[\![\hbar]\!], \partial\Big) \underset{p}{\overset{i}{\rightleftarrows}} (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], \partial + [-J, \cdot\,]) \circlearrowleft h \tag{3-10}$$

where p and i denote the obvious projection and inclusion. This means that one has $pi = \mathrm{id}$ and $\mathrm{id} - ip = [h, \partial + [-J, \cdot\,]]$. Also, the identities $hi = 0 = ph$ hold.

Remark 3.4. Note that one has $h^2 \neq 0$, i.e., the above retract is not a special deformation retract. However, by the results of [Huebschmann 2011b, Remark 2.1] we know that this could also be achieved.

The reduction works now in two steps. At first, we use the homological perturbation lemma from Proposition A.1 to deform the differential on $D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!]$, and in the second step we use the homotopy transfer theorem, see Theorem B.2, to extend the deformed projection to an L_∞-morphism. This will possibly give us higher brackets on $\big(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}[\![\hbar]\!]$ that we have to discuss.

3B. ***Application of the homological perturbation lemma.*** In our setting, the bundle $C \times \mathfrak{g} \to C$ can be equipped with the structure of a Lie algebroid since $\mathfrak{g}$ acts on C by the fundamental vector fields. The bracket of this *action Lie algebroid* is given by

$$[\xi, \eta]_{C \times \mathfrak{g}}(p) = [\xi(p), \eta(p)] - (\mathscr{L}_{\xi_C}\eta)(p) + (\mathscr{L}_{\eta_C}\xi)(p) \tag{3-11}$$

for $\xi, \eta \in \mathscr{C}^\infty(C, \mathfrak{g})$. The anchor is given by $\rho(p, \xi) = -\xi_C|_p$. In particular, one can check that π_{KKS} is the negative of the linear Poisson structure on its dual $C \times \mathfrak{g}^*$ in the convention of [Neumaier and Waldmann 2009].

For Lie algebroids there is a well-known construction of universal enveloping algebras [Moerdijk and Mrčun 2010; Neumaier and Waldmann 2009; Rinehart 1963]. It turns out that in our special case we get a simpler description of the universal enveloping algebra:

Proposition 3.5. *The universal enveloping algebra $U(C \times \mathfrak{g})$ of the action Lie algebroid $C \times \mathfrak{g}$ is isomorphic to $\mathscr{C}^\infty(C) \rtimes U(\mathfrak{g})$ with product*

$$(f, x) \cdot (g, y) = \sum (f\mathscr{L}(x_{(1)})(g), x_{(2)}y). \tag{3-12}$$

Here $y_{(1)} \otimes y_{(2)} = \Delta(y)$ denotes the coproduct on $U(\mathfrak{g})$ induced by extending $\Delta(\xi) = 1 \otimes \xi + \xi \otimes 1$ as an algebra morphism. Also, $\mathscr{L} : U(\mathfrak{g}) \to \mathrm{Diffop}(\mathscr{C}^\infty(C))$

is the extension of the anchor of the action algebroid, that is, of the negative fundamental vector fields, to the universal enveloping algebra. The same holds also in the formal setting of $U_\hbar(\mathfrak{g})$ *with bracket rescaled by* $\hbar$. *Note that in this case one has to rescale* $\mathscr{L}$ *by powers of* $\hbar$, *that is,* $\mathscr{L}_\xi = -\hbar\mathscr{L}_{\xi_C}$ *for* $\xi \in \mathfrak{g}$.

Proof. Note that the product is associative since

$$\begin{aligned}((f,x)\cdot(g,y))\cdot(h,z) &= \sum (f\mathscr{L}(x_{(1)})g, x_{(2)}y)\cdot(h,z)\\ &= \sum (f\mathscr{L}(x_{(1)})g\mathscr{L}(x_{(2)}y_{(1)})h, x_{(3)}y_{(2)}z)\\ &= \sum (f,x)\cdot(g\mathscr{L}(y_{(1)})h, y_{(2)}z) = (f,x)\cdot((g,y)\cdot(h,z)),\end{aligned}$$

where the penultimate identity follows with the coassociativity of Δ and the identity $\mathscr{L}(x)(fg) = \mathscr{L}(x_{(1)})(f)\mathscr{L}(x_{(2)})(g)$. The inclusions $\kappa_C : \mathscr{C}^\infty(C) \to \mathscr{C}^\infty(C) \rtimes U(\mathfrak{g})$ and $\kappa : \mathscr{C}^\infty(C)\otimes\mathfrak{g} \to \mathscr{C}^\infty(C)\rtimes U(\mathfrak{g})$ satisfy

$$[\kappa(s), \kappa_C(f)] = \kappa(\rho(s)f) \quad \text{and} \quad \kappa_C(f)\kappa(s) = \kappa(fs).$$

Thus the universal property gives the desired morphism $U(C\times\mathfrak{g}) \to \mathscr{C}^\infty(C)\rtimes U(\mathfrak{g})$. Recursively we can show that the right-hand side is generated by $u \in \mathscr{C}^\infty(C)$ and $\xi \in \mathscr{C}^\infty(C)\otimes\mathfrak{g}$ which gives the surjectivity of the morphism. Concerning injectivity, suppose $(f^{i_1}, e_{i_1})\cdots(f^{i_n}, e_{i_n}) = 0$ in $\mathscr{C}^\infty(C)\rtimes U(\mathfrak{g})$. We have to show that also $(f^{i_1}e_{i_1})\cdots(f^{i_1}e_{i_1}) = 0$ in $U(C\times\mathfrak{g})$. But this follows from a direct comparison of the terms in the corresponding associated graded algebras. □

It is worth mentioning that in [Huebschmann 1990] the above smashed product (used for Hopf algebras) is studied in a more general context.

Recall that by the Poincaré–Birkhoff–Witt theorem the map

$$\mathrm{S}(\mathfrak{g}) \ni x_1 \vee\cdots\vee x_n \longmapsto \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}\cdots x_{\sigma(n)} \in U(\mathfrak{g})$$

is a coalgebra isomorphism with respect to the usual coalgebra structures induced by extending $\Delta(\xi) = \xi\otimes 1 + 1\otimes\xi$ for $\xi\in\mathfrak{g}$; see, for example, [Berezin 1967; Higgins 1969]. This statement holds also in the case of formal power series in $\hbar$ whence we can transfer the product on the universal enveloping algebra as in Proposition 3.5 to an associative product $\star_{\mathrm{G}} = \mu + \hbar m_{\mathrm{G}}$ on $\mathscr{C}^\infty(C)\otimes\mathrm{S}(\mathfrak{g})[[\hbar]]$.

Lemma 3.6. *The* ***Gutt product*** $\star_{\mathrm{G}}$ *on* $\mathscr{C}^\infty(C)\otimes\mathrm{S}(\mathfrak{g})[[\hbar]]$ *is* G*-invariant and* $J = \mathrm{pr}_{\mathfrak{g}^*} : M = C\times\mathfrak{g}^* \to \mathfrak{g}^*$ *is a momentum map, i.e.,*

$$-\mathscr{L}_{\xi_M} = \frac{1}{\hbar}\,\mathrm{ad}_{\star_{\mathrm{G}}}(J(\xi)). \tag{3-13}$$

Proof. The lemma follows directly from the explicit formula in Proposition 3.5. □

We deform the differential $\partial + [-J, \cdot\,]$ by $[\hbar m_{\mathrm{G}}, \cdot\,]$, that is, exactly by the higher orders of this product. The perturbed differential $\partial^{\mathfrak{g}} + [\hbar m_{\mathrm{G}} - J, \cdot\,] = [\star_{\mathrm{G}} - J, \cdot\,]$ squares indeed to zero since we have with the above lemma

$$[\star_{\mathrm{G}} - J, \cdot\,]^2 = \tfrac{1}{2}[[\star_{\mathrm{G}} - J, \star_{\mathrm{G}} - J], \cdot\,] = [-\hbar\lambda, \cdot\,] = 0,$$

where again $\lambda = e^i \otimes (e_i)_M$. By the homological perturbation lemma as formulated in Section A1 this yields a homotopy retract

$$\Big(\big(\textstyle\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}[\![\hbar]\!], \partial_\hbar\Big) \underset{p_\hbar}{\overset{i_\hbar}{\rightleftarrows}} (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[\![\hbar]\!], [\star_{\mathrm{G}} - J, \cdot\,]) \circlearrowleft h_\hbar \tag{3-14}$$

with $B = [\hbar m_{\mathrm{G}}, \cdot\,]$ and

$$\begin{gathered} A = (\mathrm{id} + Bh)^{-1}B, \quad \partial_\hbar = \partial + pAi, \quad i_\hbar = i - hAi, \\ p_\hbar = p - pAh, \quad h_\hbar = h - hAh; \end{gathered} \tag{3-15}$$

compare with Proposition A.1. More explicitly, we have

$$i_\hbar = \sum_{k=0}^{\infty}(\widetilde{\Phi}\circ B)^k \circ i \quad \text{and} \quad h_\hbar = h \circ \sum_{k=0}^{\infty}(-Bh)^k, \tag{3-16}$$

where $\widetilde{\Phi}$ is the combination of Φ with the degree-counting coefficient from h from (3-9). We want to take a closer look at the induced differential:

Proposition 3.7. *One has*

$$p_\hbar = p \quad \textit{and} \quad \partial_\hbar = \partial + \delta \tag{3-17}$$

with

$$\delta(P\otimes D) = (-1)^d P_{(1)} \otimes D \cup \mathscr{L}_{P_{(2)}} - (-1)^d P \otimes D \cup \mathrm{id}$$

for homogeneous $P \otimes D \in \mathrm{S}\mathfrak{g} \otimes D^d_{\mathrm{poly}}(C)$.

Proof. The fact that $p_\hbar = p$ follows since Bh always adds differentials in the $\mathfrak{g}$-direction. For the deformed differential we compute for homogeneous $P\otimes D \in \mathrm{S}\mathfrak{g}\otimes D^d_{\mathrm{poly}}(C)$ and $f_i \in \mathscr{C}^\infty(C)$

$$\begin{aligned} (\delta(P\otimes D))(f_0, f_1, \dots, f_{d+1}) &= \bigg(p\circ\sum_{k=0}^{\infty}(B\circ\widetilde{\Phi})^k B\circ i(P\otimes D)\bigg)(f_0, f_1, \dots, f_{d+1}) \\ &= p(B(P\otimes D)(f_0, f_1, \dots, f_{d+1})) \\ &= (-1)^d p(\hbar m_{\mathrm{G}}(P\otimes D(f_0, \dots, f_d), f_{d+1}) \\ &= (-1)^d P_{(1)}\otimes D(f_0, \dots, f_d)\cdot\mathscr{L}_{P_{(2)}} f_{d+1} \end{aligned}$$

for all $P_{(2)} \neq 1$. Here we used the explicit form of the Gutt product as in Proposition 3.5 and the fact that $\mathrm{S}(\mathfrak{g})[\![\hbar]\!]$ and $U_\hbar(\mathfrak{g})$ are isomorphic coalgebras. □

Since the classical homotopy equivalence data (3-10) is not a special deformation retract, the perturbed one is also not a special one. But it still has some nice properties.

Proposition 3.8. *One has*

$$p_\hbar \circ h_\hbar = 0 = h_\hbar \circ i_\hbar \quad \textit{and} \quad p_\hbar \circ i_\hbar = \mathrm{id}\,. \tag{3-18}$$

Proof. The properties follow from $p \circ h = 0 = h \circ i$, $p \circ i = \mathrm{id}$ and $\widetilde{\Phi}^2 = 0$. □

Thus the deformation retract (3-14) satisfies all properties of a special deformation retract except for $h_\hbar \circ h_\hbar = 0$, and we can still apply the homotopy transfer theorem.

3C. ***Application of the homotopy transfer theorem.*** We use the homotopy transfer theorem to extend $p_\hbar$ to an L_∞-morphism. We denote the L_∞-structure on the Taylor expansion by Q and the extension of $h_\hbar$ to the symmetric algebra as in (B-2) by H. Then applying the homotopy transfer theorem in the form of Theorem B.2 to the deformation retract (3-14) induces higher brackets $(Q_C)^1_k$ on $\big(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}[\![\hbar]\!]$:

Proposition 3.9. *The maps*

$$(Q_C)^1_1 = -\partial_\hbar, \quad (Q_C)^1_{k+1} = P^1_k \circ Q^k_{k+1} \circ i_\hbar^{\vee(k+1)}, \tag{3-19}$$

where

$$\begin{aligned} P^1_1 &= p_\hbar = p, \\ P^1_{k+1} &= \bigg(\sum_{\ell=2}^{k+1} Q^1_{C,\ell} \circ P^\ell_{k+1} - P^1_k \circ Q^k_{k+1}\bigg) \circ H_{k+1} \quad \textit{for } k \geq 1, \end{aligned} \tag{3-20}$$

induce a codifferential Q_C on the symmetric coalgebra of

$$\bigg(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\bigg)^{\mathrm{G}}[\![\hbar]\!][1]$$

and an L_∞-quasi-isomorphism

$$P : (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow \Bigg(\bigg(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\bigg)^{\mathrm{G}}[\![\hbar]\!], Q_C\Bigg).$$

Proof. The proposition follows directly from the homotopy transfer theorem as in Theorem B.2. Note that we do not need $h_\hbar \circ h_\hbar = 0$, only the other properties of a special deformation retract from Proposition 3.8. □

Let us take a closer look at the higher brackets Q_C induced by the homotopy transfer theorem. One can check that they vanish:

Proposition 3.10. *One has*

$$(Q_C)^1_{k+1} = 0 \quad \textit{for all } k \geq 2. \tag{3-21}$$

Proof. In the higher brackets with $k \geq 2$ one has

$$H_k \circ Q^k_{k+1} \circ i_\hbar^{\vee(k+1)},$$

where in H_k one component consists of the application of $\widetilde{\Phi}$, that is, contains an insertion of a linear coordinate function e_t. We claim that it has to vanish. At first, it is clear that the image of i vanishes if one argument is e_t. Let us now show that $i_\hbar$ satisfies the same property, which directly gives the proposition since then also the bracket vanishes if one inserts a $\mathfrak{g}^*$-coordinate.

For homogeneous $D \in D^d_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ and $f_0, \ldots, f_d \in \prod_i (\mathrm{S}^i \mathfrak{g} \otimes \mathscr{C}^\infty(C))$, we can compute

$$\begin{aligned}
&\Phi \circ B(D)(f_0, \ldots, f_d) \\
&= \sum_{t=1}^n \sum_{j=1}^d \sum_{i=0}^j (-1)^i (B(D))\Big(f_0, \ldots, f_{i-1}, e_t, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d\Big) \\
&= \sum_{t=1}^n \sum_{j=1}^d \sum_{i=0}^j (-1)^i \Big(\hbar m_{\mathrm{G}}\Big(f_0, D(f_1, \ldots, f_{i-1}, e_t, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d)\Big) \\
&\qquad\qquad - D\Big(\hbar m_{\mathrm{G}}(f_0, f_1), \ldots, f_{i-1}, e_t, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d\Big) + \cdots \\
&\qquad\qquad + (-1)^d \hbar m_{\mathrm{G}}\Big(D\Big(f_0, \ldots, f_{i-1}, e_t, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_{d-1}\Big), f_d\Big)\Big).
\end{aligned}$$

If D vanishes if one of the arguments is a $\mathfrak{g}^*$-coordinate, then this simplifies to

$$\begin{aligned}
&\Phi \circ B(D)(f_0, \ldots, f_d) \\
&\quad = \sum_{j=0}^d \Big(\hbar m_{\mathrm{G}}\Big(e_t, D\Big(f_0, \ldots, f_{i-1}, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d\Big)\Big) \\
&\qquad\qquad - D\Big(\hbar m_{\mathrm{G}}(e_t, f_0), \ldots, f_{i-1}, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d\Big)\Big) \\
&\qquad\qquad + \sum_{j=1}^d D\Big(\hbar m_{\mathrm{G}}(f_0, e_t), \ldots, f_{i-1}, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d\Big) + \cdots,
\end{aligned}$$

where e_t is always an argument of $\hbar m_{\mathrm{G}}$. In particular, we know $\hbar m_{\mathrm{G}}(e_i, e_j) = \frac{\hbar}{2}[e_i, e_j]$ and we see that the above sum vanishes if one of the functions f_i is a $\mathfrak{g}^*$-coordinate, that is, $\Phi \circ B(D)$ has the same vanishing property as D. The same holds for $\widetilde{\Phi} \circ B(D)$; hence by induction the image of $i_\hbar$ has the same property and the proposition is shown. □

Considering $(Q_C)^1_2$, we can simplify (3-19) to

$$(Q_C)^1_2 = \sum_{k=1}^{\infty} p \circ Q^1_2 \circ ((\widetilde{\Phi} \circ B)^k \circ i \vee i + i \vee (\widetilde{\Phi} \circ B)^k \circ i) + p \circ Q^1_2 \circ (i \vee i),$$

where the last term is the usual Gerstenhaber bracket. This is clear since $\widetilde{\Phi}$ adds a differential in the $\mathfrak{g}^*$-direction and the bracket can only eliminate it on one argument. Recall that we also have the canonical projection $\mathrm{pr} : \big(\prod_{i=0}^{\infty}(\mathrm{S}^i \mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}} \to D_{\mathrm{poly}}(M_{\mathrm{red}})$ which projects first to symmetric degree zero and then restricts to $\mathscr{C}^\infty(C)^{\mathrm{G}} \cong \mathscr{C}^\infty(M_{\mathrm{red}})$. It is a DGLA morphism with respect to classical structures, namely, Hochschild differentials and Gerstenhaber brackets. We extend it $\hbar$-linearly and can show that it is also a DLGA morphism with respect to the deformed DGLA structure Q_C:

Proposition 3.11. *The projection induces a DGLA morphism*

$$\mathrm{pr} : \Bigg(\bigg(\prod_{i=0}^{\infty}(\mathrm{S}^i \mathfrak{g} \otimes D_{\mathrm{poly}}(C))\bigg)^{\mathrm{G}}[[\hbar]], Q_C\Bigg) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}). \tag{3-22}$$

Proof. By the explicit form of the differential $(Q_C)^1_1 = -\partial_\hbar = -(\partial + \delta)$ from Proposition 3.7 we know that $\mathrm{pr} \circ \partial_\hbar = \mathrm{pr} \circ \partial = \partial \circ \mathrm{pr}$. Thus it only remains to show that $\mathrm{pr} \circ (Q_C)^1_2 = Q^1_2 \circ \mathrm{pr}^{\vee 2}$, which is equivalent to showing

$$\mathrm{pr} \circ \sum_{k=1}^{\infty} p \circ Q^1_2 \circ ((\widetilde{\Phi} \circ B)^k \circ i \vee i + i \vee (\widetilde{\Phi} \circ B)^k \circ i) = 0. \tag{$*$}$$

In the proof of Proposition 3.10 we computed $\Phi \circ B(D)$ of some $D \in D^d_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ and we saw that the image of i vanishes if one inserts a $\mathfrak{g}^*$-coordinate and that $\Phi \circ B$ preserves this property. Therefore, we got for such a D that vanishes if one of the arguments is e_t

$$\begin{aligned} &\Phi \circ B(D)(f_0, \dots, f_d) \\ &\quad = \sum_{j=0}^{d} \bigg(\hbar m_{\mathrm{G}}\Big(e_t, D\Big(f_0, \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big)\Big) \\ &\qquad\qquad - D\Big(\hbar m_{\mathrm{G}}(e_t, f_0), \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big)\bigg) \\ &\qquad\qquad + \sum_{j=1}^{d} D\Big(\hbar m_{\mathrm{G}}(f_0, e_t), \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big) - \cdots \\ &\qquad\qquad\qquad - D\Big(f_0, \dots, , f_{d-1}, \hbar m_{\mathrm{G}}\Big(e_t, \frac{\partial}{\partial e_t} f_d\Big)\Big), \end{aligned} \tag{$**$}$$

where $f_0, \dots, f_d \in \prod_i (\mathrm{S}^i \mathfrak{g} \otimes \mathscr{C}^\infty(C))$. Let us consider now $(*)$ applied to homogeneous $P \otimes D \vee Q \otimes D'$, where $P, Q \in \mathrm{S}\mathfrak{g}$ and $D, D' \in D_{\mathrm{poly}}(C)[\![\hbar]\!]$. At first we note that this is zero if both $P \neq 1 \neq Q$ since the Gerstenhaber bracket can cancel at most one term. Similarly, it is zero if both $P = 1 = Q$. Thus we consider without loss of generality D, $Q \otimes D'$ with $Q \neq 1$ and $D \in (D^d_{\mathrm{poly}}(C))^{\mathrm{G}}[\![\hbar]\!]$, where the only possible contributions are

$$\mathrm{pr} \circ p \circ Q_2^1\big(((\widetilde{\Phi} \circ B)^k D) \vee (Q \otimes D')\big) = (-1)^{d+(dd')}\, \mathrm{pr} \circ p\big(((\widetilde{\Phi} \circ B)^k D) \circ (Q \otimes D')\big)$$

for all $k \geq 1$. Note that, up to a sign, this is $((\widetilde{\Phi} \circ B)^k D) \circ (Q \otimes D')$ applied to invariant functions $\mathscr{C}^\infty(C)^{\mathrm{G}}[\![\hbar]\!]$ and then projected to $\mathrm{S}^0\mathfrak{g}$. But on invariant functions the vertical vector fields and the differentials in the $\mathfrak{g}^*$-direction vanish, and we have only one slot where they can give a nontrivial contribution, namely $Q \otimes D'$. We fix the symmetric degree $Q \in \mathrm{S}^i\mathfrak{g}$ and get

$$\begin{aligned}
\mathrm{pr} \circ p \circ Q_2^1\big(((\widetilde{\Phi} \circ B)^k D) \vee (Q \otimes D')\big)&\\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\big((\Phi(B(\widetilde{\Phi}B)^{k-1}D)_i) \circ (Q \otimes D')\big)\\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\big((\Phi B(\widetilde{\Phi}B)^{k-1}D) \circ (Q \otimes D')\big).
\end{aligned}$$

Here $(B(\widetilde{\Phi}B)^{k-1}D)_i$ denotes the component of $B(\widetilde{\Phi}B)^{k-1}D$ with i differentiations in the $\mathfrak{g}^*$-direction. The $1/i$ comes from the degree of the homotopy (3-9) since we have no $\mathrm{S}\mathfrak{g}^*$-degree and since the only term that can be nontrivial is the one with i differentiations in the $\mathfrak{g}^*$-direction applied to Q. We compute with $(**)$

$$\begin{aligned}
\mathrm{pr} \circ p \circ & Q_2^1\big(((\widetilde{\Phi} \circ B)^k D) \vee (Q \otimes D')\big)\\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\big((\Phi B(\widetilde{\Phi}B)^{k-1}D) \circ (Q \otimes D')\big)\\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\Big(\Big(-\hbar \mathscr{L}_{(e_t)_C} \circ \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D \circ \frac{\partial}{\partial e_t}\Big) \circ (Q \otimes D')\\
&\qquad\qquad - \Big(\mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D \circ \Big(\hbar m_{\mathrm{G}}\Big(e_t, \frac{\partial}{\partial e_t}\cdot\Big)\Big)\Big) \circ (Q \otimes D')\Big)\\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\Big((-\hbar \mathscr{L}_{(e_t)_C} \circ \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D) \circ \Big(\frac{\partial}{\partial e_t} Q \otimes D'\Big)\\
&\qquad\qquad - (\mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D) \circ \Big(\Big(\hbar m_{\mathrm{G}}\Big(e_t, \frac{\partial}{\partial e_t}\cdot\Big)\Big) \circ (Q \otimes D')\Big)\Big).
\end{aligned}$$

But we know $\hbar m_{\mathrm{G}}(e_t, \cdot) = -\hbar \mathscr{L}_{(e_t)_C} + \hbar m_{\mathfrak{g}}(e_t, \cdot)$, where $\hbar m_{\mathfrak{g}}$ denotes the higher components of the Gutt product on $\mathfrak{g}^*$. Moreover, we have by the invariance

$$-[\mathscr{L}_{(e_t)_C}, \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D]_{\mathrm{G}} = \Big[-f_{tk}^j e_j \frac{\partial}{\partial e_k}, \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D\Big]_{\mathrm{G}}$$

and thus

$$
\begin{aligned}
&\hbar \operatorname{pr} \circ p\Big(\big(-[\mathscr{L}_{(e_t)_C}, \operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi} \circ B)^{k-1} D]_{\mathrm{G}}\big) \circ \Big(\frac{\partial}{\partial e_t} Q \otimes D'\Big)\Big) \\
&\qquad = \hbar \operatorname{pr} \circ p\Big(\Big(\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi} \circ B)^{k-1} D \circ \Big(f_{tk}^j e_j \frac{\partial}{\partial e_k}\Big)\Big) \circ \Big(\frac{\partial}{\partial e_t} Q \otimes D'\Big)\Big) \\
&\qquad = \hbar \operatorname{pr} \circ p\Big((\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi} \circ B)^{k-1} D) \circ \Big(f_{tk}^j e_j \frac{\partial}{\partial e_k} \frac{\partial}{\partial e_t} Q \otimes D'\Big)\Big) = 0.
\end{aligned}
$$

The only remaining terms are

$$
\begin{aligned}
&\operatorname{pr} \circ p \circ Q_2^1\big(((\widetilde{\Phi} \circ B)^k D) \vee (Q \otimes D')\big) \\
&\qquad = (-1)^{d+(dd')} \operatorname{pr} \circ p\big((\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi} \circ B)^k D) \circ (Q \otimes D')\big) \\
&\qquad = -\frac{(-1)^{d+(dd')}}{i} \operatorname{pr} \circ p\Big((\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi} \circ B)^{k-1} D) \circ \Big(\hbar m_{\mathfrak{g}}(e_t \frac{\partial}{\partial e_t} Q) \otimes D'\Big)\Big).
\end{aligned}
$$

We know that $\hbar m_{\mathfrak{g}}(e_t, (\partial/\partial e_t) Q)$ is either zero or in $\mathrm{S}^{>0}\mathfrak{g}$ and the statement follows by induction. □

In particular, we can compose this projection pr with the L_∞-projection from Proposition 3.9 that we constructed with the homotopy transfer theorem. Summarizing, we have shown:

Theorem 3.12. *There exists an L_∞-morphism*

$$
D_{\mathrm{red}} = \operatorname{pr} \circ P : (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!], \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}).
$$

Finally, as in the polyvector field case in [Esposito et al. 2022b], we can twist the above morphism to obtain an L_∞-morphism from the curved equivariant polydifferential operators into the Cartan model and therefore also into the polydifferential operators on M_{red}, see Proposition 2.2 for the basics of the twisting procedure.

Proposition 3.13. *Twisting the reduction L_∞-morphism D_{red} from Theorem 3.12 with $-\hbar m_{\mathrm{G}}$ yields an L_∞-morphism*

$$
D_{\mathrm{red}}^{-\hbar m_{\mathrm{G}}} : (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], \hbar\lambda, \partial + [-J, \cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!], \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}),
$$

where $\lambda = \sum_i e^i \otimes (e_i)_M$ denotes the curvature.

Proof. At first we check that the curvature is indeed given by

$$
e^i \otimes [-e_i, -\hbar m_{\mathrm{G}}]_{\mathrm{G}} = e^i \otimes -[e_i, \cdot\,]_{\star_{\mathrm{G}}} = e^i \otimes (\hbar \mathscr{L}_{(e_i)_C} - \hbar \operatorname{ad}(e_i)) = \hbar\lambda; \tag{3-23}
$$

see Lemma 3.6. The only thing left to show is that the DGLA structure on M_{red} is not changed, which is equivalent to

$$
\sum_{k=1}^{\infty} \frac{(-\hbar)^k}{k!} (D_{\mathrm{red}})_k^1 (m_{\mathrm{G}} \vee \cdots \vee m_{\mathrm{G}}) = 0. \tag{3-24}
$$

But using the explicit form of P from Proposition 3.9 we see inductively that P vanishes if every argument has a differential in the $\mathfrak{g}^*$-direction and the statement is shown. □

Remark 3.14. In the polyvector field case from [Esposito et al. 2022b, Proposition 4.29] we saw that the structure maps of the twisted morphism coincide with the structure maps of the original one. In our case it is not clear, that is, one might indeed have $D_{\mathrm{red}}^{-\hbar m_{\mathrm{G}}} \neq D_{\mathrm{red}}$.

This reduction morphism can be used to obtain a reduction morphism of the equivariant polydifferential operators $D_{\mathfrak{g}}^{\bullet}(M)$ of more general manifolds $M \neq C \times \mathfrak{g}^*$. More explicitly, assuming that the action is proper around C and free on C, we can restrict at first to $M_{\mathrm{nice}} \cong U_{\mathrm{nice}} \subset C \times \mathfrak{g}^*$, that is, we have

$$\begin{aligned} \cdot|_{U_{\mathrm{nice}}} : (D_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial^{\mathfrak{g}} - [J, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \\ \longrightarrow (D_{\mathfrak{g}}(U_{\mathrm{nice}})[[\hbar]], \hbar\lambda|_{U_{\mathrm{nice}}}, \partial^{\mathfrak{g}} - [J|_{U_{\mathrm{nice}}}, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}). \end{aligned}$$

But on U_{nice} we can perform the Taylor expansion that is a morphism of curved DGLAs

$$\begin{aligned} D_{\mathfrak{g}^*} : (D_{\mathfrak{g}}(U_{\mathrm{nice}})[[\hbar]], \hbar\lambda|_{U_{\mathrm{nice}}}, \partial^{\mathfrak{g}} - [J|_{U_{\mathrm{nice}}}, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \\ \longrightarrow \big(D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], \hbar\lambda, \partial - [J, \cdot\,], [\,\cdot\,,\cdot\,]\big). \end{aligned}$$

Finally, we can compose it with $D_{\mathrm{red}}^{-\hbar m_{\mathrm{G}}}$ and obtain the following statement:

Theorem 3.15. *The composition of the above morphisms is an L_∞-morphism*

$$D_{\mathrm{red}} : (D_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial^{\mathfrak{g}} - [J, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], 0, \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}),$$

called the ***reduction L_∞-morphism****.*

Remark 3.16 (choices). Note that the only noncanonical choice we made is an open neighborhood of C in M which is diffeomorphic to a star shaped open neighborhood of C in $C \times \mathfrak{g}*$. Recall that the choice of this neighborhood works as follows. Take an arbitrary G-equivariant tubular neighborhood embedding $\psi : \nu(C) \to U \subseteq M$, where $\nu(C)$ denotes the normal bundle. Then define

$$\phi : \nu(C) \ni [v_p] \longmapsto \big(p, J(\psi([v_p]))\big) \in C \times \mathfrak{g}^*, \tag{3-25}$$

which is a diffeomorphism in a neighborhood of C. After some suitable restriction we obtain the identification. Nevertheless, we had to choose a G-equivariant tubular neighborhood and any two choices differ by a G-equivariant local diffeomorphism around C

$$A : C \times \mathfrak{g}^* \longrightarrow C \times \mathfrak{g}^*,$$

which is the identity when restricted to C. One can show that in the Taylor expansion

$$D_{\mathfrak{g}^*}(A^* f) = \mathrm{e}^X D_{\mathfrak{g}^*}(f)$$

for a vector field $X \in \prod_{i\geq 1}(\mathrm{S}^i\mathfrak{g} \otimes \mathfrak{X}(C))^{\mathrm{G}} \subseteq D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$. Since any vector field is closed, X does not derive in the $\mathfrak{g}^*$-direction and λ is central, we obtain an inner automorphism

$$\begin{aligned} \mathrm{e}^{[X,\,\cdot\,]} : (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], \hbar\lambda, \partial - [J, \cdot\,], [\,\cdot\,,\cdot\,]) \\ \longrightarrow (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], \hbar\lambda, \partial - [J, \cdot\,], [\,\cdot\,,\cdot\,]) \end{aligned}$$

of curved Lie algebras which acts trivially on the level of equivalence classes of Maurer–Cartan elements. We are certain that the two reduction L_∞-morphisms are homotopic in a suitable curved setting, which, to our knowledge, is not developed yet.

As a last remark of this section, we want to mention a very interesting observation, which is not directly connected to the rest of this paper. Nevertheless, we felt that it can be interesting from many other perspectives.

Remark 3.17 (Cartan model). One can show that the DGLA structure Q_C from Proposition 3.9 on $\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}[\![\hbar]\!]$ restricts to $(\mathrm{S}\mathfrak{g}\otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}[\hbar]$ and hence can be evaluated at $\hbar = 1$. We still have the DGLA map

$$\mathrm{pr} : (\mathrm{S}\mathfrak{g} \otimes D_{\mathrm{poly}}(C))^{\mathrm{G}} \longrightarrow D_{\mathrm{poly}}(M_{\mathrm{red}}).$$

We want to sketch the proof of the fact that this is a quasi-isomorphism, which motivates us to interpret $(\mathrm{S}\mathfrak{g}\otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}$ as a Cartan model for equivariant polydifferential operators, generalizing the Cartan model for equivariant polyvector fields from [Esposito et al. 2022b, Section 4.2].

Picking a G-invariant covariant derivative (not necessarily torsion-free) for which the fundamental vector fields are flat in the fiber direction one can, using the PBW-isomorphism for Lie algebroids (see [Laurent-Gengoux et al. 2021; Nistor et al. 1999]), prove that there is an equivariant cochain map $K : D_{\mathrm{poly}}(C) \to T_{\mathrm{poly}}(C)$ and an equivariant homotopy $h : D^{\bullet}_{\mathrm{poly}}(C) \to D^{\bullet-1}_{\mathrm{poly}}(C)$, such that

$$\begin{tikzcd} T_{\mathrm{poly}}(C) \arrow[r, shift left, "\mathsf{hkr}"] & (D_{\mathrm{poly}}(C), \partial) \arrow[l, shift left, "K"] \arrow[loop right, "h"] \end{tikzcd} \tag{3-26}$$

is a special deformation retract. Additionally, one can show that

$$K(D_1 \cup D_2) = K(D_1) \wedge K(D_2) \quad \text{and} \quad K(\mathscr{L}_P) = \begin{cases} -P_C & \text{for } P \in \mathfrak{g} \subseteq \mathrm{S}\mathfrak{g}, \\ 0 & \text{else,} \end{cases}$$

for $D_1, D_2 \in D_{\mathrm{poly}}(C)$ and $P \in \mathrm{S}\mathfrak{g}$. We extend now (3-26) to

$$\begin{tikzcd} ((\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}, 0) \arrow[r, shift left, "\mathsf{hkr}"] & ((\mathrm{S}\mathfrak{g} \otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}, \partial) \arrow[l, shift left, "K"] \arrow[loop right, "h"] \end{tikzcd}$$

to obtain a special deformation retract. Now we include δ as in Proposition 3.7 and see it as a perturbation of ∂. One can show that the perturbation is small in the sense of the homological perturbation lemma as in [Crainic 2004], and we obtain

$$\begin{tikzcd} ((\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}, \delta) \arrow[r, shift left, "\widehat{\mathsf{hkr}}"] & ((\mathrm{S}\mathfrak{g} \otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}, \partial + \delta) \arrow[l, shift left, "\widehat{K}"] \arrow[loop right, "\hat{h}"] \end{tikzcd}$$

where δ is the differential

$$\delta(P \otimes X) = \mathrm{i}(e^i) P \otimes (e_i)_C \wedge X$$

obtained in [Esposito et al. 2022b, Definition 4.14] on $(\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}$. Finally, one can show that

$$\begin{tikzcd} ((\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}, \delta) \arrow[r, "\widehat{\mathsf{hkr}}"] \arrow[d] & ((\mathrm{S}\mathfrak{g} \otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}, \partial + \delta) \arrow[d] \\ (T_{\mathrm{poly}}(M_{\mathrm{red}}), 0) \arrow[r, "\mathsf{hkr}"] & (D_{\mathrm{poly}}(M_{\mathrm{red}}), \partial) \end{tikzcd}$$

commutes and both the horizontal maps, as well as the left-vertical map, are quasi-isomorphisms, which implies the claim.

4. Comparison of the reduction procedures

At the level of Maurer–Cartan elements, we know that the L_∞-morphism D_{red} from Theorem 3.15 induces a map from equivariant star products $(\star, H)$ with quantum momentum map $H = J + O(\hbar)$ on M to star products $\star_{\mathrm{red}}$ on the reduced manifold M_{red}. We conclude with a comparison of this reduction procedure with the reduction of formal Poisson structures via the quantized Koszul complex as in [Bordemann et al. 2000; Gutt and Waldmann 2010]; see also our adapted version in Appendix A.

We assume for simplicity $M = C \times \mathfrak{g}^*$ and work in the Taylor expansion of the equivariant polydifferential operators. We identify $\mathscr{C}^\infty(C)$ with $\mathrm{prol}\,\mathscr{C}^\infty(C) \subset \mathscr{C}^\infty(C \times \mathfrak{g}^*)$. Let us start with an equivariant star product $(\star, H = J + \hbar H')$ on $C \times \mathfrak{g}^*$, which means that $\hbar\pi_\star - \hbar H' = \star - \star_{\mathrm{G}} - (H - J)$ is a Maurer–Cartan element in

$$(D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,]).$$

Proposition 4.1. *Defining $I_1^1 = i_\hbar$ and $I_k^1 = h_\hbar \circ Q_2^1 \circ I_{k+1}^2$ gives an L_∞-morphism*

$$I : \left(\left(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[[\hbar]],\, Q_C\right) \longrightarrow (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]],\, [\star_{\mathrm{G}}-J,\,\cdot\,],\, [\,\cdot\,,\,\cdot\,]).$$

Moreover, one I is a quasi-inverse of the L_∞-projection P from Proposition 3.9 and one has $P\circ I = \mathrm{id}$.

Proof. Note that we have in general $h_\hbar^2 \neq 0$, but the only part of the homotopy that appears in the above recursions is $\widetilde{\Phi}$, where we know $\widetilde{\Phi}\circ\widetilde{\Phi} = 0$. Therefore, the statement follows from Proposition B.3. $\square$

We get with Corollary B.5:

Corollary 4.2. *The L_∞-morphism I is compatible with the filtration induced by $\hbar$ and*

$$\hbar\widetilde{\pi}_\star = (I\circ P)^1(\overline{\exp}(\hbar\pi_\star - \hbar H')) \in (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]],\, [\star_{\mathrm{G}} - J,\,\cdot\,],\, [\,\cdot\,,\,\cdot\,])$$

is a well-defined Maurer–Cartan element that is equivalent to $\hbar\pi_\star - \hbar H'$. In particular, $(\tilde{\star} = \star_{\mathrm{G}} + \hbar\widetilde{\pi}_\star,\, J)$ is a strongly invariant star product, that is, an equivariant star product such that the quantum momentum map is just the classical momentum map, and it is equivariantly equivalent to $(\star, H)$.

The reduction of $(\tilde{\star}, J)$ via the reduction L_∞-morphism D_{red} is now easy:

Lemma 4.3. *The reduction L_∞-morphism*

$$D_{\mathrm{red}} = \mathrm{pr}\circ P : (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]],\, [\star_{\mathrm{G}}-J,\,\cdot\,],\, [\,\cdot\,,\,\cdot\,]) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]],\, \partial,\, [\,\cdot\,,\,\cdot\,]_{\mathrm{G}})$$

from Theorem 3.12 maps $\hbar\widetilde{\pi}_\star$ to a Maurer–Cartan element $\hbar m_{\mathrm{red}} = \mathrm{pr}\circ P^1(\exp\hbar\widetilde{\pi}_\star)$ in the polydifferential operators on M_{red}. The corresponding star product $\tilde{\star}_{\mathrm{red}} = \mu + \hbar m_{\mathrm{red}}$ is given by

$$\mathrm{pr}^*(u_1\tilde{\star}_{\mathrm{red}}u_2) = \iota^*(\mathrm{prol}(\mathrm{pr}^*u_1)\tilde{\star}\,\mathrm{prol}(\mathrm{pr}^*u_2)) \tag{4-1}$$

for all $u_1, u_2 \in \mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]]$.

Proof. By definition of $\hbar\widetilde{\pi}_\star$ we know $h_\hbar\hbar\widetilde{\pi}_\star = \widetilde{\Phi}(\hbar\widetilde{\pi}_\star) = 0$, and thus

$$\hbar m_{\mathrm{red}} = \mathrm{pr}\circ P^1(\exp\hbar\widetilde{\pi}_\star) = \mathrm{pr}\circ p(\hbar\widetilde{\pi}_\star).$$

Equation (4-1) follows since $\hbar m_{\mathrm{G}}(\mathrm{prol}(\mathrm{pr}^*u_1), \mathrm{prol}(\mathrm{pr}^*u_2)) = 0$. $\square$

Moreover, we know by Lemma A.5 that the BRST reduction of $\mu + \hbar m_{\mathrm{G}} + \hbar\widetilde{\pi}_\star$ coincides with (4-1), and we have shown:

Theorem 4.4. *Let $(\star, H)$ be an equivariant star product on M. Then the reduced star product induced by D_{red} from Theorem 3.12 and the reduced star product via the formal Koszul complex* (A-14) *are equivalent.*

Proof. We know that both reduction procedures map equivalent equivariant star products to equivalent reduced star products. Moreover, we saw above that both reduction procedures coincide on $(\tilde{\star} = \star_{\mathrm{G}} + \hbar \widetilde{\pi}_{\star},\, J)$ which is equivariantly equivalent to $(\star,\, H)$. □

Appendix A: BRST reduction of equivariant star products

We recall a slightly modified version of the reduction of equivariant star products as introduced in [Bordemann et al. 2000; Gutt and Waldmann 2010]; see also [Esposito et al. 2020] for a discussion of this reduction scheme in the context of Hermitian star products. It relies on the quantized Koszul complex and the homological perturbation lemma.

A1: *Homological perturbation lemma.* At first we recall from [Crainic 2004, Theorem 2.4; Reichert 2017, Chapter 2.4] a version of the homological perturbation lemma that is adapted to our setting. Let

$$
\begin{tikzcd}
(C, \mathrm{d}_C) \arrow[r, shift left, "i"] & (D, \mathrm{d}_D) \arrow[l, shift left, "p"] \arrow[loop right, "h"]
\end{tikzcd}
$$

be a homotopy retract (also called homotopy equivalence data), i.e., let (C, d_C) and (D, d_D) be two chain complexes together with two quasi-isomorphisms

$$
i : C \longrightarrow D \quad \text{and} \quad p : D \longrightarrow C \tag{A-1}
$$

and a chain homotopy

$$
h : D \longrightarrow D \quad \text{with} \;\; \mathrm{id}_D - ip = \mathrm{d}_D h + h \mathrm{d}_D \tag{A-2}
$$

between id_D and ip. Then we say that a graded map $B : D_\bullet \longrightarrow D_{\bullet-1}$ with $(\mathrm{d}_D + B)^2 = 0$ is a *perturbation* of the homotopy retract. The perturbation is called *small* if $\mathrm{id}_D + Bh$ is invertible, and the homological perturbation lemma states that in this case the perturbed homotopy retract is a again a homotopy retract; see [Crainic 2004, Theorem 2.4] for a proof.

Proposition A.1 (homological perturbation lemma). *Let*

$$
\begin{tikzcd}
(C, \mathrm{d}_C) \arrow[r, shift left, "i"] & (D, \mathrm{d}_D) \arrow[l, shift left, "p"] \arrow[loop right, "h"]
\end{tikzcd}
$$

be a homotopy retract and let B be small perturbation of d_D. *Then the perturbed data*

$$
\begin{tikzcd}
(C, \hat{\mathrm{d}}_C) \arrow[r, shift left, "I"] & (D, \hat{\mathrm{d}}_D) \arrow[l, shift left, "P"] \arrow[loop right, "H"]
\end{tikzcd} \tag{A-3}
$$

with

$$
\begin{aligned}
A &= (\mathrm{id}_D + Bh)^{-1} B, & \hat{\mathrm{d}}_D &= \mathrm{d}_D + B, & \hat{\mathrm{d}}_C &= \mathrm{d}_C + pAi, \\
I &= i - hAi, & P &= p - pAh, & H &= h - hAh,
\end{aligned}
\tag{A-4}
$$

is again a homotopy retract.

Remark A.2. In [Crainic 2004] it is shown that perturbations of special deformation retracts are again special deformation retracts, which is in general not true for deformation retracts; see Appendix B for the different notions.

We are interested in even simpler complexes of the form

$$
\begin{tikzcd}
0 & D_0 \arrow[l] \arrow[d, shift right, "p"'] \arrow[r, shift right, "h_0"'] & D_1 \arrow[l, shift right, "\mathrm{d}_{D,1}"'] \arrow[r, shift right, "h_1"'] & \cdots \arrow[l, shift right, "\mathrm{d}_{D,2}"'] \\
0 & C_0 \arrow[l] \arrow[u, shift right, "i"'] & 0 \arrow[l]
\end{tikzcd}
\tag{A-5}
$$

In this case, the perturbed homotopy retract corresponding to a small perturbation B according to (A-4) is given by

$$
I = i, \quad P = p - p(\mathrm{id}_D + B_1 h_0)^{-1} B_1 h_0, \quad H = h - h(\mathrm{id}_D + Bh)^{-1} Bh
$$

and, using the geometric power series, this can be simplified to

$$
I = i, \quad P = p(\mathrm{id}_D + B_1 h_0)^{-1}, \quad H = h(\mathrm{id}_D + Bh)^{-1}. \tag{A-6}
$$

Here we denote by $B_1 : D_1 \longrightarrow D_0$ the degree one component of B, analogously for h. By Remark A.2 we know that deformation retracts are in general not preserved under perturbations. However, in this case we see that, starting with a deformation retract, the additional condition $h_0 i = 0$ suffices to guarantee

$$
PI = p(\mathrm{id}_D + B_1 h_0)^{-1} i = pi = \mathrm{id}_{C_0}.
$$

A2: *Quantized Koszul complex.* Let now $(M, \{\,\cdot\,,\,\cdot\,\})$ be a smooth Poisson manifold with a left action of the Lie group G. Moreover, let $J : M \to \mathfrak{g}^*$ be a classical (equivariant) momentum map. As usual, we assume that $0 \in \mathfrak{g}^*$ is a value and a regular value of J and set $C = J^{-1}(\{0\})$. In addition, we require the action to be proper on M (or at least around C) and free on C, which implies that $M_{\mathrm{red}} = C/\,\mathrm{G}$ is a smooth manifold. The reduction via the classical Koszul complex $\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)$ is one way to show that M_{red} is even a Poisson manifold, but we need the quantum version to show that we have an induced star product on M_{red}. The Koszul differential ∂ is given by

$$
\partial : \Lambda^q \mathfrak{g} \otimes \mathscr{C}^\infty(M) \longrightarrow \Lambda^{q-1} \mathfrak{g} \otimes \mathscr{C}^\infty(M), \quad a \mapsto \mathrm{i}(J)a = J_i \, \mathrm{i}_{\mathrm{a}}(e^i)a, \tag{A-7}
$$

where i denotes the left insertion and $J = J_i e^i$ the decomposition of J with respect to a basis $e^1, \dots, e^n$ of $\mathfrak{g}^*$. Then $\partial^2 = 0$ follows immediately with the commutativity of the pointwise product in $\mathscr{C}^\infty(M)$. The differential ∂ is also a derivation with respect to the associative and supercommutative product on the Koszul complex, consisting of the $\wedge$-product on $\Lambda^\bullet \mathfrak{g}$ tensored with the pointwise product on the functions. Also, it is invariant with respect to the induced $\mathfrak{g}$-representation

$$\mathfrak{g} \ni \xi \mapsto \rho(\xi) = \operatorname{ad}(\xi) \otimes \operatorname{id} - \operatorname{id} \otimes \mathscr{L}_{\xi_M} \in \operatorname{End}(\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)) \tag{A-8}$$

as we have

$$\begin{aligned} \partial \rho(e_a)(x \otimes f) &= f_{aj}^k e_k \wedge \mathrm{i}(e^j) \wedge \mathrm{i}(e^i) x \otimes J_{0,i} f + f_{aj}^i \, \mathrm{i}(e^j) x \otimes J_{0,i} f \\ &\qquad\qquad + \mathrm{i}(e^i) x \otimes J_{0,i} \{J_{0,a}, f\}_0 \\ &= \rho(e_a) \partial (x \otimes f) \end{aligned}$$

for all $x \in \Lambda^\bullet \mathfrak{g}$ and $f \in \mathscr{C}^\infty(M)$.

One can show that the Koszul complex is acyclic in positive degree with homology $\mathscr{C}^\infty(C)$ in order zero, and that one has a G-equivariant homotopy

$$h : \Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M) \longrightarrow \Lambda^{\bullet+1} \mathfrak{g} \otimes \mathscr{C}^\infty(M); \tag{A-9}$$

see [Bordemann et al. 2000, Lemma 6; Gutt and Waldmann 2010]. In other words, this means that

$$\operatorname{prol} : (\mathscr{C}^\infty(C), 0) \rightleftarrows (\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M), \partial) : \iota^*, h$$

is a HE data of the special type of (A-5), that is, we have the diagram

$$\begin{tikzcd} 0 & \mathscr{C}^\infty(M) \arrow[l] \arrow[r, shift right, "h_0"'] \arrow[d, shift right, "\iota^*"'] & \Lambda^1 \mathfrak{g} \otimes \mathscr{C}^\infty(M) \arrow[l, shift right, "\partial_1"'] \arrow[r, shift right, "h_1"'] & \cdots \arrow[l, shift right, "\partial_2"'] \\ 0 & \mathscr{C}^\infty(C) \arrow[l] \arrow[u, shift right, "\operatorname{prol}"'] & 0 \arrow[l] \end{tikzcd}$$

For the reduction of equivariant star products, we need to deform it to the *quantized Koszul complex*. The *quantized Koszul differential*

$$\boldsymbol{\partial} : \Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M_{\text{nice}})[[\hbar]] \longrightarrow \Lambda^{\bullet-1} \mathfrak{g} \otimes \mathscr{C}^\infty(M_{\text{nice}})[[\hbar]]$$

is defined by

$$\begin{aligned} &\boldsymbol{\partial}^{(\kappa)}(x \otimes f) = \\ &\qquad \mathrm{i}(e^a) x \otimes H_a \star f - \frac{\hbar}{2} f_{ab}^c e_c \wedge \mathrm{i}(e^a)\, \mathrm{i}(e^b) x \otimes f + \hbar \kappa f_{ab}^b \, \mathrm{i}(e^a)\, (x \otimes f) \end{aligned} \tag{A-10}$$

for $\kappa \in \mathbb{C}[[\hbar]]$, $x \in \Lambda^\bullet \mathfrak{g}[[\hbar]]$ and $f \in \mathscr{C}^\infty(M_{\text{nice}})[[\hbar]]$, where $\Delta = f_{ab}^b e^a$ is the modular one-form of $\mathfrak{g}$.

Remark A.3. Note that in the literature [Bordemann et al. 2000; Gutt and Waldmann 2010] a different convention is used:

$$\boldsymbol{\partial}'^{(\kappa)}(x\otimes f)=\mathrm{i}(e^a)x\otimes f\star H_a+\frac{\hbar}{2}f_{ab}^c e_c\wedge\mathrm{i}(e^a)\,\mathrm{i}(e^b)x\otimes f+\hbar\kappa\,\mathrm{i}(\Delta)\,(x\otimes f)$$

for $\kappa\in\mathbb{C}[\![\hbar]\!]$. In particular, $\boldsymbol{\partial}'^{(\kappa)}$ is left $\star$-linear. However, in order to simplify the comparison of the BRST reduction with the reduction via D_{red} in Section 4, we want the quantized Koszul differential to be right $\star$-linear, which leads to our convention in (A-10).

The reduction of the star product in our convention works analogously to [Bordemann et al. 2000; Gutt and Waldmann 2010] since $\boldsymbol{\partial}^{(\kappa)}$ satisfies all the desired properties:

Lemma A.4. *Let* $(\star, H)$ *be an equivariant star product and* $\kappa\in\mathbb{C}[\![\hbar]\!]$.

(i) *One has* $\boldsymbol{\partial}^{(0)}\circ\mathrm{i}(\Delta)+\mathrm{i}(\Delta)\circ\boldsymbol{\partial}^{(0)}=0$.

(ii) $\boldsymbol{\partial}^{(\kappa)}$ *is right* $\star$*-linear.*

(iii) $\boldsymbol{\partial}^{(\kappa)}=\partial+O(\hbar)$.

(iv) $\boldsymbol{\partial}^{(\kappa)}$ *is* G*-equivariant.*

(v) *One has* $\boldsymbol{\partial}^{(\kappa)}\circ\boldsymbol{\partial}^{(\kappa)}=0$.

Proof. The proof is analogous to [Gutt and Waldmann 2010, Lemma 3.4]. □

Assume that we have chosen a value $\kappa\in\mathbb{C}[\![\hbar]\!]$ and write $\boldsymbol{\partial}=\boldsymbol{\partial}^{(\kappa)}$. Then by the homological perturbation lemma one gets a perturbed homotopy retract

$$\begin{tikzcd}
0 & \mathscr{C}^\infty(M_{\mathrm{nice}})[\![\hbar]\!] \arrow[l] \arrow[r, shift right, "\boldsymbol{h}_0"'] \arrow[d, shift right, "\boldsymbol{\iota}^*"'] & \Lambda^1\mathfrak{g}\otimes\mathscr{C}^\infty(M_{\mathrm{nice}})[\![\hbar]\!] \arrow[l, shift right, "\boldsymbol{\partial}_1"'] \arrow[r, shift right, "\boldsymbol{h}_1"'] & \cdots \arrow[l, shift right, "\boldsymbol{\partial}_2"'] \\
0 & \mathscr{C}^\infty(C)[\![\hbar]\!] \arrow[l] \arrow[u, shift right, "\mathbf{prol}"'] & 0, \arrow[l] &
\end{tikzcd}$$

where

$$\mathbf{prol}=\mathrm{prol},\quad \boldsymbol{\iota}^*=\iota^*(\mathrm{id}+B_1h_0)^{-1},\quad \boldsymbol{h}=h(\mathrm{id}+Bh)^{-1}, \tag{A-11}$$

and where $\boldsymbol{\partial}-\partial=B$; see (A-6). One can show that the *deformed restriction map* $\boldsymbol{\iota}^*$ is given by

$$\boldsymbol{\iota}^*=\iota^*\circ S=\sum_{r=0}\hbar^r\boldsymbol{\iota}^*{}_r:\mathscr{C}^\infty(M_{\mathrm{nice}})[\![\hbar]\!]\longrightarrow\mathscr{C}^\infty(C)[\![\hbar]\!] \tag{A-12}$$

with a G-equivariant formal series of differential operators $S=\mathrm{id}+\sum_{r=1}^\infty\hbar^r S_r$ on $\mathscr{C}^\infty(M_{\mathrm{nice}})$ and with S_r vanishing on constants. Also, it is uniquely determined by the properties

$$\boldsymbol{\iota}^*{}_0 = \iota^*, \quad \boldsymbol{\iota}^* \boldsymbol{\partial}_1 = 0 \quad \text{and} \quad \boldsymbol{\iota}^* \operatorname{prol} = \mathrm{id}_{\mathscr{C}^\infty(C)[[\hbar]]}. \tag{A-13}$$

The reduced star product $\star_{\mathrm{red}}$ on $M_{\mathrm{red}} = C/\mathrm{G}$ is then given by

$$\mathrm{pr}^*(u_1 \star_{\mathrm{red}} u_2) = \boldsymbol{\iota}^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)) \tag{A-14}$$

for all $u_1, u_2 \in \mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]]$; compare with [Bordemann et al. 2000, Theorem 32]. In [Reichert 2017, Lemma 4.3.1] it has been shown that equivariantly equivalent star products reduce to equivalent star products on M_{red}.

For the comparison of the reduction procedures in Section 4 we need the following observation:

Lemma A.5. *Let* $(\star = \mu + \hbar\pi_\star + \hbar m_{\mathrm{G}}, J)$ *be an equivariant star product on* $C \times \mathfrak{g}^*$, *and choose* $\kappa = -1$ *for the quantized Koszul differential. If one has* $\widetilde{\Phi}(\hbar\pi_\star) = 0 = \Phi(\hbar\pi_\star)$, *then it follows for all* $u_1, u_2 \in \mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]]$

$$\mathrm{pr}^*(u_1 \star_{\mathrm{red}} u_2) = \boldsymbol{\iota}^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)) = \iota^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)).$$

Proof. We have for a polynomial function $f = P \otimes \phi \in \mathrm{S}^j \mathfrak{g} \otimes \mathscr{C}^\infty(C) \subset \mathscr{C}^\infty(C \times \mathfrak{g}^*)$

$$\begin{aligned}
(\boldsymbol{\partial} - \partial) h_0(P \otimes \phi) &= \frac{1}{j}\big(\hbar(\pi_\star + m_{\mathrm{G}})(e_i, \mathrm{i}(e^i) P \otimes \phi) + \hbar\kappa f_{ib}^b \,\mathrm{i}(e^i) P \otimes \phi\big) \\
&= \frac{1}{j}(\Phi(\hbar\pi_\star + \hbar m_{\mathrm{G}})(P \otimes \phi) + \hbar\kappa f_{ib}^b \,\mathrm{i}(e^i) P \otimes \phi) \\
&= \frac{1}{j}\big(\hbar m_{\mathrm{G}}(e_i, \mathrm{i}(e^i) P \otimes \phi) + \hbar\kappa f_{ib}^b \,\mathrm{i}(e^i) P \otimes \phi\big) \\
&= \frac{1}{j}\big(\hbar m_{\mathfrak{g}}(e_i, \mathrm{i}(e^i) P) \otimes \phi - \mathrm{i}(e^i) P \otimes \hbar \mathscr{L}_{(e_i)_C} \phi + \hbar\kappa f_{ib}^b \,\mathrm{i}(e^i) P \otimes \phi\big),
\end{aligned}$$

where $\hbar m_{\mathfrak{g}}$ denotes the nontrivial part of the Gutt product on $\mathfrak{g}^*$. We know that $\mathrm{im}(\hbar m_{\mathfrak{g}}(e_i, \cdot)) \in \mathrm{S}^{>0}\mathfrak{g}[[\hbar]]$, hence it follows

$$\iota^* \circ (\boldsymbol{\partial} - \partial) h_0(P \otimes \phi) = \frac{1}{j}\iota^*(-\mathrm{i}(e^i) P \otimes \hbar \mathscr{L}_{(e_i)_C} \phi + \hbar\kappa f_{ib}^b \,\mathrm{i}(e^i) P \otimes \phi). \tag{$*$}$$

On an invariant polynomial $P \otimes \phi \in (\mathrm{S}^j \mathfrak{g} \otimes \mathscr{C}^\infty(C))^{\mathrm{G}}$ we have

$$-\mathrm{i}(e^i) P \otimes \hbar \mathscr{L}_{(e_i)_C} \phi = -\hbar \,\mathrm{i}(e^i) \operatorname{ad}(e_i) P \otimes \phi = -\hbar f_{ij}^i \,\mathrm{i}(e^j) P \otimes \phi,$$

hence $(*)$ vanishes for $\kappa = -1$. Thus we have in this case

$$\mathrm{pr}^*(u_1 \star_{\mathrm{red}} u_2) = \boldsymbol{\iota}^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)) = \iota^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2))$$

and the statement is shown. □

Appendix B: Explicit formulas for the homotopy transfer theorem

In is well-known that L_∞-quasi-isomorphisms always admit L_∞-quasi-inverses. It is also well-known that given a homotopy retract one can transfer L_∞-structures; see, for instance, [Loday and Vallette 2012, Section 10.3]. Explicitly, a homotopy retract (also called homotopy equivalence data) consists of two cochain complexes (A, d_A) and (B, d_B) with chain maps i, p and homotopy h such that

$$
\begin{tikzcd}
(A, \mathrm{d}_A) \arrow[r, shift left, "i"] & (B, \mathrm{d}_B) \arrow[l, shift left, "p"] \arrow[loop right, "h"]
\end{tikzcd}
\tag{B-1}
$$

with $h \circ \mathrm{d}_B + \mathrm{d}_B \circ h = \mathrm{id} - i \circ p$, and such that i and p are quasi-isomorphisms. Then the homotopy transfer theorem states that if there exists a flat L_∞-structure on B, then one can transfer it to A in such a way that i extends to an L_∞-quasi-isomorphism. By the invertibility of L_∞-quasi-isomorphisms there also exists an L_∞-quasi-isomorphism into A denoted by P; see, for example, [Loday and Vallette 2012, Proposition 10.3.9].

In this section we state a version of this statement adapted to our applications. For simplicity, we assume that we have a deformation retract satisfying

$$p \circ i = \mathrm{id}_A .$$

By [Huebschmann 2011b, Remark 2.1] we can assume that we have even a special deformation retract, also called *contraction*, where

$$h^2 = 0, \quad h \circ i = 0 \quad \text{and} \quad p \circ h = 0.$$

Assume now that (B, Q_B) is an L_∞-algebra with $(Q_B)_1^1 = -\mathrm{d}_B$. In the following we give a more explicit description of the transferred L_∞-structure Q_A on A and of the L_∞-projection $P : (B, Q_B) \to (A, Q_A)$ inspired by the symmetric tensor trick [Berglund 2014; Huebschmann 2011a; 2011b; Manetti 2010]. The map h extends to a homotopy $H_n : \mathrm{S}^n(B[1]) \to \mathrm{S}^n(B[1])[-1]$ with respect to $Q^n_{B,n} : \mathrm{S}^n(B[1]) \to \mathrm{S}^n(B[1])[1]$; see, for instance, [Loday and Vallette 2012, p. 383] for the construction on the tensor algebra, which we adapt to our setting as follows. We define the operator

$$K_n : \mathrm{S}^n(B[1]) \longrightarrow \mathrm{S}^n(B[1])$$

by

$$K_n(x_1 \vee \cdots \vee x_n) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \frac{\epsilon(\sigma)}{n-i} ipX_{\sigma(1)} \vee \cdots \vee ipX_{\sigma(i)} \vee X_{\sigma(i+1)} \vee X_{\sigma(n)}.$$

Note that here we sum over the whole symmetric group and not the shuffles, since in this case the formulas are easier. We extend $-h$ to a coderivation to $\mathrm{S}(B[1])$, i.e.,

$$\widetilde{H}_n(x_1 \vee \cdots \vee x_n) := -\sum_{\sigma \in \mathrm{Sh}(1,n-1)} \epsilon(\sigma)\, hx_{\sigma(1)} \vee x_{\sigma(2)} \vee \cdots \vee x_{\sigma(n)}$$

and define

$$H_n = K_n \circ \widetilde{H}_n = \widetilde{H}_n \circ K_n. \tag{B-2}$$

Since i and p are chain maps, we have $K_n \circ Q^n_{B,n} = Q^n_{B,n} \circ K_n$, where $Q^n_{B,n}$ is the extension of the differential $Q^1_{B,1} = -\mathrm{d}_B$ to $\mathrm{S}^n(B[1])$ as a coderivation. Hence we have

$$Q^n_{B,n} H_n + H_n Q^n_{B,n} = (n \cdot \mathrm{id} - ip) \circ K_n,$$

where ip is extended as a coderivation to $\mathrm{S}(B[1])$. A combinatorial and not very enlightening computation shows that finally

$$Q^n_{B,n} H_n + H_n Q^n_{B,n} = \mathrm{id} - (ip)^{\vee n}. \tag{B-3}$$

Now assume that we have a codifferential Q_A and a morphism of coalgebras P with structure maps $P^1_\ell : \mathrm{S}^\ell(B[1]) \to A[1]$ such that P is an L_∞-morphism up to order k, that is,

$$\sum_{\ell=1}^{m} P^1_\ell \circ Q^\ell_{B,m} = \sum_{\ell=1}^{m} Q^1_{A,\ell} \circ P^\ell_m$$

for all $m \leq k$. Then we have the following statement, whose proof can be found in [Esposito et al. 2022b].

Lemma B.1. *Let $P : \mathrm{S}(B[1]) \to \mathrm{S}(A[1])$ be an L_∞-morphism up to order $k \geq 1$. Then*

$$L_{\infty,k+1} = \sum_{\ell=2}^{k+1} Q^1_{A,\ell} \circ P^\ell_{k+1} - \sum_{\ell=1}^{k} P^1_\ell \circ Q^\ell_{B,k+1} \tag{B-4}$$

satisfies

$$L_{\infty,k+1} \circ Q^{k+1}_{B,k+1} = -Q^1_{A,1} \circ L_{\infty,k+1}. \tag{B-5}$$

This allows us to prove one version of the homotopy transfer theorem.

Theorem B.2 (homotopy transfer theorem). *Let (B, Q_B) be a flat L_∞-algebra with differential $(Q_B)^1_1 = -\mathrm{d}_B$ and contraction*

$$\begin{tikzcd} (A, \mathrm{d}_A) \arrow[r, shift left, "i"] & (B, \mathrm{d}_B) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd} \tag{B-6}$$

Then

$$
\begin{aligned}
(Q_A)_1^1 &= -\mathrm{d}_A, & (Q_A)_{k+1}^1 &= \sum_{i=1}^{k} P_i^1 \circ (Q_B)_{k+1}^i \circ i^{\vee(k+1)}, \\
P_1^1 &= p, & P_{k+1}^1 &= L_{\infty,k+1} \circ H_{k+1} \qquad \text{for } k \geq 1
\end{aligned}
$$

turns (A, Q_A) *into an* L_∞*-algebra with* L_∞*-quasi-isomorphism* $P : (B, Q_B) \to (A, Q_A)$*. In addition, one has* $P_k^1 \circ i^{\vee k} = 0$ *for* $k \neq 1$*.*

Proof. We observe $P_{k+1}^1(ix_1 \vee \cdots \vee ix_{k+1}) = 0$ for all $k \geq 1$ and $x_i \in A$, which directly follows from $h \circ i = 0$, and thus $H_{k+1} \circ i^{\vee(k+1)} = 0$. Suppose that Q_A is a codifferential up to order $k \geq 1$, i.e., $\sum_{\ell=1}^m (Q_A)_\ell^1 (Q_A)_m^\ell = 0$ for all $m \leq k$, and that P is an L_∞-morphism up to order $k \geq 1$. We know that these conditions are satisfied for $k = 1$ and we show that they hold for $k+1$. Starting with Q_A we compute

$$
\begin{aligned}
(Q_A Q_A)_{k+1}^1 &= (Q_A Q_A)_{k+1}^1 \circ P_{k+1}^{k+1} \circ i^{\vee(k+1)} \\
&= \sum_{\ell=1}^{k+1} (Q_A Q_A)_\ell^1 P_{k+1}^\ell i^{\vee(k+1)} \\
&= (Q_A Q_A P)_{k+1}^1 i^{\vee(k+1)} \\
&= \sum_{\ell=2}^{k+1} (Q_A)_\ell^1 (Q_A P)_{k+1}^\ell i^{\vee(k+1)} + (Q_A)_1^1 (Q_A P)_{k+1}^1 i^{\vee(k+1)} \\
&= \sum_{\ell=2}^{k+1} (Q_A)_\ell^1 (P Q_B)_{k+1}^\ell i^{\vee(k+1)} + (Q_A)_1^1 (Q_A)_{k+1}^1 \\
&= (Q_A P Q_B)_{k+1}^1 i^{\vee(k+1)} - (Q_A)_1^1 (Q_A)_{k+1}^1 + (Q_A)_1^1 (Q_A)_{k+1}^1 \\
&= \sum_{\ell=1}^{k} (Q_A P)_\ell^1 (Q_B)_{k+1}^\ell i^{\vee(k+1)} + (Q_A P)_{k+1}^1 (Q_B)_{k+1}^{k+1} i^{\vee(k+1)} \\
&= \sum_{\ell=1}^{k} (P Q_B)_\ell^1 (Q_B)_{k+1}^\ell i^{\vee(k+1)} + (Q_A P)_{k+1}^1 i^{\vee(k+1)} (Q_A)_{k+1}^{k+1} \\
&= -(P Q_B)_{k+1}^1 i^{\vee(k+1)} (Q_A)_{k+1}^{k+1} + (Q_A P)_{k+1}^1 i^{\vee(k+1)} (Q_A)_{k+1}^{k+1} \\
&= -(Q_A)_{k+1}^1 (Q_A)_{k+1}^{k+1} + (Q_A)_{k+1}^1 (Q_A)_{k+1}^{k+1} = 0.
\end{aligned}
$$

By the same computation as in Lemma B.1, where one in fact only needs that Q_A is a codifferential up to order $k+1$, it follows that

$$
L_{\infty,k+1} \circ Q_{B,k+1}^{k+1} = -Q_{A,1}^1 \circ L_{\infty,k+1}.
$$

It remains to show that P is an L_∞-morphism up to order $k+1$. We have

$$
\begin{aligned}
P_{k+1}^1 \circ (Q_B)_{k+1}^{k+1} &= L_{\infty,k+1} \circ H_{k+1} \circ (Q_B)_{k+1}^{k+1} \\
&= L_{\infty,k+1} - L_{\infty,k+1} \circ (Q_B)_{k+1}^{k+1} \circ H_{k+1} - L_{\infty,k+1} \circ (i \circ p)^{\vee(k+1)} \\
&= L_{\infty,k+1} + (Q_A)_1^1 \circ P_{k+1}^1
\end{aligned}
$$

since

$$
\begin{aligned}
L_{\infty,k+1} \circ (i \circ p)^{\vee(k+1)} &= \Big(\sum_{\ell=2}^{k+1} Q^1_{A,\ell} \circ P^\ell_{k+1} - \sum_{\ell=1}^{k} P^1_\ell \circ Q^\ell_{B,k+1} \Big) \circ (i \circ p)^{\vee(k+1)} \\
&= (Q_A)^1_{k+1} \circ p^{\vee(k+1)} - (Q_A)^1_{k+1} \circ p^{\vee(k+1)} = 0.
\end{aligned}
$$

Therefore

$$
P^1_{k+1} \circ (Q_B)^{k+1}_{k+1} - (Q_A)^1_1 \circ P^1_{k+1} = L_{\infty,k+1},
$$

i.e., P is an L_∞-morphism up to order $k+1$. The statement follows inductively. □

A special case of the above theorem, for i being a DGLA morphism, was proven in [Esposito et al. 2022b, Proposition 3.2]. We also want to give an explicit formula for a L_∞-quasi-inverse of P, generalizing [Esposito et al. 2022b, Proposition 3.3].

Proposition B.3. *The coalgebra map $I : \mathrm{S}^\bullet(A[1]) \to \mathrm{S}^\bullet(B[1])$ recursively defined by the maps $I^1_1 = i$ and $I^1_{k+1} = h \circ L_{\infty,k+1}$ for $k \geq 1$ is an L_∞-quasi inverse of P. Since $h^2 = 0 = h \circ i$, one even has $I^1_{k+1} = h \circ \sum_{\ell=2}^{k+1} Q^1_{B,\ell} \circ I^\ell_{k+1}$ and $P \circ I = \mathrm{id}_A$.*

Proof. We proceed by induction. Assume that I is an L_∞-morphism up to order k; then we have

$$
\begin{aligned}
I^1_{k+1} Q^{k+1}_{A,k+1} - Q^1_{B,1} I^1_{k+1} &= -Q^1_{B,1} \circ h \circ L_{\infty,k+1} + h \circ L_{\infty,k+1} \circ Q^{k+1}_{A,k+1} \\
&= -Q^1_{B,1} \circ h \circ L_{\infty,k+1} - h \circ Q^1_{B,1} \circ L_{\infty,k+1} \\
&= (\mathrm{id} - i \circ p) L_{\infty,k+1}.
\end{aligned}
$$

We used that $Q^1_{B,1} = -\mathrm{d}_B$ and the homotopy equation of h. Moreover, we get with $p \circ h = 0$

$$
\begin{aligned}
p \circ L_{\infty,k+1} &= p \circ \Big(\sum_{\ell=2}^{k+1} Q^1_{B,\ell} \circ I^\ell_{k+1} - \sum_{\ell=1}^{k} I^1_\ell \circ Q^\ell_{A,k+1} \Big) \\
&= \sum_{\ell=2}^{k+1} (P \circ Q_B)^1_\ell \circ I^\ell_{k+1} - \sum_{\ell=2}^{k+1} \sum_{i=2}^{\ell} P^1_i \circ Q^i_{B,\ell} \circ I^\ell_{k+1} - Q^1_{A,k+1} \\
&= \sum_{\ell=2}^{k+1} (Q_A \circ P)^1_\ell \circ I^\ell_{k+1} - \sum_{i=2}^{k+1} \sum_{\ell=i}^{k+1} P^1_i \circ Q^i_{B,\ell} \circ I^\ell_{k+1} - Q^1_{A,k+1} \\
&= Q^1_{A,k+1} - \sum_{i=2}^{k+1} \sum_{\ell=i}^{k+1} P^1_i \circ I^i_\ell \circ Q^\ell_{A,k+1} - Q^1_{A,k+1} = 0,
\end{aligned}
$$

and therefore I is an L_∞-morphism. □

Remark B.4. In the homotopy transfer theorem the property $h^2 = 0$ is not needed, and that one can also adapt the above construction of I to this more general case.

Note that there exists a homotopy equivalence relation $\sim$ between L_∞-morphisms, see, for example, [Dolgushev 2007], such that equivalent L_∞-morphisms map

Maurer–Cartan elements to equivalent Maurer–Cartan elements; see, for instance, [Bursztyn et al. 2012, Lemma B.5] for the case of DGLAs and [Kraft 2021, Proposition 1.4.6] for the case of flat L_∞-algebras.

Corollary B.5. *In the above setting one has $P \circ I = \mathrm{id}_A$ and $I \circ P \sim \mathrm{id}_B$. In particular, assume that one has complete descending filtrations on A, B such that all the maps are compatible. Then every Maurer–Cartan element $\pi \in \mathcal{F}^1 B$ is equivalent to $(I \circ P)^1(\overline{\exp}(\pi))$.*

Proof. By [Kraft and Schnitzer 2021, Proposition 3.8] P admits a quasi-inverse I' such that $P \circ I' \sim \mathrm{id}_A$ and $I' \circ P \sim \mathrm{id}_B$, which implies

$$I \circ P = \mathrm{id}_B \circ I \circ P \sim I' \circ P \circ I \circ P = I' \circ P \sim \mathrm{id}_B .$$

The rest of the statement is then clear. □

Acknowledgements

The authors are grateful to Ryszard Nest and Boris Tsygan for helpful comments. This work was supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA - INdAM). Schnitzer is supported by the DFG research training group “gk1821: Cohomological Methods in Geometry”.

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Received April 2, 2022. Revised July 17, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 325, No. 1, 2023
https://doi.org/10.2140/pjm.2023.325.85

THE MAXIMAL SYSTOLE OF HYPERBOLIC SURFACES WITH MAXIMAL S^3-EXTENDABLE ABELIAN SYMMETRY

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We study the maximal systole of hyperbolic surfaces with certain symmetries. We give the formula for the maximal systole of the surfaces that admit the largest S^3-extendable abelian group symmetry. The result is obtained by parametrizing such surfaces and enumerating all possible systoles.

1. Introduction

The systole is an important topic in the study of hyperbolic surfaces. The systole has applications in various areas on surfaces, e.g, the Mumford's classical compactness criterion [Mumford 1971] and the Weil–Peterson metric [Wolpert 2017; Wu 2019] in Teichmüller theory, and the spectrum of the Laplacian [Ballmann et al. 2016; 2018; Mondal 2014] and the optimal systolic ratio [Chen and Li 2015; Croke and Katz 2003; Gromov 1983] in differential geometry. For a survey on the study of the systole, see Parlier [2014].

We use the term "systole" to refer to either the minimal length of a closed geodesic on a hyperbolic surface, or a closed geodesic realizing this length, by abuse of notation. The systole can also be regarded as a real-valued function on the moduli space $\mathcal{M}_g$ of all closed hyperbolic surfaces of genus g or the Teichmüller space $\mathcal{T}_g$. The maximal value of the systole function on $\mathcal{M}_g$ is called the maximal systole in genus g. The maximal systole can be realized by Mumford's compactness criterion. It is quite difficult to compute the exact value of the maximal systole. The only known case is genus 2, for which the maximal systole is realized by the Bolza surface [Jenni 1984].

It is also interesting to study the maximal value of the systole function on certain subspaces of $\mathcal{M}_g$. Bavard [1992] obtained the maximal systole of genera 2 and 5 on hyperelliptic surfaces. Schmutz [1993] gave a necessary and sufficient condition for the local maxima of the systole function and he gave some examples of local maxima with polyhedral symmetry. Fortier Bourque and Rafi [2022] constructed surfaces with locally maximal systoles and trivial symmetry.

Buser and Sarnak [1994] constructed surfaces with systoles larger than $\frac{4}{3}\log g$ by arithmetic methods for infinitely many genera. Katz, Schaps, and Vishne [Katz

MSC2020: 30F45.
Keywords: systole, hyperbolic surfaces, S^3-extendable symmetry.

et al. 2007] obtained more surfaces with this lower bound. Hurwitz surfaces are among their examples. Petri and Walker [2018] and Petri [2018] gave concrete examples with systoles larger than $\frac{4}{7}\log g - K$.

Inspired by [Katz et al. 2007; Schmutz 1993], we are interested in the maximal systole of hyperbolic surfaces with certain symmetries. We consider hyperbolic surfaces with the largest S^3-extendable abelian symmetry. The S^3-extendable symmetry on a topological surface was introduced in [Wang et al. 2013; 2015]. A group action G on the genus-g topological surface Σ_g is *S^3-extendable* if there exist an embedding $i : \Sigma_g \to S^3$ and an injective homomorphism $\phi : G \to \mathrm{SO}(4)$ such that for any $g \in G$, the following diagram commutes:

$$\begin{tikzcd} \Sigma_g \arrow[r, "g"] \arrow[d, "i"] & \Sigma_g \arrow[d, "i"] \\ S^3 \arrow[r, "\phi(g)"] & S^3 \end{tikzcd}$$

When restricted to finite abelian groups, the maximal order of an S^3-extendable group action on Σ_g is $2g+2$ [Wang et al. 2013]. Such a group action can be realized as an isometry group action on a hyperbolic surface, and we say such a hyperbolic surface has the *maximal S^3-extendable abelian symmetry* or call the surface a *hyperbolic $\Gamma(2,n)$ surface* [Wang et al. 2013] where $n = g+1$.

Hyperbolic surfaces that admit an isometric S^3-extendable abelian group action of maximal order form a 2-dimensional subset of $\mathcal{M}_g$, and we consider the systole function on this subspace. Our main result is the following:

Theorem 1. *The maximal value of the systole function on hyperbolic $\Gamma(2,n)$ surfaces is*

$$2\,\mathrm{arccosh}\, K,$$

where

$$K = \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} + \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} + \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} - \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} + \tfrac{1}{6}(L+3),$$

and $L = 4\cos^2\frac{\pi}{n}$. The maximal value is obtained when

$$(c,t) = \left(\mathrm{arccosh}\, K, 2\,\mathrm{arccosh}\,\frac{K+1}{2\cos\frac{\pi}{n}}\right).$$

(*The symbols c and t are defined in Section* 2).

The maximal value of systoles of the $\Gamma(2,n)$-surfaces for small genera are shown in Table 1. We remark that the $\Gamma(2,3)$ surface is exactly the Bolza surface that realizes the maximal systole in genus 2.

genus	$\Gamma(2, n)$ maximal systole
2	3.0571
3	3.6478
4	3.9078
5	4.0464
6	4.1291

Table 1. Maximal systole of surface with largest S^3-extendable abelian symmetry.

Compared with the work [Bai et al. 2021] on the systole of surfaces with large cyclic symmetry for which the surfaces with large cyclic symmetry have a unique geometric structure, the surfaces studied in this paper form a two-dimensional subspace of $\mathcal{M}_g$, and the methods in the papers are quite different. We obtain our result by classifying the family of curves that are potentially the shortest geodesics on surfaces admitting this symmetry, and then determine when the systole is maximal.

The paper is organized as follows. In Section 2 we describe how to construct all hyperbolic $\Gamma(2, n)$ surfaces and determine their symmetry. In Section 3, we give a useful lemma (Lemma 3) on the intersection properties of systoles. In Section 4, we prove that for any $\Gamma(2, n)$ surface, there are only four closed geodesics in the quotient orbifold of the surface by its symmetric group that can lift to systoles of the $\Gamma(2, n)$ surface (Proposition 7). In the last section, by calculating the length of these four curves and the differentials of these lengths, we get a condition for when the $\Gamma(2, n)$ surface has the maximal systole (Proposition 9) and calculate its length.

2. The symmetry of $\Gamma(2, n)$ surfaces

In this section, we construct the hyperbolic $\Gamma(2, n)$ surface and describe the geometry and topology of its quotient by its symmetry group.

Let $\Sigma_{0,n}$ be the surface of genus 0 with n boundaries, endowed with a hyperbolic structure so that its boundaries are geodesics and $\Sigma_{0,n}$ admits an isometric rotation of order n, as indicated in Figure 1. Each boundary circle is called a *cuff* of Σ. The shortest geodesic connecting two adjacent boundary circles or its image under the isometric rotation is called a *seam*. A seam is perpendicular to the two boundary circles it connects. The n seams cut $\Sigma_{0,n}$ into two isometric right-angled hyperbolic $2n$-gons with geodesic boundary edges. The hyperbolic structure of $\Sigma_{0,n}$ is parametrized by the length of its cuffs, called the *cuff length*. We denote the cuff length by $2c$, where $c \in \mathbb{R}_+$ is called the *half cuff length*.

Two copies of $\Sigma_{0,n}$ with the same cuff length can be glued together along the cuffs to form a closed surface, so that the two rotations on each copy can be extended

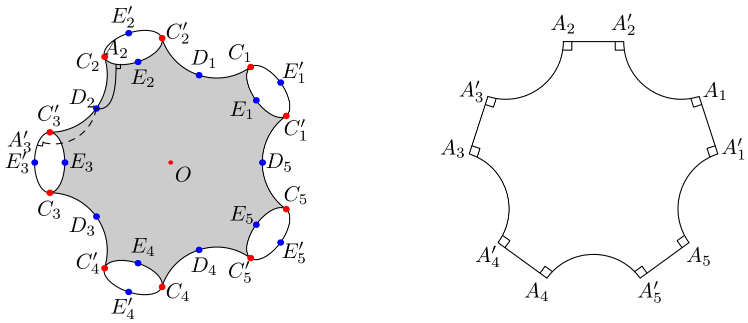

Figure 1. $\Sigma_{0,n}$ and the right-angled $2n$-gon.

to the glued surface. We call the resulting surface a *hyperbolic* $\Gamma(2, n)$*-surface*. Two seams on the surface are *paired* if they connect the same two cuffs. Similar to the Fenchel–Nielsen coordinates on the Teichmüller space, a hyperbolic $\Gamma(2, n)$-surface can be parametrized by (c, t), where c is the half cuff length and t is the "*twist parameter*". The twist parameter t equals 0 if (any) two paired seams form a closed geodesic. The hyperbolic $\Gamma(2, n)$ surface with parameter (c, t) is obtained from the hyperbolic $\Gamma(2, n)$ surface with parameter $(c, 0)$ by performing a Fenchel–Nielsen deformation of length t simultaneously along each cuff. Here the *Fenchel–Nielsen deformation* on a hyperbolic surface X along a simple closed geodesic $\alpha \subset X$ with length t is constructed by cutting X along α and then regluing the boundary curves with a left twist of length t. We may assume $0 \le t \le c$ since the surface with parameter (c, t) is isometric to the surface with parameter $(c, t + 2c)$ while the surface with parameter (c, t) is the reflection of the surface with parameter $(c, 2c - t)$ when $0 \le t \le 2c$.

The symmetry group of a hyperbolic $\Gamma(2, n)$ surface is $D_n \oplus (\mathbb{Z}/2\mathbb{Z})$, where D_n is the order n dihedral group. This symmetric group is generated by three rotations σ, τ, and ρ. As illustrated in Figure 2, σ is the order n rotation that maps each n-holed sphere to itself, τ is the order 2 rotation of each n-holed sphere, and ρ is the order 2 rotation exchanging the two n-holed spheres.

For a $\Gamma(2, n)$ surface X, the quotient $X/\langle\rho\rangle$ is a spherical orbifold with $2n$ singular points of order 2, denoted as $S^2(2, \dots, 2)_X$ (Figure 3, top left). The quotient $X/\langle\rho, \sigma\rangle$ is a spherical orbifold with four singular points of order 2, 2, n, n, respectively, denoted as $S^2(2, 2, n, n)_X$ (Figure 3, top right). The quotient $X/\langle\rho, \sigma, \tau\rangle$ is a spherical orbifold with four singular points of order 2, 2, 2, n, respectively, denoted as $S^2(2, 2, 2, n)_X$ (Figure 3, bottom). In Figure 3, top right, C and C' are the two order 2 singular points and O and O' are the order n singular points. In Figure 3, bottom, C, D, E are the order 2 singular points and O is the order n singular point. We will abbreviate the subscript X when there is no confusion. Denote the quotient

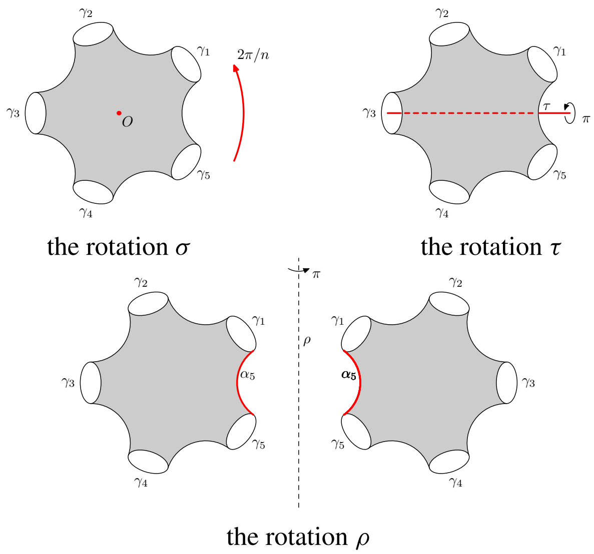

Figure 2. Generators for the isometry group of the hyperbolic $\Gamma(2, n)$ surface.

branched covering maps as

$$
\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[rr, bend left, "p"] \arrow[rrr, bend right, "q"'] & S^2(2,2,\dots,2)_X \arrow[r, "\pi'"] & S^2(2,2,n,n)_X \arrow[r, "\pi''"] & S^2(2,2,2,n)_X
\end{tikzcd}
\tag{2-1}
$$

For a cuff γ_i in X, let A and $\rho(A)$ be the endpoints of two paired seams between γ_i and γ_{i-1} on the cuff γ_i. The Fenchel–Nielsen deformation gives a geodesic of length t between A and $\rho(A)$, and we let C be the midpoint. The other two paired

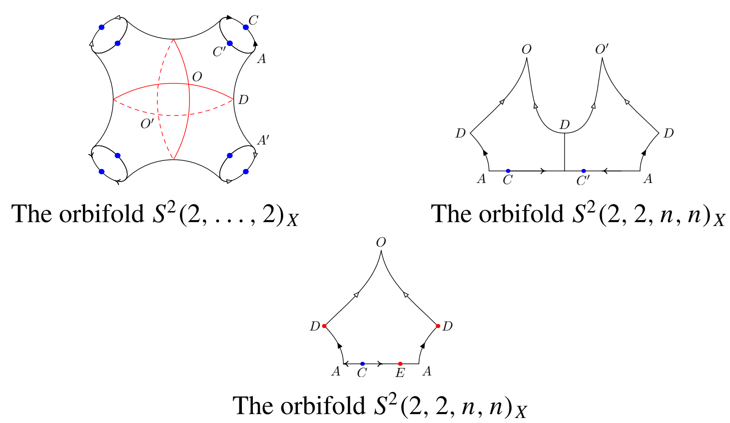

Figure 3. Orbifolds for the hyperbolic $\Gamma(2, n)$ surface.

seams give another midpoint C'. Then C and C' are fixed points of the rotation ρ on γ_i and $|AC|$ is half the twist parameter t. It follows that in $S^2(2,2,n,n)_X$ (Figure 3, top right), we have

$$|CC'| = c, \quad |AC| = \tfrac{1}{2}t$$

and in $S^2(2,2,2,n)_X$ (Figure 3, bottom) we have

$$|CE| = \tfrac{1}{2}c, \quad |AC| = \tfrac{1}{2}t.$$

3. The intersection properties of the systoles

In this section, we rule out curves on $\Gamma(2,n)$ surface that cannot be the systole. The following lemma is classical and well known.

Lemma 2. *Any systole on a closed hyperbolic surface is simple and any two systoles intersect at most once. On the orbifold $S^2(2,2,\dots,2)$, $S^2(2,2,n,n)$ or $S^2(2,2,2,n)$, any simple closed curve is separating, and any two simple closed curves are either disjoint or intersect at least twice.*

The following lemma is used to rule out curves on $\Gamma(2,n)$ that cannot be the systole.

Lemma 3. *Given a hyperbolic $\Gamma(2,n)$ surface X, let π, π', π'' denote the branched covering maps*

$$X \xrightarrow{\pi} S^2(2,2,\dots,2)_X \xrightarrow{\pi'} S^2(2,2,n,n)_X \xrightarrow{\pi''} S^2(2,2,2,n)_X$$

Then under the maps π, $\pi'\circ\pi$ and $\pi''\circ\pi'\circ\pi$, the image of a systole on X has no self-intersection at any regular point on the targeting orbifold, and the images of two systoles do not intersect at any regular point on the targeting orbifold.

Proof. (1) If α is a simple closed curve in X, $\pi(\alpha)$ has a self-intersection point p. Then $\pi^{-1}(p)$ consists of two points, both are the intersection points of $\pi^{-1}(\pi(\alpha))$. By the definition of double branched cover, $\pi^{-1}(\pi(\alpha))$ consists of either one curve or two curves with equal length. Since α is simple, $\pi^{-1}(\pi(\alpha))$ consists of two curves. These two curves intersect at least twice, therefore cannot be systole.

We assume α and β are two simple closed curves with equal length on X, Hence the shape of $\pi(\alpha)$ and $\pi(\beta)$ has two possibilities: S^1 or a segment whose endpoints

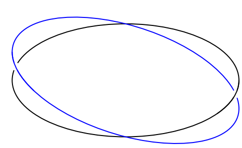

Figure 4. Case (a): $\pi(\alpha)\cup\pi(\beta)$.

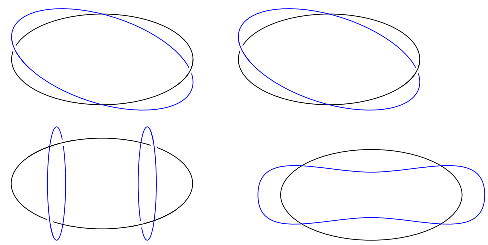

Figure 5. Case (a): The double covers of $\pi(\alpha)\cup\pi(\beta)$. The three types are shown in the top, the bottom left, and the bottom right, respectively.

are branched points of the branched cover π. We also assume p is the intersection point of $\pi(\alpha)$ and $\pi(\beta)$.

(a) If $\pi(\alpha)$ and $\pi(\beta)$ are simple closed curves, then $\pi(\alpha)$ intersects $\pi(\beta)$ at least twice by Lemma 2. Recall that there are two types of double covers of S^1, namely S^1 and $S^1 \coprod S^1$. Then there are three types of double covers of $\pi(\alpha)\cup\pi(\beta)$, shown in Figure 5.

In all the cases, α intersects β at least twice, which contradicts to Lemma 2.

(b) If $\pi(\alpha)$ is a segment while $\pi(\beta)$ is a simple closed curve (Figure 6, left), then there are two types of the double (branched) covers of $\pi(\alpha)\cup\pi(\beta)$ shown in Figure 6, middle and right.

If the double branched cover of $\pi(\alpha)\cup\pi(\beta)$ is the case shown in Figure 6, middle, then it is clear that the curve α and β have at least two intersections. Therefore, α and β cannot be systoles.

If the double branched cover of $\pi(\alpha)\cup\pi(\beta)$ is the case shown in Figure 6, right, we assume $\tilde{p}$ is one of the branched point of π in Figure 6, right. Therefore $\pi_*([\tilde{\beta}])=\pi_*([\tilde{\beta}'])$ in $\pi_1(\pi(X),\pi(\tilde{p}))$. Here $[\tilde{\beta}]$ and $[\tilde{\beta}']$ are elements of $\pi_1(X,\tilde{p})$ represented by $\tilde{\beta}$ and $\tilde{\beta}'$. It contradicts the injectivity of π_* (π is a covering map).

(c) If both $\pi(\alpha)$ and $\pi(\beta)$ are segments (Figure 7), then $|\pi^{-1}(\pi(\alpha))\cap\pi^{-1}(\pi(\beta))|\geq 2$ since the intersection point of $\pi(\alpha)$ and $\pi(\beta)$ is a regular point. However, both $\pi^{-1}(\pi(\alpha))$ and $\pi^{-1}(\pi(\beta))$ are connected. Therefore $|\alpha\cap\beta|\geq 2$, so that α and β cannot be systole.

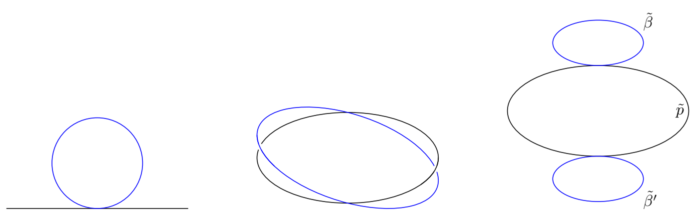

Figure 6. Case (b): $\pi(\alpha)\cup\pi(\beta)$ (left) and the double covers of $\pi(\alpha)\cup\pi(\beta)$ (middle and right).

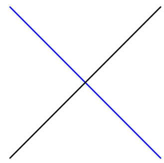

Figure 7. Case (c): $\pi(\alpha)\cup\pi(\beta)$.

(2) Let α be a systole of X; then by (1), $\pi(\alpha)$ has no self-intersection and won't intersect the image of another systole at regular points. Therefore, if $\pi'\pi(\alpha)$ has self-intersection at regular points, then it implies that either $\pi(\alpha)$ intersects itself or it intersects another lift of $\pi'\pi(\alpha)$. Therefore $\pi'\circ\pi(\alpha)$ has no self-intersections.

By exactly the same argument, we can prove that the images of two systoles of X on $S^2(2,2,n,n)$ do not intersect at any regular point of the orbifold.

The case for $\pi''\circ\pi'\circ\pi$ is similar to the case for $\pi'\circ\pi$. □

4. The image of systoles on $S^2(2, 2, 2, n)$

For a $\Gamma(2,n)$ surface X, we find geodesics in $S^2(2,2,2,n)_X$ that lift to the systoles in X in this section.

Lemma 4. *For a $\Gamma(2,n)$ surface X, a systole's image in the orbifold $S^2(2,2,2,n)_X$ (Figure 3, bottom) has only two possibilities*:

(1) *A geodesic segment joining two order-two singular points (C and D, C and E or D and E).*

(2) *A simple closed geodesic passing through C.*

Proof. By Lemma 3, the image of a systole of X is a simple closed geodesic or a geodesic segment joining two singular points.

(1) Image of a systole of X cannot be a simple closed curve not passing through any singular point of the orbifold. Such a curve separates $S^2(2,2,2,n)_X$ by Lemma 2. On each side of the curve, there are two singular points; otherwise, the curve lifts to null-homotopic curves in X. The order of both singular points on one side is two; hence the geodesic homotopic to this curve is the geodesic joining these two points.

(2) No systole's image passes through the order n singular point O. This point lifts to a regular point in $S^2(2,2,\dots,2)_X$, and a segment through O lifts to n segments intersecting at the preimage of O. Then by Lemma 3, this conclusion holds.

(3) The simple closed curve passing through D or E cannot lift to a systole of X, since D and E lift to regular points in $S^2(2,2,n,n)_X$, and such curves lift to nonsimple curves (Figure 8).

By (1), (2), (3), this lemma holds. □

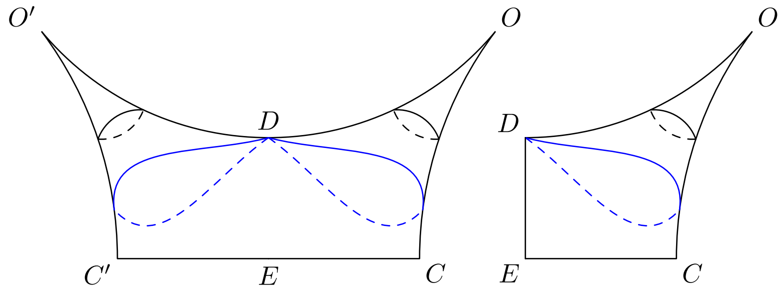

Figure 8. A simple closed curve passing through D in $S^2(2,2,2,n)$ and its lift.

In order to obtain the systole of a $\Gamma(2, n)$ surface, the next step is to find the geodesic in $S^2(2, 2, 2, n)_X$ joining C and D (or C and E or D and E or the simple closed geodesic through C), whose lift in X is the shortest one among all geodesics joining C and D (or C and E or D and E or the simple closed geodesic through C, respectively).

Lemma 5. *For a $\Gamma(2, n)$ surface X, let l and l' be two geodesics in $S^2(2, 2, 2, n)_X$ joining C and D (or joining C and E or D and E or the simple closed geodesics passing through C), and $\tilde{l}$ and $\tilde{l}'$ are their preimages in X, respectively. Then the covering $\tilde{l} \to l$ and $\tilde{l}' \to l'$ are topologically equivalent. More precisely, a homeomorphism $f : l \to l'$ can lift to $\tilde{f} : \tilde{l} \to \tilde{l}'$, letting this diagram commute*:

$$\begin{tikzcd} \tilde{l} \arrow[r, "\tilde{f}"] \arrow[d, "q"'] & \tilde{l}' \arrow[d, "q"] \\ l \arrow[r, "f"] & l' \end{tikzcd}$$

Proof. We provide the proof for the geodesics joining C and D only, since the proofs for other cases are exactly the same.

For any $l \subset S^2(2, 2, 2, n)_X$ joining C and D, there is a curve l'' joining D and E, intersecting l only at D. The double branched cover $\pi'' : S^2(2, 2, n, n)_X \to S^2(2, 2, 2, n)_X$ can be constructed by gluing two copies of $S^2(2, 2, 2, n)_X \backslash l''$ along their boundaries. Hence the preimage of l in $S^2(2, 2, n, n)_X$ is a segment (denoted as $\tilde{l}_1$) joining C and C' for any l. Therefore the coverings of any two segments joining C and D are equivalent (Figure 9).

Similarly, for any $\tilde{l}_1 \subset S^2(2, 2, n, n)_X$ joining C and C', there is a segment l_1'' joining O and O', intersecting $\tilde{l}_1$ at exactly one point. Thus we can construct the n-fold cyclic branched cover of $S^2(2, 2, n, n)_X$ by gluing n-copies of $S^2(2, 2, n, n)_X \backslash l_1''$. Since for any $\tilde{l}_1$, we always choose a curve l_1'' intersecting $\tilde{l}_1$ once, the covering

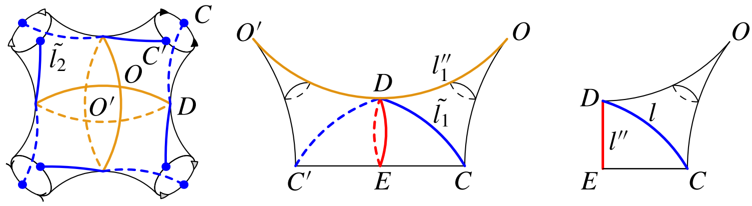

Figure 9. l and its lifts.

of $\tilde{l}_1$ by its preimage in $S^2(2, 2, \dots, 2)_X$ (denoted by $\tilde{l}_2$) is topologically unique (Figure 9).

The multicurve $\tilde{l}_2 \subset S^2(2, 2, \dots, 2)_X$ consists of segments joining the singular points. Therefore its preimage in the $\Gamma(2, n)$ surface X (a manifold with no singular points) is topologically unique. □

Corollary 6. *Let $l, l' \subset S^2(2, 2, 2, n)_X$ be geodesic segments joining C and D, and $\alpha, \alpha' \subset X$ be simple closed geodesics lifted from l and l', respectively. If $|l| < |l'|$, then $|\alpha| < |\alpha'|$.*

This conclusion also holds for geodesics joining C and E, geodesics joining D and E, or simple closed geodesics passing through C.

Proof. By Lemma 5,

$$\frac{|\alpha|}{|l|} = \frac{|\alpha'|}{|l'|}.$$ □

Proposition 7. *For a $\Gamma(2, n)$ surface X, there are only four possible geodesics in $S^2(2, 2, 2, n)_X$ that lift to systoles in X. They are the shortest geodesics joining C and D, joining C and E and joining D and E, and the shortest simple closed geodesic passing through C, denoted as l_{CD}, l_{CE}, l_{DE} and l_C, respectively. Figure 10 describes the geometry of these four curves.*

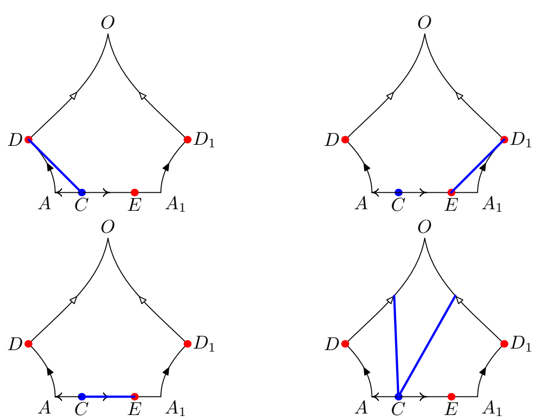

Figure 10. The four geodesics that possibly lift to systoles of X.

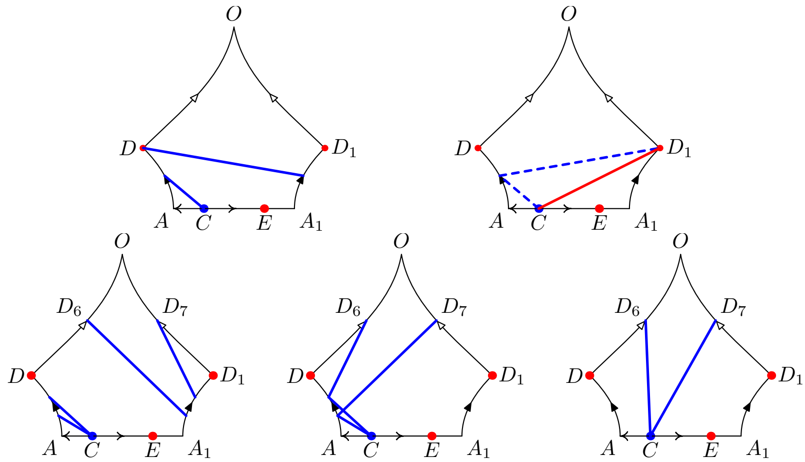

Figure 11. Looking for the shortest geodesic joining C and D (top) and looking for the shortest simple closed geodesic passing through C (bottom).

Proof. By Corollary 6, the goal of this proposition is to describe the shortest geodesics in $S^2(2,2,2,n)_X$ joining C and D, joining C and E, joining D and E and the simple closed geodesic passing through C.

(1) The shortest geodesic joining C and D: Let's consider the pentagon shown in Figure 11, a fundamental domain of $S^2(2,2,2,n)_X$. If a geodesic $l \subset S^2(2,2,2,n)_X$ joining C and D consists of more than one segment in the pentagon (Figure 11, top left), then by reflecting some of its segments, we obtain a bending geodesic segment joining C and D or C and D_1 with equal length to l and show that l is longer than the segment CD or CD_1.

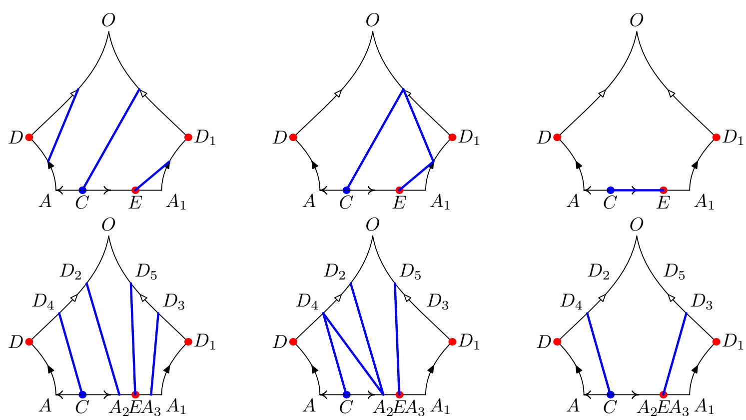

Figure 12. Looking for the shortest geodesic joining C and D (I) (top) and looking for the shortest geodesic joining C and D (II) (bottom).

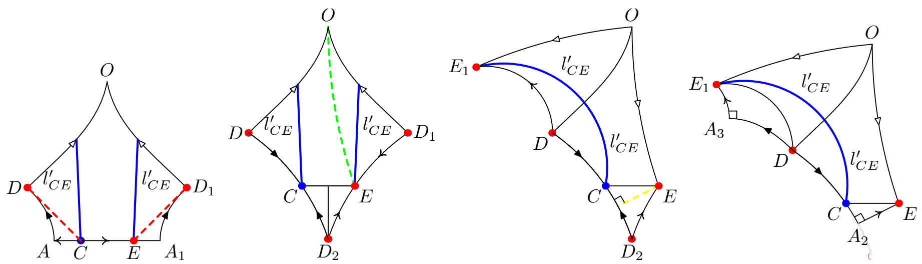

Figure 13. Cut and paste.

The segment CD is shorter than CD_1, because

$$|AD| = |AD_1| \quad \text{and} \quad |AC| < \frac{c}{2} < |A_1C|,$$

and it follows by the hyperbolic cosine law [Buser 2010, p. 454, 2.2.2 (i)]. Therefore, the shortest geodesic joining C and D is the segment l_{CD} shown in Figure 10, top left.

(2) The shortest geodesic joining D and E is the segment l_{DE} shown in Figure 10, top right, by the same argument.

(3) The shortest simple closed geodesic passing through C is the geodesic l_C shown in Figure 10, bottom right, by the same argument; see Figure 11.

(4) The shortest geodesic joining C and E: By reflecting some segments, we get the shortest geodesic is either the geodesic in Figure 12, top right, (denoted by l_{CE}) or the geodesic in Figure 12, bottom right (denoted by l'_{CE}). By the cut-and-paste shown in Figure 13, we see that l_{CE} is shorter than l'_{CE}, hence the shortest geodesic joining C and E is l_{CE} in Figure 10, bottom left. □

5. Calculations

In this section, we represent the length of the curves in Figure 10 by the parameters c and t. Then by these formulae, we give a condition of the $\Gamma(2,n)$ surface having the longest systole. For convenience, we call this surface a *maximal surface*. Finally, we calculate the systole length of this surface.

Recall that in the pentagons in Figure 10, $|CE| = c/2$, $|AC| = t/2$, $|A_1E| = (c-t)/2$ and $\angle DOD_1 = 2\pi/n$. We assume $AD = A_1D_1 = s/2$. Then in one of the two half pieces of the pentagon (Figure 14), by hyperbolic trigonometry [Buser 2010, p. 454, 2.3.1 (i)],

$$\sinh\frac{c}{2}\sinh\frac{s}{2} = \cos\frac{\pi}{n}. \tag{5-1}$$

Therefore, directly, for the lengths of the geodesics in Figure 10, we have

$$|l_{CE}| = \frac{c}{2}, \tag{5-2}$$

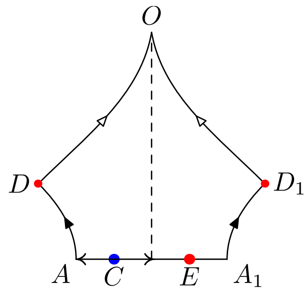

Figure 14. The pentagon.

and by the hyperbolic cosine law in right-angled triangles [Buser 2010, p. 454, 2.2.2 (i)]

$$\cosh|l_{CD}| = \cosh|CD| = \cosh\frac{t}{2}\cosh\frac{s}{2}; \tag{5-3}$$

$$\cosh|l_{DE}| = \cosh|D_1E| = \cosh\frac{c-t}{2}\cosh\frac{s}{2}. \tag{5-4}$$

To calculate the length of l_C (Figure 10, bottom right), we treat the pentagon as a fundamental domain for the orbifold $S^2(2, 2, 2, n)$ in $\mathbb{H}^2$. Then in the joining of two pentagons shown in Figure 15, left, l_C is realized by the segment CC'. Hence its length is

$$\begin{aligned}\cosh|l_C| &= \cosh|CC'| \\ &= \cosh|AA_1'|\cosh|AC|\cosh|A_1'C'| - \sinh|AC|\sinh|A_1'C'| \\ &= \cosh s\cosh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big) - \sinh\frac{t}{2}\sinh\Big(c-\frac{t}{2}\Big)\end{aligned} \tag{5-5}$$

by a trigonometric formula [Buser 2010, p. 38, 2.3.2].

Now we are ready to prove the following:

Proposition 8. *In the $\Gamma(2, n)$ surface X_0 with maximal systole among all the $\Gamma(2, n)$ surfaces, l_{DE} in $S^2(2, 2, 2, n)_{X_0}$ cannot lift to a systole of this surface.*

Proof. Recall that by Proposition 7, in $S^2(2, 2, 2, n)_{X_0}$, there are only four geodesics that can lift to systoles of X_0, namely l_{CD}, l_{DE}, l_{CE} and l_C in Figure 10.

If l_{DE} lifts to a systole, then l_{CD} and l_{CE} cannot lift to systoles. This is because, when lifting to $S^2(2, 2, n, n)$, l_{DE} intersects l_{CD} and l_{CE} at regular points D and E of the orbifold (Figure 15, right), therefore by Lemma 3, they cannot simultaneously become systoles.

Then we calculate the differentials of $|l_{DE}|$ and $|l_C|$, showing that there is vector $(A(c, t), B(c, t))$ such that $\mathrm{d}|l_C|(A, B) > 0$ and $\mathrm{d}|l_{DE}|(A, B) > 0$ simultaneously. Since only l_{DE} and l_C can lift to systoles, a surface with a systole lifted from l_{DE} cannot be a maximal surface.

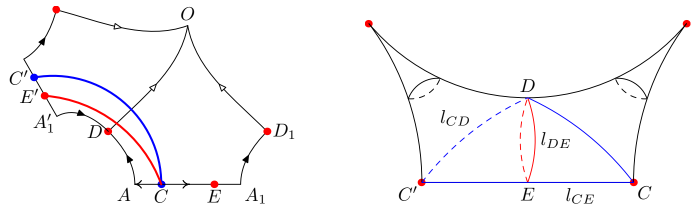

Figure 15. The geodesic l_C (left) and the lift of l_{DE}, l_{CD} and l_{CE} (right).

As a preparation, we differentiate both sides of (5-1) and get

$$\frac{\mathrm{d}s}{\mathrm{d}c} = -\frac{\cosh\frac{c}{2}\sinh\frac{s}{2}}{\cosh\frac{s}{2}\sinh\frac{c}{2}}. \tag{5-6}$$

Then for l_{DE}

$$\begin{aligned}
\frac{\partial|l_{DE}|}{\partial t} &= \frac{\partial}{\partial t}\Big(\cosh\frac{s}{2}\cosh\frac{c-t}{2}\Big) = -\frac{1}{2}\cosh\frac{s}{2}\sinh\frac{c-t}{2},\\
\frac{\partial|l_{DE}|}{\partial c} &= \frac{\partial}{\partial c}\Big(\cosh\frac{s}{2}\cosh\frac{c-t}{2}\Big)\\
&= \frac{1}{2}\Big(\sinh\frac{s}{2}\frac{\mathrm{d}s}{\mathrm{d}c}\cosh\frac{c-t}{2}+\cosh\frac{s}{2}\sinh\frac{c-t}{2}\Big),\\
\mathrm{d}|l_{DE}| &= \frac{\partial|l_{DE}|}{\partial t}\mathrm{d}t+\frac{\partial|l_{DE}|}{\partial c}\mathrm{d}c.
\end{aligned}$$

For l_C

$$\begin{aligned}
\frac{\partial|l_C|}{\partial t} &= \frac{\partial}{\partial t}\Big(\cosh s\cosh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big)-\sinh\frac{t}{2}\sinh\Big(c-\frac{t}{2}\Big)\Big)\\
&= \frac{1}{2}(\cosh s+1)\sinh(t-c),
\end{aligned} \tag{5-7}$$

and

$$\begin{aligned}
\frac{\partial|l_C|}{\partial c} &= \frac{\partial}{\partial c}\Big(\cosh s\cosh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big)-\sinh\frac{t}{2}\sinh\Big(c-\frac{t}{2}\Big)\Big)\\
&= -\frac{\cosh\frac{c}{2}\sinh\frac{s}{2}}{\cosh\frac{s}{2}\sinh\frac{c}{2}}\sinh s\cosh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big)+\cosh s\cosh\frac{t}{2}\sinh\Big(c-\frac{t}{2}\Big)\\
&\qquad -\sinh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big),\\
\mathrm{d}|l_C| &= \frac{\partial|l_C|}{\partial t}\mathrm{d}t+\frac{\partial|l_C|}{\partial c}\mathrm{d}c.
\end{aligned}$$

The two tangent vectors $\mathrm{d}|l_{DE}|$, $\mathrm{d}|l_C|$ are nonzero vectors. When $c>0$, $0\le t\le c$,

$$\frac{\partial|l_{DE}|}{\partial t}<0, \qquad \frac{\partial|l_C|}{\partial t}<0.$$

For any $k\le 0$, $\mathrm{d}|l_{DE}|\ne k\mathrm{d}|l_C|$. Then there is a vector $(A(c,t), B(c,t))$ such that

$$\mathrm{d}|l_{DE}|(A(c,t), B(c,t))>0, \qquad \mathrm{d}|l_C|(A(c,t), B(c,t))>0.$$

By the assumption that l_{DE} lifts to a systole of X_0, only l_{DE} and l_C can lift to a systole of the surface. Then there is another surface with systole bigger than X_0. Therefore X_0 is not maximal. □

From Propositions 7 and 8, we know that only l_{CE}, l_{CD} and l_C in the orbifold $S^2(2, 2, 2, n)$ can lift to a systole of the maximal surface.

By the symmetry of $\Gamma(2, n)$ surfaces and Lemma 3, the preimage of the geodesic $l_{CE} \subset S^2(2, 2, 2, n)$ (l_{CD}, l_C, respectively) on the $\Gamma(2, n)$ surface consists of pairwise disjoint geodesics with equal length.

Proposition 9. *On the maximal $\Gamma(2, n)$ surface X_0, a simple closed geodesic is a systole if and only if it is lifted from l_{CE}, l_{CD} or l_C.*

Proof. It is sufficient to prove that in X_0, every geodesic lifted from l_{CE}, l_{CD} or l_C is a systole.

The proof is divided into two steps.

(1) If there is only one curve among l_{CE}, l_{CD} and l_C that lifts to the systoles of X_0, then X_0 is not maximal.

Without loss of generality, we assume that l_{CE} lifts to the systole of X_0, while l_{CD} and l_C do not lift to systoles of X_0. On the orbifold $S^2(2, 2, 2, n)_{X_0}$, there are deformations increasing or decreasing the length of the curve l_{CE}. A deformation increasing the length of l_{CE} increases the length of geodesics lifted from l_{CE} in X_0. If the deformation is small enough, then we get a new $\Gamma(2, n)$ surface, whose systoles are lifted from l_{CE} and the length of these curves are longer than the corresponding curves in X_0. Hence X_0 is not maximal.

(2) If there are exactly two curves among l_{CE}, l_{CD} and l_C lifting to the systole of the $\Gamma(2, n)$ surface, then the surface is not maximal.

(2a) We assume l_{CE} and l_{CD} lift to the systoles of X_0, while l_C does not lift to systoles of X_0. Then in the Fenchel–Nielsen coordinate (c, t), the length $|l_{CD}|(c, t)$ is monotonely increasing with respect to t by (5-3), while $l_{CE} = c/2$. We pick a sufficiently small $\varepsilon > 0$ and deform the Fenchel–Nielsen coordinate from (c, t) to $(c, t + \varepsilon)$; then we get a new surface X'. The systoles of X' are exactly the geodesics lifted from l_{CE}, and this surface has the same systole length to X_0. Then by (1), there exists a surface with longer systole than these two surfaces.

(2b) If l_{CE} and l_C lift to the systoles of X_0, while l_{CD} does not lift to systoles of X_0, the proof is similar. By (5-7), l_C is decreasing with respect to t when $t \le c$. Thus using the deformation $(c, t) \mapsto (c, t - \varepsilon)$, we get a surface whose systoles are all lifted from l_{CE} and whose systole length is equal to X_0's. Thus by (1), we know X_0 is not maximal.

(2c) The last case is that l_{CD} and l_C lift to the systoles of X_0, while l_{CE} does not lift to systoles of X_0. Similarly, some of the Fenchel–Nielsen deformations along

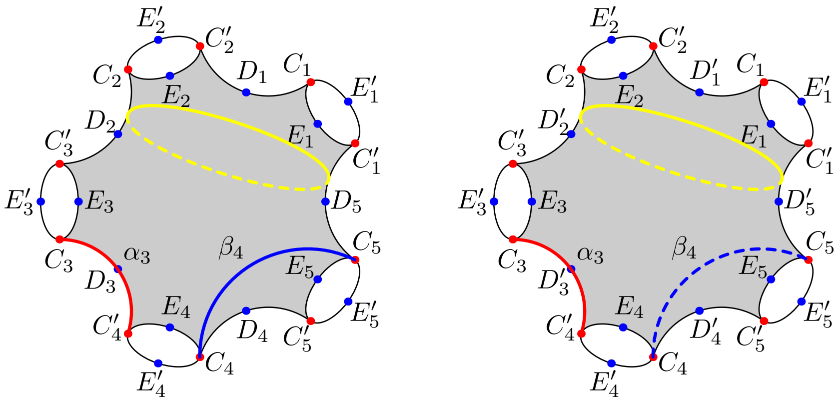

Figure 16. Cutting off a subsurface with signature (1, 2).

l_{CD} increase the length of l_C and let l_{CD} become the unique geodesic that can lift to systoles of X_0. Then by (1), this surface is not maximal.

By (1) and (2), all three curves l_{CD}, l_{CE} and l_C lift to the systoles of the maximal $\Gamma(2, n)$ surface. □

We are now ready to calculate the maximal systole on hyperbolic $\Gamma(2, n)$ surfaces.

Proof of Theorem 1. First we describe the lift of l_{CE}, l_{CD} and l_C in the $\Gamma(2, n)$ surface, respectively.

In the $\Gamma(2, n)$ surface shown in Figure 16, C_i and C_i' $(i = 1, \dots, n)$ are the fixed points of the order-two involution τ (recall its definition in Section 2) and are the lifts of the singular point C in $S^2(2, 2, 2, n)$; D_i and D_i' are the mid-points of the seams, and the lifts of the point D in $S^2(2, 2, 2, n)$; E_i and E_i' are the mid-points of C_iC_i' and the lifts of the point E in $S^2(2, 2, 2, n)$.

The curve l_{CE} lifts to cuffs of the surface, denoted as γ_i $(i = 1, \dots, 5\ )$; l_{CD} lifts to geodesics passing through $C_iD_iC_{i+1}'D_i'$ denoted as α_i; l_C lifts to geodesics passing through C_iC_{i+1}, denoted as β_i.

To calculate the systole length, we cut off a subsurface with signature (1, 2) from the $\Gamma(2, n)$ surface containing γ_1, γ_2 and α_1 (Figures 16 and 17), the boundary length of this surface is given by [Buser 2010, p. 454, 2.4.1 (i)]. We take common perpendiculars between cuffs and boundary components as in Figure 17. Then in the hexagon $H_1H_2A_1A_2A_3A_4$, we have

$$\begin{aligned}(5\text{-}8)\quad \cosh|H_1H_2| &= \sinh|A_1A_2|\sinh|A_3A_4|\cosh|A_2A_3| - \cosh|A_1A_2|\cosh|A_3A_4| \\ &= \sinh^2 c\cosh s - \cosh^2 c.\end{aligned}$$

In this subsurface, C_1, C_1', C_2, C_2' are branched points of the hyperelliptic involution and α_1, β_1, γ_1, γ_2 are the systoles of the $\Gamma(2, n)$ surface (Figure 18). To calculate the systole length, we redraw the (1, 2)-subsurface as Figure 19. When the systole

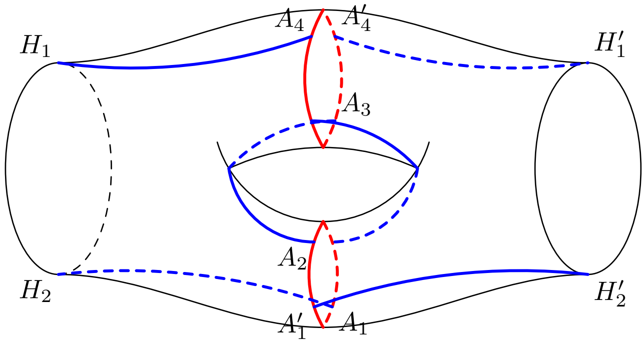

Figure 17. The red curves are cuffs. The blue curves are seams.

is maximal, the lengths of γ_1 and β_1 are the same, namely $|C_1C_1'| = |C_1C_2|$. In this case, the subsurface is shown in Figure 20.

When the surface is maximal, $|H_5H_6| = \frac{1}{2}|\alpha_1| = c$, $|C_1C_1'| = \frac{1}{2}\gamma_1 = c$. We assume $|H_3H_4| = l$ and $|H_5C_1'| = h$. Then in the hexagon $H_4H_3C_2'H_6H_5C_1$, by the symmetry of this hexagon, $|H_6C_2'| = |H_5C_1'| = h$, and by [Buser 2010, p. 454, 2.4.1(i)] we have

$$\cosh|H_3H_4| = \sinh|C_1H_5|\sinh|C_2'H_6|\cosh|H_5H_6| - \cosh|C_1H_5|\cosh|C_2'H_6|$$

and

$$\cosh l = \sinh^2 h \cosh c - \cosh^2 h. \tag{5-9}$$

In the triangle $\triangle C_1H_5C_1'$, by [Buser 2010, p. 454, 2.2.2 (i)], we have

$$\cosh|C_1C_1'| = \cosh|C_1H_5|\cosh|C_1'H_5|\cosh c = \cosh h \cosh\frac{c}{2}. \tag{5-10}$$

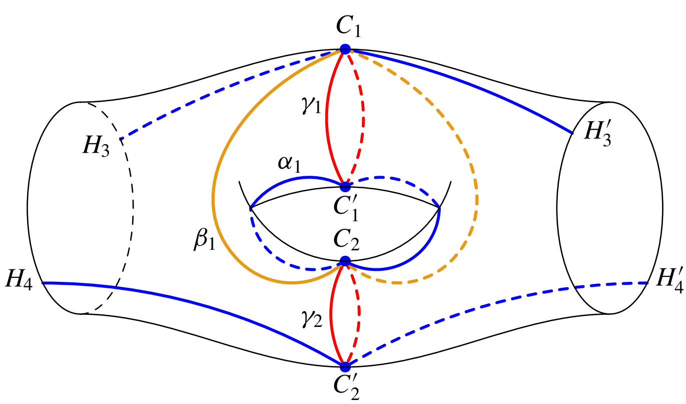

Figure 18. Systoles of X in the $(1,2)$-subsurface.

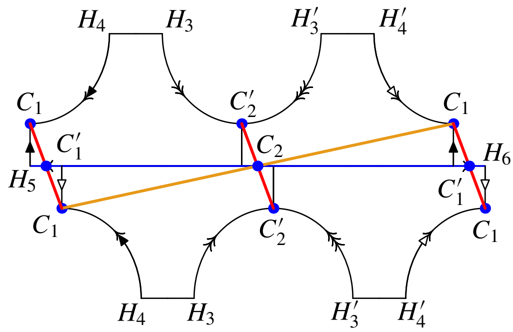

Figure 19. Redrawing the (1, 2)-subsurface.

For convenience, we denote $\cosh c$ by K. Then combining (5-9) and (5-10), we eliminate h and get

$$\frac{2K^2}{K+1} = \frac{K+\cosh l}{K-1}. \tag{5-11}$$

Recall that $l = |H_3H_4|$. Then combining (5-11), (5-8) and (5-1), we eliminate l and s, and get

$$2K^3 - 3K^2 + 1 - 4\cos^2\frac{\pi}{n}(K+1)^2 = 0.$$

The unique real solution of this equation is

$$K = \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} + \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} \\ + \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} - \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} + \tfrac{1}{6}(L+3),$$

where $L = 4\cos^2\frac{\pi}{n}$.

At last, we calculate the twist parameter t of the maximal surface, using (5-1) and (5-3):

$$\cosh\frac{t}{2} = \frac{\cosh\frac{c}{2}}{\cosh\frac{s}{2}} = \frac{\cosh^2\frac{c}{2}}{\cos\frac{\pi}{n}} = \frac{\cosh c+1}{2\cos\frac{\pi}{n}} = \frac{K+1}{2\cos\frac{\pi}{n}}.$$

Theorem 1 follows. □

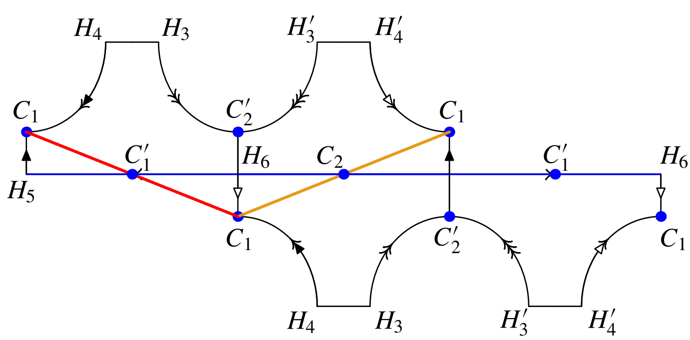

Figure 20. When $|C_1C_1'| = |C_1C_2|$.

Acknowledgements

We would like to thank Professors Sheng Bai and Shicheng Wang for their helpful discussions and suggestions. Gao thanks Professor Shicheng Wang for his supervision, help and support during his PhD studies. We thank Professor Ying Zhang for his helpful comments.

Wang is partially supported by NSFC 12131009 and National Key R&D Program of China 2020YFA0712800.

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Received November 28, 2021. Revised June 18, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 325, No. 1, 2023
https://doi.org/10.2140/pjm.2023.325.105

STABLE SYSTOLES OF HIGHER RANK IN RIEMANNIAN MANIFOLDS

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This paper introduces the stable systoles of higher rank of a Riemannian manifold as a generalization of the usual stable systoles. Several inequalities involving these higher rank systoles are proved.

1. Introduction

Let M be a smooth compact orientable manifold of dimension n. A Riemannian metric g on M induces an associated stable mass norm $\|\cdot\|$ on the real homology groups $H_p(M, \mathbb{R})$. Because the image of the p-th integral homology group $H_p(M, \mathbb{Z})$ in $H_p(M, \mathbb{R})$ is a lattice, denoted $H_p(M, \mathbb{Z})_{\mathbb{R}}$, in $H_p(M, \mathbb{R})$, we thereby obtain for each p a lattice in a normed vector space. Such structures are the central objects of study in the geometry of numbers [6], and their various numerical invariants thus give rise to a host of invariants of the Riemannian manifold (M, g).

For example, the p-dimensional stable systole $\operatorname{stsys}_p(M, g)$ is the minimum norm of the nonzero elements in the lattice $H_p(M, \mathbb{Z})_{\mathbb{R}}$. These have been studied extensively [8; 9; 12]. One can also consider the successive minimums of the lattice or its whole length spectrum [10]. The volume of the Jacobian variety $J_p = H_p(M, \mathbb{R})/H_p(M, \mathbb{Z})_{\mathbb{R}}$ with the (Finsler) metric induced from the stable norm gives an additional invariant. There are various natural ways to define the volume of the quotient tori [15]. In [8], the mass and mass* measures were used to define the volume of J_p. In this paper we will use the Busemann–Hausdorff measure to define the volume of J_p as well as the higher rank systoles of a Riemannian manifold.

Given a positive integer k less than or equal to the p-th Betti number b_p of M, we define $\operatorname{stsys}_{p,k}(M, g)$ to be the minimum Hausdorff–Busemann volume of the fundamental region of sublattices of $H_p(M, \mathbb{Z})_{\mathbb{R}}$ of rank k. This can be interpreted as the k-th systole of J_p. In particular $\operatorname{stsys}_{p,1}(M, g)$ is the ordinary stable p-th systole, and $\operatorname{stsys}_{p,b_p}(M, g)$ is the Hausdorff–Busemann volume of J_p.

MSC2020: primary 53C20; secondary 53C22.

Keywords: Riemannian manifolds, systoles, systolic inequalities, lattices, Hausdorff–Busemann measure.

Among the results of this paper are sharp stable systolic inequalities. The first of these generalizes an inequality due to Bangert and Katz [1].

Theorem 5.1. *Let (M, g) be a compact oriented manifold of dimension n whose first Betti number is b. Then*

$$\mathrm{stsys}_{1,b}(M, g)\, \mathrm{stsys}_{n-1,b}(M, g) \leq \mathrm{Vol}(M, g).$$

Equality holds if and only if there exists a Riemannian submersion of M onto a flat torus of dimension b with connected minimal fibers.

A further generalization is Theorem 5.2 which is stated and proved in Section 5. In addition, we prove a sharp inequality for conformally flat metrics on the 4-dimensional torus for the 2-dimensional stable systole of rank 6.

Theorem 6.7. *Let (M, g) be a conformally flat 4-dimensional torus. Then*

$$\mathrm{stsys}_{2,6}(M, g)^2 \leq \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \mathrm{Vol}(M, g).$$

Equality holds if and only if (M, g) is flat.

This paper is organized as follows. In Section 2, we discuss lattices in normed vector spaces and their invariants, as well as the behavior of Hausdorff–Busemann volume under linear transformations. Section 3 reviews some properties of the mass, comass and L^2 norms on the (co)homology of a compact oriented manifold. The formal definition of the stable systoles of higher rank is given in Section 4. This section provides a number of inequalities involving them related to the properties of the cap product. In Section 5, we prove a sharp $(1, n-1)$-inequality that generalizes that of Bangert and Katz [1]. In Section 6, we calculate the 2-dimensional systole of rank 6 in flat 4-dimensional tori, and prove a sharp inequality for conformally flat metrics on the 4-dimensional torus. Finally, in the Appendix, we prove a result needed in Section 5 that the dual k-extreme lattices are dual k-perfect.

2. Normed vector spaces

Hausdorff measure. Let $(V, \|\cdot\|)$ be an n-dimensional normed vector space over the real numbers. Let $K \subset V$ be the unit ball in $(V, \|\cdot\|)$. According to a theorem of Busemann [5, (2.3)], the Hausdorff n-dimensional measure $\mathrm{Vol}_n(\,\cdot\,, \|\cdot\|)$ is the unique translation invariant (Haar) measure on V normalized such that

$$\mathrm{Vol}_n(K, \|\cdot\|) = \omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)}.$$

Here ω_n denotes the Euclidean volume of the Euclidean unit ball, and Γ is the gamma function.

Proposition 2.1 (cf. [15]). *Suppose that $T : V \to W$ is a linear isomorphism between the n-dimensional normed spaces $(V, \|\cdot\|)$ and $(W, \|\cdot\|')$ such that*

$$\|T(x)\|' \leq C\,\|x\|$$

for all $x \in V$ for some constant C. Then

$$\mathrm{Vol}_n(T(E), \|\cdot\|') \leq C^n\,\mathrm{Vol}_n(E, \|\cdot\|) \tag{2-1}$$

for all Borel sets $E \subset V$. Moreover, if equality holds in (2-1) *for some E with nonzero volume, then equality holds in* (2-1) *for all E and*

$$\|T(x)\|' = C\,\|x\|$$

for all $x \in V$.

Proof. Let K and K' denote the unit balls of V and W, respectively. By assumption $T(K) \subset CK' = \{Cx : x \in K'\}$. We may choose inner products on V and W so that T is an isometry. Let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure on V and W for the Euclidean metrics induced from the inner products. Thus $\mathcal{L}^n(T(E)) = \mathcal{L}^n(E)$ for all Borel sets E in V. Hence $\mathcal{L}^n(K) = \mathcal{L}^n(T(K)) \leq \mathcal{L}^n(CK') = C^n\mathcal{L}^n(K')$. Therefore

$$\begin{aligned}\mathrm{Vol}_n(T(E), \|\cdot\|') &= \frac{\omega_n}{\mathcal{L}^n(K')}\mathcal{L}^n(T(E)) \\ &\leq C^n \frac{\omega_n}{\mathcal{L}^n(K)}\mathcal{L}^n(E) = C^n\,\mathrm{Vol}_n(E, \|\cdot\|)\end{aligned} \tag{2-2}$$

proving (2-1). If equality holds for some E with $\mathcal{L}^n(E) \neq 0$, then, by the proof of (2-2), $\mathcal{L}^n(TK) = \mathcal{L}^n(CK')$. Now $TK \subset CK'$ are both closed bounded convex sets of W. Thus, if $TK \neq CK'$, there would exist an open set contained in $CK' \backslash TK$ which would imply the contradiction $\mathcal{L}^n(CK' \backslash TK) > 0$. Hence $TK = CK'$, and therefore $\|T(x)\|' = C\,\|x\|$ for all $x \in V$. □

Lattices. Suppose that Λ is a lattice in $(V, \|\cdot\|)$. The Hausdorff measure in V passes down to the Hausdorff–Busemann measure on the n-dimensional torus V/Λ. Its volume is equal to the measure of a fundamental domain for Λ and will be denoted $\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)$. The Hausdorff–Busemann volume has the following asymptotic interpretation. Let $N(R)$ equal the number of lattice points $x \in \Lambda$ with $\|x\| \leq R$. Then

$$\lim_{R\to\infty} \frac{N(R)}{R^n} = \frac{\omega_n}{\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)}.$$

Thus $\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)$ depends only on the length spectrum of Λ. By the second Minkowski inequality [6, p. 218], the Hausdorff–Busemann volume also satisfies the inequality

$$\frac{2^n}{n!}\operatorname{Vol}_n(V/\Lambda, \|\cdot\|) \le \lambda_1 \cdots \lambda_n \omega_n \le 2^n \operatorname{Vol}_n(V/\Lambda, \|\cdot\|),$$

where the λ_i are the successive minimums of the lattice Λ.

Let V^* be the dual space of V with dual norm $\|\cdot\|^*$. The polar set $K^\circ \subset V^*$ is defined to be the unit ball in $(V^*, \|\cdot\|^*)$. Let $\Lambda^* \subset V^*$ denote the dual lattice of Λ.

Lemma 2.2. *Let Λ and Λ^* be dual lattices in the respective dual normed spaces $(V, \|\cdot\|)$ and $(V^*, \|\cdot\|^*)$. There exists a universal constant $c > 0$ such that*

$$1 \le \operatorname{Vol}_n(V/\Lambda, \|\cdot\|)\operatorname{Vol}_n(V^*/\Lambda^*, \|\cdot\|^*) \le \frac{1}{c^n}.$$

Proof. Let K° be the polar of K. Fix an inner product on V with its dual inner product, and let $\mathcal{L}^n$ denote the corresponding n-dimensional Lebesgue measures. Thus

$$\operatorname{Vol}_n(V/\Lambda, \|\cdot\|)\operatorname{Vol}_n(V^*/\Lambda^*, \|\cdot\|^*) = \omega_n^2 \frac{\mathcal{L}^n(V/\Lambda)}{\mathcal{L}^n(K)}\frac{\mathcal{L}^n(V^*/\Lambda^*)}{\mathcal{L}^n(K^\circ)}.$$

But by the Santalo and the Bourgain–Milman [4] inequalities,

$$\omega_n^2 \ge \mathcal{L}^n(K)\mathcal{L}^n(K^\circ) \ge c^n\omega_n^2$$

for some universal constant c, and by Lemma 5 of [6, p. 24],

$$\mathcal{L}^n(V/\Lambda)\mathcal{L}^n(V^*/\Lambda^*) = 1.$$

The proof is completed by combining these three formulas. □

Remark 2.3. It is conjectured that $c = \frac{2}{\pi}$. Kuperberg [13] has shown $c \ge \frac{1}{2}$.

Sublattices. Let k be an integer with $1 \le k \le n$. By definition, a sublattice of Λ of rank k is a lattice Λ' in a k-dimensional vector subspace V' of V such that $\Lambda' = \Lambda \cap V'$. Let $\|\cdot\|'$ be the restriction of $\|\cdot\|$ to V'. Then the Hausdorff–Busemann volume of Λ' is $\operatorname{Vol}_k(V'/\Lambda', \|\cdot\|')$. Define

$$\Delta_k(\Lambda, \|\cdot\|) = \inf_{\Lambda'} \operatorname{Vol}_k(V'/\Lambda', \|\cdot\|'),$$

where Λ' runs over all sublattices of rank k in Λ. In particular $\Delta_1(\Lambda, \|\cdot\|)$ is just the length of the shortest nonzero element of Λ.

In the special case when $|\cdot|$ is a Euclidean norm obtained from an inner product on V,

$$\operatorname{Vol}_n(V/\Lambda, |\cdot|) = \det(\Lambda) \equiv |e_1 \wedge \cdots \wedge e_n|,$$

where $e_1, \ldots, e_n$ is a basis for Λ. Thus if k is an integer with $1 \le k \le n$, the numbers $\Delta_k(\Lambda, |\cdot|)$ are exactly the *carcans* of flat tori defined by Berger in [3, §7].

Following [2], define the Hermite–Rankin constants

$$\gamma_{n,k} = \sup_{\Lambda} \frac{\Delta_k(\Lambda, |\cdot|)^2}{\det(\Lambda)^{2k/n}}$$

and the Bergé–Martinet constants

$$\gamma'_{n,k} = \sup_{\Lambda} \Delta_k(\Lambda, |\cdot|)\Delta_k(\Lambda^*, |\cdot|^*),$$

where Λ runs over all lattices in the n-dimensional Euclidean space $(V, |\cdot|)$.

Proposition 2.4. *Let $(V, \|\cdot\|)$ be a normed vector space of dimension n. Then*

$$\frac{\Delta_k(\Lambda, \|\cdot\|)^2}{\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)^{2k/n}} \le n^k \gamma_{n,k}, \tag{2-3}$$

and

$$\Delta_k(\Lambda, \|\cdot\|)\Delta_k(\Lambda^*, \|\cdot\|^*) \le n^{\frac{k}{2}} \gamma'_{n,k}. \tag{2-4}$$

Proof. Let E be the John ellipsoid for the unit ball in V [11]. Then E determines a Euclidean norm $|\cdot|_E$ on V that satisfies

$$\|\cdot\| \le |\cdot|_E \le \sqrt{n}\, \|\cdot\|. \tag{2-5}$$

Thus by Proposition 2.1

$$\mathrm{Vol}_n(V/\Lambda, |\cdot|_E) \le n^{\frac{n}{2}}\, \mathrm{Vol}_n(V/\Lambda, \|\cdot\|),$$

and, for any sublattice Λ' of rank k in a k-dimensional subspace V',

$$\mathrm{Vol}_k(V'/\Lambda', \|\cdot\|) \le \mathrm{Vol}_k(V'/\Lambda', |\cdot|_E).$$

Hence,

$$\Delta_k(\Lambda, \|\cdot\|) \le \Delta_k(\Lambda, |\cdot|_E),$$

and therefore

$$\frac{\Delta_k(\Lambda, \|\cdot\|)^2}{\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)^{2k/n}} \le n^k \gamma_{n,k}.$$

Passing to the dual norms, inequality (2-5) implies

$$\|\cdot\|^* \ge |\cdot|_E^* \ge \frac{1}{\sqrt{n}}\, \|\cdot\|^*. \tag{2-6}$$

Thus

$$\Delta_k(\Lambda^*, \|\cdot\|^*) \le n^{\frac{k}{2}}\, \Delta_k(\Lambda^*, |\cdot|_E^*),$$

and therefore

$$\Delta_k(\Lambda, \|\cdot\|)\Delta_k(\Lambda^*, \|\cdot\|^*) \le n^{\frac{k}{2}} \gamma'_{n,k}. \qquad \square$$

Linear maps. If $T : V \to W$ is a linear isomorphism such that $T(\Lambda) \subset \Gamma$ where Λ and Γ are lattices in V and W, respectively, then T induces a covering map

$$\overline{T} : V/\Lambda \to W/\Gamma$$

between two n-dimensional tori. The degree of $\overline{T}$ is clearly equal to the index of $T(\Lambda)$ in Γ as free abelian groups. Let $\deg(T)$ denote this degree.

Proposition 2.5. *Suppose that $T : V \to W$ is a linear isomorphism between the n-dimensional normed spaces $(V, \|\cdot\|)$ and $(W, \|\cdot\|')$ such that*

$$\|T(x)\|' \leq C\,\|x\|$$

for all $x \in V$. Suppose also that $T(\Lambda) \subset \Gamma$ where Λ and Γ are lattices in V and W, respectively. Then

$$\deg(T)\,\mathrm{Vol}_n(W/\Gamma, \|\cdot\|') \leq C^n\,\mathrm{Vol}_n(V/\Lambda, \|\cdot\|). \tag{2-7}$$

Equality holds if and only if

$$\|T(x)\|' = C\,\|x\|$$

for all $x \in V$.

Proof. Let $E \subset V$ be the fundamental domain for the cover V over V/Λ and $F \subset W$ the fundamental domain for the cover W over W/Γ. Then $T(E)$ can be expressed as a union of the translates of $\deg(T)$ copies of F. By Proposition 2.1 and the translation invariance of the Hausdorff measure,

$$\deg(T)\,\mathrm{Vol}_n(F, \|\cdot\|') = \mathrm{Vol}_n(T(E), \|\cdot\|') \leq C^n\,\mathrm{Vol}_n(E, \|\cdot\|).$$

The case of equality follows from Proposition 2.1 as well. $\square$

Suppose now that $\dim(V) = n$, $\dim(W) = m$, and $T : V \to W$ is a linear transformation of rank k such that $T(\Lambda) \subset \Gamma$. Let $\overline{V} = V/\ker(T)$ be the cokernel of T, and $\widehat{W} = T(V) \subset W$ be the image of V under T. Also, let $\bar{\Lambda} \subset \overline{V}$ be the quotient lattice of Λ, and $\hat{\Gamma} = \Gamma \cap \widehat{W}$ the rank-k sublattice of Γ. Then T induces a linear isomorphism

$$\overline{T} : \overline{V} \to \widehat{W}$$

such that $\overline{T}(\bar{\Lambda}) \subset \hat{\Gamma}$. Let $\|\cdot\|_{\mathsf{q}}$ denote the quotient norm on $\overline{V}$, and let $\|\cdot\|'_{\mathsf{r}}$ denote the restriction of $\|\cdot\|'$ to $\widehat{W}$.

Corollary 2.6. *With this notation, if $\|T(x)\|' \leq C\,\|x\|$ for all $x \in V$, then*

$$\deg(\overline{T})\,\mathrm{Vol}_k(\widehat{W}/\hat{\Gamma}, \|\cdot\|'_{\mathsf{r}}) \leq C^k\,\mathrm{Vol}_k(\overline{V}/\bar{\Lambda}, \|\cdot\|_{\mathsf{q}}). \tag{2-8}$$

Equality holds if and only if

$$\|\overline{T}(x)\|'_{\mathsf{r}} = C\,\|x\|_{\mathsf{q}}$$

for all $x \in \overline{V}$ where $\|\cdot\|'_{\mathsf{r}}$ denotes the restriction of $\|\cdot\|'$ to $\widehat{W}$.

Proof. By hypothesis and the definition of the quotient norm, $\|\overline{T}(x)\|' \leq C\|x\|_{\mathsf{q}}$ for all $x \in \overline{V}$. The result follows immediately from Proposition 2.5. □

The next lemma identifies the dual norm of a quotient norm.

Lemma 2.7. *Let $(V, \|\cdot\|)$ be a normed vector space. Let $(\overline{V}, \|\cdot\|_{\mathsf{q}})$ be a quotient space of V with quotient norm. Let $(V^*, \|\cdot\|^*)$ be the dual space of V with the dual norm. Then the dual space $\overline{V}^*$ of $\overline{V}$ can be identified with a subspace of V^* and the dual norm $\|\cdot\|_{\mathsf{q}}^*$ is equal to the restriction of $\|\cdot\|^*$ to $\overline{V}^*$.*

Proof. Let V_0 be the kernel of the quotient map $\mathsf{q} : V \to \overline{V}$. Clearly

$$\overline{V}^* = \{\lambda \in V^* : \lambda(V_0) = 0\}.$$

By definition $\|\bar{v}\|_{\mathsf{q}} = \inf\{\|v\| : v \in V, \mathsf{q}(v) = \bar{v}\}$ for every $\bar{v} \in \overline{V}$. Let $\lambda \in \overline{V}^*$. Then

$$\|\lambda\|_{\mathsf{q}}^* = \sup\{\lambda(\bar{v}) : \bar{v} \in \overline{V}, \|\bar{v}\|_{\mathsf{q}} \leq 1\}$$

and

$$\|\lambda\|^* = \sup\{\lambda(v) : v \in V, \|v\| \leq 1\}.$$

Now, there exists $v_0 \in V$ such that $\|v_0\| \leq 1$ and $\lambda(v_0) = \|\lambda\|^*$. Thus $\|\mathsf{q}(v_0)\|_{\mathsf{q}} \leq \|v_0\| \leq 1$ and $\lambda(\mathsf{q}(v_0)) = \lambda(v_0) = \|\lambda\|^*$ which implies $\|\lambda_{\mathsf{q}}\|^* \geq \|\lambda\|^*$. But there also exists $\bar{v}_0 \in \overline{V}$ such that $\|\bar{v}_0\|_{\mathsf{q}} \leq 1$ and $\lambda(\bar{v}_o) = \|\lambda\|_{\mathsf{q}}^*$. Thus there exists $v_0 \in V$ with $\mathsf{q}(v_0) = \bar{v}_0$ and $\|v_0\| = \|\bar{v}_0\|_{\mathsf{q}} \leq 1$. Hence $\lambda(v_0) = \lambda(\mathsf{q}(v_0)) = \lambda(\bar{v}_0) = \|\lambda\|_{\mathsf{q}}^*$ which implies $\|\lambda\|^* \geq \|\lambda\|_{\mathsf{q}}^*$. Therefore $\|\lambda\|^* = \|\lambda\|_{\mathsf{q}}^*$. □

3. Norms on homology and cohomology

Throughout this section (M, g) is a compact oriented Riemannian manifold of dimension n.

Mass and comass. Recall that the comass norm $\|\cdot\|^*$ of a cohomology class $\alpha \in H^p(M, \mathbb{R})$ is defined by

$$\|\alpha\|^* = \inf\{\text{comass}(\omega) : \omega \text{ a closed } p\text{-form representing } \alpha\},$$

where $\text{comass}(\omega) = \max\{\omega_x(e_1, \ldots, e_p) : x \in M, e_i \in T_xM, |e_i| = 1\}$, and that the stable mass norm $\|\cdot\|$ of a homology class $h \in H_p(M, \mathbb{R})$ is defined by

$$\|h\| = \inf\left\{\sum_i |r_i| \text{vol}_p(\sigma_i) : \sum_i r_i\sigma_i \text{ is a Lipschitz cycle representing } h\right\}.$$

It is well known that comass and mass are dual norms relative to the Kronecker pairing

$$\langle \cdot, \cdot \rangle : H_p(M, \mathbb{R}) \times H^p(M, \mathbb{R}) \to \mathbb{R}.$$

Thus for all $h \in H_p(M, \mathbb{R})$ and $\alpha \in H^p(M, \mathbb{R})$

$$\langle h, \alpha\rangle \le \|h\|\,\|\alpha\|^*. \tag{3-1}$$

Moreover, for all $\alpha \in H^p(M, \mathbb{R})$ and $\beta \in H^q(M, \mathbb{R})$,

$$\|\alpha \smile \beta\|^* \le C(n;\, p, q)\|\alpha\|^*\|\beta\|^*, \tag{3-2}$$

where $\smile$ denotes the cup product; see [7, 1.8.1]. (Note that $C(n;\, p, q) \le \binom{p+q}{p}$ and $C(n;\, p, q) = 1$ if $p \in \{0, 1, n-1, n\}$. We will see in Section 6 that $C(4;\, 2, 2) = 2$.)

Lemma 3.1. *If $h \in H_{p+q}(M, \mathbb{R})$ and $\alpha \in H^p(M, \mathbb{R})$, then*

$$\|h \frown \alpha\| \le C(n;\, p, q)\|h\|\,\|\alpha\|^*,$$

where $\frown$ denotes the cap product.

Proof. If $\beta \in H^q(M, \mathbb{R})$, then using (3-1),

$$\langle h \frown \alpha, \beta\rangle = \langle h, \alpha \smile \beta\rangle \le \|h\|\,\|\alpha \smile \beta\|^*.$$

Applying inequality (3-2) gives

$$\langle h \frown \alpha, \beta\rangle \le \|h\|\,C(n;\, p, q)\,\|\alpha\|^*\,\|\beta\|^*,$$

and taking the supremum for all β with $\|\beta\|^* \le 1$ gives the result since $\|\cdot\|$ is dual to $\|\cdot\|^*$. □

The L^2 norm. According to Hodge theory the cohomology classes in $H^p(M, \mathbb{R})$ can be represented by the harmonic p-forms on (M, g). Moreover $H^p(M, \mathbb{R})$ is endowed with an inner product which for two harmonic p-forms φ and ψ is given by

$$\langle\langle \varphi, \psi\rangle\rangle = \int_M \varphi \wedge \star\psi,$$

where $\star$ denotes the Hodge star operator. We will denote the corresponding Euclidean norm as $|\cdot|_2^*$.

We have need of the following proposition proved in [9, Corollary 3].

Proposition 3.2. *Let $h \in H_p(M, \mathbb{R})$ be the Poincaré dual of the cohomology class $\alpha \in H^{n-p}(M, \mathbb{R})$. Then*

$$\|h\| \le \mathrm{Vol}(M, g)^{\frac{1}{2}}\, C(n, p)\,|\alpha|_2^*, \tag{3-3}$$

where $C(n, p)$ is a constant depending only on n and p. Moreover, if equality holds then α can be represented by a harmonic p-form of constant norm.

Remark 3.3. Conversely, the proof of [9, Corollary 3] also shows that equality holds in (3-3) for $p \in \{1, n-1\}$ when α is represented by a harmonic p-form of constant norm. Note that if p equals 0, 1, $n-1$, or n, then $C(n, p) = 1$, and that in any case $C(n, p) \le \binom{n}{p}^{\frac{1}{2}}$ always.

4. Higher rank systoles

Let (M, g) be a compact oriented Riemannian manifold of dimension n, and let $\|\cdot\|$ denote the stable mass norm on $H_p(M, \mathbb{R})$ induced from g. The image of the p-th integral homology group $H_p(M, \mathbb{Z})$ in $H_p(M, \mathbb{R})$ is a lattice in $H_p(M, \mathbb{R})$ which will be denoted $H_p(M, \mathbb{Z})_{\mathbb{R}}$. The following definition generalizes the well-known p-dimensional stable systole $\operatorname{stsys}_p(M, g)$.

Definition 4.1. For any integer k between 1 and the p-th Betti number b_p of M, the p-dimensional stable systole of rank k is defined to be

$$\operatorname{stsys}_{p,k}(M, g) = \Delta_k(H_p(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|)^{\frac{1}{k}}.$$

Clearly $\operatorname{stsys}_{p,1}(M, g) = \operatorname{stsys}_p(M, g) = \inf\{\|x\| : 0 \neq x \in H_p(M, \mathbb{Z})_{\mathbb{R}}\}$.

The dual lattice $H^p(M, \mathbb{Z})_{\mathbb{R}}$ in $H^p(M, \mathbb{R})$ can be identified with the set of cohomology classes of degree p with integral periods [12, Lemma 15.4.2]. Thus

$$H^p(M, \mathbb{Z})_{\mathbb{R}} = \{\alpha \in H^p(M, \mathbb{R}) : \langle x, \alpha\rangle \in \mathbb{Z} \ \ \forall x \in H_p(M, \mathbb{Z})_{\mathbb{R}}\}.$$

As is Section 3, $\|\cdot\|^*$ is the comass norm and $|\cdot|_2^*$ is the L^2 norm on $H^p(M, \mathbb{R})$.

In general, the existence of nonzero cap products give rise to inequalities involving higher rank stable systoles. The results that follow give examples of such inequalities under various hypotheses. Proposition 4.2 is used in the proofs of Corollaries 4.3 and 4.4 and Theorems 4.5 and 4.6.

Proposition 4.2. *Suppose $h \in H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}$, and $h \frown : H^p(M, \mathbb{R}) \to H_q(M, \mathbb{R})$ has rank k. Then*

$$\deg(\overline{h \frown})^{\frac{1}{k}} \operatorname{stsys}_{q,k}(M, g) \operatorname{stsys}_{p,k}(M, g) \leq \frac{C(n; p, q)}{c} \|h\|,$$

where $C(n; p, q)$ is the constant in (3-2) *and c is the constant in Lemma 2.2*

Proof. Using Lemma 3.1, apply Corollary 2.6 to obtain

$$\deg(\overline{h \frown}) \operatorname{Vol}_k(\widehat{W}/\hat{\Gamma}, \|\cdot\|'_{\mathrm{r}}) \leq C(n; p, q)^k \|h\|^k \operatorname{Vol}_k(\bar{V}/\bar{\Lambda}, \|\cdot\|_{\mathrm{q}}), \tag{4-1}$$

where $\widehat{W} \subset H_q(M, \mathbb{R})$ is the image of $H^p(M, \mathbb{R})$ under $h \frown$, $\hat{\Gamma} = H_q(M, \mathbb{Z})_{\mathbb{R}} \cap \widehat{W}$, and $\bar{V}$ and $\bar{\Lambda}$ are the quotients of $H^p(M, \mathbb{R})$ and $H^p(M, \mathbb{Z})_{\mathbb{R}}$ by the kernel of $h \frown$. Multiply both sides of (4-1) by $\operatorname{Vol}_k(\bar{V}^*/\bar{\Lambda}^*, \|\cdot\|_{\mathrm{q}}^*)$ and use Lemma 2.2 to obtain

$$\deg(\overline{h \frown}) \operatorname{Vol}_k(\widehat{W}/\hat{\Gamma}, \|\cdot\|'_{\mathrm{r}}) \operatorname{Vol}_k(\bar{V}^*/\bar{\Lambda}^*, \|\cdot\|_{\mathrm{q}}^*) \leq C(n; p, q)^k \|h\|^k \frac{1}{c^k}. \tag{4-2}$$

But $\bar{V}^*$ is a k-dimensional subspace of $H_p(M, \mathrm{R})$ and $\bar{\Lambda}^* = H_p(M, \mathrm{Z})_{\mathrm{R}} \cap \bar{V}^*$. On taking k-th roots and using the Definition 4.1 we obtain the stated inequality. □

Here is a simple application. Let T^4 be the 4-dimensional torus. It is easily checked that $h\frown\colon H^1(T^4,\mathbb{R})\to H_2(T^4,\mathbb{R})$ has rank 3 for every nonzero $h\in H_3(T^4,\mathbb{R})$. Thus for nonzero $h\in H_3(T^4,\mathbb{Z})_{\mathbb{R}}$ and any Riemannian metric g on T^4 we have by Proposition 4.2,

$$\operatorname{stsys}_{1,3}(T^4,g)\operatorname{stsys}_{2,3}(T^4,g)\le 2\,\|h\|$$

because $C(4;1,2)=1$, $c\ge\frac{1}{2}$ and $\deg(\overline{h\frown})\ge 1$. This leads to the following intersystolic inequality.

Corollary 4.3. *For every Riemannian metric g on T^4,*

$$\operatorname{stsys}_{1,3}(T^4,g)\operatorname{stsys}_{2,3}(T^4,g)\le 2\operatorname{stsys}_{3,1}(T^4,g).$$

By taking $h=[M]\in H_n(M,\mathbb{Z})$, the fundamental class of M, in Proposition 4.2 we obtain:

Corollary 4.4. *Let $0<p<n$, and let b_p the p-th Betti number of M. Then*

$$\operatorname{stsys}_{p,b_p}(M,g)\operatorname{stsys}_{n-p,b_p}(M,g)\le\frac{C(n;p,n-p)}{c}\operatorname{Vol}(M,g).$$

Proof. Capping by $[M]$ is the Poincaré duality map which is a linear isomorphism of rank b_p from $H^p(M,\mathbb{R})$ to $H_{n-p}(M,\mathbb{R})$ with degree 1. Also observe that $\|[M]\|=\operatorname{Vol}(M,g)$. □

Theorem 4.5. *Suppose for every nonzero $h\in H_{p+q}(M,\mathbb{R})$ there exists an $\alpha\in H^p(M,\mathbb{R})$ such that $h\frown\alpha\ne 0$. Then*

$$\min_{1\le k\le b}\operatorname{stsys}_{q,k}(M,g)\operatorname{stsys}_{p,k}(M,g)\le\frac{C(n;p,q)}{c}\operatorname{stsys}_{p+q,1}(M,g),$$

where $b=\min(b_p,b_q)$.

Proof. Take $h\in H_{p+q}(M,\mathbb{Z})_{\mathbb{R}}$ with $\|h\|=\operatorname{stsys}_{p+q,1}(M,g)$. By assumption,

$$h\frown\colon H^p(M,\mathbb{R})\to H_q(M,\mathbb{R})$$

has rank k for some $1\le k\le b=\min(b_p,b_q)$, and $\deg(\overline{h\frown})\ge 1$. We apply Proposition 4.2 to obtain the stated inequality. □

The next result shows that the existence of just one nonzero cap product places a bound on the stable systoles in appropriate dimensions.

Theorem 4.6. *Suppose there exist $h\in H_{p+q}(M,\mathbb{R})$ and $\alpha\in H^p(M,\mathbb{R})$ such that $h\frown\alpha\ne 0$. Then*

$$\min_{1\le k\le b}\operatorname{stsys}_{q,k}(M,g)\operatorname{stsys}_{p,k}(M,g)\le\frac{C(n;p,q)}{c}\,\lambda_{b_{p+q}},$$

where $b=\min(b_p,b_q)$ and $\lambda_{b_{p+q}}$ is the b_{p+q}-th successive minimum of the lattice $(H_{p+q}(M,\mathbb{Z})_{\mathbb{R}},\|\cdot\|)$.

Proof. There is a basis $h_1, \dots, h_{b_{p+q}}$ of $H_{p+q}(M, \mathbb{R})$ consisting of the successive minimums of $(H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|)$. Write $h = t_1 h_1 + \cdots + t_\beta h_\beta$. Since

$$h \frown \alpha = t_1 h_1 \frown \alpha + \cdots + t_\beta h_\beta \frown \alpha \neq 0,$$

there is at least one successive minimum $h_i \in H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}$ with $h_i \frown \alpha \neq 0$. Thus capping by h_i has rank k for some $1 \leq k \leq b$ and $\deg(\overline{h_i \frown}) \geq 1$. By definition $\|h_i\| = \lambda_i \leq \lambda_{b_{p+q}}$. The result now follows by applying Proposition 4.2 □

Proposition 4.7. *Let $\alpha \in H^p(M, \mathbb{Z})_{\mathbb{R}}$. Suppose that $\frown \alpha : H_{p+q}(M, \mathbb{R}) \to H_q(M, \mathbb{R})$ is injective. Then*

$$\deg(\overline{\frown \alpha})^{\frac{1}{b_{p+q}}} \operatorname{stsys}_{q, b_{p+q}}(M, g) \leq C(n; p, q) \operatorname{stsys}_{p+q, b_{p+q}}(M, g) \|\alpha\|^*, \tag{4-3}$$

where b_{p+q} is the $(p+q)$-th Betti number.

Proof. By hypothesis the rank of $T = \frown \alpha$ is b_{p+q}. Using Lemma 3.1 and the injectivity of $T = \frown \alpha$, apply Corollary 2.6 to obtain

$$\deg(\overline{\frown \alpha}) \operatorname{Vol}_{b_{p+q}}(\widehat{W}/\hat{\Gamma}, \|\cdot\|') \leq (C(n; p, q)\|\alpha\|^*)^{b_{p+q}} \operatorname{Vol}_{b_{p+q}}(V/\Lambda, \|\cdot\|), \tag{4-4}$$

where $\widehat{W} \subset H_q(M, \mathbb{R})$ is the image of $H_{p+q}(M, \mathbb{R})$ under $\frown \alpha$, $\hat{\Gamma} = H_q(M, \mathbb{Z})_{\mathbb{R}} \cap \widehat{W}$, $V = H_{p+q}(M, \mathbb{R})$, and $\Lambda = H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}$. Since $\widehat{W}$ has dimension b_{p+q} we have

$$\begin{aligned} \deg(\overline{\frown \alpha}) \Delta_{b_{p+q}}&(H_q(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|) \\ &\leq (C(n; p, q)\|\alpha\|^*)^{b_{p+q}} \Delta_{b_{p+q}}(H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|). \end{aligned} \tag{4-5}$$

Finally (4-3) follows from (4-5) by taking b_{p+q}-th roots. □

The following lemma is needed to show, in Proposition 4.9, that the product of two stable systoles of the same rank in complementary dimensions is bounded from above in terms of the volume of the manifold.

Lemma 4.8. *Let $0 < p < n$, let b_p be the p-th Betti number of M, and let $1 \leq k \leq b_p$. Then*

$$\operatorname{stsys}_{p,k}(M, g) \leq C(n, p) \operatorname{Vol}(M, g)^{\frac{1}{2}} \Delta_k(H^{n-p}(M, \mathbb{Z})_{\mathrm{R}}, |\cdot|_2^*)^{\frac{1}{k}},$$

where $C(n, p)$ is the constant in Proposition 3.2.

Proof. Let Λ be a sublattice of $H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}$ of rank k in a k-dimensional subspace $V \subset H^{n-p}(M, \mathbb{R})$. Then Poincaré duality maps Λ onto a sublattice Γ of $H_p(M, \mathbb{Z})_{\mathbb{R}}$ of rank k in a k-dimensional subspace $W \subset H_p(M, \mathbb{R})$ (because Poincaré duality is an isomorphism). Thus Poincaré duality is of rank k and degree 1. Applying Propositions 2.5 and 3.2, we obtain

$$\operatorname{Vol}_k(W/\Gamma, \|\cdot\|) \leq (\operatorname{Vol}(M, g))^{\frac{k}{2}} C(n, p)^k \operatorname{Vol}_k(V/\Lambda, |\cdot|_2^*). \tag{4-6}$$

The result follows by taking the infima over all rank-k sublattices and k-th roots. □

Proposition 4.9. *Let $0 < p < n$, and let b_p be the p-th Betti number of M and $1 \le k \le b_p$. Then*

$$\operatorname{stsys}_{p,k}(M, g) \operatorname{stsys}_{n-p,k}(M, g) \le C(n, p)^2 \operatorname{Vol}(M, g)(\gamma'_{b_p,k})^{\frac{1}{k}}.$$

Proof. Multiply the inequalities of Lemma 4.8 with p equal to p and to $n-p$. Observe that $C(n, p) = C(n, n-p)$, and that, because $H^p(M, \mathbb{Z})_{\mathbb{R}}$ and $H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}$ are dual lattices,

$$\Delta_k(H^p(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*)\, \Delta_k(H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*) \le \gamma'_{b_p,k}. \qquad \square$$

5. A sharp inequality in dimensions 1 and $n-1$

The two theorems in this section are analogs of the main theorem in [1].

Theorem 5.1. *Let (M, g) be a compact oriented manifold of dimension n whose first Betti number is b. Then*

$$\operatorname{stsys}_{1,b}(M, g) \operatorname{stsys}_{n-1,b}(M, g) \le \operatorname{Vol}(M, g). \tag{5-1}$$

Equality holds in (5-1) *if and only if there exists a Riemannian submersion of M onto a flat torus of dimension b with connected minimal fibers.*

Proof. The inequality (5-1) follows from Proposition 4.9 because $C(n, 1) = 1$ and $\gamma'_{b,b} = 1$.

Suppose now that equality holds in (5-1). Then by the proof of Proposition 4.9, equality holds in Lemma 4.8. Thus inequality (4-6) is an equality for $p \in \{1, n-1\}$ and $k = b$, that is,

$$\operatorname{Vol}_b(H_p(M, \mathbb{R})/H_p(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|) = \operatorname{Vol}(M, g)^{\frac{b}{2}} \operatorname{Vol}_b(H^{n-p}(M, \mathbb{R})/H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*)$$

for $p \in \{1, n-1\}$. Consequently, by Propositions 2.5 and 3.2,

$$\|[M] \frown \alpha\| = \operatorname{Vol}(M, g)^{\frac{1}{2}} |\alpha|_2^*$$

for all $\alpha \in H^p(M, \mathbb{R})$ with $p \in \{1, n-1\}$. Hence by Proposition 3.2, the first (and $(n-1)$-st) degree cohomology classes of M with integral periods are represented by harmonic 1-forms (and $(n-1)$-forms) of constant norm. Applying [12, Proposition 16.7.3], there exists a Riemannian submersion of M onto a flat torus of dimension b with minimal fibers. In fact the submersion is the Abel–Jacobi map using a basis of harmonic 1-forms from $H^1(M, \mathbb{Z})_{\mathbb{R}}$ which induces an epimorphism on the fundamental groups. Thus the fibers are connected.

Conversely, if there exists a Riemannian submersion of M onto a flat torus of dimension b with connected minimal fibers, then each step in this argument is reversible with equality holding at every step. Therefore equality holds in (5-1). $\square$

This result can be generalized to stable systoles of rank k, $1 \le k \le b$.

Theorem 5.2. *Let (M, g) be a compact oriented manifold of dimension n whose first Betti number is b. Then, for each $1 \le k \le b$,*

$$\operatorname{stsys}_{1,k}(M, g)\operatorname{stsys}_{n-1,k}(M, g) \le (\gamma'_{b,k})^{\frac{1}{k}} \operatorname{Vol}(M, g). \tag{5-2}$$

Equality holds if and only if there exists a Riemannian submersion with connected minimal fibers from M onto a flat b-dimensional torus $\mathbb{R}^b/\Lambda$ such that Λ is dual k-critical.

Proof. The inequality (5-2) follows from Proposition 4.9 because $C(n, 1) = 1$.

Suppose that equality holds in (5-2). Since $p = 1$, the proof of Proposition 4.9 implies

$$\Delta_k(H^1(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*)\, \Delta_k(H^{n-1}(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*) = \gamma'_{b,k}.$$

This means that the lattice $\Lambda = H^1(M, \mathbb{Z})_{\mathbb{R}}$ is dual k-critical (Definition A.1), and thus, by Lemma A.4, it is dual k-perfect (Definition A.3). Observe that the dual lattice $H^{n-1}(M, \mathbb{Z})_{\mathbb{R}}$ can be identified under the Hodge star operator with the lattice

$$\Lambda^* = \{\varphi \in H^1(M, \mathbb{R}) : \langle\langle \varphi, \psi \rangle\rangle \in \mathbb{Z}, \forall \psi \in \Lambda\}.$$

Let Q denote the vector space of quadratic forms on $H^1(M, \mathbb{R})$. That Λ is dual k-perfect implies that Q^* is generated by the linear functionals of the form $q \mapsto q(\alpha)$ where, in the notation of the Appendix, $\alpha \in W \in S(\Lambda) \cup S(\Lambda^*)$. We next need to prove that every $\alpha \in W \in S(\Lambda) \cup S(\Lambda^*)$ can be represented by a harmonic 1-form of constant norm. For then arguing as in [12, Remark 16.11.6], dual k-perfection implies that every harmonic 1-form on M has constant norm. This reduces us to the situation in Theorem 5.1, so that by [12, Proposition 16.7.3], the Abel–Jacobi map defines a Riemannian submersion with connected minimal fibers of M onto $\mathbb{R}^b/\Lambda$.

Let $V \in S(\Lambda)$ and set $\Lambda' = V \cap \Lambda$. Thus $\operatorname{Vol}_k(V/\Lambda', |\cdot|_2^*) = \Delta_k(\Lambda)$. The Poincaré duality map

$$T = [M] \frown : V \to H_{n-1}(M, \mathbb{R})$$

restricted to V has rank k and degree 1. Set $W = T(V)$ and $\Gamma = W \cap H_{n-1}(M, \mathbb{Z})_{\mathbb{R}}$. Since we are assuming equality in (5-2), equality holds in Proposition 4.9 which implies that equality holds in Lemma 4.8. Thus equality holds in (4-6) with $p = 1$, that is,

$$\operatorname{Vol}_k(W/\Gamma, \|\cdot\|) \le (\operatorname{Vol}(M, g))^{\frac{k}{2}} \operatorname{Vol}_k(V/\Lambda', |\cdot|_2^*).$$

Hence by Propositions 2.5 and 3.2,

$$\|T(\alpha)\| = \operatorname{Vol}(M, g)^{\frac{1}{2}} |\alpha|_2^*$$

for all $\alpha \in V$, and thus by Proposition 3.2, every such α is represented by a harmonic 1-form of constant norm. A similar argument shows that if $V \in S(\Lambda^*)$, then every $\alpha \in V$ also can be represented by a harmonic $(n-1)$-form of constant norm, so that its Hodge star $\star\alpha \in \Lambda^*$ is represented by a harmonic 1-form of constant norm.

Conversely, if there exists a Riemannian submersion with connected minimal fibers from M onto a flat b-dimensional torus $\mathbb{R}^b/\Lambda$ such that Λ is dual k-critical, the steps of the above argument are reversible with equality holding at each step so that equality holds in (5-2). $\square$

6. Example

Unless the manifold is nice enough, computing a stable systole of higher rank for a general Riemannian manifold is a difficult task. The purpose of this section is to illustrate the computation of a stable systole of higher rank in a case where the manifold is simple and nice enough to effect such a computation. In particular we compute the two dimensional stable systole of rank 6 in flat 4-dimensional tori. As a consequence we obtain a sharp stable systolic inequality for conformally flat 4-dimensional tori (Theorem 6.7).

Here we will consider a 4-dimensional flat torus $M = \mathbb{R}^4/\Lambda$ where Λ is a lattice in $\mathbb{R}^4$. According to [14] there are natural isomorphisms in cohomology

$$H^*(M, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^4) \tag{6-1}$$

and

$$H^*(M, \mathbb{Z})_{\mathbb{R}} \cong \Lambda^*_{\mathbb{Z}}(\Lambda^*), \tag{6-2}$$

as well as in homology

$$H_*(M, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^4) \tag{6-3}$$

and

$$H_*(M, \mathbb{Z})_{\mathbb{R}} \cong \Lambda^*_{\mathbb{Z}}(\Lambda). \tag{6-4}$$

Lemma 6.1. *The mass norm for $\xi \in \Lambda^2(\mathbb{R}^4)$ is given by*

$$\|\xi\| = (|\xi|^2 + |\xi \wedge \xi|)^{\frac{1}{2}}, \tag{6-5}$$

where $|\cdot|$ is the norm on the exterior algebra $\Lambda^*(\mathbb{R}^4)$ induced from the Euclidean norm of $\mathbb{R}^4$.

Proof. Under an orthogonal change of coordinates in $\mathbb{R}^4$ any given ξ can be put in the form

$$\xi = Ae_1 \wedge e_2 + Be_3 \wedge e_4.$$

Whitney [17, equation (13), p. 54] has proved that for such ξ, $\|\xi\| = |A| + |B|$. Thus

$$\|\xi\|^2 = |A|^2 + |B|^2 + 2|A|\,|B| = |\xi|^2 + |\xi \wedge \xi|.$$

This completes the proof because the expression (6-5) is invariant under orthogonal changes of coordinates. □

Lemma 6.2. *The comass norm for $\phi \in \Lambda^2(\mathbb{R}^4)$ is given by*

$$\|\phi\|^* = \left(\frac{|\phi|^2 + \sqrt{|\phi|^4 - |\phi \wedge \phi|^2}}{2}\right)^{\frac{1}{2}}. \tag{6-6}$$

Thus

$$\|\phi\|^* \le |\phi| \le \sqrt{2}\,\|\phi\|^*. \tag{6-7}$$

In particular the constant $C(4, 2) = \sqrt{2}$.

Proof. By the invariance of the expression (6-6) under orthogonal changes of coordinates in $\mathbb{R}^4$, it suffices to consider the case $\phi = Ae_1 \wedge e_2 + Be_3 \wedge e_4$. In this case the left-hand side of (6-6) is $\|\phi\|^* = \max(|A|, |B|)$ according to [17, equation (12), p. 54]. The right-hand side of (6-6) becomes

$$\left(\frac{A^2 + B^2 + \sqrt{(A^2+B^2)^2 - (2AB)^2}}{2}\right)^{\frac{1}{2}} = \left(\frac{A^2 + B^2 + |A^2 - B^2|}{2}\right)^{\frac{1}{2}},$$

which is equal to $\max(|A|, |B|)$. The inequality (6-7) follows easily from (6-6). □

Lemma 6.3. *Let K be the unit mass ball of $\Lambda^2(\mathbb{R}^4)$. Then*

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{2\pi^2}{9}.$$

Proof. Setting

$$\xi = x_1 e_2 \wedge e_3 - x_2 e_1 \wedge e_3 + x_3 e_1 \wedge e_2 + y_1 e_1 \wedge e_4 + y_2 e_2 \wedge e_4 + y_3 e_3 \wedge e_4$$

gives an isomorphism between $\Lambda^2(\mathbb{R}^4)$ and $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ with the Euclidean norms. If $dV_{\mathbb{R}^6}$ and dV_{S^5} are the volume elements in $\mathbb{R}^6$ and S^5, respectively, we have, on switching to spherical coordinates,

$$\mathrm{Vol}_6(K, |\cdot|) = \int_{\|\xi\| \le 1} dV_{\mathbb{R}^6} = \int_{\xi \in S^5} \int_0^{\frac{1}{\|\xi\|}} r^5\, dr\, dV_{S^5} = \frac{1}{6} \int_{\xi \in S^5} \frac{1}{\|\xi\|^6}\, dV_{S^5}.$$

The mapping $S^2 \times \left[0, \frac{\pi}{2}\right] \times S^2 \to S^5$ given by sending the ordered triplet (X, t, Y) to $\xi = (\cos(t)X, \sin(t)Y) \in \mathbb{R}^3 \times \mathbb{R}^3$ reparameterizes S^5 except on a set of measure 0. Making this change of variables in the integral gives us

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{1}{6} \int_{Y \in S^2} \int_0^{\frac{\pi}{2}} \int_{X \in S^2} \frac{\cos^2(t)\sin^2(t)}{(1 + 2\cos(t)\sin(t)|X \cdot Y|)^3}\, dV_{S^2}\, dt\, dV_{S^2}.$$

By the invariance of the inner product under isometries of S^2, the value of the inner double integral is independent of $Y \in S^2$. Because the area of S^2 is 4π, integrating over Y gives the value

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{4\pi}{6} \int_0^{\frac{\pi}{2}} \int_{X\in S^2} \frac{\cos^2(t)\sin^2(t)}{(1+2\cos(t)\sin(t)|X\cdot N|)^3}\, dV_{S^2}\, dt,$$

where

$$N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is the north pole of S^2. Now change to spherical coordinates (ϕ, θ) with $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, on S^2. Since $X \cdot N = \cos(\phi)$ we obtain

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{4\pi}{6} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\pi} \frac{\cos^2(t)\sin^2(t)}{(1+2\cos(t)\sin(t)|\cos(\phi)|)^3} \sin(\phi)\, d\phi\, d\theta\, dt.$$

Using the double angle formula for $\sin(2t)$, the symmetry of the integrand in ϕ about $\frac{\pi}{2}$, and the independence of the integrand in θ we obtain

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{4\pi^2}{6} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin^2(2t)}{(1+\sin(2t)\cos(\phi))^3} \sin(\phi)\, d\phi\, dt.$$

This can be easily evaluated by iterated integration to obtain

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{\pi^2}{3} \int_0^{\frac{\pi}{2}} \sin(2t) - \frac{\sin(2t)}{(1+\sin(2t))^2}\, dt = \frac{2\pi^2}{9}.$$

□

Corollary 6.4. *The Hausdorff–Busemann measure in* $\Lambda^2(\mathrm{R}^4)$ *is given by*

$$\mathrm{Vol}_6(-, \|\cdot\|) = \frac{3\pi}{4} \mathrm{Vol}_6(-, |\cdot|).$$

Proof. Let K be the unit mass ball in $\Lambda^2(\mathbb{R}^4)$. By definition of the Hausdorff-Busemann measure and Lemma 6.3,

$$\mathrm{Vol}_6(-, \|\cdot\|) = \frac{\omega_6}{\mathrm{Vol}_6(K, \|\cdot\|)} \mathrm{Vol}_6(-, |\cdot|) = \frac{\pi^3/3!}{2\pi^2/9} \mathrm{Vol}_6(-, |\cdot|). \qquad \square$$

Lemma 6.5.

$$\mathrm{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), |\cdot|) = \mathrm{Vol}_4(\mathbb{R}^4/\Lambda, |\cdot|)^3.$$

Proof. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^4$ be a set of generators for Λ over $\mathbb{Z}$. Then

$$\mathrm{Vol}_4(\mathbb{R}^4/\Lambda, |\cdot|) = |\det(v_1, v_2, v_3, v_4)|,$$

and

$$v_1 \wedge v_2, \quad v_1 \wedge v_3, \quad v_1 \wedge v_4, \quad v_2 \wedge v_3, \quad v_2 \wedge v_4, \quad v_3 \wedge v_4$$

is a set of generators for $\Lambda^2(\Lambda)$ over $\mathbb{Z}$. Recall that $\det(v_1, v_2, v_3, v_4)$ may be computed by keeping track of the sequence of elementary row operations that convert v_1, v_2, v_3, v_4 to the standard basis e_1, e_2, e_3, e_4. An elementary row operation either (i) adds one of the vectors to another, (ii) interchanges two vectors, or (iii) factors out a constant multiple c from one of the vectors. If v_1', v_2', v_3', v_4' is the result of applying an elementary row operation to v_1, v_2, v_3, v_4 we have $\det(v_1', v_2', v_3', v_4') = \det(v_1, v_2, v_3, v_4)$ in case (i), $-\det(v_1', v_2', v_3', v_4') = \det(v_1, v_2, v_3, v_4)$ in case (ii), and $c\det(v_1', v_2', v_3', v_4') = \det(v_1, v_2, v_3, v_4)$ in case (iii). Since $\det(e_1, e_2, e_3, e_4) = 1$, we see that $\det(v_1, v_2, v_3, v_4)$ is equal to the product of the constants c that we factored out in operations of type (ii) times ± 1 depending on whether there were an even or odd number of operations of type (ii). Now consider what happens to the generators of $\Lambda^2(\Lambda)$ under this sequence of operations. The operation that takes v_1, v_2, v_3, v_4 to v_1', v_2', v_3', v_4' will correspondingly take

$$v_1 \wedge v_2, \quad v_1 \wedge v_3, \quad v_1 \wedge v_4, \quad v_2 \wedge v_3, \quad v_2 \wedge v_4, \quad v_3 \wedge v_4$$

to

$$v_1' \wedge v_2', \quad v_1' \wedge v_3', \quad v_1' \wedge v_4', \quad v_2' \wedge v_3', \quad v_2' \wedge v_4', \quad v_3' \wedge v_4'.$$

If the operation is of type (i), the corresponding operation has the same effect as two operations of type (i). If the operation is of type (ii), the corresponding operation has the same effect as two operations of type (ii) and multiplying one vector by -1. If the operation is of type (iii), then the corresponding operation has the same effect as factoring out the same constant c from three of the vectors. As the result of the sequence of corresponding operations is the orthonormal basis

$$e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_1 \wedge e_4, \quad e_2 \wedge e_3, \quad e_2 \wedge e_4, \quad e_3 \wedge e_4,$$

the result follows because the sequence of elementary row operations gives a power of 3 times those computing the determinant of v_1, v_2, v_3, v_4. □

Theorem 6.6. *Let (M, g) be a 4-dimensional flat torus $\mathbb{R}^4/\Lambda$. Then*

$$\operatorname{stsys}_{2,6}(M, g)^2 = \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \operatorname{Vol}(M, g)$$

Proof. By Corollary 6.4 and Lemma 6.5,

$$\begin{aligned}\operatorname{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|) \\ &= \frac{3\pi}{4}\operatorname{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), |\cdot|) = \frac{3\pi}{4}\operatorname{Vol}_4(\mathbb{R}^4/\Lambda, |\cdot|)^3. \quad \square\end{aligned}$$

The paper [16] proved systolic inequalities for metrics on real projective spaces which are conformal to the constant curvature metric. Similar ideas combined with Theorem 6.6 lead to the following result about conformally flat metrics on 4-dimensional tori.

Theorem 6.7. *Let (M, g) be a conformally flat 4-dimensional torus. Then*

$$\mathrm{stsys}_{2,6}(M, g)^2 \leq \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \mathrm{Vol}(M, g).$$

Equality holds if and only if (M, g) is flat.

Proof. Since (M, g) is conformally flat, we may assume that there exists a lattice Λ in $\mathbb{R}^4$ such that $M = \mathbb{R}^4/\Lambda$ and that $g = f^2 g_0$ for some positive real-valued function f on M where g_0 is the flat metric on M. Let G be the group of isometries of the flat metric g_0 with Haar measure da normalized so that $\int_{a\in G} da = 1$. Set $\bar{f}(x) = \left(\int_{a\in G} f(ax)^2\, da\right)^{\frac{1}{2}}$. Since G acts transitively on M, $\bar{f}$ would be a constant function. Set $\bar{g} = \bar{f}^2 g_0$. Then $(M, \bar{g})$ is flat. Thus if dx is the volume form for (M, g_0),

$$\begin{aligned} \text{(6-8)} \qquad \mathrm{Vol}(M, \bar{g}) &= \int_{x\in M} \bar{f}(x)^4\, dx = \int_{x\in M} \left(\int_{a\in G} f(ax)^2\, da\right)^2 dx \\ &\leq \int_{x\in M}\int_{a\in G} f(ax)^4\, da\, dx \\ &= \int_{a\in G}\int_{x\in M} f(ax)^4\, dx\, da = \int_{a\in G}\int_{x\in M} f(ax)^4\, dx\, da \\ &= \int_{a\in G} \mathrm{Vol}(M, g)\, da = \mathrm{Vol}(M, g), \end{aligned}$$

where we have used successively Jensen's inequality, Fubini's theorem, the change of variables formula, that a is an isometry of g_0, and that G has unit measure. Note on account of Jensen's inequality, if equality holds then f is a constant function, and thus (M, g) would be flat.

On the other hand, given a homology class $h \in H_2(M, R)$ taking a 2-chain S in M representing h that gives the least mass (area) $\|h\|_{\bar{g}}$ in the homology class, one has

$$\|h\|_{\bar{g}} = \mathrm{Area}(S, \bar{g}) = \int_{a\in G} \mathrm{Area}(aS, g) \geq \|h\|_g.$$

As an explanation, suppose that S is a surface and $j : S \to M$ is the inclusion mapping. Then $j^* g_0$ induces an area form ds on S. Thus $j^* g$ induces the area form $(f \circ j)^2 ds$ and $j^*\bar{g}$ induces $(\bar{f} \circ j)^2 ds$. Thus

$$\begin{aligned} \text{(6-9)} \qquad \mathrm{Area}(S, \bar{g}) &= \int_{s\in S} (\bar{f} \circ j)^2\, ds = \int_{s\in S}\int_{a\in G} f(aj(s))^2\, da\, ds \\ &= \int_{a\in G}\int_{s\in S} f(aj(s))^2\, ds\, da = \int_{a\in G} \mathrm{Area}(aS, g)\, da. \end{aligned}$$

Therefore by Proposition 2.1

$$\text{(6-10)} \quad \mathrm{Vol}_6(H_2(M, \mathbb{R})/H_2(M, \mathbb{Z}), \|\cdot\|_g) \leq \mathrm{Vol}_6(H_2(M, \mathbb{R})/H_2(M, \mathbb{Z}), \|\cdot\|_{\bar{g}}).$$

Thus, since $(M, \bar{g})$ is flat,

$$\mathrm{stsys}_{2,6}(M, g) \leq \mathrm{stsys}_{2,6}(M, \bar{g}) = \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \mathrm{Vol}(M, \bar{g}) \leq \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \mathrm{Vol}(M, g).$$

If equality holds, then f must be constant and thus (M, g) would be flat. □

It is an open question whether this inequality also holds for metrics on 4-dimensional tori which are not conformally flat.

Theorem 6.6 gives information about the conformal volume norm of flat 4-dimensional tori. The conformal volume norm is an invariant of a conformal class of Riemannian metrics; see [8, 7.4; 12, 15.8]. When M is a 4-dimensional manifold, $h \in H_2(M, \mathbb{R})$, and $\mathcal{G}$ is a conformal class of Riemannian metrics on M, the conformal volume norm satisfies

$$\|h\|_{L^2} = \sup\left\{ \frac{\|h\|_g}{\sqrt{\operatorname{Vol}(M, g)}} : g \in \mathcal{G} \right\},$$

where $\|h\|_g$ is the stable mass norm for the Riemannian metric g. Thus for any $g \in \mathcal{G}$ and $h \in H_2(M, \mathbb{R})$, one has

$$\|h\|_g \leq \sqrt{\operatorname{Vol}(M, g)}\, \|h\|_{L^2}.$$

Corollary 6.8. *Let (M, g) be a 4-dimensional flat torus $\mathbb{R}^4/\Lambda$. Then*

$$\left(\frac{3\pi}{4}\right)^{\frac{1}{6}} \leq \operatorname{confsys}_{2,6}(M, g).$$

Proof. Applying Proposition 2.5, the Hausdorff–Busemann volumes satisfy

$$\operatorname{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|) \leq \operatorname{Vol}(\mathbb{R}^4/\Lambda, g)^3 \operatorname{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|_{L^2}).$$

Dividing by $\operatorname{Vol}(\mathbb{R}^4/\Lambda, g)^3$, extracting 6-th roots, and using Theorem 6.6 gives

$$\left(\frac{3\pi}{4}\right)^{\frac{1}{6}} = \frac{\operatorname{stsys}_{2,6}(M, g)}{\operatorname{Vol}(M, g)^{\frac{1}{2}}} \leq \operatorname{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|_{L^2})^{\frac{1}{6}},$$

where the right side of the inequality is by the definition of $\operatorname{confsys}_{2,6}(M, g)$. □

Appendix

The following proof that a dual k-extreme lattice is dual k-perfect is a modification of the argument in [2] that a dual extreme lattice is dual perfect.

Let V be a Euclidean space of dimension n with Euclidean norm $|\cdot|$, and let Λ be a lattice in V. Define $S_k(\Lambda)$ to be the collection of all k-dimensional subspaces W of V for which $\Gamma = \Lambda \cap W$ is a rank-k sublattice of Λ such that $\operatorname{Vol}_k(W/\Gamma, |\cdot|) = \Delta_k(\Lambda)$.

Definition A.1. A lattice Λ of rank k is dual k-extreme if it is a local maximum of the function

$$\Lambda \mapsto \Delta_k(\Lambda)\, \Delta_k(\Lambda^*)$$

and is dual k-critical if it is an absolute maximum, that is, if

$$\Delta_k(\Lambda)\, \Delta_k(\Lambda^*) = \gamma'_{n,k}.$$

Lemma A.2. *Suppose there exists a hyperspace H of V such that $W \subset H$ for all $W \in S_k(\Lambda)$. Then there exists a lattice M near Λ such that $\Delta_k(\Lambda)\,\Delta_k(\Lambda^*) < \Delta_k(\mathsf{M})\,\Delta_k(\mathsf{M}^*)$. In other words, Λ is not dual k-extreme.*

Proof. Let $0 < r < 1$, and consider the linear transformation $\tau : V \to V$ such that τ is the identity on H and contracts by a factor of r on $H^\perp$. As $W \subset H$ for all $W \in S(\Lambda)$, by continuity, if r is chosen close enough to 1, then $S_k(\tau(\Lambda)) = S_k(\Lambda)$ and $\Delta_k(\Lambda) = \Delta_k(\tau(\Lambda))$. Note that the adjoint map $(\tau^\dagger)^{-1}$ is the identity on H and expands by a factor of $\frac{1}{r} > 1$ on $H^\perp$ and that $(\tau\Lambda)^* = (\tau^\dagger)^{-1}\Lambda^*$. Thus if W is a k-dimensional subspace such that $\Gamma^* = \Lambda^* \cap W$ is a rank-k sublattice of Λ^* which is not contained in H, then

$$\text{(A-1)} \qquad \mathrm{Vol}_k((\tau^\dagger)^{-1}(W)/(\tau^\dagger)^{-1}(\Gamma^*), |\cdot|) > \mathrm{Vol}_k(W/\Gamma^*, |\cdot|).$$

Thus if no $W \in S_k(\Lambda^*)$ is contained in H, we have $\Delta_k((\tau\Lambda)^*) > \Delta_k(\Lambda^*)$ and we may take $\mathsf{M} = \tau(\Lambda)$. However if $W \subset H$ for some $W \in S_k(\Lambda^*)$, then on account of our hypothesis and (A-1), $W \subset H$ for all $W \in S_k(\tau\Lambda) \cup S_k((\tau\Lambda)^*)$ and

$$\Delta_k(\tau(\Lambda))\,\Delta_k((\tau\Lambda)^*) = \Delta_k(\Lambda)\,\Delta_k(\Lambda^*).$$

Now proceed by taking a hyperplane F in H which contains no $W \in S_k((\tau\Lambda)^*)$. We have the orthogonal decomposition

$$V = H^\perp \oplus H = H^\perp \oplus F^\perp \oplus F,$$

where $F^\perp$ is the orthogonal complement to F in H. Consider a linear transformation $\sigma : V \to V$ whose matrix, relative to an orthonormal basis which respects this orthogonal decomposition, can be written in the block form

$$\sigma = \begin{pmatrix} q & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

where $q > 1$ and $a \neq 0$. Then

$$(\sigma^\dagger)^{-1} = \begin{pmatrix} \frac{1}{q} & -\frac{a}{q} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

Observe that σ is the identity on H, so that if q is sufficiently close to 1 and a to 0, $S_k(\sigma\tau\Lambda) = S_k(\tau\Lambda)$ and $\Delta_k(\sigma\tau\Lambda) = \Delta_k(\tau\Lambda) = \Delta_k(\Lambda)$. However, σ increases lengths of vectors in H which are not in F. Thus $\Delta_k((\sigma\tau\Lambda)^*) > \Delta_k((\tau\Lambda)^*) = \Delta_k(\Lambda^*)$. Therefore $\Delta_k(\Lambda)\,\Delta_k(\Lambda^*) < \Delta_k(\mathsf{M})\,\Delta_k(\mathsf{M}^*)$ where $\mathsf{M} = \sigma\tau\Lambda$. □

Definition A.3. Let Q be the vector space of quadratic forms on V. We say that a lattice Λ is dual k-perfect if the linear functionals $q \mapsto q(x)$ for $x \in W \in S_k(\Lambda) \cup S_k(\Lambda^*)$ generate Q^*.

Clearly Λ is k-perfect if $q(x) = 0$ for all $x \in W \in S_k(\Lambda) \cup S_k(\Lambda^*)$ implies $q = 0$. Note that the elements q of Q correspond to symmetric endomorphisms υ of V such that $q(x) = \upsilon(x) \cdot x$ for $x \in V$ where $\cdot$ denotes the inner product on V.

Lemma A.4. *If Λ is dual k-extreme, then Λ is dual k-perfect.*

Proof. If Λ is not dual k-perfect, then there exists a nonzero symmetric endomorphism υ of V such that $\upsilon(x) \cdot x = 0$ for all $x \in W \in S_k(\Lambda) \cup S_k(\Lambda^*)$. Consider the linear isomorphism $\tau = \mathrm{id} + \epsilon\upsilon$. Since υ is symmetric, $\tau^\dagger = \tau$. If $\epsilon > 0$ is sufficiently small, $S_k(\tau\Lambda) \subset S_k(\Lambda)$ and $S_k((\tau^\dagger)^{-1}\Lambda^*) \subset S_k(\Lambda^*)$. Thus if $x \in W \in S_k(\Lambda)$, then $\tau(x) = x + \epsilon\upsilon(x)$ and thus, using $\upsilon(x) \cdot x = 0$,

$$|\tau(x)|^2 = |x|^2 + \epsilon^2 |\upsilon(x)|^2.$$

Hence τ increases lengths of vectors $x \in W$ for $W \in S_k(\Lambda)$. Therefore $\Delta_k(\Lambda^*) \geq \Delta_k(\Lambda)$, with equality if and only if $\upsilon(x) = 0$ for all $x \in W \in S_k(\tau\Lambda)$.

Choosing $\epsilon > 0$ sufficiently small so that $\epsilon|\upsilon| < 1$, where $|\upsilon|$ is the operator norm, one has the series expansion

$$(\tau^\dagger)^{-1} = \tau^{-1} = \mathrm{id} - \epsilon\upsilon + \epsilon^2\upsilon^2 - \epsilon^3\upsilon^3 + \cdots .$$

Thus if $y \in W \in S_k(\Lambda^*)$, then

$$(\tau^\dagger)^{-1}(y) = y - \epsilon\upsilon(y) + \epsilon^2\upsilon^2(y) - \epsilon^3\upsilon^3(y) + \cdots .$$

Using the symmetry of υ, it follows that

$$|(\tau^\dagger)^{-1}(y)|^2 = |y|^2 - 2\epsilon\upsilon(y) \cdot y + 3\epsilon^2\upsilon(y) \cdot \upsilon(y) - 4\epsilon^3\upsilon^2(y) \cdot \upsilon(y) + \cdots .$$

But by assumption, $\upsilon(y) \cdot y = 0$. Hence, using the symmetry of υ,

$$\begin{aligned} |(t^\dagger)^{-1}(y)|^2 &= |y|^2 + 3\epsilon^2\upsilon(y) \cdot \upsilon(y) - 4\epsilon^3\upsilon^2(y) \cdot \upsilon(y) + 5\epsilon^4\upsilon^3(y) \cdot \upsilon(y) - \cdots \\ &= |y|^2 + 3\epsilon^2|\upsilon(y)|^2 - 4\epsilon^3\upsilon^2(y) \cdot \upsilon(y) + 5\epsilon^4|\upsilon^2(y)|^2 - \cdots \\ &\geq |y|^2 + \epsilon^2|\upsilon(y)|^2(3 - 4\epsilon|\upsilon| - 6\epsilon^3|\upsilon|^3 - \cdots), \end{aligned} \tag{A-2}$$

since the terms with odd coefficients are nonnegative. If ϵ is sufficiently small, inequality (A-2) shows that $(\tau^\dagger)^{-1}$ increases lengths of vectors $y \in W \in S_k(\Lambda^*)$. Thus $\Delta_k((\tau\Lambda)^*) \geq \Delta_k(\Lambda^*)$ with equality if and only if $\upsilon(y) = 0$ for all $y \in W \in S_k((\tau\Lambda)^*)$.

Since Λ is assumed to be dual k-extreme, these inequalities show

$$\Delta_k(\tau\Lambda)\,\Delta_k((\tau\Lambda)^*) \leq \Delta_k(\Lambda)\,\Delta_k(\Lambda^*) \leq \Delta_k(\tau\Lambda)\,\Delta_k((\tau\Lambda)^*).$$

Thus $\Delta_k((\tau\Lambda)) = \Delta_k(\Lambda)$ and $\Delta_k((\tau\Lambda)^*) = \Delta_k(\Lambda^*)$. Consequently, as υ is nonzero, every $W \in S_k(\Lambda) \cup S_k(\Lambda^*)$ is contained in the hyperplane $\upsilon(x) = 0$ contradicting the dual k-extremality of Λ by Lemma A.2. Therefore Λ is dual k-perfect. $\square$

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Received January 18, 2021. Revised March 24, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 325, No. 1, 2023
https://doi.org/10.2140/pjm.2023.325.127

SPIN KOSTKA POLYNOMIALS AND VERTEX OPERATORS

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An algebraic iterative formula for the spin Kostka–Foulkes polynomial $K^-_{\xi\mu}(t)$ is given using vertex operator realizations of Hall–Littlewood symmetric functions and Schur Q-functions. Based on the operational formula, more favorable properties are obtained parallel to the Kostka polynomial. In particular, we obtain some formulae for the number of (unshifted) marked tableaux. As an application, we confirmed a conjecture of Aokage on the expansion of the Schur P-function in terms of Schur functions. Tables of $K^-_{\xi\mu}(t)$ for $|\xi| \le 6$ are listed.

1. Introduction

The Hall–Littlewood symmetric functions $P_\mu(x;t)$ and the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ both have played an active role in algebraic combinatorics and representation theory. On one hand, the Hall–Littlewood symmetric functions $P_\mu(x;t)$ are certain deformations of the Schur functions $s_\lambda(x)$, and the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ are the transition coefficients between the two bases. On the other hand, $K_{\lambda\mu}(t)$ have the following representation theoretic interpretation. Let $\mathfrak{B}_\mu$ be the variety of flags preserved by a nilpotent matrix with Jordan block of shape μ. The cohomology group $H^\bullet(\mathfrak{B}_\mu)$ affords a graded $\mathfrak{S}_n$-module structure. Set

$$C_{\lambda\mu}(t)=\sum_{i\ge 0} t^i(\dim \mathrm{Hom}_{\mathfrak{S}_n}(S^\lambda, H^{2i}(\mathfrak{B}_\mu)),$$

where S^λ denotes the Specht module of $\mathfrak{S}_n$ associated with λ. Garsia and Procesi [1992] proved that

$$K_{\lambda\mu}(t)=C_{\lambda\mu}(t^{-1})t^{n(\mu)}, \tag{1-1}$$

which confirms geometrically the positivity of the Kostka–Foulkes polynomials [Lascoux and Schützenberger 1978].

Recently, Wan and Wang [2013] have introduced the spin Kostka–Foulkes polynomials $K^-_{\xi\mu}(t)$ as the transition coefficients between the Hall–Littlewood functions

MSC2020: primary 05E05, 17B69; secondary 20C25, 20C30.
Keywords: spin Kostka polynomials, vertex operators.

$P_\mu(x;t)$ and Schur Q-functions Q_ξ with interesting representation theoretic interpretations. As is well-known, the Schur Q-functions are indexed by strict partitions and were used by Schur [Stembridge 1989] in generalizing the Frobenius character formula for projective irreducible characters of the symmetric group $\mathfrak{S}_n$. Schur Q-functions form a distinguished basis in the subring of symmetric functions generated by $p_1, p_3, \dots$. Yamaguchi [1999] has shown that the category of irreducible $\mathfrak{S}_n$-supermodules is equivalent to that of supermodules of the Hecke–Clifford algebra $\mathcal{H}_n = \mathcal{C}_n \rtimes \mathbb{C}\mathfrak{S}_n$ and the irreducible objects D^ξ are parametrized by strict partitions $\xi \in \mathcal{SP}_n$. Wan and Wang [2013] have shown that the spin Kostka polynomials admit the interpretation

$$K^-_{\lambda\mu}(t) = 2^{[l(\xi)/2]} C_{\xi\mu}(t^{-1}) t^{n(\mu)}, \tag{1-2}$$

and

$$C^-_{\xi\mu}(t) = \sum_{i\geq 0} t^i (\dim \mathrm{Hom}_{\mathcal{H}_n}(D^\xi, \mathcal{C}_n \otimes H^{2i}(\mathfrak{B}_\mu)).$$

Let $\mathfrak{q}(n)$ be the queer Lie superalgebra containing the general linear Lie algebra $\mathfrak{gl}(n)$ as its even subalgebra. Sergeev [1984] has shown that the irreducible $\mathfrak{q}(n)$-modules $V(\xi)$ are also parametrized by strict partitions $\xi \in \mathcal{SP}_n$. It turns out that the q-weight multiplicity $\gamma^-_{\xi\mu}(t)$ associated with the weight space $V(\xi)_\mu$ also appears as the spin Kostka polynomial [Wan and Wang 2013]:

$$K^-_{\lambda\mu}(t) = 2^{[l(\xi)/2]} \gamma^-_{\xi\mu}(t). \tag{1-3}$$

The purpose of this paper is to give an operational algebraic formula for the spin Kostka–Foulkes polynomials $K^-_{\xi\mu}(t)$. The method we adopt is similar to that of [Bryan and Jing 2021], in which the vertex operator realizations of the Hall–Littlewood polynomials and Schur functions were employed. However, there is some subtlety in the spin situation.

In the usual vertex realization of Schur Q-functions [Jing 1991b], only the modes of odd indices (of the twisted Heisenberg algebra) were used in the definition. Should this vertex operator be employed, the commutation relations of its components with those of the vertex operator for the Hall–Littlewood symmetric functions would have infinitely many terms in the quadratic relations. To salvage the situation, we introduce a new vertex operator realization of Schur Q-functions using a larger Heisenberg algebra graded by all integers (see (2-8) and (2-9)). The new vertex operator realization enables us to get a finite quadratic relation between the operators realizing both the Hall–Littlewood and Schur Q-functions and then the matrix coefficients express the spin Kostka polynomials.

As matrix coefficients, the spin Kostka–Foulkes polynomials can be computed in general, and exact formulas are given in some special cases. We also prove a stability formula for the spin Kostka polynomials. We have clarified some questions regarding them (in Example 3.11, we disproved the symmetric property) and obtained counting formulas for the Stembridge coefficients [Stembridge 1989] between the Schur P-functions and Schur functions. As applications, we answer a recent conjecture of Aokage and are able to derive a tensor decomposition in the general situation.

The paper is organized as follows. In Section 2 we recall the vertex operator realization of the Hall–Littlewood functions and give a new vertex operator construction of the Schur Q-functions, which is specifically tailored for taming the commutation relation between the two vertex operators. In Section 3 we express the spin Hall–Littlewood polynomials as matrix coefficients of vertex operators and derive an iterative formula (see Theorem 3.5). Finally in Section 4 we use the iterative formulas to verify Aokage's conjecture on multiplicities of tensor products of spin modules, and a formula is also obtained for the general case.

2. Vertex operator realization of Hall–Littlewood and Schur Q-functions

A partition (resp. strict partition) $\lambda = (\lambda_1, \lambda_2, \ldots)$, denoted $\lambda \vdash n$, is a weakly (resp. strictly) decreasing sequence of positive integers such that $\sum_i \lambda_i = n$. The sum $|\lambda| = \sum_i \lambda_i$ is called the weight and the number $l(\lambda)$ of nonzero parts is called the length. We also define $\lambda \models n$ if λ is a composition of n when the past λ_i are not necessarily ordered. The set of partitions (resp. strict partitions) of weight n will be denoted by $\mathcal{P}_n$ (resp. $\mathcal{SP}_n$). The dominance order $\lambda \geq \mu$ is defined by $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for each i.

Let m_i be the multiplicity of i in λ and set $z_\lambda = \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!$; we define the parity $\varepsilon_\lambda = (-1)^{|\lambda| - l(\lambda)}$ and

$$z_\lambda(t) = \frac{z_\lambda}{\prod_{i\geq 1}(1 - t^{\lambda_i})}, \quad n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i. \tag{2-1}$$

A partition λ can be visualized by its Young diagram when λ is identified with $\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. To each cell $(i, j) \in \lambda$, we define its content $c_{ij} = j - i$ and hook length $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, where the partition $\lambda' = (\lambda'_1, \ldots, \lambda'_{\lambda_1})$ is the dual partition of λ obtained by reflecting the Young diagram of λ along the diagonal.

In this paper, we use the t-integer $[n] = t^{n-1} + t^{n-2} + \cdots + t + 1$. Similarly $[n]! = [n] \cdots [1]$, and the Gauss t-binomial symbol $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \frac{[n]!}{[k]![n-k]!}$.

Let Λ_F be the ring of symmetric functions over $F = \mathbb{Q}(t)$, the field of rational functions in t. We also consider Λ over the ring of integers. The space Λ_F is graded and decomposes into a direct sum:

$$\Lambda_F = \bigoplus_{n=0}^{\infty} \Lambda_F^n, \tag{2-2}$$

where Λ_F^n is the subspace of degree n, spanned by the elements $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$ with $|\lambda| = n$. Here p_r is the degree r power sum symmetric function.

Let $\Gamma_{\mathbb{Q}}$ be the subring of $\Lambda_{\mathbb{Q}}$ generated by the p_{2r-1}, $r \in \mathbb{N}$. Then

$$\Gamma_{\mathbb{Q}} = \mathbb{Q}[p_r : r \text{ odd}]. \tag{2-3}$$

The Schur Q-functions Q_ξ, ξ strict, form a $\mathbb{Q}$-basis of $\Gamma_{\mathbb{Q}}$ [Macdonald 1979]. Also, Γ is a graded ring $\Gamma = \oplus_{n \geq 0} \Gamma^n$, where $\Gamma^n = \Gamma \cap \Lambda^n$.

The space Λ_F is equipped with the bilinear form $\langle \cdot , \cdot \rangle$ defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda(t). \tag{2-4}$$

As $\{z_\lambda(t)^{-1} p_\lambda\}$ is the dual basis of the power sum basis, the adjoint operator of the multiplication operator p_n is the differential operator $p_n^* = (n/(1-t^n))\, \partial/\partial p_n$ of degree $-n$.

We recall the vertex operator realization of the Hall–Littlewood symmetric functions [Jing 1991a] and construct a variant vertex operator for the Schur Q-function on the space Λ_F. The *vertex operators* $H(z)$ and its adjoint $H^*(z)$ are t-parametrized linear maps, $\Lambda_F \longrightarrow \Lambda_F[\![z, z^{-1}]\!] = \Lambda_F \otimes F[z, z^{-1}]$, defined by

$$H(z) = \exp\biggl(\sum_{n \geq 1} \frac{1-t^n}{n} p_n z^n\biggr) \exp\biggl(-\sum_{n \geq 1} \frac{\partial}{\partial p_n} z^{-n}\biggr) = \sum_{n \in \mathbb{Z}} H_n z^n, \tag{2-5}$$

and

$$H^*(z) = \exp\biggl(-\sum_{n \geq 1} \frac{1-t^n}{n} p_n z^n\biggr) \exp\biggl(\sum_{n \geq 1} \frac{\partial}{\partial p_n} z^{-n}\biggr) = \sum_{n \in \mathbb{Z}} H_n^* z^{-n}. \tag{2-6}$$

Note that $*$ is $\mathbb{Q}(t)$-linear and anti-involutive satisfying

$$\langle H_n u, v \rangle = \langle u, H_n^* v \rangle \tag{2-7}$$

for $u, v \in \Lambda_F$.

We now introduce the *vertex operators* $Q(z)$ and its *adjoint* $Q^*(z)$ as the linear maps, $\Lambda_F \longrightarrow \Lambda_F[\![z, z^{-1}]\!]$, defined by

$$Q(z) = \exp\biggl(\sum_{n \geq 1, \text{odd}} \frac{2}{n} p_n z^n\biggr) \exp\biggl(-\sum_{n \geq 1} \frac{\partial}{\partial p_n} z^{-n}\biggr) = \sum_{n \in \mathbb{Z}} Q_n z^n, \tag{2-8}$$

and

$$Q^*(z) = \exp\biggl(-\sum_{n \geq 1} \frac{1-t^n}{n} p_n z^n\biggr) \exp\biggl(\sum_{n \geq 1, \text{odd}} \frac{2}{1-t^n} \frac{\partial}{\partial p_n} z^{-n}\biggr) = \sum_{n \in \mathbb{Z}} Q_n^* z^{-n}. \tag{2-9}$$

The components $H_n, H_{-n}^* \in \mathrm{End}_F(\Lambda)$ are of degree n, and so are the annihilation operators for $n > 0$. Similarly $Q_n, Q_{-n}^* \in \mathrm{End}_{\mathbb{Q}}(\Lambda)$. We remark that the second exponential factor of $Q(z)$ is different from the usual construction in [Jing 1991b], and this will be crucial for our later discussion. In particular, note that $Q(-z) \neq Q^*(z)$ in the current situation due to different inner product.

We collect the relations of the vertex operators as follows.

Proposition 2.1 [Jing 1991a; 1991b]. (1) *The operators H_n and H_n^* satisfy the relations*

$$H_m H_n - t H_n H_m = t H_{m+1} H_{n-1} - H_{n-1} H_{m+1}, \tag{2-10}$$

$$H_m^* H_n^* - t H_n^* H_m^* = t H_{m-1}^* H_{n+1}^* - H_{n+1}^* H_{m-1}^*, \tag{2-11}$$

$$H_m H_n^* - t H_n^* H_m = t H_{m-1} H_{n-1}^* - H_{n-1}^* H_{m-1} + (1-t)^2 \delta_{m,n}, \tag{2-12}$$

$$H_{-n}.1 = Q_{-n}.1 = \delta_{n,0}, \quad H_n^*.1 = Q_n^*.1 = \delta_{n,0}, \tag{2-13}$$

where $\delta_{m,n}$ is the Kronecker delta function.

(2) *The operators Q_n satisfy the Clifford algebra relations*

$$\{Q_m, Q_n\} = (-1)^n 2\delta_{m,-n}, \tag{2-14}$$

where $\{A, B\} = AB + BA$.

Proof. Commutation relations (2-10)–(2-13) were from [Jing 1991a]. We focus on (2). Define the normal ordering product by

$$:Q(z)Q(w): = \exp\biggl(\sum_{n\geq 1,\mathrm{odd}} \frac{2}{n} p_n (z^n + w^n)\biggr) \exp\biggl(-\sum_{n\geq 1} \frac{\partial}{\partial p_n}(z^{-n} + w^{-n})\biggr).$$

Then we have for $|z| < |w|$

$$Q(z)Q(w) = :Q(z)Q(w): \exp\biggl(-\sum_{n\geq 1,\mathrm{odd}} \frac{2}{n}\Bigl(\frac{w}{z}\Bigr)^n\biggr) = :Q(z)Q(w): \frac{z-w}{z+w}.$$

The rest of the argument is similar to Proposition 4.15 in [Jing 1991b]. □

Note that the vacuum vector 1 is annihilated by p_n^*, so

$$H(z).1 = \exp\biggl(\sum_{n=1}^{\infty} \frac{1-t^n}{n} p_n z^n\biggr) = \sum_{n=0}^{\infty} q_n z^n = q(z), \tag{2-15}$$

where q_n is the Hall–Littlewood polynomial of one-row partition (n), and clearly

$$q_n = H_n.1 = \sum_{\lambda\vdash n} \frac{1}{z_\lambda(t)} p_\lambda. \tag{2-16}$$

We also introduce a spin analogue $h(z)$ by

$$\tilde{h}(z) = \exp\biggl(\sum_{n=1}^{\infty} \frac{t^n - (-1)^n}{n} p_n z^n\biggr) = \sum_{n\geq 0} \tilde{h}_n z^n; \tag{2-17}$$

then

$$\tilde{h}_n = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda(-t)} p_\lambda. \tag{2-18}$$

Moreover,

$$\tilde{h}_n(-t) = \sum_{\lambda \vdash n} \varepsilon_\lambda u_\lambda q_\lambda, \tag{2-19}$$

where $\varepsilon_\lambda = (-1)^{|\lambda|-l(\lambda)}$ and $u_\lambda = l(\lambda)!/\prod_{i\geq 1} m_i(\lambda)!$.

As consequences of the proposition, one also has that

$$H_n H_{n+1} = t H_{n+1} H_n, \tag{2-20}$$

$$H_n^* H_{n-1}^* = t H_{n-1}^* H_n^*, \tag{2-21}$$

$$\langle H_n.1, H_n.1\rangle = \sum_{\lambda \vdash n} \frac{1}{z_\lambda(t)} = 1 - t, \quad n > 0, \tag{2-22}$$

$$\langle H_n.1, H_{-n}^*.1\rangle = \sum_{\lambda \vdash n} \frac{(-1)^{l(\lambda)}}{z_\lambda(t)} = t^n - t^{n-1}, \quad n > 0, \tag{2-23}$$

where the last two identities follow from (2-12) and (2-10) by induction.

In general, expressing H_μ for any composition μ in terms of the basis elements H_λ, $\lambda \in \mathcal{P}$, can be formulated as follows. Let $S_{i,a}$ be the transformation $(\lambda_1, \dots, \lambda_i, \lambda_{i+1}, \dots) \mapsto (\lambda_1, \dots, \lambda_{i+1} - a, \lambda_i + a, \dots)$, where $\lambda_{i+1} > \lambda_i$. Define

$$C(S_{i,a}) = \begin{cases} t, & a = 0, \\ t^{a+1} - t^{a-1}, & 1 \leq a < \left[\frac{\lambda_{i+1}-\lambda_i}{2}\right], \\ t^{a+\epsilon} - t^{a-1}, & 1 \leq a = \left[\frac{\lambda_{i+1}-\lambda_i}{2}\right], \end{cases} \tag{2-24}$$

where $\epsilon \equiv \lambda_{i+1} - \lambda_i \pmod 2$. For $\underline{i} = (i_1, \dots, i_r)$ and $\underline{a} = (a_1, \dots, a_r)$ let

$$C(S_{\underline{i},\underline{a}}) = C(S_{i_1,a_1}) C(S_{i_2,a_2}) \cdots C(S_{i_r,a_r}), \tag{2-25}$$

where the product order follows that of $S_{i_1,a_1} S_{i_2,a_2} \cdots S_{i_r,a_r} \lambda$, i.e., from the right to the left. In particular, when $t = 0$, $C(S_{\underline{i},\underline{a}}) = 0$ unless all $a_i = 1$; in that case, $C(S_{\underline{i},\underline{1}}) = (-1)^r$ which is possible only when $\lambda_{i+1} - \lambda_i \geq 2$. When $t = -1$, $C(S_{\underline{i},\underline{a}}) = 0$ unless all $a_i = 0$ and $C(S_{\underline{i},\underline{0}}) = (-1)^r$.

Let μ be a composition and λ be a partition. Define

$$B(\lambda, \mu) \triangleq \sum_{\underline{i},\underline{a}} C(S_{\underline{i},\underline{a}}) \tag{2-26}$$

summed over $\underline{i} = (i_1, i_2, \dots, i_r)$, $\underline{a} = (a_1, a_2, \dots, a_r)$ such that $S_{\underline{i},\underline{a}} \mu = \lambda$.

Proposition 2.2 [Jing and Liu 2022]. *Suppose μ is a composition. Then*

$$H_\mu = \sum_{\lambda \vdash |\mu|} B(\lambda, \mu) H_\lambda. \tag{2-27}$$

We remark that λ appears only when $\lambda \geq \mu$ in (2-27). Let μ be a composition and λ be a partition. If there exists $\underline{i} = (i_1, i_2, \dots, i_r)$, $\underline{a} = (a_1, a_2, \dots, a_r)$ such that $S_{\underline{i},\underline{a}}\mu = \lambda$, then $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$, $k = 1, 2, \dots$.

Proposition 2.3 [Jing 1991a; 1991b]. (1) *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition. The vertex operator products $H_{\lambda_1} \cdots H_{\lambda_l}.1$ is the Hall–Littlewood function $Q_\lambda(t)$:*

$$H_{\lambda_1} \cdots H_{\lambda_l}.1 = Q_\lambda(t) = \prod_{i<j} \frac{1 - R_{ij}}{1 - tR_{ij}} q_{\lambda_1} \cdots q_{\lambda_l}, \tag{2-28}$$

where the raising operator is given by $R_{ij} q_\lambda = q_{(\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j - 1, \dots, \lambda_l)}$.

(2) *Let $\xi = (\xi_1, \xi_2, \dots, \xi_l)$ be a strict partition. Then*

$$Q_\xi = Q_{\xi_1} Q_{\xi_2} \cdots Q_{\xi_l}.1 \tag{2-29}$$

is the Schur Q-function indexed by ξ. Moreover, $Q_\xi.1$, where ξ ranges over strict partitions, form an orthogonal $\mathbb{Z}$-base of Γ under the specialized inner product $\langle \cdot, \cdot \rangle_{t=-1}$, explicitly

$$\langle Q_\lambda.1, Q_\xi.1 \rangle|_{t=-1} = 2^{l(\lambda)} \delta_{\lambda\xi}, \quad \lambda, \xi \in \mathcal{SP}. \tag{2-30}$$

Proof. Part (1) is from [Jing 1991a]. Since our vertex operator $Q(z)$ is different from that of [Jing 1991b], we explain why the new vertex operator also realizes the Schur Q-functions. From the argument in proving (2-14) in Proposition 2.1 it follows that

$$\begin{aligned} Q(z_1)Q(z_2) \cdots Q(z_l).1 &= \prod_{i<j} \frac{z_i - z_j}{z_i + z_j} {:}Q(z_1)Q(z_2) \cdots Q(z_l){:}.1 \\ &= \prod_{i<j} \frac{z_i - z_j}{z_i + z_j} \exp\biggl(\sum_{n \geq 1, \text{odd}} \frac{2p_n}{n} (z_1^n + \cdots + z_l^n) \biggr). \end{aligned}$$

Taking coefficients of $z_1^{\xi_1} \cdots z_l^{\xi_l}$, we obtain that Q_ξ is exactly the Schur Q-function indexed by ξ (cf. [Jing 1991b]). □

3. Spin Hall–Littlewood polynomials and vertex operators

Wan and Wang [2013] have introduced an extremely interesting spin analogue of Kostka(–Foulkes) polynomials and shown that these polynomials enjoy favorable properties parallel to those of the Kostka polynomials.

Definition 3.1 [Wan and Wang 2013]. The spin Kostka polynomials $K^-_{\xi\mu}(t)$ for $\xi \in \mathcal{SP}$ and $\mu \in \mathcal{P}$ are defined by

$$Q_\xi(x) = \sum_\mu K^-_{\xi\mu}(t) P_\mu(x;t), \tag{3-1}$$

where $Q_\xi(x)$ (resp. $P_\mu(x;t)$) are Schur Q-functions (resp. Hall–Littlewood functions).

From the above discussion and Proposition 2.3, it is clear that the spin Kostka polynomials can be expressed as matrix coefficients:

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle Q_\mu(x;t), Q_\xi(x)\rangle \\ &= \langle H_{\mu_1}H_{\mu_2}\cdots H_{\mu_l}.1,\, Q_{\xi_1}Q_{\xi_2}\cdots Q_{\xi_k}.1\rangle. \end{aligned}$$

To compute the matrix coefficients, we first get the commutation relations by usual techniques of vertex operators:

$$H^*(z)Q(w)(w-tz) + Q(w)H^*(z)(z+w) = 2(1-t)z\delta\Big(\frac{w}{z}\Big)\tilde{h}(z), \tag{3-2}$$

$$\tilde{h}^*(z)H(w) = H(w)\tilde{h}^*(z)\frac{w+z}{w-tz}, \tag{3-3}$$

$$Q(z)\tilde{h}(w) = \tilde{h}(w)Q(z)\frac{z-tw}{z+w}. \tag{3-4}$$

We remark that if the old vertex operator $\tilde{Q}(w)$ from [Jing 1991b] were used, then the commutation relations between $H^*(z)$ and $\tilde{Q}(w)$ would have been an infinite quadratic relation.

Taking coefficients we obtain the following commutation relations.

Proposition 3.2. *The commutation relations between the Hall–Littlewood vertex operators and Schur Q-function operators are*

$$H^*_nQ_m = t^{-1}H^*_{n-1}Q_{m-1} + t^{-1}Q_mH^*_n + t^{-1}Q_{m-1}H^*_{n-1} + 2(1-t^{-1})\tilde{h}_{m-n}, \tag{3-5}$$

$$\tilde{h}^*_mH_n = H_n\tilde{h}^*_m + (1+t)\sum_{k=0}^{m-1} t^{m-k-1}H_{n-m+k}\tilde{h}^*_k, \tag{3-6}$$

$$Q_n\tilde{h}_m = \tilde{h}_mQ_n + (1+t)\sum_{k=0}^{m-1}(-1)^{m-k}\tilde{h}_kQ_{n-k+m}. \tag{3-7}$$

Now we can state our formulas to compute the spin Kostka polynomials. To this end, we prepare some necessary notation. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ be (strict) partitions. We define $\lambda^{[i]} = (\lambda_{i+1}, \dots, \lambda_l)$, $\lambda^{\hat{i}} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_l)$, and $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots)$.

Theorem 3.3. *For an integer k, strict partition $\xi = (\xi_1, \xi_2, \dots, \xi_l)$ and partition $\mu = (\mu_1, \mu_2, \dots, \mu_m)$,*

$$H_k^* Q_\xi = \sum_{i=1}^{l} (-1)^{i-1} 2\tilde{h}_{\xi_i - k} Q_{\xi^{\hat{i}}}, \tag{3-8}$$

$$\tilde{h}_k^* H_\mu = \sum_{\tau \models k} t^{k-l(\tau)} (1+t)^{l(\tau)} H_{\mu-\tau}. \tag{3-9}$$

Proof. We show the first relation by induction on $k + |\xi|$. The case of $k + |\xi| = 1$ is clear. Assume that (3-8) holds for $k + |\xi| = n - 1$. Using the induction hypothesis and (3-5) we have that

$$\begin{aligned}
&H_k^* Q_{\xi_1} Q_{\xi_2} \cdots Q_{\xi_l} \\
&\quad = t^{-1} H_{k-1}^* Q_{\xi_1 - 1} Q_{\xi_2} \cdots Q_{\xi_l} + t^{-1} Q_{\xi_1} H_k^* Q_{\xi_2} \cdots Q_{\xi_l} + t^{-1} Q_{\xi_1 - 1} H_{k-1}^* Q_{\xi_2} \cdots Q_{\xi_l} \\
&\qquad\qquad + 2(1 - t^{-1}) \tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} \\
&\quad = t^{-1} \big(2\tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_2 - k + 1} Q_{\xi_1 - 1} Q_{\xi_3} \cdots Q_{\xi_l} \\
&\qquad\quad + 2\tilde{h}_{\xi_3 - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} + \cdots + (-1)^{l+1} 2\tilde{h}_{\xi_l - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad\quad + t^{-1} Q_{\xi_1} \big(2\tilde{h}_{\xi_2 - k} Q_{\xi_3} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_3 - k} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\qquad + 2\tilde{h}_{\xi_4 - k} Q_{\xi_2} Q_{\xi_3} Q_{\xi_5} \cdots Q_{\xi_l} + \cdots + (-1)^l 2\tilde{h}_{\xi_l - k} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad\qquad + t^{-1} Q_{\xi_1 - 1} \big(2\tilde{h}_{\xi_2 - k + 1} Q_{\xi_3} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_3 - k + 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\qquad\quad + 2\tilde{h}_{\xi_4 - k + 1} Q_{\xi_2} Q_{\xi_3} Q_{\xi_5} \cdots Q_{\xi_l} \\
&\qquad\qquad\quad + \cdots + (-1)^l 2\tilde{h}_{\xi_l - k + 1} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad\qquad\quad + 2(1 - t^{-1}) \tilde{h}_{\xi_1 - k} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_l}.
\end{aligned}$$

Simplifying the expression, we see the above is

$$\begin{aligned}
&t^{-1} \big(2\tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_2 - k + 1} Q_{\xi_1 - 1} Q_{\xi_3} \cdots Q_{\xi_l} \\
&\qquad + 2\tilde{h}_{\xi_3 - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad + \cdots + (-1)^{l+1} 2\tilde{h}_{\xi_l - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad + 2t^{-1} \big(\tilde{h}_{\xi_2 - k + 1} Q_{\xi_1 - 1} Q_{\xi_3} \cdots Q_{\xi_l} - \tilde{h}_{\xi_3 - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\quad + \tilde{h}_{\xi_4 - k + 1} Q_{\xi_1 - 1} \cdots Q_{\xi_l} \\
&\qquad\quad + \cdots + (-1)^l \tilde{h}_{\xi_l - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad\quad - 2 \big(\tilde{h}_{\xi_2 - k} Q_{\xi_1} Q_{\xi_3} \cdots Q_{\xi_l} - \tilde{h}_{\xi_3 - k} Q_{\xi_1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\qquad + \tilde{h}_{\xi_4 - k} Q_{\xi_1} Q_{\xi_2} Q_{\xi_3} Q_{\xi_5} \cdots Q_{\xi_l} \\
&\qquad\qquad + \cdots + (-1)^l \tilde{h}_{\xi_l - k} Q_{\xi_1} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad\qquad + 2(1 - t^{-1}) \tilde{h}_{\xi_1 - k} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_l} \quad \text{(by (3-7))} \\
&= 2\tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_2 - k} Q_{\xi_1} Q_{\xi_3} \cdots Q_{\xi_l} + \cdots + 2(-1)^{l-1} \tilde{h}_{\xi_l - k} Q_{\xi_1} \cdots Q_{\xi_{l-1}},
\end{aligned}$$

which has proved (3-8). The second relation is similarly shown by (3-6) and induction on $l(\mu)$. □

Example 3.4. Let $\mu = (2, 2)$ and $\xi = (3, 1)$. Then by Theorem 3.3

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle H_2 H_2.1,\, Q_3 Q_1.1\rangle \\ &= \langle H_2.1,\, 2h_1 Q_1.1\rangle \\ &= 2\langle t(1+t^{-1})H_1.1,\, Q_1.1\rangle \\ &= 4t+4. \end{aligned}$$

By Theorem 3.3, we now obtain an algebraic formula for $K^-_{\xi\mu}(t)$.

Theorem 3.5. *For $\xi = (\xi_1, \dots, \xi_l) \in \mathcal{SP}_n$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathcal{P}_n$, $K^-_{\xi\mu}(t)$ is given by the iterative formula*

$$K^-_{\xi\mu}(t) = \sum_{i=1}^{l} \sum_{\tau \models \xi_i - \mu_1} \sum_{\lambda \vdash n-\xi_i} (-1)^{i-1} 2t^{\xi_i-\mu_1}(1+t^{-1})^{l(\tau)} B(\lambda, \mu^{[1]}-\tau) K^-_{\xi^{\hat{i}}\lambda}(t). \tag{3-10}$$

Proof. It follows readily from (3-8), (3-9) and (2-27). □

Equation (3-10) shows that all spin Kostka polynomials are integral polynomials, and it also gives an effective recurrence of $K^-_{\xi\mu}(t)$ as shown by the following example.

Example 3.6. Let $\xi = (4, 3, 1)$ and $\mu = (3, 3, 2)$. Then

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle H_3 H_3 H_2.1,\, Q_4 Q_3 Q_1.1\rangle \\ &= \langle H_3 H_2.1,\, 2\tilde{h}_1 Q_3 Q_1.1\rangle - \langle H_3 H_2.1,\, 2\tilde{h}_0 Q_4 Q_1.1\rangle \\ &= 2\langle t(1+t^{-1})(H_2 H_2.1 + H_3 H_1.1),\, Q_3 Q_1.1\rangle - 2\langle H_3 H_2.1,\, Q_4 Q_1.1\rangle \\ &= 2(t+1)(K^-_{(3,1)(2,2)}(t) + K^-_{(3,1)(3,1)}(t)) - 2K^-_{(4,1)(3,2)}(t). \end{aligned}$$

The spin Kostka polynomials have quite a few remarkable properties resembling those of the Kostka–Foulkes polynomials. As a consequence of the recurrence we have the following.

Corollary 3.7. *Let ξ be a strict partition and μ be a partition. We have*:

(1) *If there exists $k \in \mathbb{N}$, such that $\xi_i = \mu_i$, $i = 1, 2, \dots, k$, then*

$$K^-_{\xi\mu}(t) = 2^k K^-_{\xi^{[k]}\mu^{[k]}}(t). \tag{3-11}$$

In particular, $K^-_{\xi\xi}(t) = 2^{l(\xi)}$.

(2) $2^{l(\xi)} \mid K^-_{\xi\mu}(t)$.

(3) $K^-_{\xi\mu}(-1) = 2^{l(\xi)}\delta_{\xi\mu}$.

Proof. They are immediate consequences of Theorem 3.5. □

Some special cases of Theorem 3.5 are listed as follows.

Example 3.8. Suppose $\xi \in \mathcal{SP}_n$, $\mu \in \mathcal{P}_n$. We have

$$K^-_{\xi(n)}(t) = 2\delta_{\xi,(n)}, \tag{3-12}$$

$$K^-_{(n)\mu}(t) = 2t^{n-\mu_1} \sum_{\tau \models n-\mu_1} (1+t^{-1})^{l(\tau)} B(\varnothing, \mu^{(1)} - \tau), \tag{3-13}$$

$$K^-_{\xi(\mu_1,\mu_2)}(t) = \begin{cases} 2^{2-\delta_{0,\xi_2}} t^{\xi_1-\mu_1}(1+t^{-1}) & \text{if } \xi > (\mu_1,\mu_2), \\ 4 & \text{if } \xi = (\mu_1,\mu_2), \\ 0 & \text{otherwise.} \end{cases} \tag{3-14}$$

There is a compact formula of $K^-_{(n)\mu}(t)$ [Wan and Wang 2013] by using a result of [Macdonald 1979]. We will come back to the Wan–Wang formula using the iteration in the next section.

The following result was first proved in [Wan and Wang 2013] using the similar property of the Kostka–Foulkes polynomials. Using our iterative formula, one can give an independent proof from that of the Kostka–Foulkes polynomials. We remark that the method can also be used to show this property for the Kostka–Foulkes polynomial by the iterative formula in [Bryan and Jing 2021].

Corollary 3.9. *Let $\xi = (\xi_1, \xi_2, \dots) \in \mathcal{SP}_n$, $\mu = (\mu_1, \mu_2, \dots) \in \mathcal{P}_n$. Then $K^-_{\xi\mu}(t) = 0$, unless $\xi \geq \mu$.*

Proof. It is equivalent to prove $K^-_{\xi\mu}(t) = 0$, if $\xi \ngeq \mu$. We argue it by induction on n. The initial step is obvious. Suppose it holds for weight $< n$. There exists a smallest $k \geq 1$, such that $\xi_1 + \xi_2 + \cdots + \xi_k < \mu_1 + \mu_2 + \cdots + \mu_k$.

If $k = 1$, then it's evident that $K^-_{\xi\mu}(t) = 0$ by the iterative formula (3-10).

If $k > 1$, then there exists $k > j \geq 1$, such that $\xi_{j+1} < \mu_1 \leq \xi_j$. We have

$$\begin{aligned} K^-_{\xi\mu}(t) &= \sum_{i=1}^{j} (-1)^{i-1} \langle H_{\mu_2} H_{\mu_3} \cdots, 2\tilde{h}_{\xi_i-\mu_1} Q_{\xi_1} \cdots \hat{Q}_{\xi_i} \cdots \rangle \\ &= \sum_{i=1}^{j} (-1)^{i-1} \sum_{\tau \models \xi_i - \mu_1} 2t^{\xi_i-\mu_1}(1+t^{-1})^{l(\tau)} \langle H_{\mu^{[1]}-\tau}, Q_{\xi^{\hat{i}}} \rangle \\ &= \sum_{i=1}^{j} (-1)^{i-1} \sum_{\tau \models \xi_i - \mu_1} 2t^{\xi_i-\mu_1}(1+t^{-1})^{l(\tau)} \sum_{\nu \vdash n-\xi_i} B(\nu, \mu^{[1]} - \tau) \langle H_\nu, Q_{\xi^{\hat{i}}} \rangle \\ &= \sum_{i=1}^{j} (-1)^{i-1} \sum_{\tau \models \xi_i - \mu_1} 2t^{\xi_i-\mu_1}(1+t^{-1})^{l(\tau)} \sum_{\nu \vdash n-\xi_i} B(\nu, \mu^{[1]} - \tau) K^-_{\xi^{\hat{i}}\nu}(t). \end{aligned}$$

By the remark below Proposition 2.2, for each $1 \le i \le j$, we have $\nu_1+\cdots+\nu_{k-1} \ge \mu_2+\cdots+\mu_k-\tau_1-\cdots-\tau_{k-1} \ge \mu_2+\cdots+\mu_k+\mu_1-\xi_i > \xi_1+\cdots+\xi_{i-1}+\xi_{i+1}+\cdots\xi_k$. By induction, we have $K^-_{\xi\mu}(t)=0$. $\square$

The Kostka–Foulkes polynomials have the stability property [Bryan and Jing 2021], which says that if $\mu_1 \ge \lambda_2$, then $K_{\lambda+(r),\mu+(r)}(t)=K_{\lambda\mu}(t)$ for all $r \ge 1$. Here, $\lambda+(r)=(\lambda_1+r, \lambda_2, \ldots)$. The spin Kostka polynomials also enjoy the same stability.

Proposition 3.10. *Let $\xi=(\xi_1, \ldots, \xi_l) \in \mathcal{SP}$, $\mu=(\mu_1, \ldots, \mu_m) \in \mathcal{P}$, and $\mu_1 > \xi_2$. Then for any $r \ge 1$, we have*

$$K^-_{\xi+(r)\mu+(r)}(t)=K^-_{\xi\mu}(t). \tag{3-15}$$

Proof. By Theorem 3.3, it follows that

$$K^-_{\xi+(r)\mu+(r)}(t)=\langle H_{\mu_2}H_{\mu_3}\cdots H_{\mu_m}.1, 2\tilde{h}_{\xi_1-\mu_1}Q_{\xi_2}\cdots Q_{\xi_l}.1\rangle = K^-_{\xi\mu}(t). \qquad \square$$

The spin Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ were conjecturally symmetric [Wan and Wang 2013, Question 4.10] in the sense that

$$K^-_{\lambda\mu}(t)=t^{m_{\lambda\mu}}K^-_{\lambda\mu}(t^{-1})$$

for some $m_{\lambda\mu} \in \mathbb{Z}$. However, the following is a counterexample.

Example 3.11. Given $\xi=(3,2)$ and $\mu=(2,1^3)$, we have

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle H_2H_1H_1H_1.1, Q_3Q_2.1\rangle \\ &= \langle H_1H_1H_1.1, 2\tilde{h}_1Q_2.1\rangle - \langle H_1H_1H_1.1, 2\tilde{h}_0Q_3.1\rangle \\ &= 2\langle t(1+t^{-1})[3]H_1H_1.1, Q_2.1\rangle - 2K^-_{(3)(1^3)}(t) \\ &= 4t(t^3+2t^2+3t+2). \end{aligned}$$

4. Marked tableaux

To study projective representations of the symmetric group, Stembridge [1989] introduced the number $g_{\xi\lambda}$ as follows:

$$Q_\xi(x)=\sum_\lambda b_{\xi\lambda}s_\lambda(x), \quad g_{\xi\lambda}=2^{-l(\xi)}b_{\xi\lambda}. \tag{4-1}$$

Note that $b_{\xi\lambda}=K^-_{\xi\lambda}(0)$, but we will see that $g_{\xi\lambda}$ can be extended to any partition ξ, so we reserve this notation in this section.

Let ξ, λ be partitions with ξ strict. The coefficient $g_{\xi\lambda}$ of s_λ in the expansion of the Schur Q-function $2^{-l(\xi)}Q_\xi$ counts the number of (unshifted) marked tableaux T of shape λ and weight ξ such that

(a) $w(T)$ has the lattice property;

(b) for each $k \geq 1$, the last occurrence of k' in $w(T)$ precedes the last occurrence of k.

Here $w(T)$ is the word of T by reading the symbols in T from right to left in successive rows, starting with the top row.

The combinatorial interpretation and the representation-theoretic interpretation of $g_{\xi\lambda}$ are known [Sagan 1987; Stembridge 1989; Wan and Wang 2013; Worley 1984]. However, no effective formula for $g_{\xi\mu}$ is available. As an application of the preceding section, we give an algebraic formula for $g_{\xi\lambda}$.

The ring $\Lambda_{\mathbb{Q}}$ of symmetric functions has the canonical bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle_0 = \langle\,\cdot\,,\,\cdot\,\rangle_{t=0}$ under which Schur functions are orthonormal:

$$\langle p_\lambda, p_\mu\rangle_0 = \delta_{\lambda,\mu} z_\lambda. \tag{4-2}$$

Thus the adjoint operator of the multiplication operator p_n is the differential operator $p_n^- = n(\partial/(\partial p_n))$.

With respect to $\langle\,\cdot\,,\,\cdot\,\rangle_0$, the *vertex operators* and their adjoint operators for Schur functions and Schur Q-functions are given by [Jing 1991b; 2000]

$$S^{\pm}(z) = \exp\biggl(\pm\sum_{n\geq 1}\frac{1}{n}p_n z^n\biggr)\exp\biggl(\mp\sum_{n\geq 1}\frac{\partial}{\partial p_n}z^{-n}\biggr) = \sum_{n\in\mathbb{Z}} S_n^{\pm} z^{\pm n}, \tag{4-3}$$

$$Q^+(z) = Q(z) = \sum_{n\in\mathbb{Z}} Q_n^+ z^n, \tag{4-4}$$

$$Q^-(z) = \exp\biggl(-\sum_{n\geq 1}\frac{1}{n}p_n z^n\biggr)\exp\biggl(\sum_{n\geq 1,\text{odd}} 2\frac{\partial}{\partial p_n}z^{-n}\biggr) = \sum_{n\in\mathbb{Z}} Q_n^- z^{-n}. \tag{4-5}$$

Note that $Q^-(z)$ is the specialized vertex operator $Q^*(z)|_{t=0}$. Here we denote the adjoint operators by S_n^+ and Q_n^+, respectively, to distinguish from the preceding section.

Therefore $g_{\xi\lambda}$ can be expressed in terms of this inner product:

$$g_{\xi\lambda} = 2^{-l(\xi)} b_{\xi\lambda} = 2^{-l(\xi)}\langle s_\lambda, Q_\xi\rangle_0 = 2^{-l(\xi)}\langle S_\lambda.1, Q_\xi.1\rangle_0. \tag{4-6}$$

Recall that the involution $\omega : \Lambda \to \Lambda$ defined by $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$ [Macdonald 1979] is an isometry with respect to the canonical inner product $\langle\,\cdot\,,\,\cdot\,\rangle_0$ such that

$$\omega(s_\lambda) = s_{\lambda'}, \quad \omega(Q_\xi) = Q_\xi.$$

Proposition 4.1. *If $\lambda \in \mathcal{P}_n$, $\xi \in \mathcal{SP}_n$, then $g_{\xi\lambda}$ or $b_{\xi\lambda}$ has the property*

$$g_{\xi\lambda} = g_{\xi\lambda'}. \tag{4-7}$$

We introduce the operators for the elementary symmetric functions e_n

$$e^{\pm}(z) = \exp\left(\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} p_n^{\pm} z^{\pm n}\right) = \sum_{n\geq 0} e_n^{\pm} z^{\pm n}, \tag{4-8}$$

where $p_n^+ = p_n$, $p_n^- = n(\partial/(\partial p_n))$, and $e^+(z) = h(z)|_{t=0}$.

Then by Theorem 3.3 we have:

Proposition 4.2. *For any strict partition $\xi = (\xi_1, \xi_2, \dots, \xi_l)$, any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and integer k,*

$$S_k^- Q_\xi = \sum_{i=1}^{l} (-1)^{i-1} 2 e_{\xi_i - k} Q_{\xi_1} Q_{\xi_2} \cdots \hat{Q}_{\xi_i} \cdots Q_{\xi_l}, \tag{4-9}$$

$$e_k^- S_\lambda = \sum_\rho S_\rho, \tag{4-10}$$

where ρ runs through the partitions such that λ/ρ are vertical k-strips.

The algebraic iterative formula for $b_{\xi\lambda}$ is then natural:

Theorem 4.3. *Let $\xi \in \mathcal{SP}_n$, $\lambda \in \mathcal{P}_n$. Then*

$$b_{\xi\lambda} = \sum_{i=1}^{l(\xi)} 2(-1)^{i-1} \sum_{\rho^i} b_{\xi^{(i)}\rho^i}, \tag{4-11}$$

where ρ^i runs through the partitions such that $\lambda^{[1]}/\rho^i$ are vertical $\xi_i - \lambda_1$-strips.

Example 4.4. Let $\lambda \in \mathcal{P}_n$. We have

$$b_{(n)\lambda} = \begin{cases} 2 & \text{if } \lambda \text{ is a hook,} \\ 0 & \text{if } \lambda \text{ is not a hook.} \end{cases} \tag{4-12}$$

Combining (3-1) and (4-1), we have

$$K_{\xi\mu}^-(t) = \sum_\lambda b_{\xi\lambda} K_{\lambda\mu}(t), \tag{4-13}$$

where $K_{\lambda\mu}(t)$ are the Kostka–Foulkes polynomials.

By (4-12), we have

$$K_{(n)\mu}^-(t) = \sum_{\lambda \text{ hook}} 2K_{\lambda\mu}(t). \tag{4-14}$$

Recall that a compact formula for the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ is known for λ being hook-shaped [Kirillov 2001; Bryan and Jing 2021]:

$$K_{(n-k,1^k)\mu}(t) = t^{n(\mu) + \frac{k(k+1-2l)}{2}} \begin{bmatrix} l-1 \\ k \end{bmatrix}, \tag{4-15}$$

where $n = |\mu|, l = l(\mu)$. Therefore, we have that for any partition $\mu \vdash n$

$$K^{-}_{(n)\mu}(t) = \sum_{k=0}^{l(\mu)-1} 2t^{n(\mu)+\frac{k(k+1-2l(\mu))}{2}} \begin{bmatrix} l(\mu)-1 \\ k \end{bmatrix}, \tag{4-16}$$

$$= t^{n(\mu)} \prod_{i=1}^{l(\mu)} (1+t^{1-i}). \tag{4-17}$$

Here the second equation follows from the t-binomial expansion [Andrews 1986, (2.9)] or an easy induction on $l(\mu)$ from (4-16). We remark that (4-17) was first given by Wan and Wang [2013] using identities of Hall–Littlewood polynomials.

For a given partition λ, we define

$$\{\lambda\}_s \doteq \{\rho \subset \lambda^{[1]} \mid \rho \text{ is a hook and } \lambda^{[1]}/\rho \text{ is a vertical } s\text{-strip}\}.$$

Set $N^{(s)}(\lambda) = \mathrm{Card}\{\lambda\}_s$. It is clear that $N^{(s)}(\lambda) = 0$ when $s < 0$ or $s > |\lambda^{[1]}|$. Now we can give a two-row formula by the iterative formula for $b_{\xi\lambda}$.

Theorem 4.5. *Let* $1 \le m < \frac{n}{2}$, $\lambda \in \mathcal{P}_n$. *We have*

$$b_{(n-m,m)\lambda} = 4(N^{(n-m-\lambda_1)}(\lambda) - N^{(m-\lambda_1)}(\lambda)). \tag{4-18}$$

To compute $N^{(s)}(\lambda)$, we denote all hook (resp. double hook) partitions of n by $\mathrm{HP}(n)$ (resp. $\mathrm{DHP}(n)$). That is, $\mathrm{HP}(n) \doteq \{(\lambda_1, 1^{m_1}) \mid \lambda_1 + m_1 = n\}$, $\mathrm{DHP}(n) \doteq \{(\lambda_1, \lambda_2, 2^{m_2}, 1^{m_1}) \mid \lambda_1+\lambda_2+2m_2+m_1 = n\}$. Clearly, $\mathrm{HP}(n) \subset \mathrm{DHP}(n)$. We remark that $N^{(s)}(\lambda) = 0$ unless $\lambda \in \mathrm{DHP}(n)$. Now let's consider $N^{(s)}(\lambda)$, for $0 \le s \le |\lambda^{[1]}|$ and $\lambda \in \mathrm{DHP}(n)$, case by case.

Case 1: If $\lambda \in \mathrm{HP}(n)$, then $N^{(s)}(\lambda) = 1$.

Before considering the case $\lambda \in \mathrm{DHP}(n) \setminus \mathrm{HP}(n)$, we look at the following special case.

Case 2: If $\lambda = (\lambda_1, \lambda_2, 1^{m_1})$ and $\lambda \notin \mathrm{HP}(n)$, then we have

$$N^{(s)}(\lambda) = \begin{cases} 0 & \text{if } s \ge m_1 + 2, \\ 1 & \text{if } s = 0 \text{ or } s = m_1 + 1, \\ 2 & \text{if } 1 \le s \le m_1. \end{cases} \tag{4-19}$$

Case 3: If $\lambda = (\lambda_1, \lambda_2, 2^{m_2}, 1^{m_1}) \in \mathrm{DHP}(n) \setminus \mathrm{HP}(n)$, then it follows from case 2 that

$$\begin{aligned} N^{(s)}(\lambda) &= N^{(s-m_2)}((\lambda_1, \lambda_2, 1^{m_1})) \\ &= \begin{cases} 0 & \text{if } 0 \le s \le m_2 - 1 \text{ or } s \ge m_1 + m_2 + 2, \\ 1 & \text{if } s = m_2 \text{ or } s = m_1 + m_2 + 1, \\ 2 & \text{if } 1 + m_2 \le s \le m_1 + m_2. \end{cases} \end{aligned} \tag{4-20}$$

Example 4.6. Given $\xi = (4, 3)$, $\lambda = (2, 2, 2, 1)$, we have $\lambda_1 = \lambda_2 = 2$, $m_1 = m_2 = 1$, and

$$b_{(4,3)(2,2,2,1)} = 4(N^{(2)}(\lambda) - N^{(1)}(\lambda)) = 4 \times (2 - 1) = 4.$$

The symmetric group $\mathfrak{S}_n$ has a two-valued representation, known as the spin representation studied by Schur, and this is actually a representation of the double covering group $\widetilde{\mathfrak{S}}_n$ of $\mathfrak{S}_n$ [Schur 1911]. It is known that the irreducible spin representations of $\mathfrak{S}_n$ are parametrized by strict partitions of n. Let ζ^λ be the irreducible spin character of the Schur double covering group $\tilde{\mathfrak{S}}_n$ afforded by the module V^λ, $\lambda \in \mathcal{SP}_n$. Stembridge [1989] obtained the irreducible decomposition for the twisted tensor product of $\widetilde{\mathfrak{S}}_n$ [Kleshchev 2005]

$$\boldsymbol{ch}(\zeta^{(n)} \otimes \zeta^\lambda) = P_\lambda(x; -1),$$

where $\boldsymbol{ch}$ is the characteristic map (cf. [Jing 1991b]).

Corollary 4.7. *Let S^λ be the Specht module corresponding to partition $\lambda \vdash n$ and $1 \le m < \frac{n}{2}$. Then we have the irreducible decomposition as $\mathfrak{S}_n$-modules*

$$V^{(n)} \otimes V^{(n-m,m)} \simeq \bigoplus_{\lambda \in \mathrm{DHP}(n)} (N^{(n-m-\lambda_1)}(\lambda) - N^{(m-\lambda_1)}(\lambda)) S^\lambda. \tag{4-21}$$

Aokage [2021b] obtained the explicit irreducible decomposition of $(V^{(n)})^{\otimes 2}$ when n is even, so (4-21) offers the formula for a general tensor product. Recall that the symmetric functions $P_\mu(x; -1)$ are well defined for all partitions μ, so $g_{\mu\lambda}$ are defined similarly as (4-1) for any partitions λ, μ:

$$P_\mu(x; -1) = \sum_\lambda g_{\mu\lambda} s_\lambda(x). \tag{4-22}$$

Note that the following identities between the Schur P-functions and the Schur functions hold by using the tensor product of the spin representations of the symmetric group [Aokage 2021a]:

$$\begin{aligned} \sum_{\lambda \in \mathrm{HP}(n) \setminus \mathrm{HOP}(n)} s_\lambda(x) &= \sum_{l(\mu) \le 2} (-1)^{\mu_2} P_\mu(x; -1), \\ \sum_{\lambda \in \mathrm{HOP}(n)} s_\lambda(x) &= \sum_{l(\mu) = 2} (-1)^{\mu_2 + 1} P_\mu(x; -1), \end{aligned} \tag{4-23}$$

where $\mathrm{HOP}(n) \doteq \{\lambda \in \mathrm{HP}(n) \mid \lambda_1 \text{ is odd}\}$ and $n = 2r$ is even.

Aokage [2021a] has this conjecture at the end of his paper:

Theorem 4.8. *For $\lambda = (n - j, 1^j) \in \mathrm{HP}(n)$,*

$$g_{(r^2)\lambda} = \begin{cases} 0 & \text{if } j < r, \\ (-1)^{r+j} & \text{if } j \ge r. \end{cases} \tag{4-24}$$

As an application of our two-row formula for $b_{\xi\lambda}$, we will present a proof of Aokage's conjecture.

Combining with the above two identities in (4-23), we have

$$P_n(x;-1)+2\sum_{i\geq 1}^{r}(-1)^i P_{(n-i,i)}(x;-1)=\sum_{j=0}^{n}(-1)^j s_{(n-j,1^j)}(x).$$

Thus,

$$P_{(r^2)}(x;-1)=\frac{1}{4}\sum_{i\geq 0}^{r-1}(-1)^{i+r+1}Q_{(n-i,i)}(x;-1)+\frac{1}{2}\sum_{j=0}^{n}(-1)^{r+j}s_{(n-j,1^j)}(x).$$

By the orthonormality of s_λ,

$$g_{(r^2)\lambda}=\frac{1}{4}\sum_{i\geq 0}^{r-1}(-1)^{i+r+1}b_{(n-i,i)\lambda}+\frac{1}{2}(-1)^{r+j}\delta_{(n-j,1^j)\lambda}.$$

It follows from the remark below Theorem 4.5, we have $g_{(r^2)\lambda}=0$ unless $\lambda\in\mathrm{DHP}(n)$. Now let's show Theorem 4.8.

Proof. Let $\lambda=(n-j,1^j)\in\mathrm{HP}(n)$. We have

$$\begin{aligned}g_{(r^2)\lambda}&=\frac{1}{2}(-1)^{r+1}+\sum_{i=1}^{r-1}(-1)^{i+r+1}(N^{(j-i)}(\lambda)-N^{(i+j-n)}(\lambda))+\frac{1}{2}(-1)^{r+j}\\&=\frac{1}{2}(-1)^{r+1}+(-1)^{r+1}\left(\sum_{i=1}^{\min\{r-1,j\}}(-1)^i-\sum_{i=n-j}^{r-1}(-1)^i\right)+\frac{1}{2}(-1)^{r+j}.\end{aligned}$$

Then the result follows immediately by a careful analysis of j and direct computation. □

We remark that there exists a quadratic expression of the P-function in terms of Schur functions [Lascoux et al. 1993]. Explicit and direct linear expansion (4-22) in general is thus needed. Indeed, we can give a compact formula of $g_{(r^2)\lambda}$ for any partition λ.

Theorem 4.9. *For $\lambda=(\lambda_1,\lambda_2,2^{m_2},1^{m_1})\in\mathrm{DHP}(n)\setminus\mathrm{HP}(n)$, we have that*

$$g_{(r^2)\lambda}=\sum_{i=1}^{r-1}(-1)^{i+r+1}(N^{(n-i-\lambda_1)}(\lambda)-N^{(i-\lambda_1)}(\lambda)). \tag{4-25}$$

By considering λ case by case, we have that

$$g_{(r^2)\lambda}=\begin{cases}1 & \text{if } \lambda_2+m_1-1\leq\lambda_1\leq\lambda_2+m_1+1,\\ 0 & \text{otherwise.}\end{cases}$$

Tables for $K^-_{\xi\mu}(t)$, $2 \le n \le 6$

Here

$$[n] = t^{n-1} + \cdots + t + 1, \quad [n]!! = [n][n-2]\cdots .$$

For completeness, we include $n = 2, 3, 4$ from [Wan and Wang 2013].

μ	$\xi =$ (2)
(2)	2
(1^2)	2[2]

Table 1. $n = 2$.

μ	$\xi =$ (3)	(2, 1)
(3)	2	0
(2, 1)	2[2]	4
(1^3)	2[4]	$4t[2]$

Table 2. $n = 3$.

μ	$\xi =$ (4)	(3, 1)
(4)	2	0
(3, 1)	2[2]	4
(2^2)	$2t[2]$	4[2]
$(2, 1^2)$	2[4]	$4[2]^2$
(1^4)	$2[6]!!/[3]!$	$4t[4]!!$

Table 3. $n = 4$.

μ	$\xi =$ (5)	(4, 1)	(3, 2)
(5)	2	0	0
(4, 1)	2[2]	4	0
(3, 2)	$2t[2]$	4[2]	4
$(3, 1^2)$	2[4]	$4[2]^2$	4[2]
$(2^2, 1)$	$2t[4]$	4[2][3]	$4[2]^2$
$(2, 1^3)$	$2[6]!!/[3]!$	4[4][3]	$4t[2]([3]+1)$
(1^5)	$2[8]!!/[4]!$	$4t[6]!!/[2]$	$4t^2[4]^2$

Table 4. $n = 5$.

μ	$\xi =$ (6)	(5, 1)	(4, 2)	(3, 2, 1)
(6)	2	0	0	0
(5, 1)	$2[2]$	4	0	0
(4, 2)	$2t[2]$	$4[2]$	4	0
$(4, 1^2)$	$2[4]$	$4[2]^2$	$4[2]$	0
(3, 3)	$2t^2[2]$	$4t[2]$	$4[2]$	0
(3, 2, 1)	$2t[4]$	$4[2][3]$	$4[2](t+2)$	8
$(3, 1^3)$	$2[6]!!/[3]!$	$4[4][3]$	$4[2]^2[3]$	$8t[2]$
(2^3)	$2t^3[4]$	$4t[4]!!$	$4[2]([4]+t^2)$	$8t[2]$
$(2^2, 1^2)$	$2t[6]!!/[3]!$	$4[4]^2$	$4[2]^2([4]+t)$	$8t[2]^2$
$(2, 1^4)$	$2[8]!!/[4]!$	$4[4][6]!!$	$4t[4]!!([4]+1)$	$8t^2[4]!!$
(1^6)	$2[10]!!/[5]!$	$4t[8]!!/[3]!$	$4t^2[5][6]!!/[3]$	$8t^4[6]!!/[3]$

Table 5. $n = 6$.

Acknowledgments

We thank Jinkui Wan and Weiqiang Wang for helpful discussions on the subject. The project is partially supported by the Simons Foundation under grant no. 523868 and NSFC grant 12171303.

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Received January 25, 2022. Revised October 2, 2022.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 325, No. 1, 2023
https://doi.org/10.2140/pjm.2023.325.147

THE STRUCTURE OF GROUPS WITH ALL PROPER QUOTIENTS VIRTUALLY NILPOTENT

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Just infinite groups play a significant role in profinite group theory. For each $c \geqslant 0$, we consider more generally $\mathrm{JNN}_c\mathrm{F}$ profinite (or, in places, discrete) groups that are Fitting-free; these are the groups G such that every proper quotient of G is virtually class-c nilpotent whereas G itself is not, and additionally G does not have any nontrivial abelian normal subgroup. When $c = 1$, we obtain the just non-(virtually abelian) groups without nontrivial abelian normal subgroups.

Our first result is that a finitely generated profinite group is virtually class-c nilpotent if and only if there are only finitely many subgroups arising as the lower central series terms $\gamma_{c+1}(K)$ of open normal subgroups K of G. Based on this we prove several structure theorems. For instance, we characterize the $\mathrm{JNN}_c\mathrm{F}$ profinite groups in terms of subgroups of the above form $\gamma_{c+1}(K)$. We also give a description of $\mathrm{JNN}_c\mathrm{F}$ profinite groups as suitable inverse limits of virtually nilpotent profinite groups. Analogous results are established for the family of hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups and, for instance, we show that a Fitting-free $\mathrm{JNN}_c\mathrm{F}$ profinite (or discrete) group is hereditarily $\mathrm{JNN}_c\mathrm{F}$ if and only if every maximal subgroup of finite index is $\mathrm{JNN}_c\mathrm{F}$. Finally, we give a construction of hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups, which uses as an input known families of hereditarily just infinite groups.

1. Introduction and main results

If $\mathscr{P}$ is a property of groups, a group G is said to be *just non-$\mathscr{P}$* when G does not have property $\mathscr{P}$ but all proper quotients of G do satisfy $\mathscr{P}$. In the case when G is a profinite group, we require instead that every quotient of G by a nontrivial *closed* normal subgroup has $\mathscr{P}$. The property $\mathscr{P}$ considered most often has been that of being finite and the more common term *just infinite* is then used. Just infinite groups are particularly important within the context of profinite — or more generally residually finite — groups, since infinite residually finite groups are never simple but instead just infinite groups can be viewed as those with all proper

MSC2020: primary 20E18; secondary 20E26, 20F19.
Keywords: profinite groups, residually finite groups, just infinite groups, just non-nilpotent-by-finite groups, virtually nilpotent groups, inverse system characterizations.

quotients *essentially* trivial from a 'residually finite' viewpoint (see, for example, the discussion in [Leedham-Green and McKay 2002, §12.1]). Important examples of just infinite groups include the Grigorchuk group [1984] and the Nottingham group [Klopsch 2000; Hegedűs 2001], but also families arising as quotients of arithmetic groups by their centers [Bass et al. 1967].

There is a dichotomy in the study of just non-$\mathscr{P}$ groups. One thread within the literature is concerned with the study of just non-$\mathscr{P}$ groups possessing a nontrivial normal abelian subgroup. In this context, a key idea is to exploit the structure of a maximal abelian normal subgroup when viewed as a module in the appropriate way. Studies of this type include [McCarthy 1968; 1970; De Falco 2002; Quick 2007] and we also refer to the monograph [Kurdachenko et al. 2002] for more examples. On the other hand, Wilson [1971; 2000] addresses the case of just infinite groups with no nontrivial abelian normal subgroup. He shows that such groups fall into two classes: (i) branch groups and (ii) certain subgroups of wreath products of a hereditarily just infinite group by a symmetric group of finite degree. The class of branch groups has been studied considerably (see, for example, [Grigorchuk 2000; Bartholdi et al. 2003], though many more articles on these groups have appeared since these surveys were written). It is known that every proper quotient of a branch group is virtually abelian (see the proof of [Grigorchuk 2000, Theorem 4]) and there are examples of branch groups that are not just infinite (see [Fink 2014], for example). It is interesting therefore to note that Wilson's methods extend to the class of groups with all proper quotients virtually abelian, as observed by Hardy in his PhD thesis [2002]. We shall use the abbreviation *JNAF groups* for these just non-(abelian-by-finite) groups.

More recently, Reid [2010a; 2010b; 2012; 2018] established various fundamental results concerning the structure and properties of just infinite groups. One might wonder to what extent JNAF groups have a similar structure to just infinite groups. In this article, we demonstrate how, for fixed $c \geqslant 0$, Reid's results may in fact be extended to the even larger class of groups with all proper quotients being virtually nilpotent of class at most c; that is, the just non-(class-c-nilpotent-by-finite) groups. We shall abbreviate this term to *JNN$_c$F group* in what follows. The case $c = 0$ essentially returns Reid's results, while the case $c = 1$ covers all JNAF groups and so, in particular, would apply to all branch groups.

We do require some additional, though rather mild, hypotheses to those appearing in Reid's work. First, the JNN$_c$F groups that we consider will be assumed to be *Fitting-free*; that is, to have no nontrivial abelian normal subgroup. This is consistent with Wilson's and Hardy's studies and with the viewpoint that says that the case with a nontrivial abelian normal subgroup should be studied through a module-theoretic lens. (As an aside, we emphasize that JNN$_c$F groups with some nontrivial abelian normal subgroup are, in particular, abelian-by-nilpotent-by-finite). Infinite

Fitting-free groups cannot be virtually nilpotent, so part of the definition of $\mathrm{JNN}_c\mathrm{F}$ group comes immediately. In addition, we shall frequently assume that the groups under consideration are finitely generated. This latter condition will enable us to control the structure of the quotients that arise.

It is interesting to note which parts of Reid's ideas adapt readily to the $\mathrm{JNN}_c\mathrm{F}$ setting and where differences occur. One example is that he implicitly uses the fact that a proper quotient of a just infinite group, being finite, has only finitely many subgroups. In contrast, any infinite (virtually nilpotent) quotient of a profinite group will necessarily have infinitely many open normal subgroups. We shall depend upon the following result as a key tool in our work. It means that, while a finitely generated virtually nilpotent profinite group typically has infinitely many closed normal subgroups, it only has finitely many that occur as corresponding lower central series subgroups of open normal subgroups.

Theorem A. *Let G be a finitely generated profinite group. Then G is virtually nilpotent of class at most c if and only if the set $\{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} G\}$ is finite.*

Theorem A is established as Theorem 2.10 in Section 2. In that section, we also give precise definitions and recall properties needed during the course of our work.

In Section 3, we fix an integer $c \geqslant 0$ and investigate the structure of $\mathrm{JNN}_c\mathrm{F}$ profinite groups G that are Fitting-free. We shall establish various descriptions that generalize those of just infinite groups in [Reid 2010a; 2012; 2018]. One point that can be noted is that the subgroups of the form $\gamma_{c+1}(K)$, for K an open normal subgroup of G, play a role in $\mathrm{JNN}_c\mathrm{F}$ groups analogous to that of open normal subgroups in just infinite groups. For example, we show that a directed graph Γ can be constructed from a suitable subcollection of $\{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{c}} G\}$ that is locally finite. This enables us to establish our first characterization of $\mathrm{JNN}_c\mathrm{F}$ groups (established as Theorem 3.3 below), which is the following analogue of Reid's "generalized obliquity theorem" [Reid 2010a, Theorem A]. Specifying $c = 0$ results in a mild weakening of Reid's theorem.

Theorem B. *Let G be a finitely generated infinite profinite group that has no nontrivial abelian closed normal subgroup. Then G is $\mathrm{JNN}_c\mathrm{F}$ if and only if the set $\mathcal{A}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} G \text{ with } \gamma_{c+1}(K) \nleqslant H\}$ is finite for every open subgroup H of G.*

This result is used to characterize, in Theorem 3.5, when a finitely generated Fitting-free profinite group is $\mathrm{JNN}_c\mathrm{F}$. The characterization is expressed in terms of properties of a descending chain of open normal subgroups H_i and the $(c+1)$-th term of their lower central series. In Theorem 3.7, we establish a further characterization of such a group as an inverse limit in a manner analogous to [Reid 2012, Theorem 4.1]. One important tool (see Lemma 3.1) that is used throughout Section 3 is that, if G is a Fitting-free $\mathrm{JNN}_c\mathrm{F}$ profinite group and N is a nontrivial closed normal subgroup, then the Melnikov subgroup $M(N)$ of N is nontrivial and so, via the Fitting-free assumption, $\gamma_i(M(N)) \neq \mathbf{1}$ for all $i \geqslant 1$.

Section 4 is concerned with the structure of profinite groups that are hereditarily $\mathrm{JNN}_c\mathrm{F}$. We establish there a similar suite of results, though the description of a finitely generated, Fitting-free hereditarily $\mathrm{JNN}_c\mathrm{F}$ group as an inverse limit is more technical (see Theorem 4.7). It shares this level of technicality with Reid's characterization of hereditarily just infinite groups.

In Section 5, we establish the following (as Corollary 5.5) which is the analogue of the main result of [Reid 2010a]. The material in this section does not depend upon Theorem A and so is more directly developed from Reid's arguments.

Theorem C. *Let G be a JNN_cF profinite or discrete group that has no nontrivial abelian normal subgroup. Then G is hereditarily JNN_cF if and only if every maximal (open) subgroup of finite index is JNN_cF.*

One reasonable conclusion from the results described so far is that there is a similarity in the structure of $\mathrm{JNN}_c\mathrm{F}$ groups when compared to just infinite groups. One might ask: just how closely are these classes linked? As $\mathrm{JNN}_c\mathrm{F}$ groups have not yet been studied systematically, there are presently rather few examples to examine when considering these links. In the final section of the paper, Section 6, we take a first step and present one way to construct hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups from hereditarily just infinite groups as semidirect products and discuss some explicit examples. We give examples of hereditarily JNAF groups of the form $G \rtimes A$ where G can be a hereditarily just infinite group suitably built as an iterated wreath product or using Wilson's Construction B [2010] and A can be selected from a rather broad range of abelian groups (see Examples 6.10 and 6.16). By exploiting the fact that every countable pro-p group can be embedded in the Nottingham group, we construct a hereditarily $\mathrm{JNN}_c\mathrm{F}$ group of the form $\mathrm{SL}_n^1(\mathbb{F}_p[\![T]\!]) \rtimes A$ where A can be any virtually nilpotent pro-p group (see Example 6.17). This last family of examples demonstrates that, for every possible choice of $c \geqslant 1$, there is a $\mathrm{JNN}_c\mathrm{F}$ pro-p group that is not just non-(virtually nilpotent of smaller class).

Since the examples constructed are built using hereditarily just infinite groups, one is drawn back to the above question concerning the link between $\mathrm{JNN}_c\mathrm{F}$ groups and just infinite groups. The results of Sections 3–5 suggest such a link and it is an open challenge to produce examples of hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups of a compellingly different flavor to those built in Section 6.

2. Preliminaries

In this section, we first give the precise definitions of the groups under consideration. We then recall some useful tools from [Reid 2012] and make a number of basic observations about $\mathrm{JNN}_c\mathrm{F}$ groups. In the last part of the section we consider the behavior of finitely generated virtually nilpotent groups and establish Theorem A which is crucial for the sections that follow.

We shall write maps on the right throughout, so $H\phi$ denotes the image of a group H under a homomorphism ϕ and x^y is the conjugate $y^{-1}xy$. If G is a profinite group, we use the usual notation $H \trianglelefteq_{\mathrm{o}} G$ and $K \trianglelefteq_{\mathrm{c}} G$ for an open normal subgroup and a closed normal subgroup, respectively. If K and L are closed subgroups of G, then $[K, L]$ will denote the *closed* subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ where $x \in K$ and $y \in L$. The *lower central series* of G is then defined by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for each $i \geqslant 1$. As usual, we also use G' for the derived subgroup $\gamma_2(G)$ of G. These concepts will, in particular, be relevant for the instances of the following definition that concern us.

Definition 2.1. Let $\mathscr{P}$ be a property of groups. A profinite (or discrete) group G is said to be *just non-$\mathscr{P}$* if G does not have property $\mathscr{P}$ but G/N does have $\mathscr{P}$ for every nontrivial closed normal subgroup N of G. It is *hereditarily just non-$\mathscr{P}$* if every closed subgroup of finite index in G is just non-$\mathscr{P}$.

(When G is discrete, the word "closed" can and should be ignored. Note that a closed subgroup of finite index is necessarily open, but the definition is phrased to enable that for discrete groups to be readily extracted).

In this paper we consider three options for the property $\mathscr{P}$:

(1) When $\mathscr{P}$ is the property of being finite, we use the more common term *just infinite* for an infinite group with every proper quotient finite.

(2) We use the abbreviation *JNAF* for just non-$\mathscr{P}$ when $\mathscr{P}$ is the property of being virtually abelian, which is the same as being abelian-by-finite. A profinite group has an abelian subgroup of finite index if and only if it has an abelian open subgroup (as the topological closure of any abelian subgroup is again abelian), so we use the term virtually abelian in this situation also.

(3) If c is an integer with $c \geqslant 0$, we use the abbreviation *JNN$_c$F* for just non-$\mathscr{P}$ when $\mathscr{P}$ is the property that there is a subgroup H of finite index such that $\gamma_{c+1}(H) = \mathbf{1}$. A profinite group has a class-c nilpotent subgroup of finite index if and only if it has an open class-c nilpotent subgroup (as the topological closure of any class-c nilpotent subgroup is again class-c nilpotent).

The case $c = 1$ for a JNN$_c$F group is then identical to it being JNAF. We shall speak of a group G being *virtually class-c nilpotent* when it has a subgroup H of finite index satisfying $\gamma_{c+1}(H) = \mathbf{1}$. More precisely such a group is "virtually (nilpotent of class at most c)". The JNN$_c$F groups G considered will usually be assumed to not have a nontrivial abelian closed normal subgroup. Consequently, such G will itself not be virtually nilpotent (of any class) and so we are studying groups that are just non-(virtually nilpotent) with an additional bound upon the nilpotency class occurring in the proper quotients. In particular, when $c = 1$ we are considering groups that are not virtually metabelian but where every proper quotient is virtually abelian.

Let G be a profinite group. In line with [Reid 2012, Definition 2.1], a *chief factor* of G is a quotient K/L where K and L are closed normal subgroups of G such that there is no closed normal subgroup M of G with $L < M < K$. Accordingly, we do not require that K be open in G in this definition, though necessarily L is open in K and hence K/L is isomorphic (under an isomorphism that commutes with the action of G) to a chief factor K_0/L_0 with K_0 an open normal subgroup of G.

The *Melnikov subgroup* $M(G)$ of G is the intersection of all maximal open normal subgroups of G. Provided G is nontrivial, this is a topologically characteristic proper closed subgroup of G. As usual, to say a subgroup of G is *topologically characteristic* means that it is invariant under all automorphisms of G that are also homeomorphisms. We follow [Reid 2012, Definition 3.1] and, for a nontrivial closed normal subgroup A of G, define $M_G(A)$ to be the intersection of all maximal G-invariant open subgroups of A. This satisfies $M(A) \leqslant M_G(A) < A$. We call A a *narrow subgroup* of G if A has a unique maximal G-invariant open subgroup (that is, when $M_G(A)$ is this unique subgroup). The first part of the following lemma is a consequence of the correspondence theorem, while the other two are, respectively, Lemmas 3.2 and 3.3 in Reid's paper [2012].

Lemma 2.2. *Let G be a profinite group.*

(i) *Let K and L be closed normal subgroups of G such that $L \leqslant M_G(K)$. Then $M_{G/L}(K/L) = M_G(K)/L$.*

(ii) *Let K and L be closed normal subgroups of G. Then $K \leqslant LM_G(K)$ if and only if $K \leqslant L$.*

(iii) *If K/L is a chief factor of G, there is a closed normal subgroup A which is narrow in G and is contained in K but not in L. This narrow subgroup satisfies $A \cap L = M_G(A)$.*

It is well-known that a finitely generated finite-by-abelian discrete group is center-by-finite. This is established by ideas related to FC-groups (see [Robinson 1996, Section 14.5], in particular, the proof of (14.5.11)). In the case of profinite groups, however, the hypothesis of finite generation is unnecessary, as observed by Detomi, Morigi and Shumyatsky (see [Detomi et al. 2020, Lemma 2.7]). In fact, a similar argument establishes the following result needed in our context and that only needs residual finiteness as a hypothesis:

Lemma 2.3. *Let G be a residually finite group with a finite normal subgroup N. If G/N is virtually class-c nilpotent, for some $c \geqslant 0$, then G is also virtually class-c nilpotent.*

Proof. For each nonidentity element x of N, there exists a normal subgroup of finite index in G that does not contain x. By intersecting these, we produce a normal subgroup K of finite index in G such that $N \cap K = \mathbf{1}$. Then G embeds in the direct product $G/N \times G/K$ of G/N and a finite group, so the result follows. □

Corollary 2.4. *Let G be a profinite group that is JNN$_c$F. Then G has no nontrivial finite normal subgroup.*

Sections 3 and 4 are concerned with profinite JNN$_c$F groups, whereas the last two sections consider both profinite and abstract JNN$_c$F groups. To state efficiently the results in Section 5, we shall adopt there the convention that "subgroup" for a profinite group means "closed subgroup" so that it remains in the same category. For the results in the current section that will be used in the discrete case, we simply bracket the word "closed" to indicate it is unnecessary in such a situation. The following lemma illustrates this convention. It is a standard elementary fact about just non-$\mathscr{P}$ groups when $\mathscr{P}$ is a property that is inherited by both finite direct products and subgroups.

Lemma 2.5. *Let G be a profinite group or discrete group that is JNN$_c$F. If K and L are nontrivial* (*closed*) *normal subgroups of G, then $K \cap L \neq \mathbf{1}$.*

Just as Reid [2010b] does, we use Wilson's concept [2000] of a basal subgroup:

Definition 2.6. A subgroup B of a group G is called *basal* if B is nontrivial, has finitely many conjugates $B_1, B_2, \ldots, B_n$ in G and the normal closure of B in G is the direct product of these conjugates: $B^G = B_1 \times B_2 \times \cdots \times B_n$.

Lemma 2.8 below is based on [Reid 2010b, Lemma 5]. The hypothesis that K has only finitely many conjugates is sufficient to adapt the proof of Reid's lemma to our needs. In its statement, and in many that follow, we shall say that a (profinite or discrete) group G is *Fitting-free* when it has no nontrivial abelian (closed) normal subgroup. This is immediately equivalent to the requirement that the Fitting subgroup $F(G)$ be trivial. Furthermore, if G is a JNN$_c$F group, one observes that G is Fitting-free if and only if G is not virtually soluble. If K is a normal subgroup of G, then $Z(K) = K \cap C_G(K)$ and we deduce the following characterization of the Fitting-free condition in JNN$_c$F groups using Lemma 2.5.

Lemma 2.7. *Let G be a profinite or discrete group that is JNN$_c$F. Then G is Fitting-free if and only if $C_G(K) = \mathbf{1}$ for every nontrivial normal* (*closed*) *subgroup K of G.*

Lemma 2.8. *Let G be a profinite or discrete group that is Fitting-free. Let K be a nontrivial* (*closed*) *subgroup of G whose conjugates $\{K_i \mid i \in I\}$ are parametrized by the finite set I and which satisfies $K \trianglelefteq K^G$. Then there exists some $J \subseteq I$ such that $\bigcap_{j \in J} K_j$ is basal.*

Proof. For $J \subseteq I$, define $K_J = \bigcap_{j \in J} K_j$. Let $\mathcal{I}$ be the set of subsets J of I such that $K_J \neq \mathbf{1}$. Certainly $\mathcal{I}$ is nonempty since it contains all singletons as $K \neq \mathbf{1}$. Choose $J \in \mathcal{I}$ of largest size and define $B = K_J$. Then B also has finitely many conjugates in G and we denote these by $B_1, B_2, \ldots, B_n$. Two distinct conjugates intersect trivially, $B_i \cap B_j = \mathbf{1}$ when $i \neq j$, since this is the intersection of more than $|J|$ conjugates of K. Since each K_i is normal in K^G, it follows that each $B_j \trianglelefteq K^G$ and therefore $[B_i, B_j] \leqslant B_i \cap B_j = \mathbf{1}$ when $i \neq j$. Set $L = B^G = B_1 B_2 \cdots B_n$. Then the center of L is the product of the centers of the B_j. Our hypothesis that G has no nontrivial abelian (closed) subgroup then forces $Z(B_j) = \mathbf{1}$ for each j. Now if $j \in \{1, 2, \ldots, n\}$, set $P_j = B_1 \cdots B_{j-1} B_{j+1} \cdots B_n$. Then $[B_j, P_j] = \mathbf{1}$ and so $P_j \cap B_j \leqslant Z(B_j) = \mathbf{1}$. Since this holds for each j, we conclude that $L = B_1 \times B_2 \times \cdots \times B_n$; that is, B is basal. □

Properties of virtually nilpotent profinite groups. If N is a closed normal subgroup of a profinite group G, we define the commutator subgroup $[N,_i G] \trianglelefteq_{\mathrm{c}} G$ recursively by $[N,_0 G] = N$ and $[N,_i G] = [[N,_{i-1} G], G]$ for $i \geqslant 1$. Thus, using left-normed commutator notation,

$$[N,_c G] = [N, \underbrace{G, G, \ldots, G}_{c \text{ times}}].$$

We also write $Z_i(G)$ for the i-th term of the upper central series of a group G.

Lemma 2.9. *Let G be a finitely generated profinite group and N be an open normal subgroup of G such that $\gamma_{c+1}(N) = \mathbf{1}$ for some $c \geqslant 0$. Then $[N,_i G]$ is an open subgroup of $\gamma_{c+1}(G)$ for all $i \geqslant c$.*

Proof. Define $k = |G/N|$. It follows from the definitions that $[N,_c G]$ is a closed normal subgroup of G with $N/[N,_c G] \leqslant Z_c(G/[N,_c G])$. Hence, this term of the upper central series is open in $G/[N,_c G]$ and a theorem of Baer — see [Robinson 1996, (14.5.1)] — shows that $\gamma_{c+1}(G/[N,_c G])$ is finite. Hence $[N,_c G]$ is an open subgroup of $\gamma_{c+1}(G)$.

Now suppose that we have shown $[N,_i G]$ is open in $\gamma_{c+1}(G)$ for some $i \geqslant c$. This subgroup is generated, modulo $[N,_{i+1} G]$, by all left-normed commutators $[x, y_1, y_2, \ldots, y_i]$ where x is selected from some finite generating set for N and $y_1, y_2, \ldots, y_i$ from a finite generating set for G. In particular, $[N,_i G]/[N,_{i+1} G]$ is a finitely generated abelian profinite group. Furthermore, standard commutator calculus shows that, modulo $[N,_{i+1} G]$,

$$[x, y_1, y_2, \ldots, y_i]^{k^i} \equiv [x, y_1^k, y_2^k, \ldots, y_i^k] \in \gamma_{i+1}(N) = \mathbf{1}.$$

Hence, every generator of $[N,_i G]/[N,_{i+1} G]$ has finite order and so this abelian group is finite. It follows that $[N,_{i+1} G]$ is an open subgroup of $[N,_i G]$. The lemma then follows by induction on $i \geqslant c$. □

The next result establishes, in particular, Theorem A, stated in the introduction.

Theorem 2.10. (i) *Let G be a finitely generated virtually class-c nilpotent profinite group, for some $c \geqslant 0$. Then the set $\{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{c}} G\}$ is finite.*

(ii) *Conversely, if G is a profinite group such that $\{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} G\}$ is finite, for some $c \geqslant 0$, then G is virtually class-c nilpotent.*

Proof. (i) Let N be an open normal subgroup of G such that $\gamma_{c+1}(N) = \mathbf{1}$. Let K be any closed normal subgroup of G and set $L = KN$. By standard commutator calculus, any element of $[N,_{2c} KN]$ can be expressed as a product of commutators $[x, y_1, y_2, \dots, y_{2c}]$ where $x \in N$ and each y_i belongs either to K or to N. Since such a commutator involves either at least $c+1$ entries from K or at least $c+1$ entries from N, we deduce

$$[N,_{2c} L] = [N,_{2c} KN] \leqslant \gamma_{c+1}(N)\,[N,_{c+1} K] \leqslant \gamma_{c+1}(K) \leqslant \gamma_{c+1}(L).$$

Furthermore, upon applying Lemma 2.9 to the profinite group L, we conclude that $[N,_{2c} L]$ is an open subgroup of $\gamma_{c+1}(L)$. Therefore, for each open normal subgroup L of G that contains N, there are at most finitely many possibilities for $\gamma_{c+1}(K)$ as K ranges over all closed normal subgroups of G with $KN = L$. Finally, since there are only finitely many possibilities for L, we conclude that $\{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{c}} G\}$ is indeed finite.

(ii) Let G be a profinite group and suppose that $\mathcal{A} = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} G\}$ is finite. If N is any open normal subgroup of G, then the set $\mathcal{L}_{G/N} = \{\gamma_{c+1}(H) \mid H \trianglelefteqslant G/N\}$ is the image of $\mathcal{A}$ under the map induced by the natural homomorphism $G \to G/N$. In particular, there exists some open normal subgroup M of G such that $|\mathcal{L}_{G/M}|$ is maximal. If N is an open normal subgroup of G contained in M, then $|\mathcal{L}_{G/N}| = |\mathcal{L}_{G/M}|$ and so, in particular, $\gamma_{c+1}(M/N)$ must coincide with $\gamma_{c+1}(N/N)$; that is, $\gamma_{c+1}(M) \leqslant N$. As this holds for all such open normal subgroups N, we conclude that $\gamma_{c+1}(M) = \mathbf{1}$. This shows that G is virtually class-c nilpotent. □

The following example demonstrates that the assumption of finite generation is necessary in Theorem 2.10(i). We construct a countably-based virtually abelian pro-p group such that the set $\{K' \mid K \trianglelefteqslant_{\mathrm{o}} G\}$ contains infinitely many subgroups.

Example 2.11. Let p be a prime and, for each $i \geqslant 0$, set V_i to be the direct product of p copies of the cyclic group C_p of order p. Take $H = C_p$ and let H act on each V_i by cyclically permuting the factors. Define $W_i = V_i \rtimes H \cong C_p \operatorname{wr} C_p$, the standard wreath product. Then $[V_i, H]$ and $[V_i, H, H]$ are normal subgroups of W_i of indices p^2 and p^3, respectively. Now take $G = \left(\prod_{i=0}^{\infty} V_i\right) \rtimes H$. This is a virtually abelian pro-p group, indeed $G = \varprojlim G_n$ where $G_n = \left(\prod_{i=0}^{n} V_i\right) \rtimes H$. Certainly G

is not finitely generated. Observe that, for each finite subset S of $\mathbb{N}_0$,

$$U_S = \Bigl(\prod_{i \in S} [V_i, H] \times \prod_{i \notin S} V_i\Bigr) \rtimes H$$

is an open normal subgroup of G and

$$U_S' = \prod_{i \in S} [V_i, H, H] \times \prod_{i \notin S} [V_i, H].$$

In particular, the set $\{K' \mid K \trianglelefteq_{\mathrm{o}} G\}$ is infinite for this group G.

3. Characterization of JNN$_c$F profinite groups

We fix the integer $c \geqslant 0$ throughout this section. In order to establish Theorem B that characterizes Fitting-free JNN$_c$F profinite groups, we shall associate a directed graph Γ to the set $\mathcal{C}_H$ that appears in the statement of Theorem 3.3 below. This graph is similar to that used by Reid [2010a]. A key difference is that the vertices of Γ correspond only to closed subgroups that have the form $\gamma_{c+1}(K)$ (where K is a closed normal subgroup of the profinite group under consideration) rather than any other nontrivial closed subgroups that the group may have. We begin by describing this graph and establishing that it is locally finite.

In the following, recall that the Melnikov subgroup $M(N)$ of N is the intersection of the maximal open normal subgroups of N.

Lemma 3.1. *Let G be a Fitting-free JNN$_c$F profinite group and let N be a nontrivial closed normal subgroup of G. Then $\gamma_i(M(N)) \neq \mathbf{1}$ for all $i \geqslant 1$.*

Proof. We shall show that the normal subgroup $M(N)$ is nontrivial, for the hypothesis that G is Fitting-free then ensures it cannot be nilpotent. Suppose for a contradiction that $M(N) = \mathbf{1}$. Let $\mathcal{L}$ be the set of open normal subgroups M of N such that N/M is cyclic of prime order and $\mathcal{M}$ be the set of open normal subgroups M of N such that N/M is a nonabelian finite simple group. Then $\bigl(\bigcap \mathcal{L}\bigr) \cap \bigl(\bigcap \mathcal{M}\bigr) = M(N) = \mathbf{1}$. By Lemma 2.5, either $\bigcap \mathcal{L} = \mathbf{1}$ or $\bigcap \mathcal{M} = \mathbf{1}$. If $\bigcap \mathcal{L} = \mathbf{1}$, then N embeds in a Cartesian product of cyclic groups of prime order and so N would be abelian, contrary to hypothesis.

Hence $\bigcap \mathcal{M} = \mathbf{1}$. Then [Ribes and Zalesskii 2000, Corollary 8.2.3] tells us that N is a Cartesian product of nonabelian finite simple groups S_R indexed by the set $\mathcal{M}$, say $N = \prod_{R \in \mathcal{M}} S_R$. Now there exists some open normal subgroup K of G such that $N \cap K < N$. Define

$$M_1 = \prod_{S_R \leqslant K} S_R \quad \text{and} \quad M_2 = \prod_{S_R \nleqslant K} S_R,$$

the products of those factors S_R contained in K and not contained in K, respectively. Any closed normal subgroup of N is the product of the factors S_R that it contains, so $M_1 = N \cap K$. Hence M_2 is nontrivial and finite. Furthermore, since K is normal in G, for $g \in G$, $S_R \nleqslant K$ if and only if $S_R^g \nleqslant K$. Therefore M_2 is a normal subgroup of G and we have a contradiction by Corollary 2.4. □

Let G be a finitely generated Fitting-free JNN$_c$F profinite group and H be an open subgroup of G. Construct a directed graph $\Gamma = \Gamma(H)$ whose vertices are the members of the set

$$\mathcal{C}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{c}} G \text{ with } \gamma_{c+1}(K) \nleqslant H\}$$

and where there is an edge from a member A of $\mathcal{C}_H$ to another member B when $B < A$ and there is no $C \in \mathcal{C}_H$ with $B < C < A$.

Lemma 3.2. *Let G be a finitely generated Fitting-free JNN$_c$F profinite group, let H be an open subgroup of G and let $\Gamma = \Gamma(H)$ be the graph defined above.*

(i) *If K and L are closed normal subgroups of G such that $\gamma_{c+1}(K), \gamma_{c+1}(L) \in \mathcal{C}_H$ and there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ, then $\gamma_{c+1}(M(\gamma_{c+1}(K))) \leqslant \gamma_{c+1}(L)$.*

(ii) *If K is a closed normal subgroup of G such that $\gamma_{c+1}(K) \in \mathcal{C}_H$, then there are at most finitely many $\gamma_{c+1}(L) \in \mathcal{C}_H$ such that there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ.*

Proof. (i) Suppose that there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ. Then $\gamma_{c+1}(L)$ is a proper subgroup of $\gamma_{c+1}(K)$, so the intersection R of the maximal open normal subgroups of $\gamma_{c+1}(K)$ that contain $\gamma_{c+1}(L)$ satisfies $R < \gamma_{c+1}(K)$. By definition, $M(\gamma_{c+1}(K)) \leqslant R$ and so $M(\gamma_{c+1}(K))\gamma_{c+1}(L) \leqslant R < \gamma_{c+1}(K)$. Take $J = M(\gamma_{c+1}(K))L$. Then J is a closed normal subgroup of G and $\gamma_{c+1}(L) \leqslant \gamma_{c+1}(J) \leqslant M(\gamma_{c+1}(K))\gamma_{c+1}(L) < \gamma_{c+1}(K)$. Since there is an edge in Γ from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$, this forces $\gamma_{c+1}(J) = \gamma_{c+1}(L)$ and hence $\gamma_{c+1}(M(\gamma_{c+1}(K))) \leqslant \gamma_{c+1}(L)$.
(ii) Define $M = \gamma_{c+1}(M(\gamma_{c+1}(K)))$. By Lemma 3.1, $M \neq \mathbf{1}$ and hence $Q = G/M$ is virtually class-c nilpotent. If there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ then, by part (i), $\gamma_{c+1}(L)$ corresponds to $\gamma_{c+1}(L/M)$ and here L/M is a closed normal subgroup of Q. Consequently, there are only finitely many possibilities for $\gamma_{c+1}(L)$ by Theorem 2.10(i). □

Theorem 3.3. *Let G be a finitely generated infinite profinite group that is Fitting-free and let c be a nonnegative integer. Then the following conditions are equivalent:*

(i) *The group G is JNN$_c$F.*

(ii) *The set $\mathcal{A}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{o}} G$ with $\gamma_{c+1}(K) \nleqslant H\}$ is finite for every open subgroup H of G.*

(iii) *The set $\mathcal{C}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{c}} G$ with $\gamma_{c+1}(K) \nleqslant H\}$ is finite for every open subgroup H of G.*

Observe that if H is any open subgroup of G with $C = \mathrm{Core}_G(H)$, then $\mathcal{A}_H = \mathcal{A}_C$ and $\mathcal{C}_H = \mathcal{C}_C$. Hence each of the conditions (ii) and (iii) is equivalent to the requirement that the given set be finite for every open *normal* subgroup H of G.

Proof. Assume that G is JNN$_c$F. Suppose that $\mathcal{C}_H$ is infinite for some open subgroup H of G. As described above, construct the graph $\Gamma = \Gamma(H)$ whose vertices are the members of $\mathcal{C}_H$. Lemma 3.2(ii) tells us that each vertex of Γ has finite out-degree. Furthermore, if $\gamma_{c+1}(K) \in \mathcal{C}_H$, then $G/\gamma_{c+1}(K)$ is a proper quotient of G and so is virtually class-c nilpotent. Hence, by Theorem 2.10(i), $G/\gamma_{c+1}(K)$ contains only finitely many subgroups of the form $\gamma_{c+1}(\bar{L})$ where $\bar{L}$ is a closed normal subgroup; that is, there are only finitely many members of $\mathcal{C}_H$ that contain $\gamma_{c+1}(K)$. Consequently there is a path of finite length in Γ from $\gamma_{c+1}(G)$ to $\gamma_{c+1}(K)$.

Thus Γ is a connected, locally finite, infinite directed graph. By König's lemma (see, for example, [Diestel 2017, Lemma 8.1.2]), Γ has an infinite directed path and this corresponds to an infinite descending chain $\gamma_{c+1}(K_1) > \gamma_{c+1}(K_2) > \cdots$ of members of $\mathcal{C}_H$. An application of [Reid 2010a, Lemma 2.4], taking $O = H$, shows that $J = \bigcap_{i=1}^{\infty} \gamma_{c+1}(K_i) \neq \mathbf{1}$. Then G/J is finitely generated and virtually class-c nilpotent but it has infinitely many subgroups of the form $\gamma_{c+1}(K_i)/J$ with $K_i \trianglelefteqslant_{\mathrm{c}} G$. This contradicts Theorem 2.10(i). We conclude therefore that $\mathcal{C}_H$ is finite for every open subgroup H of G.

Since $\mathcal{A}_H \subseteq \mathcal{C}_H$ for every H, it is certainly the case that the third condition in the statement implies the second.

Suppose finally that $\mathcal{A}_H$ is finite for every open subgroup H of G. As G is Fitting-free, it is not virtually nilpotent. Let N be a nontrivial closed normal subgroup of G. Then $\gamma_{c+1}(N) \neq \mathbf{1}$ and so there exists an open normal subgroup H of G such that $\gamma_{c+1}(N) \not\leqslant H$. By hypothesis, $\mathcal{A}_H = \{\gamma_{c+1}(L_1), \gamma_{c+1}(L_2), \dots, \gamma_{c+1}(L_r)\}$ for some open normal subgroups $L_1, L_2, \dots, L_r$ of G. Set $L = \bigcap_{i=1}^{r} L_i$. If K is an open normal subgroup of G with $N \leqslant K$, then necessarily $\gamma_{c+1}(K) \not\leqslant H$ and so $\gamma_{c+1}(K) = \gamma_{c+1}(L_i)$ for some i. Therefore

$$\gamma_{c+1}(L) \leqslant \bigcap\{\gamma_{c+1}(K) \mid N \leqslant K \trianglelefteqslant_{\mathrm{o}} G\} \leqslant \bigcap\{K \mid N \leqslant K \trianglelefteqslant_{\mathrm{o}} G\} = N.$$

Hence LN/N is a class-c nilpotent open subgroup of G/N, as required. □

We shall now use Theorem 3.3 to establish further information about finitely generated Fitting-free JNN$_c$F groups, including a description of them as inverse limits of suitable virtually nilpotent groups (see Theorem 3.7 below).

Suppose that G is a finitely generated Fitting-free JNN$_c$F group. We start with any open normal subgroup H_0. Then certainly $\gamma_{c+1}(H_0) \neq \mathbf{1}$ since G is Fitting-free. Now assume, as an inductive hypothesis, that we have constructed a sequence of open normal subgroups $G \geqslant H_0 > H_1 > \cdots > H_{n-1}$ such that for each $i \in \{1, 2, \dots, n-1\}$ the following holds: $\gamma_{c+1}(H_i) \leqslant M_G(\gamma_{c+1}(H_{i-1}))$ and if N is an open normal subgroup of G either $\gamma_{c+1}(N) \leqslant H_{i-1}$ or $\gamma_{c+1}(H_i) \leqslant \gamma_{c+1}(N)$. By Theorem 3.3, the set $\mathcal{A}_{H_{n-1}} = \{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{o}} G \text{ with } \gamma_{c+1}(K) \nleqslant H_{n-1}\}$ is finite. Also $M_G(\gamma_{c+1}(H_{n-1})) \neq \mathbf{1}$ since it contains $M(\gamma_{c+1}(H_{n-1}))$ which is nontrivial by Lemma 3.1. Let

$$R = M_G(\gamma_{c+1}(H_{n-1})) \cap \bigcap \mathcal{A}_{H_{n-1}}.$$

Since this is a finite intersection of nontrivial closed normal subgroups, R is also a nontrivial closed normal subgroup of G by Lemma 2.5. Then G/R is virtually class-c nilpotent, so there exists an open normal subgroup S of G with $\gamma_{c+1}(S) \leqslant R$. Take $H_n = H_{n-1} \cap S$, so that H_n is open in G, $H_n \leqslant H_{n-1}$ and

$$\gamma_{c+1}(H_n) \leqslant R \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1}).$$

Furthermore, if N is an open normal subgroup of G, then either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(N) \in \mathcal{A}_{H_{n-1}}$. In the latter case, $\gamma_{c+1}(H_n) \leqslant R \leqslant \gamma_{c+1}(N)$ according to our definition of R.

By repeated application of these steps, we obtain a descending sequence of open normal subgroups H_n. Let $J = \bigcap_{n=0}^{\infty} H_n$. If $J \neq \mathbf{1}$, then necessarily $\gamma_{c+1}(J) \neq \mathbf{1}$ so $G/\gamma_{c+1}(J)$ is virtually class-c nilpotent. By Theorem 2.10(i), the set $\{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{o}} G/\gamma_{c+1}(J)\}$ is finite but each term $\gamma_{c+1}(H_i)/\gamma_{c+1}(J)$ is a member of this set. This contradiction shows that $J = \mathbf{1}$.

In conclusion, we have established the following observation:

Lemma 3.4. *Let G be a finitely generated profinite group that is Fitting-free and JNN$_c$F. Then there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that*:

(i) *For each $n \geqslant 1$, $\gamma_{c+1}(H_n) \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1})$.*

(ii) $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

(iii) *If N is an open normal subgroup of G and $n \geqslant 1$, then either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(N)$.*

The conditions appearing in the lemma are sufficient to ensure that the group G is JNN$_c$F. In fact, we can make them marginally weaker as the following shows:

Theorem 3.5. *Let G be a finitely generated Fitting-free profinite group and let c be a nonnegative integer. Then G is JNN$_c$F if and only if there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that*:

(i) $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

(ii) *If N is an open normal subgroup of G and $n \geqslant 1$, then either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(N)$.*

When these conditions are satisfied, the group G is just infinite if and only if $\gamma_{c+1}(H_n)$ has finite index in G for all $n \geqslant 0$.

Proof. If G is JNN$_c$F, the existence of the descending sequence of open subgroups H_n is provided by Lemma 3.4. Suppose conversely that G possesses a descending chain H_n, $n \geqslant 0$, of open normal subgroups satisfying (i) and (ii). Since G is Fitting-free, it cannot be virtually nilpotent. Let K be a nontrivial closed normal subgroup of G. Then, for the same reason, $\gamma_{c+1}(K) \neq \mathbf{1}$. Therefore, since condition (i) holds, there exists some $m \geqslant 0$ such that $\gamma_{c+1}(K) \nleqslant H_m$. Let N be any open normal subgroup of G with $K \leqslant N$. Since $\gamma_{c+1}(N) \nleqslant H_m$, condition (ii) shows that $\gamma_{c+1}(H_{m+1}) \leqslant \gamma_{c+1}(N) \leqslant N$. It follows that

$$\gamma_{c+1}(H_{m+1}) \leqslant \bigcap\{N \mid K \leqslant N \trianglelefteqslant_{\mathrm{o}} G\} = K,$$

and hence G/K is virtually nilpotent of class c, as required.

Finally, observe that if $|G : \gamma_{c+1}(H_n)|$ is infinite for some $n \geqslant 0$, then $G/\gamma_{c+1}(H_n)$ is an infinite quotient, and therefore G is not just infinite. On the other hand, if $|G : \gamma_{c+1}(H_n)| < \infty$ for all $n \geqslant 0$, then in the previous paragraph it follows that if K is a nontrivial closed normal subgroup of G then $\gamma_{c+1}(H_{m+1}) \leqslant K$ for some $m \geqslant 0$, and so K has finite index. Hence G is in fact just infinite under this assumption. □

Reid [2012, Theorem 3.6] presents a condition which guarantees the existence of a just infinite quotient of a profinite group. The condition is expressed in terms of the relation $\succ_{\mathrm{nar}}$ concerning chief factors of the profinite group G under consideration. Notice, however, that with use of [Reid 2012, Proposition 3.5(iii)], the assumption that $K_1/L_1 \succ_{\mathrm{nar}} K_2/L_2 \succ_{\mathrm{nar}} \cdots$ is a descending sequence of open chief factors (as appears in [Reid 2012, Theorem 3.6]) is equivalent to the existence of open normal subgroups $G \geqslant K_1 > L_1 \geqslant K_2 > L_2 \geqslant \cdots$ with $L = \bigcap_{n=1}^{\infty} L_n$ such that, for each n, K_n/L is a narrow subgroup of G/L and $M_{G/L}(K_n/L) = L_n/L$. Theorem 3.6 below can consequently be viewed as an analogous result for the existence of JNN$_c$F quotients of a profinite group.

The application of Zorn's Lemma in our proof is more delicate than for Reid's result. Under the hypotheses and notation of [Reid 2012, Theorem 3.6], the quotient G/L_n would be finite and so would have only finitely many subgroups. However, our corresponding quotient G/L_n is a finitely generated virtually nilpotent group and this does not necessarily even possess the ascending chain condition on closed normal subgroups.

Theorem 3.6. *Let G be a finitely generated profinite group and let c be a nonnegative integer.*

(i) *For each $n \geqslant 1$, let K_n and L_n be closed normal subgroups of G and define $L = \bigcap_{n=1}^{\infty} L_n$. Suppose that*

$$G \geqslant K_1 > L_1 \geqslant \gamma_{c+1}(L_1)L \geqslant K_2 > L_2 \geqslant \gamma_{c+1}(L_2)L \geqslant \cdots$$

and that, for each n, K_n/L is a narrow subgroup of G/L with $M_{G/L}(K_n/L) = L_n/L$ and G/L_n is virtually class-c nilpotent. Then there exists a closed normal subgroup K of G that is maximal subject to the conditions that $K \geqslant L$ and $K_n \nleqslant L_n K$ for all n. Furthermore, such a closed normal subgroup K has the property that G/K is JNN$_c$F.

(ii) *Every Fitting-free JNN$_c$F quotient G/K of G arises in the manner described in (i) with $L = K$.*

Proof. (i) When $c = 0$, this follows from [Reid 2012, Theorem 3.6(i)]. We shall assume that $c \geqslant 1$ in the following argument. Let $\mathcal{N}$ be the set of all closed normal subgroups N of G which contain L and such that $K_n \nleqslant L_n N$ for all $n \geqslant 1$. We shall order $\mathcal{N}$ by inclusion. Observe that $L \in \mathcal{N}$ since $L_n L = L_n < K_n$. Let $\mathcal{C}$ be a chain in $\mathcal{N}$ and define $R = \overline{\bigcup \mathcal{C}}$. Suppose that $R \notin \mathcal{N}$. Then there exists some $m \geqslant 1$ such that $K_m \leqslant L_m R$. If $C \in \mathcal{C}$, then $K_m \nleqslant L_m C$, so $(K_m \cap C)L_m = K_m \cap L_m C < K_m$ and therefore $K_m \cap C \leqslant L_m$ since L_m is maximal among G-invariant open subgroups of K_m. Hence $[C, K_m] \leqslant L_m$ and so $C \leqslant C_G(K_m/L_m)$ for all $C \in \mathcal{C}$. Since this centralizer is an open subgroup of G, it follows that $R \leqslant C_G(K_m/L_m)$. Hence $K_m \leqslant L_m R \leqslant C_G(K_m/L_m)$ and so the chief factor K_m/L_m is abelian; that is, it is an elementary abelian q-group for some prime q.

Since G/L_m is virtually class-c nilpotent, there is an open normal subgroup A with $L_m \leqslant A$ such that $\gamma_{c+1}(A) \leqslant L_m$. If $K_m \nleqslant A$, then $K_m \cap A = L_m$ and so $K_m A/L_m \cong K_m/L_m \times A/L_m$ is also class-c nilpotent. Consequently, if necessary, we may replace A by $K_m A$ and hence assume $K_m \leqslant A$. For each prime p, write $A[p]/L_m$ for the Sylow pro-p subgroup of A/L_m. Then A/L_m is the product $\prod_p A[p]/L_m$ of these pro-p groups. Furthermore, for each $C \in \mathcal{C}$ and prime p, let $C[p]/L_m$ be the Sylow pro-p subgroup of $(C \cap A)L_m/L_m$. Since $\mathcal{C}$ is a chain, so is the set $\mathcal{S}_p = \{C[p]/L_m \mid C \in \mathcal{C}\}$. As a finitely generated nilpotent pro-p group, $A[p]/L_m$ satisfies the ascending chain condition on closed subgroups and so there exists some maximal member $M[p]/L_m$ of $\mathcal{S}_p$.

If it were the case that $K_m/L_m \leqslant M[q]/L_m$, then $K_m \leqslant (C \cap A)L_m \leqslant C L_m$ for some $C \in \mathcal{C}$, contrary to the fact that $C \in \mathcal{N}$. Define M to be the closed subgroup of A defined by $M/L_m = \prod_p M[p]/L_m$. Then $K_m \nleqslant M$ since we have observed that K_m/L_m is not contained in the Sylow pro-q subgroup of M/L_m.

On the other hand, $C \cap A \leqslant M$ for all $C \in \mathcal{C}$, since by construction $C[p]/L_m \leqslant M[p]/L_m$ for each prime p. Furthermore, since A is a clopen subset of G,

$$\overline{\bigcup_{C \in \mathcal{C}} (C \cap A)} = \overline{\left(\bigcup \mathcal{C}\right) \cap A} = R \cap A$$

and so we conclude that $R \cap A \leqslant M$. Therefore, $K_m \leqslant L_m R \cap A = (R \cap A)L_m \leqslant M$, which contradicts our previous observation.

In conclusion, we have shown that $R = \overline{\bigcup \mathcal{C}} \in \mathcal{N}$ and so every chain in $\mathcal{N}$ has an upper bound. Therefore, by Zorn's lemma, there is a maximal member $K \in \mathcal{N}$; that is, K is maximal subject to the condition that $K_n \nleqslant L_n K$ for all $n \geqslant 1$. Suppose that G/K is virtually class-c nilpotent. By Theorem 2.10(i), the set $\{\gamma_{c+1}(J) \mid J \trianglelefteq_{\mathrm{c}} G/K\}$ is finite. Hence $\gamma_{c+1}(L_m)K = \gamma_{c+1}(L_{m+1})K$ for some $m \geqslant 1$ and so

$$K_{m+1} \leqslant \gamma_{c+1}(L_m)L \leqslant \gamma_{c+1}(L_m)K = \gamma_{c+1}(L_{m+1})K \leqslant L_{m+1}K,$$

contrary to the fact that $K \in \mathcal{N}$. We deduce that G/K is not virtually class-c nilpotent.

Now let N be a closed normal subgroup of G that strictly contains K. Then $N \notin \mathcal{N}$ by maximality of K, so there exists some $m \geqslant 1$ such that $K_m \leqslant L_m N$; that is, $K_m/L \leqslant M_{G/L}(K_m/L) \cdot (N/L)$. Lemma 2.2(ii) then tells us that $K_m \leqslant N$. Hence G/N is a quotient of G/L_m and so is virtually class-c nilpotent. This shows that G/K is indeed JNN$_c$F.

(ii) Assume that G/K is a JNN$_c$F quotient of G and that it is Fitting-free. We define the sequences of closed normal subgroups K_n and L_n as follows. First take any chief factor of G/K and let K_1/K be a narrow subgroup as provided by Lemma 2.2(iii) and define L_1 by $L_1/K = M_{G/K}(K_1/K)$. Note that $L_1 > K$ by use of Corollary 2.4 and hence $\gamma_{c+1}(L_1)K > K$ by the hypothesis that G/K is Fitting-free. Assuming that, for some $n \geqslant 2$, we have defined K_{n-1} and L_{n-1} with $\gamma_{c+1}(L_{n-1})K > K$, use Lemma 2.2 again to produce a narrow subgroup K_n/K of G/K with $K_n \leqslant \gamma_{c+1}(L_{n-1})K$. Define L_n by $L_n/K = M_{G/K}(K_n/K)$ and note $\gamma_{c+1}(L_n)K > K$. This produces the required chain of closed normal subgroups

$$G \geqslant K_1 > L_1 \geqslant \gamma_{c+1}(L_1)K \geqslant K_2 > L_2 \geqslant \gamma_{c+1}(L_2)K \geqslant \cdots.$$

Now $L = \bigcap_{n=1}^{\infty} L_n$ certainly contains K, while the quotient G/L cannot be virtually class-c nilpotent by Theorem 2.10(i) as the subgroups $\gamma_{c+1}(L_n/L)$ are distinct. Hence $L = K$. Finally, if N is a closed normal subgroup of G with $N > K$, then G/N is virtually class-c nilpotent and so, by use of Theorem 2.10(i), there exists $m \geqslant 1$ such that $\gamma_{c+1}(L_m)N = \gamma_{c+1}(L_{m+1})N$. The same argument as used in part (i) shows that $K_{m+1} \leqslant L_{m+1}N$. This shows that, amongst closed normal subgroups, K is indeed maximal subject to $K_n \nleqslant L_n K$ for all n; that is, arises as in part (i). $\square$

Our final result of this section is a characterization of finitely generated Fitting-free JNN$_c$F profinite groups as inverse limits. The natural inverse system to associate to such a group is of virtually nilpotent profinite groups rather than of some class of finite groups. The properties possessed by this inverse system are analogous to those in [Reid 2018, Theorem 4.1].

Theorem 3.7. *Let G be a finitely generated profinite group that is Fitting-free and let c be a nonnegative integer. If G is JNN$_c$F, then it is the inverse limit of a family G_n, for $n \geqslant 0$, of profinite groups with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ with the following properties. For every $n \geqslant 0$, G_n has an open normal subgroup P_n such that, upon setting $Q_n = P_{n+1}\rho_n$:*

(i) *G_n is virtually class-c nilpotent.*

(ii) $P_n > Q_n$.

(iii) $\gamma_{c+1}(P_n) > M_{G_n}(\gamma_{c+1}(P_n)) \geqslant \ker \rho_{n-1} \geqslant \gamma_{c+1}(Q_n) > \mathbf{1}$.

(iv) *If N is an open normal subgroup of G_n, then either*

$$\gamma_{c+1}(N) \leqslant P_n \quad \text{or} \quad \gamma_{c+1}(Q_n) \leqslant \gamma_{c+1}(N).$$

Conversely, suppose, for some integer $d \geqslant 1$, that $G = \varprojlim G_n$ is an inverse limit of a countable family of d-generator profinite groups with respect to surjective continuous homomorphisms ρ_n such that G is Fitting-free and the above conditions hold. For each n, let $\pi_n : G \to G_n$ be the natural map associated to the inverse limit. Then if K is a nontrivial closed normal subgroup of G, there exists $n_0 \geqslant 0$ such that $\ker \pi_{n_0} \leqslant K$. *In particular, G is JNN$_c$F.*

In the case of finitely generated profinite groups, it is known that *any* homomorphism is necessarily continuous. Consequently, the word "continuous" could be omitted from the statement without affecting its validity. For arbitrary finitely generated profinite groups, this follows by the work of Nikolov and Segal [2007] (and depends upon the classification of finite simple groups). However, as the groups G_n are assumed to be virtually nilpotent, it is easy to reduce to the case of finitely generated (nilpotent) pro-p groups which was already covered by Serre; compare with [Anderson 1976].

Proof. Suppose G is JNN$_c$F. Then, as observed in Lemma 3.4, there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$ and, for each $n \geqslant 1$, $\gamma_{c+1}(H_n) \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1})$ and if $N \trianglelefteqslant_{\mathrm{o}} G$ either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(N)$. To simplify notation, write $M_n = M_G(\gamma_{c+1}(H_{n+1}))$ for each $n \geqslant 0$. Then define $G_n = G/M_n$, $P_n = H_n/M_n$ and $Q_n = H_{n+1}/M_n$. Let $\rho_n : G_{n+1} \to G_n$ be the natural map. Since $\bigcap_{n=0}^{\infty} M_n \leqslant \bigcap_{n=0}^{\infty} H_n = \mathbf{1}$, it follows that $G = \varprojlim G_n$, while the conditions stated in the theorem all hold. Indeed, using Lemma 2.2(i), $\ker \rho_{n-1} = M_{n-1}/M_n = M_{G_n}(\gamma_{c+1}(P_n))$.

Conversely, suppose that $G = \varprojlim G_n$ is an inverse limit of d-generator profinite groups G_n, for $n \geqslant 0$, with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ such that G is Fitting-free and conditions (i)–(iv) hold where $P_n \trianglelefteqslant_{\mathrm{o}} G_n$ and $Q_n = P_{n+1}\rho_n$. Then G is also d-generated (by [Ribes and Zalesskii 2000, Lemma 2.5.3]). Let $\pi_n : G \to G_n$ be the natural maps associated to the inverse limit.

Observe first that the open normal subgroups P_1 and Q_1 of G_1 satisfy $\gamma_{c+1}(P_1) > \gamma_{c+1}(Q_1) > \mathbf{1}$. Suppose that G_n, for some $n \geqslant 1$, possesses open normal subgroups $C_0, C_1, \ldots, C_n$ such that the subgroups $\gamma_{c+1}(C_i)$ are distinct and nontrivial. Upon taking the inverse images under the homomorphism ρ_n, we obtain open normal subgroups $C_0\rho_n^{-1}, C_1\rho_n^{-1}, \ldots, C_n\rho_n^{-1}$ with $\gamma_{c+1}(C_i\rho_n^{-1}) \nleqslant \ker\rho_n$. When taken together with Q_{n+1}, these give $n+1$ open normal subgroups K of G_{n+1} such that the corresponding $\gamma_{c+1}(K)$ are distinct and nontrivial. By induction, we conclude that $\{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} G_n\}$ contains at least $n+1$ subgroups for all n. The corresponding set for G must therefore be infinite and hence G is not virtually class-c nilpotent by Theorem 2.10(i).

Now let K be a nontrivial closed normal subgroup of G. Since G is Fitting-free, $\gamma_{c+1}(\gamma_{c+1}(K)) \neq \mathbf{1}$. If $\gamma_{c+1}(K)\pi_{n+2} \leqslant P_{n+2}$ for some $n \geqslant 1$, then we see that $\gamma_{c+1}(\gamma_{c+1}(K))\pi_n = \mathbf{1}$ because $P_{n+2}\rho_{n+1} = Q_{n+1}$ and $\gamma_{c+1}(Q_{n+1}) \leqslant \ker\rho_n$. Hence there exists $n_0 \geqslant 1$ such that $\gamma_{c+1}(K)\pi_n \nleqslant P_n$ for all $n \geqslant n_0$. Let N be any open normal subgroup of G with $K \leqslant N$. If $n \geqslant n_0$, then $N\pi_n$ is an open normal subgroup of G_n with $\gamma_{c+1}(N\pi_n)) \nleqslant P_n$ and so $\gamma_{c+1}(Q_n) \leqslant \gamma_{c+1}(N\pi_n)$ by condition (iv). Hence

$$\gamma_{c+1}(P_{n+1}) \leqslant \gamma_{c+1}(N\pi_{n+1}) \ker\rho_n \leqslant \gamma_{c+1}(N\pi_{n+1})\, M_{G_{n+1}}(\gamma_{c+1}(P_{n+1})),$$

and so we deduce $\gamma_{c+1}(P_{n+1}) \leqslant \gamma_{c+1}(N\pi_{n+1})$ by Lemma 2.2(ii). Consequently $\ker\rho_n \leqslant \gamma_{c+1}(N\pi_{n+1})$ for all $n \geqslant n_0$; that is, $\ker\pi_n \leqslant \gamma_{c+1}(N)\ker\pi_{n+1}$ for all $n \geqslant n_0$. This implies

$$\ker\pi_{n_0} \leqslant \bigcap_{n \geqslant n_0} \gamma_{c+1}(N)\ker\pi_n = \gamma_{c+1}(N) \leqslant N$$

since $\gamma_{c+1}(N)$ is closed. Now K is the intersection of all such open normal subgroups N and therefore $\ker\pi_{n_0} \leqslant K$. Consequently, G/K is a quotient of G_{n_0} and so is virtually class-c nilpotent. This demonstrates that G is JNN$_c$F, as required. □

4. Characterization of hereditarily JNN$_c$F profinite groups

In this section, we fix again the integer $c \geqslant 0$ and we shall provide various descriptions of Fitting-free profinite groups that are hereditarily JNN$_c$F. The results that we present parallel those of the previous section and indicate what additional properties ensure that not only is the group itself JNN$_c$F, but also every open subgroup is JNN$_c$F.

Let G be a profinite group. Analogous to the sets appearing in Theorem 3.3, we define, for an open subgroup H of G,

$$\mathcal{A}_H^* = \{\gamma_{c+1}(K) \mid K \leqslant_{\mathrm{o}} G \text{ with } H \leqslant N_G(K) \text{ and } \gamma_{c+1}(K) \nleqslant H\},$$
$$\mathcal{C}_H^* = \{\gamma_{c+1}(K) \mid K \leqslant_{\mathrm{c}} G \text{ with } H \leqslant N_G(K) \text{ and } \gamma_{c+1}(K) \nleqslant H\}.$$

If H and L are open subgroups of G with $H \leqslant L$, we also set

$$\mathcal{A}_H(L) = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} L \text{ with } \gamma_{c+1}(K) \nleqslant H\},$$
$$\mathcal{C}_H(L) = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{c}} L \text{ with } \gamma_{c+1}(K) \nleqslant H\}.$$

The following observation is straightforward:

Lemma 4.1. *Let G be a profinite group and H be an open subgroup of G. Then*

(i) $\mathcal{A}^*_H = \bigcup\{\mathcal{A}_H(L) \mid L \leqslant_{\mathrm{o}} G \textit{ with } H \leqslant L\}$;

(ii) $\mathcal{C}^*_H = \bigcup\{\mathcal{C}_H(L) \mid L \leqslant_{\mathrm{o}} G \textit{ with } H \leqslant L\}$.

In order to establish Theorem 4.4, which is the analogue of Theorem 3.3 for hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups, we need to know that the condition that the group is Fitting-free is inherited by open subgroups. We establish this in Lemma 4.3 below. We shall use the following analogue of an observation made in the proof of [Wilson 2010, (2.1)]. The argument is similar but included for completeness.

Lemma 4.2. *Let G be a Fitting-free $\mathrm{JNN}_c\mathrm{F}$ profinite group. Then*:

(i) *Every nonidentity element of G has infinitely many conjugates in G.*

(ii) *If H is a nontrivial finite subgroup of G, then H has infinitely many conjugates in G.*

Proof. (i) Suppose that x is a nonidentity element of G with finitely many conjugates in G. Let X be the closed normal subgroup of G generated by the conjugates of x and C be the intersection of the centralizers in G of each conjugate of x. Since x has finitely many conjugates, C is open in G and, in particular, nontrivial. Since $[C, X] = \mathbf{1}$, it follows that $C \cap X$ is an abelian closed normal subgroup of G and so $C \cap X = \mathbf{1}$ by assumption. This contradicts Lemma 2.5.

(ii) Let H be a nontrivial finite subgroup of G with finitely many conjugates in G. If x is a nonidentity element of H, then every conjugate of x belongs to one of the conjugates of H. It follows that x has finitely many conjugates in G, which contradicts (i). □

Lemma 4.3. *Let G be a Fitting-free $\mathrm{JNN}_c\mathrm{F}$ profinite group. If H is any open subgroup of G, then H is also Fitting-free.*

Proof. Suppose that A is an abelian closed normal subgroup of H. Let $B = A \cap \mathrm{Core}_G(H)$. Note that B has finitely many conjugates in G and each of them is a normal subgroup of $\mathrm{Core}_G(H)$. Hence the normal closure B^G is the product of these subgroups and this is nilpotent by Fitting's Theorem. Since G is Fitting-free, it follows that $B = \mathbf{1}$. Therefore A is finite and so, by Lemma 4.2(ii), $A = \mathbf{1}$. □

Theorem 4.4. *Let G be a finitely generated infinite profinite group that is Fitting-free and let c be a nonnegative integer. Then the following conditions are equivalent*:

(i) *The group G is hereditarily JNN$_c$F.*

(ii) *The set $\mathcal{A}_H^*$ is finite for every open subgroup H of G.*

(iii) *The set $\mathcal{C}_H^*$ is finite for every open subgroup H of G.*

Proof. Suppose first that $\mathcal{A}_H^*$ is finite for every open subgroup H of G. Since $\mathcal{A}_H \subseteq \mathcal{A}_H^*$, it follows that G is JNN$_c$F by Theorem 3.3. Let L be an open subgroup of G. Then L is Fitting-free by Lemma 4.3 and $\mathcal{A}_H(L)$ is finite for every open subgroup H of L as it is contained in $\mathcal{A}_H^*$. Hence L is also JNN$_c$F by Theorem 3.3. This establishes (ii) $\Rightarrow$ (i).

Since $\mathcal{A}_H^* \subseteq \mathcal{C}_H^*$, certainly (iii) $\Rightarrow$ (ii). Finally assume that G is hereditarily JNN$_c$F and let H be an open subgroup of G. There are finitely many open subgroups L of G with $H \leqslant L$. If L is such an open subgroup, then L is JNN$_c$F, so $\mathcal{C}_H(L)$ is finite by Theorem 3.3 together with Lemma 4.3. Hence $\mathcal{C}_H^*$ is a finite union of finite sets, by Lemma 4.1, and so is finite. This establishes the final implication (i) $\Rightarrow$ (iii). □

Wilson [2010, (2.1)] characterizes when a just infinite group is not hereditarily just infinite. The following is our analogue for JNN$_c$F groups. The same method is used to construct the basal subgroup K and a few additional steps establish its properties.

Proposition 4.5. *Let G be a Fitting-free JNN$_c$F profinite group that is not hereditarily JNN$_c$F. Then G has an infinite closed basal subgroup K such that $N_G(K)/K$ is not virtually class-c nilpotent and K has no nontrivial abelian closed subgroup that is topologically characteristic in K. In particular, K is not normal in G.*

Proof. Since G is not hereditarily JNN$_c$F, there is an open subgroup H of G and a nontrivial closed normal subgroup L of H such that H/L is not virtually class-c nilpotent. Let C be the core of H in G. If $C \cap L = \mathbf{1}$, then L is finite, which is a contradiction by Lemma 4.2(ii). Hence $C \cap L \neq \mathbf{1}$. Note that CL/L is a subgroup of finite index in H/L and is isomorphic to $C/(C \cap L)$. If $C/(C \cap L)$ were virtually class-c nilpotent, then as H/L is a finite extension we would obtain another contradiction. Hence $C/(C \cap L)$ is not virtually class-c nilpotent and we may replace H and L by C and $C \cap L$, respectively, and assume that H is an open *normal* subgroup of G with a nontrivial closed normal subgroup L such that H/L is not virtually class-c nilpotent.

Now L has finitely many conjugates in G and these are all contained in H. Hence $L \trianglelefteqslant L^G$ and Lemma 2.8 tells us that we can construct a basal subgroup K of G by intersecting a suitable collection of the conjugates of L. We may assume that L is one of these conjugates so that $K \leqslant L$. Note that K is infinite by use of Lemma 4.2(ii). If $N_G(K)/K$ were virtually class-c nilpotent, then so would be H/L since $K \leqslant L \leqslant H \leqslant N_G(K)$, contrary to our hypothesis. Note then that K cannot be normal in G since if it were then $N_G(K)/K = G/K$ would be virtually class-c nilpotent. Finally if A were a nontrivial abelian closed subgroup that is topologically characteristic in K, then as conjugation is a homeomorphism there would be precisely one G-conjugate of A in each conjugate of K. Hence A would also be basal and its normal closure A^G would be a nontrivial abelian normal subgroup of G, contrary to assumption. This establishes the claimed conditions. □

Using the characterization given in Theorem 4.4, we are able to give a description of hereditarily JNN$_c$F groups of a similar form to our earlier Theorem 3.5.

Theorem 4.6. *Let G be a finitely generated profinite group that is Fitting-free and let c be a nonnegative integer. Then G is hereditarily JNN$_c$F if and only if there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that*:

(i) $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

(ii) *If L is an open subgroup of G that is normalized by H_{n-1} for some $n \geqslant 1$, then either $\gamma_{c+1}(L) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(L)$.*

Proof. Suppose first that G is hereditarily JNN$_c$F. We start with any open normal subgroup H_0 of G. Suppose then, as an inductive hypothesis, that we have constructed open normal subgroups $G \geqslant H_0 > H_1 > \cdots > H_{n-1}$ such that, for each $i \in \{1, \ldots, n-1\}$, $\gamma_{c+1}(H_{i-1}) > \gamma_{c+1}(H_i)$ and if L is normalized by H_{i-1} then either $\gamma_{c+1}(L) \leqslant H_{i-1}$ or $\gamma_{c+1}(H_i) \leqslant \gamma_{c+1}(L)$. By Theorem 4.4, the set

$$\mathcal{A}^*_{H_{n-1}} = \{\gamma_{c+1}(K) \mid K \leqslant_{\mathrm{o}} G \text{ with } H_{n-1} \leqslant N_G(K) \text{ and } \gamma_{c+1}(K) \nleqslant H_{n-1}\}$$

is finite. Use of Lemma 3.1 shows that $M_G(\gamma_{c+1}(H_{n-1})) \neq \mathbf{1}$. Hence

$$R = M_G(\gamma_{c+1}(H_{n-1})) \cap \bigcap \mathcal{A}^*_{H_{n-1}}$$

is a nontrivial closed normal subgroup of G (by Lemma 2.5). The quotient G/R is then virtually class-c nilpotent, so there exists an open normal subgroup S with $\gamma_{c+1}(S) \leqslant R$. Take $H_n = H_{n-1} \cap S$, so that H_n is an open normal subgroup of G contained in H_{n-1} with $\gamma_{c+1}(H_n) \leqslant R < \gamma_{c+1}(H_{n-1})$. If L is an open subgroup normalized by H_{n-1}, then either $\gamma_{c+1}(L) \leqslant H_{n-1}$ or $\gamma_{c+1}(L) \in \mathcal{A}^*_{H_{n-1}}$. In the latter case, $\gamma_{c+1}(H_n) \leqslant R \leqslant \gamma_{c+1}(L)$.

Repeating this process constructs a descending sequence of open normal subgroups H_n such that condition (ii) holds. If the intersection $J = \bigcap_{n=0}^{\infty} H_n$ were nontrivial, then $G/\gamma_{c+1}(J)$ would be virtually class-c nilpotent, but would have infinitely many distinct subgroups $\gamma_{c+1}(H_n/\gamma_{c+1}(J))$ contrary to Theorem 2.10. Hence condition (i) also holds.

Conversely suppose that G is a finitely generated profinite group that has no nontrivial abelian closed normal subgroup with a descending sequence of open normal subgroups H_n satisfying conditions (i) and (ii). In particular, G satisfies the conditions appearing in Theorem 3.5 and so is $\mathrm{JNN}_c\mathrm{F}$. Suppose that it is not hereditarily $\mathrm{JNN}_c\mathrm{F}$. By Proposition 4.5, G has a closed basal subgroup K with no nontrivial abelian topologically characteristic subgroup such that $N_G(K)/K$ is not virtually class-c nilpotent. Then $\gamma_{c+1}(K) \neq \mathbf{1}$, so there exists $m \geqslant 0$ such that $\gamma_{c+1}(K) \nleqslant H_m$. Since $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$, it follows that every open subgroup of G contains some H_n. Hence, by increasing m if necessary, we can assume $H_m \leqslant N_G(K)$. Let U be any open normal subgroup of G and $L = KU$. Then H_m normalizes L and $\gamma_{c+1}(L) \nleqslant H_m$. Hence, by condition (ii), $\gamma_{c+1}(H_{m+1}) \leqslant \gamma_{c+1}(L)$. It follows that

$$\gamma_{c+1}(H_{m+1}) \leqslant \bigcap_{U \trianglelefteqslant_o G} KU = K.$$

Therefore $N_G(K)/K$ is isomorphic to a quotient of a subgroup of $G/\gamma_{c+1}(H_{m+1})$ and hence is virtually class-c nilpotent. This is a contradiction and we conclude that G is indeed hereditarily $\mathrm{JNN}_c\mathrm{F}$, as claimed. □

We complete the section by giving a suitable description of a hereditarily $\mathrm{JNN}_c\mathrm{F}$ profinite group as an inverse limit of virtually nilpotent groups in a manner analogous to the description appearing in Theorem 3.7.

Theorem 4.7. *Let G be a finitely generated profinite group that is Fitting-free and let c be a nonnegative integer. If G is hereditarily $\mathrm{JNN}_c\mathrm{F}$, then it is the inverse limit of a family G_n, for $n \geqslant 0$, of profinite groups with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ with the following properties. For every $n \geqslant 0$, G_n has an open normal subgroup P_n such that, upon setting $Q_n = P_{n+1}\rho_n$:*

(i) *G_n is virtually class-c nilpotent.*

(ii) $P_n > Q_n$.

(iii) $\gamma_{c+1}(P_n) > M_{G_n}(\gamma_{c+1}(P_n)) \geqslant \ker \rho_{n-1} \geqslant \gamma_{c+1}(Q_n) > \mathbf{1}$.

(iv) *If N is an open normal subgroup of G_n, then either*

$$\gamma_{c+1}(N) \leqslant P_n \quad \textit{or} \quad \gamma_{c+1}(Q_n) \leqslant \gamma_{c+1}(N).$$

(v) *There is no nonnormal closed subgroup V of G_n with at most n conjugates such that any pair of distinct conjugates of V centralize each other and such that the normal closure $W = V^G$ satisfies $\gamma_{c+1}(P_n) \leqslant \gamma_{c+1}(\gamma_{c+1}(W))$.*

Conversely, if, for some integer $d \geqslant 1$, $G = \varprojlim G_n$ is an inverse limit of a countable family of d-generator profinite groups with respect to surjective continuous homomorphisms ρ_n such that G is Fitting-free and the above conditions hold, then G is hereditarily $\mathrm{JNN}_c\mathrm{F}$.

Proof. Suppose that G is hereditarily $\mathrm{JNN}_c\mathrm{F}$. Since G is finitely generated it has finitely many open subgroups of each index and so we can enumerate a sequence of open normal subgroups U_n of G such that, for each $n \geqslant 1$, every open subgroup of index at most n contains U_n. Take H_0 to be any open normal subgroup of G. Certainly $\gamma_{c+1}(H_0) \neq \mathbf{1}$. Now assume, as an inductive hypothesis, that we have constructed a sequence of open normal subgroups $G \geqslant H_0 > H_1 > \cdots > H_{n-1}$. By Theorem 4.4, the set $\mathcal{A}^*_{H_{n-1}}$ is finite while $M_G(\gamma_{c+1}(H_{n-1}))$ is nontrivial by Lemma 3.1. Hence, by Lemma 2.5,

$$R = M_G(\gamma_{c+1}(H_{n-1})) \cap \big(\bigcap \mathcal{A}^*_{H_{n-1}}\big)'$$

is a nontrivial closed subgroup of G, so G/R is virtually class-c nilpotent and there exists an open normal subgroup S of G with $\gamma_{c+1}(S) \leqslant R$. Take $H_n = H_{n-1} \cap U_n \cap S$. In particular, $\gamma_{c+1}(H_n) \leqslant R \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1})$. By repeated application, we conclude there is a descending sequence of open normal subgroups $G \geqslant H_0 > H_1 > H_2 > \cdots$ such that $H_n \leqslant U_n$ and

$$\gamma_{c+1}(H_n) \leqslant M_G(\gamma_{c+1}(H_{n-1})) \cap \big(\bigcap \mathcal{A}^*_{H_{n-1}}\big)' < \gamma_{c+1}(H_{n-1})$$

for all $n \geqslant 1$. Since $H_n \leqslant U_n$ for each n, it immediately follows that $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

Now, for $n \geqslant 0$, write $M_n = M_G(\gamma_{c+1}(H_{2n+2}))$ and define $G_n = G/M_n$, $P_n = H_{2n}/M_n$ and $Q_n = H_{2n+2}/M_n$. Let $\rho_n : G_{n+1} \to G_n$ be the natural map. Since $\bigcap_{n=0}^{\infty} M_n = \mathbf{1}$, it is the case that $G = \varprojlim G_n$. Since each $M_n \neq \mathbf{1}$, the assumption that G is JNN$_c$F ensures each G_n is virtually class-c nilpotent and conditions (ii) and (iii) follow immediately from the construction. Indeed $\ker \rho_{n-1} = M_{n-1}/M_n = M_{G_n}(\gamma_{c+1}(P_n))$ using Lemma 2.2(i). If $N \trianglelefteqslant_{\mathrm{o}} G_n$, say $N = K/M_n$, such that $\gamma_{c+1}(N) \nleqslant P_n$, then $\gamma_{c+1}(K) \in \mathcal{A}_{H_{2n}} \subseteq \mathcal{A}^*_{H_{2n}}$. Hence $\gamma_{c+1}(H_{2n+2}) < \gamma_{c+1}(H_{2n+1}) \leqslant \bigcap \mathcal{A}^*_{H_{2n}} \leqslant \gamma_{c+1}(K)$ and this establishes condition (iv).

Suppose there is a nonnormal closed subgroup V of G_n with at most n conjugates such that $[V^g, V^h] = \mathbf{1}$ when $gh^{-1} \notin N_{G_n}(V)$ and such that the normal closure $W = V^{G_n}$ satisfies $\gamma_{c+1}(P_n) \leqslant \gamma_{c+1}(\gamma_{c+1}(W))$. Since elements from distinct conjugates of V commute, $\gamma_{c+1}(\gamma_{c+1}(W))$ is the product of the conjugates of $\gamma_{c+1}(\gamma_{c+1}(V))$. Write $V = K/M_n$ and $W = L/M_n$. Then observe $L = K^G$, $\gamma_{c+1}(\gamma_{c+1}(L)) \leqslant \gamma_{c+1}(\gamma_{c+1}(K))^G M_n$ and $\gamma_{c+1}(H_{2n}) \leqslant \gamma_{c+1}(\gamma_{c+1}(L))M_n$, which implies $\gamma_{c+1}(H_{2n}) \leqslant \gamma_{c+1}(\gamma_{c+1}(L))$ with use of Lemma 2.2(ii). Also K has at most n conjugates in G, so it must be the case that $H_{2n+1} \leqslant U_n \leqslant N_G(K)$. Now $\gamma_{c+1}(H_{2n+1}) < \gamma_{c+1}(\gamma_{c+1}(L))$, so $\gamma_{c+1}(\gamma_{c+1}(K)) \nleqslant \gamma_{c+1}(H_{2n+1})$ and therefore $\gamma_{c+1}(K) \nleqslant H_{2n+1}$. In conclusion, for each $i \geqslant 0$, KH_i is an open subgroup of G with the property that $\gamma_{c+1}(KH_i) \in \mathcal{A}^*_{H_{2n+1}}$. Thus

$$\bigcap \mathcal{A}^*_{H_{2n+1}} \leqslant \bigcap_{i \geqslant 0} \gamma_{c+1}(K)H_i = \gamma_{c+1}(K).$$

Since $\bigcap \mathcal{A}^*_{H_{2n+1}}$ is a normal subgroup, it is contained in all conjugates of K and therefore

$$\gamma_{c+1}(H_{2n+2}) \leqslant \big(\bigcap \mathcal{A}^*_{H_{2n+1}}\big)' \leqslant [K^g, K^h]$$

for all $g, h \in G$. Consequently, $\mathbf{1} \neq \gamma_{c+1}(Q_n) \leqslant [V^g, V^h]$ for all $g, h \in G_n$. However, as V is not normal in G_n there exists $g, h \in G_n$ such that V^g and V^h are distinct and these satisfy $[V^g, V^h] = \mathbf{1}$. This contradiction establishes condition (v).

Conversely, suppose that $G = \varprojlim G_n$ is an inverse limit of d-generator profinite groups G_n, for $n \geqslant 0$, with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ such that G has no nontrivial abelian closed normal subgroup and that conditions (i)–(v) hold where $P_n \trianglelefteq_o G_n$ and $Q_n = P_{n+1}\rho_n$. In particular, the conditions of Theorem 3.7 are satisfied and so G is JNN$_c$F. Let $\pi_n : G \to G_n$ be the natural maps associated to the inverse limit. Suppose that G is not hereditarily JNN$_c$F. Then by Proposition 4.5, G has some closed nonnormal basal subgroup K. Take n_0 to be a positive integer such that K has fewer than n_0 conjugates in G and set $L = K^G$, the direct product of the conjugates of K.

Since $\gamma_{c+1}(\gamma_{c+2}(L)) \neq \mathbf{1}$, it is the case that $\ker \pi_n \leqslant \gamma_{c+1}(\gamma_{c+2}(L))$ for all sufficiently large n by Theorem 3.7. Hence, increasing n_0 if necessary, we may assume that $\ker \pi_n < \gamma_{c+1}(\gamma_{c+2}(L)) \leqslant L'$ for all $n \geqslant n_0$. The subgroup K has at least two conjugates in G and any distinct pair commutes as K is basal. If $K\pi_n$ were normal in G_n, then the images of these conjugates would coincide and so $L\pi_n = K\pi_n$ would be abelian. This is impossible since $\ker \pi_n < L'$. Since the number of conjugates cannot increase in the image, we deduce that, when $n \geqslant n_0$, $K\pi_n$ is a closed subgroup of G_n that is not normal and has at most n_0 conjugates in G_n. For such n, if $x \in \gamma_{c+1}(P_{n+2})$, write $x = g\pi_{n+2}$ for some $g \in G$. Using the fact that $\gamma_{c+1}(Q_{n+1}) \leqslant \ker \rho_n$, one observes $g \in \ker \pi_n \leqslant \gamma_{c+1}(\gamma_{c+1}(L))$ and therefore $\gamma_{c+1}(P_{n+2}) \leqslant \gamma_{c+1}(\gamma_{c+1}(L\pi_{n+2}))$ for $n \geqslant n_0$. In particular, for such n, taking $V = K\pi_{n+2}$ and $W = L\pi_{n+2}$ in G_{n+2} contradicts the hypothesis in condition (v). $\square$

When comparing the above description of hereditarily JNN$_c$F groups with the corresponding result of Reid [2018, Theorem 5.2] for hereditarily just infinite groups, one notices the bound on the number of conjugates appearing in our condition (v). There seems to be no analogue in the corresponding description of hereditarily just infinite groups. However, note that the bound of n for the number of conjugates could, with only minor adjustment to the proof, be replaced by some bound $f(n)$ where $f : \mathbb{N} \to \mathbb{N}$ is any strictly increasing function. In [Reid 2018], the hereditarily just infinite group is isomorphic to an inverse limit $G = \varprojlim G_n$ of *finite* groups and there is therefore an implicit bound on the number of conjugates for subgroups of G_n. Consequently, this condition is quite reasonable.

5. Subgroups of finite index in JNN$_c$F groups

In this section we shall establish Theorem C (see Corollary 5.5) and so consider both profinite groups and discrete groups. We shall adopt the common convention that, in the case of profinite groups, all subgroups are assumed to be within the same category and so “subgroup” means “closed subgroup” in this case. This enables our results to be more streamlined in their statement and the proofs correspondingly cleaner. We fix the integer $c \geqslant 0$ throughout and begin with an observation that is, modulo our standard assumption about abelian normal subgroups, an improvement on Corollary 2.4.

Lemma 5.1. *Let G be a profinite group or discrete group that is JNN$_c$F and Fitting-free. Then G has no nontrivial normal subgroup that is virtually nilpotent.*

Proof. Suppose that N is a nontrivial normal subgroup of G with a nilpotent normal subgroup of finite index in N. The Fitting subgroup $F(N)$ of N is then a product of finitely many nilpotent normal subgroups of N and so is a nilpotent normal subgroup of G. Since G is Fitting-free, it follows that N is finite. Then $C_G(N)$ has finite index in G, which contradicts Lemma 2.7. □

Lemma 5.2. *Let G be a profinite group or a discrete group that is Fitting-free. Suppose that every normal subgroup of finite index is JNN$_c$F. Then G is hereditarily JNN$_c$F.*

Proof. Suppose that H is a subgroup of finite index in G and that N is a nontrivial normal subgroup of H. Let $K = \mathrm{Core}_G(H)$, so that K is a normal subgroup of G also of finite index and hence JNN$_c$F by hypothesis. If it were the case that $K \cap N = \mathbf{1}$, then $[K, N] = \mathbf{1}$ since both K and N are normal subgroups of H. Then $N \leqslant C_G(K)$, in contradiction to Lemma 2.7. We deduce therefore that $K \cap N \neq \mathbf{1}$. Then H/N is a finite extension of $KN/N \cong K/(K \cap N)$, which is virtually class-c nilpotent. Hence H is JNN$_c$F, as required. □

Recall that the *finite radical* $\mathrm{Fin}(G)$ of a group G is the union of all finite normal subgroups of G. The following is a JNN$_c$F analogue of [Reid 2010b, Lemma 4].

Lemma 5.3. (i) *Let G be a group with* $\mathrm{Fin}(G) = \mathbf{1}$. *If H is a subgroup of finite index, then* $\mathrm{Fin}(H) = \mathbf{1}$.

(ii) *Let G be a profinite or discrete group with* $\mathrm{Fin}(G) = \mathbf{1}$ *and H be a subgroup of finite index that is JNN$_c$F. Then every subgroup of G containing H is JNN$_c$F.*

Proof. (i) This is established in [Reid 2010b, Lemma 4].

(ii) Suppose that $H \leqslant L \leqslant G$. First note that L is not virtually class-c nilpotent as it contains H. Let K be a nontrivial normal subgroup of L. Since $\mathrm{Fin}(L) = \mathbf{1}$ by part (i), K is infinite. As $H \cap K$ has finite index in K, it follows that $H \cap K$ is nontrivial and so $H/(H \cap K)$ is virtually class-c nilpotent. We conclude that L/K is a finite extension of $HK/K \cong H/(H \cap K)$, so L/K is virtually class-c nilpotent. Hence L is JNN$_c$F. □

We are now in a position to establish a theorem for JNN$_c$F groups that is an analogue of the main theorem of [Reid 2010b]:

Theorem 5.4. *Let G be a profinite group or a discrete group and let c be a non-negative integer. Suppose that G is JNN$_c$F and Fitting-free, and that H is a normal subgroup of finite index in G. Then the following are equivalent*:

(i) *The subgroup H is JNN$_c$F.*

(ii) *Every subgroup of G containing H is JNN$_c$F.*

(iii) *Every maximal subgroup of G containing H is JNN$_c$F.*

Proof. By Lemma 5.1, $\mathrm{Fin}(G) = \mathbf{1}$. Hence an application of Lemma 5.3(ii) shows that condition (i) implies condition (ii). It is trivial that condition (ii) implies condition (iii).

Now assume condition (iii). Let K be a nontrivial normal subgroup of H. Since H is a normal subgroup of G, we observe that $K^g \trianglelefteqslant H \leqslant N_G(K)$ for all $g \in G$ and hence $K \trianglelefteqslant K^G$. By Lemma 2.8, there is a basal subgroup B that is an intersection of some conjugates of K and, conjugating if necessary, we may assume $B \leqslant K$. Note also that $H \leqslant N_G(B)$ since each conjugate of K is normal in H. We shall show that B is normal in G. For then, G/B is virtually class-c nilpotent by hypothesis and hence H/K is also virtually class-c nilpotent since $B \leqslant K$. This will establish that H is indeed JNN$_c$F.

Suppose, for a contradiction, that B is not a normal subgroup of G. Consequently, $N_G(B)$ is a proper subgroup of G and there is some maximal subgroup M of G with $N_G(B) \leqslant M$. Now B^G is the direct product of the conjugates of B and it is not virtually nilpotent by Lemma 5.1. Observe that B has fewer conjugates in M than in the group G, so B^G/B^M is isomorphic to a direct product of some copies of B and so is not virtually nilpotent. On the other hand, M is JNN$_c$F by assumption, so the quotient M/B^M of M by the normal closure of B in M is a virtually nilpotent group. Hence $(M \cap B^G)/B^M$ is virtually nilpotent and this implies B^G/B^M is also virtually nilpotent since $M \cap B^G$ has finite index in B^G. This is a contradiction and completes the proof of the theorem. □

With use of Lemma 5.2, we then immediately conclude:

Corollary 5.5. *Let G be a profinite or discrete group that is JNN$_c$F and Fitting-free. Then G is hereditarily JNN$_c$F if and only if every maximal* (*open*) *subgroup of finite index is JNN$_c$F.*

6. A construction of hereditarily JNN$_c$F groups

The work of the preceding sections suggests that JNN$_c$F groups are quite closely related to just infinite groups. Similarly, Wilson's classification [1971; 2000] of just infinite groups has the same dichotomy as Hardy's [2002] for JNAF groups, namely branch groups and subgroups of wreath products built from hereditarily just infinite or JNAF groups. To fully investigate the class of JNN$_c$F groups, one would like a good supply of examples of hereditarily JNN$_c$F groups. In Theorem 6.2, we provide one method for constructing such a group. At first sight the construction may appear somewhat unspectacular since it merely consists of a semidirect product of a hereditarily just infinite group H by some group A of (outer) automorphisms. However, by applying it to a variety of known hereditarily just infinite groups H and observing that the range of possible groups A that could be used is rather wide, we manufacture interesting examples of JNN$_c$F groups. In both Examples 6.10 and 6.16, we shall observe that, with suitable choices of ingredients for H, among abelian profinite groups the options for A are about as wide as could be hoped for. For example, one can take A to be any closed subgroup of the Cartesian product of countably many copies of the profinite completion $\hat{\mathbb{Z}}$ of the integers. In Example 6.17, we are able to take A to be any finitely generated virtually nilpotent pro-p group and so again this permits a wide range of possible choices.

Lemma 6.1. *Let H be a group and A be a group of automorphisms of H such that $A \cap \operatorname{Inn} H = \mathbf{1}$. Define $G = H \rtimes A$ to be the semidirect product of H by A via its natural action on H. Then $C_G(H) = Z(H)$.*

Proof. Let $x = h\alpha \in C_G(H)$ with $h \in H$ and $\alpha \in A$. If τ_h denotes the inner automorphism of H induced by h on H, then we observe $\tau_h\alpha = 1$ in $\operatorname{Aut} H$, so $\alpha \in \operatorname{Inn} H$. Hence $\alpha = 1$, so $x = h$ and necessarily $h \in Z(H)$. The reverse inclusion is trivial. □

Theorem 6.2. *Let H be a hereditarily just infinite (discrete or profinite) group that is Fitting-free. Let A be a (discrete or profinite, respectively) group of (continuous) automorphisms of H that is virtually class-c nilpotent, for some $c \geqslant 0$, and satisfies $A \cap \operatorname{Inn} H = \mathbf{1}$. Then the semidirect product of H by A is hereditarily JNN$_c$F.*

The only discrete hereditarily just infinite groups that are virtually abelian are the infinite cyclic group and the infinite dihedral group. The only profinite hereditarily just infinite groups that are virtually abelian are semidirect products of the p-adic integers by a finite (and consequently cyclic) subgroup of its automorphism group. Consequently, the hypothesis that H is Fitting-free in the above theorem excludes only a small number of possibilities. Moreover, this hypothesis on H is also necessary since the semidirect product $H \rtimes A$ can otherwise be virtually abelian.

Proof. Let H be a hereditarily just infinite discrete group that is Fitting-free and $A \leqslant \operatorname{Aut} H$ be virtually class-c nilpotent with $A \cap \operatorname{Inn} H = \mathbf{1}$. We shall first show that the semidirect product $G = H \rtimes A$ is JNN$_c$F. We shall view H and A as subgroups of G in the natural way. Note that as H is Fitting-free, it is not virtually nilpotent and therefore neither is G.

Let N be a nontrivial normal subgroup of G. If $H \cap N = \mathbf{1}$, then $[H, N] = \mathbf{1}$, so $N \leqslant C_G(H) = Z(H)$ by use of Lemma 6.1. This is a contradiction and so $H \cap N \neq \mathbf{1}$. Thus $H \cap N$ is of finite index in H. Then $G/(H \cap N)$ has a copy of the group A

as a subgroup of finite index and is therefore also virtually class-c nilpotent. We deduce that G/N is virtually class-c nilpotent and hence G is $\mathrm{JNN}_c\mathrm{F}$, as claimed.

Let L be a normal subgroup of finite index in G and let N be a nontrivial normal subgroup of L. If $H \cap N = \mathbf{1}$, then $[H \cap L, N] \leqslant H \cap N = \mathbf{1}$, so $N \leqslant C_G(H \cap L)$. By Lemma 2.7, this is impossible since $H \cap L$ is a normal subgroup of G that is nontrivial (since it has finite index in H) and we have already observed G is $\mathrm{JNN}_c\mathrm{F}$.

Therefore $H \cap N \neq \mathbf{1}$. Since H is hereditarily just infinite, $H \cap N$ has finite index in $H \cap L$. Moreover, $H \cap N$ is normalized by L and hence has finitely many conjugates in G, each of which also has finite index in H. We deduce that $R = \mathrm{Core}_G(H \cap N)$ is nontrivial, so G/R is virtually class-c nilpotent. Since $R \leqslant N$, we conclude that L/N is virtually class-c nilpotent.

We have shown that every normal subgroup of finite index in G is $\mathrm{JNN}_c\mathrm{F}$ and therefore G is hereditarily $\mathrm{JNN}_c\mathrm{F}$ by Lemma 5.2.

The situation when H is profinite and A consists of continuous automorphisms of H is established by the same argument. The only difference is that one needs A to have the structure of a profinite group under the topology induced from the group $\mathrm{Aut_c}\, H$ of topological automorphisms of H so that $G = H \rtimes A$ is a profinite group. □

Hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups via iterated wreath products. We shall now construct abelian groups of automorphisms of some just infinite groups that arise as iterated wreath products of nonabelian finite simple groups. We permit two possible options for the action used for the permutational wreath product at each step. The just infinite groups constructed are closely related to those in Wilson's Construction A [2010], though he uses two applications of the permutational wreath product at each stage. If one employs the product action option (P) at each step of our construction, then the inverse limit constructed would be a special case of what Vannacci terms a *generalized Wilson group* (see [Matteo 2016, Definition 3]). Vannacci makes use of [Reid 2012, Theorem 6.2] to determine that the profinite groups concerned are hereditarily just infinite (and his groups also satisfy the hypotheses of the corrected version in [Reid 2018]). Since we also wish to construct discrete examples of hereditarily just infinite groups via a direct limit, we shall present a direct verification as the discrete and profinite cases are closely linked. This verification is somewhat general since it only requires the action employed to be transitive and subprimitive (in the sense of [Reid 2012]). We shall then specialize to regular actions and product actions in Example 6.6 when constructing automorphisms of the resulting hereditarily just infinite groups so as to apply Theorem 6.2.

We first recall the definition of what is meant by a subprimitive action:

Definition 6.3 [Reid 2012, Definition 1.4]. Let Ω be a set and H be a permutation group on Ω. We shall say that H acts *subprimitively* on Ω if every normal subgroup K of H acts faithfully on every K-orbit.

Let X_0, X_1, X_2, ... be a sequence of nonabelian finite simple groups. Define $W_0 = X_0$. Suppose that for some $n \geqslant 1$, we have constructed a group W_{n-1} and choose some faithful, transitive and subprimitive action of W_{n-1} on a finite set Ω_{n-1}. We define $W_n = X_n \operatorname{wr}_{\Omega_{n-1}} W_{n-1}$ to be the wreath product of X_n by W_{n-1} and write $B_n = X_n^{\Omega_{n-1}}$ for its base group. We shall assume at this point that such an action always exists, while in Example 6.6 below we describe possible examples. Write $\rho_n : W_n \to W_{n-1}$ for the natural surjective homomorphism associated to the wreath product and also note that W_{n-1} occurs as a subgroup of W_n so we have a chain of inclusions $W_0 \leqslant W_1 \leqslant W_2 \leqslant \cdots$. We shall write W to denote the direct limit $\varinjlim W_n$ of these wreath products and $\widehat{W}$ to denote the inverse limit $\varprojlim W_n$. It will be convenient to view W as the union of the groups W_n.

The following is the key observation required to show that W is a hereditarily just infinite (discrete) group and $\widehat{W}$ is a hereditarily just infinite profinite group.

Lemma 6.4. *Let X be a nonabelian simple group and H be a permutation group on a finite set Ω that acts transitively and subprimitively. Define $W = X \operatorname{wr}_\Omega H$ to be the wreath product of X by H with respect to this action and B to be the base group of W. Let K be a normal subgroup of W and N be a normal subgroup of K such that $N \nleqslant B$. Then $B \leqslant N$.*

Proof. Write $\pi : W \to H$ for the natural map associated to the wreath product. Since H acts transitively and faithfully on Ω, it easily follows that B is the unique minimal normal subgroup of W. Therefore $B \leqslant K$, so we may write $K = B \rtimes L$ where L is a normal subgroup of H. Write $\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$ as the disjoint union of the orbits of L. Since H is assumed to act subprimitively, L acts faithfully on each Γ_i.

Since $N \nleqslant B$ by hypothesis, $M = N\pi$ is a nontrivial normal subgroup of L, so the orbits of M on Γ_i form a block system for L. Consequently, M must act without fixed points on each Γ_i, as otherwise M would fix all points of Γ_i and then lie in the kernel of the action of L on Γ_i. Therefore M acts without fixed points on Ω. Let us write

$$B = Q_1 \times Q_2 \times \cdots \times Q_m,$$

where each $Q_j = X^{\Delta_j}$ corresponds to an orbit Δ_j of M on Ω. Let us suppose, for a contradiction, that $B \nleqslant N$. Then $Q_j \nleqslant N$ for some j. Since M permutes the factors of Q_j transitively, Q_j is a minimal normal subgroup of $BM = BN$. However, $B \leqslant K$ so B normalizes N and hence $Q_j \cap N$ is normal in BN. We deduce that $Q_j \cap N = \mathbf{1}$ and hence $[Q_j, N] = \mathbf{1}$. This implies that BN fixes all the direct factors of Q_j, which is a contradiction. This establishes that $B \leqslant N$, as claimed. □

Corollary 6.5. (i) *The group $W = \varinjlim W_n$ is hereditarily just infinite.*

(ii) *The profinite group $\widehat{W} = \varprojlim W_n$ is hereditarily just infinite.*

Proof. (i) Let K be a normal subgroup of finite index in W and N be a nontrivial normal subgroup of K. Then $N \cap W_k \neq \mathbf{1}$ for some k. Consequently $N \cap W_n \nleqslant B_n$ for all $n \geqslant k+1$. Applying Lemma 6.4 with $W = W_n$, we deduce $B_n \leqslant N \cap W_n$ for each $n \geqslant k+1$. Hence $\langle B_{k+1}, B_{k+2}, \dots \rangle$ is contained in N and the former is the kernel of the surjective homomorphism $W \to W_k$. It follows that K/N is finite and this shows that W is hereditarily just infinite.

(ii) We shall write $\pi_n : \widehat{W} \to W_n$ for the surjective homomorphisms associated with the inverse limit. Let K be an open normal subgroup of $\widehat{W}$ and N be a nontrivial closed normal subgroup of K. Then $N\pi_k \neq \mathbf{1}$ for some k. Now $N\pi_n \trianglelefteq K\pi_n \trianglelefteq W_n$ and $N\pi_n \nleqslant B_n$ for all $n \geqslant k+1$. Hence by Lemma 6.4, $B_n \leqslant N\pi_n$ for all $n \geqslant k+1$; that is, $\ker \rho_{n-1} \leqslant N\pi_n$ for all $n \geqslant k+1$. It follows that $\ker \pi_{n-1} \leqslant N \ker \pi_n$ for all $n \geqslant k+1$. As the kernels form a neighborhood base for the identity in $\widehat{W}$, we conclude that

$$\ker \pi_k \leqslant \bigcap_{n=0}^{\infty} N \ker \pi_n = \overline{N} = N.$$

Since $\widehat{W}/\ker \pi_k \cong W_k$ is finite, it follows that K/N is finite. This establishes that $\widehat{W}$ is hereditarily just infinite. □

We now specify the examples of subprimitive actions that we shall use and construct abelian groups of automorphisms of the iterated wreath products.

Example 6.6. As before, let X_0, X_1, X_2, ... be a sequence of nonabelian finite simple groups. Define $W_0 = X_0$, $\Omega_0 = X_0$, and let W_0 act regularly on Ω_0. We also define $B_0 = W_0$ for use later. Suppose that, for $n \geqslant 1$, we have constructed W_{n-1} with a specified action on a set Ω_{n-1}. As above, define $W_n = X_n \operatorname{wr}_{\Omega_{n-1}} W_{n-1}$ and write $B_n = X_n^{\Omega_{n-1}}$ for its base group. There are then two options for the action of W_n on some set Ω_n:

(R) take $\Omega_n = W_n$ and let W_n act regularly upon Ω_n; or

(P) let X_n act regularly on itself and use the *product action* of W_n on $\Omega_n = B_n = X_n^{\Omega_{n-1}}$.

For more information upon the product action of a wreath product, see, for example, [Dixon and Mortimer 1996, Section 2.7]. In the case (P) of the product action, the elements of B_n act regularly on the set Ω_n while the elements of W_{n-1} act to permute the factors; that is, the action of W_{n-1} on Ω_n coincides with the conjugation action of W_{n-1} on the base group B_n of W_n. It is immediate from the definition that the regular action of W_n is subprimitive. The product action is faithful and transitive and the following ensures shows that it is a valid choice for our construction.

Lemma 6.7. *Let X be a nonabelian finite simple group acting regularly upon itself and H be a transitive permutation group on a finite set Ω. Then the product action of $W = X \operatorname{wr}_\Omega H$ on the base group $B = X^\Omega$ is subprimitive.*

Proof. By transitivity of H on Ω, B is the unique minimal normal subgroup of W. Consequently, if K is a normal subgroup of W then $B \leqslant K$. In the product action, B acts regularly and hence K is transitive on B. Thus, as the product action is faithful, it follows that the action of K on the only K-orbit is also faithful. □

Corollary 6.5 therefore applies and tells us that $W = \varinjlim W_n$ and $\widehat{W} = \varprojlim W_n$ are hereditarily just infinite. We shall now construct some examples of abelian subgroups of the automorphism groups of these groups. There has been much study of automorphism groups of wreath products (see, for example, [Mohammadi Hassanabadi 1978]), but our requirement is simply to produce some automorphisms that commute and so we choose not to use the full power of such studies.

Suppose that, for each $i \geqslant 0$, ϕ_i is an automorphism of the simple group X_i. We take $\psi_0 = \phi_0$. Suppose that at stage $n-1$, we have constructed an automorphism ψ_{n-1} of W_{n-1}. Since the action of W_{n-1} on Ω_{n-1} is either regular or the product action (with $\Omega_{n-1} = B_{n-1}$ in the latter case), ψ_{n-1} induces a permutation of Ω_{n-1} (that we also denote by ψ_{n-1}) with the property that

$$(\omega^y)\psi_{n-1} = (\omega\psi_{n-1})^{y\psi_{n-1}} \tag{1}$$

for all $\omega \in \Omega_{n-1}$ and $y \in W_{n-1}$. We define a bijection $\psi_n : W_n \to W_n$ by

$$\psi_n : (x_\omega)y \mapsto ((x_{\omega\psi_{n-1}^{-1}})\phi_n)(y\psi_{n-1})$$

where $x_\omega \in X_n$ for each $\omega \in \Omega_{n-1}$ and $y \in W_{n-1}$. (Here we are writing elements of the base group B_n as sequences (x_ω) indexed by Ω_{n-1} with $x_\omega \in X_n$ in the ω-coordinate). Thus the effect of ψ_n on elements in the base group is to apply ϕ_n to each coordinate *and* permute the coordinates using the permutation ψ_{n-1} of Ω_{n-1}, while we simply apply the previous automorphism ψ_{n-1} to elements in the complement W_{n-1}. It is a straightforward calculation to verify that the resulting map is an automorphism of W_n and by construction it restricts to ψ_{n-1} on the subgroup W_{n-1}. (Indeed, in the case (R), the group W_n is the standard wreath product of X_n by W_{n-1}. If we write $\phi = \phi_n$ and $\beta = \psi_{n-1}$, then $\psi_n = \phi^*\beta^*$ is the composite of the automorphisms ϕ^* and β^* introduced on pages 474 and 476, respectively, of [Neumann and Neumann 1959]. The verification for the product action case (P) is similarly straightforward and depends primarily on (1)).

The final result is that, for each n, we have constructed an automorphism ψ_n of W_n that extends all the previous automorphisms. As a consequence, we certainly have determined an automorphism ψ of W whose restriction to each W_n coincides with ψ_n and an automorphism $\hat{\psi}$ of the group $\widehat{W}$ such that $\hat{\psi}\pi_n = \pi_n\psi_n$ for each

n (where, as above, we write $\pi_n : \widehat{W} \to W_n$ for the surjective homomorphism determined by the inverse limit). The key properties of the automorphisms that we have constructed are as follows:

Lemma 6.8. *Let (ϕ_i), (ϕ_i') be sequences of automorphisms with $\phi_i, \phi_i' \in \operatorname{Aut} X_i$ for each i. Define ψ and $\hat{\psi}$ to be the automorphisms of W and $\widehat{W}$ determined by the sequence (ϕ_i) and ψ' and $\hat{\psi}'$ those determined by (ϕ_i'). Then:*

(i) *$\hat{\psi}$ is a continuous automorphism of $\widehat{W}$.*

(ii) *$\psi\psi'$ and $\hat{\psi}\hat{\psi}'$ are the automorphisms of W and $\widehat{W}$ determined by the sequence $(\phi_i\phi_i')$.*

(iii) *If, for some $n \geqslant 0$, $\phi_0, \phi_1, \dots, \phi_{n-1}$ are the identity maps and ϕ_n is an outer automorphism of X_n, then ψ is an outer automorphism of W and $\hat{\psi}$ is an outer automorphism of $\widehat{W}$.*

Proof. (i) By construction, $\hat{\psi}$ fixes the kernels $\ker \pi_n$ associated to the inverse limit. These form a neighborhood base for the identity and so we deduce that $\hat{\psi}$ is continuous.

(ii) For each n, write ψ_n and ψ_n' for the automorphisms of W_n determined by the sequences (ϕ_i) and (ϕ_i'). One computes that, for $n \geqslant 1$, the composite $\psi_n\psi_n'$ is given by

$$(x_\omega)y \mapsto ((x_{\omega(\psi_{n-1}')^{-1}\psi_{n-1}^{-1}})\phi_n\phi_n')(y\psi_{n-1}\psi_{n-1}') = ((x_{\omega(\psi_{n-1}\psi_{n-1}')^{-1}})\phi_n\phi_n')(y\psi_{n-1}\psi_{n-1}').$$

A straightforward induction argument then shows that $\psi_n\psi_n'$ is the automorphism of W_n determined by the sequence $(\phi_i\phi_i')$. The claim appearing in the lemma then follows.

(iii) Suppose that $\phi_0, \phi_1, \dots, \phi_{n-1}$ are the identity and that $\phi_n \notin \operatorname{Inn} X_n$. We claim that $\psi_m \notin \operatorname{Inn} W_m$ for all $m \geqslant n$. The first of these automorphisms is given by $((x_\omega)_{\omega\in\Omega_{n-1}} \cdot y)\psi_n = (x_\omega\phi_n)_{\omega\in\Omega_{n-1}} \cdot y$ for $x_\omega \in X_n$ and $y \in W_{n-1}$. Suppose that ψ_n is produced by conjugating by the element bz where $b \in B_n$ and $z \in W_{n-1}$. Note that ψ_n fixes W_{n-1} and hence b normalizes W_{n-1}. Since $y^b = [b, y^{-1}]y$ for all $y \in W_{n-1}$, we determine that b centralizes W_{n-1}. Therefore $z \in Z(W_{n-1}) = \mathbf{1}$. We then determine that $b = (b_\omega)_{\omega\in\Omega_{n-1}}$ is the constant sequence and ϕ_n coincides with conjugation by the element b_ω, contrary to assumption. Hence ψ_n is an outer automorphism of W_n.

Now suppose, as an induction hypothesis, that $\psi_m \notin \operatorname{Inn} W_m$ for some $m \geqslant n$. Suppose that ψ_{m+1} is produced by conjugating by bz where $b \in B_{m+1}$ and $z \in W_m$. Then b fixes W_m and hence centralizes this subgroup. Consequently, ψ_m, which is the restriction of ψ_{m+1} to W_m is given by conjugating by z. This contradicts the inductive hypothesis. We conclude that ψ_m is an outer automorphism for all $m \geqslant n$. It now immediately follows that ψ is an outer automorphism of W and $\hat{\psi}$ is an outer automorphism of $\widehat{W}$. $\square$

Theorem 6.9. *Let $X_0, X_1, \dots$ be a sequence of nonabelian finite simple groups. Define W to be the direct limit and $\widehat{W}$ to be the inverse limit of the wreath products W_n constructed as in Example 6.6. Suppose that, for each $i \geqslant 0$, ϕ_i is an automorphism of X_i such that $\langle \phi_i \rangle \cap \operatorname{Inn} X_i = \mathbf{1}$. Then the group $A = \prod_{i=0}^{\infty} \langle \phi_i \rangle$ embeds naturally*

(i) *as a subgroup of* $\operatorname{Aut} W$ *such that* $A \cap \operatorname{Inn} W = \mathbf{1}$;

(ii) *as a profinite subgroup of* $\operatorname{Aut_c} \widehat{W}$ *such that* $A \cap \operatorname{Inn} \widehat{W} = \mathbf{1}$.

Combining this theorem with Theorem 6.2 and Corollary 6.5 produces examples of hereditarily JNAF discrete and profinite groups.

Proof. (i) Each element g of $A = \prod_{i=1}^{\infty} \langle \phi_i \rangle$ is a sequence $(\phi_i^{k_i})$ of automorphisms. Let ψ_g denote the automorphism of W determined by this sequence. By Lemma 6.8(ii), the map $g \mapsto \psi_g$ is a homomorphism $\theta : A \to \operatorname{Aut} W$. It is clearly injective while part (iii) of the lemma ensures that the image satisfies $A\theta \cap \operatorname{Inn} W = \mathbf{1}$.

(ii) As with the first part, each g in $A = \prod_{i=1}^{\infty} \langle \phi_i \rangle$ determines a continuous automorphism $\hat{\psi}_g$ of $\widehat{W}$. Hence there is an injective homomorphism $\theta : A \to \operatorname{Aut_c} \widehat{W}$ given by $g \mapsto \hat{\psi}_g$. The subgroups $\Gamma_n = \{\gamma \in A\theta \mid [\widehat{W}, \gamma] \leqslant \ker \pi_n\}$, for $n \geqslant 0$, form a neighborhood base for the identity in the subspace topology on $A\theta$ (see [Dixon et al. 1999, Section 5.2]) and the inverse image of Γ_n under θ is $\prod_{i \geqslant n+1} \langle \phi_i \rangle$, which is open in the product topology on A. Hence θ is continuous and so its image is a profinite subgroup of $\operatorname{Aut_c} \widehat{W}$ that is topologically isomorphic to A and satisfies $A\theta \cap \operatorname{Inn} \widehat{W} = \mathbf{1}$ by Lemma 6.8(iii). □

Example 6.10. As a concrete example to finish our discussion of iterated wreath products, fix a prime number p and let (n_i) be a sequence of positive integers. Take $X_i = \mathrm{PSL}_2(p^{n_i})$, so that X_i has an outer automorphism ϕ_i of order n_i induced by the Frobenius automorphism of the finite field $\mathbb{F}_{p^{n_i}}$. Then Theorem 6.9 shows that the group $A = \prod_{i=0}^{\infty} C_{n_i}$ appears as a subgroup of the automorphism group of the direct limit W with $A \cap \operatorname{Inn} W = \mathbf{1}$ and as a profinite subgroup of $\operatorname{Aut_c} \widehat{W}$ with $A \cap \operatorname{Inn} \widehat{W} = \mathbf{1}$.

Many examples of profinite groups occur as closed subgroups of such a Cartesian product. For example, by taking a suitable enumeration (n_i) of prime-powers, we can embed the Cartesian product of countably many copies of the profinite completion $\hat{\mathbb{Z}}$ of the integers in some suitable product A and hence use Theorem 6.2 to construct a hereditarily JNAF profinite group of the form

$$(\varprojlim W_n) \rtimes \prod_{i=0}^{\infty} \hat{\mathbb{Z}}.$$

Hereditarily JNN$_c$F groups via Wilson's Construction B. The next examples of hereditarily just infinite groups that we shall consider are those introduced by Wilson [2010] in his Construction B. We recall this construction here in order that we can describe some automorphisms of these groups. We make one notational adjustment to Wilson's recipe. When constructing the group G_n, he defines $s = |U_{n-1}|$ and views $G_{n-1} = U_{n-1} \rtimes L_{n-1}$ as a subgroup of the symmetric group of degree s via its action upon U_{n-1}. Accordingly, various elements in his construction have an integer i as a parameter with $1 \leqslant i \leqslant s$. In our description, we shall index using the elements of U_{n-1} since this will aid our defining automorphisms of the constructed groups. We refer to [Wilson 2010] for justification of the assertions made when describing the construction.

Let (p_n), for $n \geqslant 1$, and (q_n), for $n \geqslant 0$, be two sequences of prime numbers such that, for every $n \geqslant 1$, $p_n \neq 2$, p_n divides $q_n - 1$ and $q_{n-1} \neq p_n$. Also let (t_n) be a sequence of positive integers. We now describe the construction of a sequence G_n of finite soluble groups.

First define $G_0 = U_0$ to be the additive group of the finite field $\mathbb{F}_{q_0}$ and take $L_0 = \mathbf{1}$. In particular, G_0 is cyclic of order q_0.

Now suppose that we have constructed a group $G_{n-1} = U_{n-1} \rtimes L_{n-1}$ where U_{n-1} is the unique minimal normal subgroup of G_{n-1} and U_{n-1} is an elementary abelian q_{n-1}-group. To simplify notation, write $U = U_{n-1}$ and let G_{n-1} act upon U by using the regular action of U_{n-1} upon itself and the conjugation action of L_{n-1} upon the normal subgroup U_{n-1}. Define

$$\Gamma = U \times \{1, 2, \ldots, t_n\} = \{(u, k) \mid u \in U,\ 1 \leqslant k \leqslant t_n\}.$$

Let A be an elementary abelian p_n-group with basis $\{a_\gamma \mid \gamma \in \Gamma\}$ and V be the group algebra $\mathbb{F}_{q_n} A$. Let ζ be an element of order p_n in the multiplicative group of the field $\mathbb{F}_{q_n}$. Define invertible linear maps x_δ, y_δ (for $\delta \in \Gamma$) and z of V by $x_\delta : v \mapsto v a_\delta$ for $v \in V$, $y_\delta : \prod a_\gamma^{r_\gamma} \mapsto \zeta^{r_\delta} \prod a_\gamma^{r_\gamma}$ for each $\prod a_\gamma^{r_\gamma} \in A$, and $z : v \mapsto \zeta v$ for $v \in V$. Then define the following subgroups of $\mathrm{GL}(V)$: $X = \langle x_\gamma \mid \gamma \in \Gamma \rangle$, $Y = \langle y_\gamma \mid \gamma \in \Gamma \rangle$ and $E = \langle X, Y \rangle$. The action of G_{n-1} upon U induces an action on Γ and hence an action on the basis of A: $a_{(u,k)}^g = a_{(u^g,k)}$ for each $u \in U$, $1 \leqslant k \leqslant t_n$ and $g \in G_{n-1}$. Hence each $g \in G_{n-1}$ determines an invertible linear transformation of V and this normalizes both X and Y (see [Wilson 2010, (4.3)]).

Now fix some element $u_0 \in U$. Set $\widetilde{\Gamma} = \{(u, k) \in \Gamma \mid u \neq u_0\}$ and, for $(u, k) \in \widetilde{\Gamma}$, let

$$\tilde{a}_{(u,k)} = a_{(u_0,k)}^{-1} a_{(u,k)}, \quad \tilde{x}_{(u,k)} = x_{(u_0,k)}^{-1} x_{(u,k)}, \quad \tilde{y}_{(u,k)} = y_{(u_0,k)}^{-1} y_{(u,k)}.$$

Define $\tilde{A} = \langle \tilde{a}_\gamma \mid \gamma \in \widetilde{\Gamma} \rangle$, $\widetilde{X} = \langle \tilde{x}_\gamma \mid \gamma \in \widetilde{\Gamma} \rangle$, $\widetilde{Y} = \langle \tilde{y}_\gamma \mid \gamma \in \widetilde{\Gamma} \rangle$, and $D = \langle \widetilde{X}, \widetilde{Y} \rangle$. In [Wilson 2010, (4.2)], it is observed that

$$[x_\gamma, y_\delta] = \begin{cases} z & \text{if } \gamma = \delta, \\ 1 & \text{if } \gamma \neq \delta. \end{cases} \tag{2}$$

Since z is central, it follows that E is nilpotent of class 2 and that $D = \widetilde{X}\widetilde{Y}\langle z \rangle$. Also set $W = \mathbb{F}_{q_n} \tilde{A}$. Then W is an irreducible D-module and the group G_{n-1}, via its action on V, normalizes D and induces automorphisms of W; see [Wilson 2010, (4.4) and (4.5)]. Finally set $G_n = (W \rtimes D) \rtimes G_{n-1}$, $U_n = W$ and $L_n = D \rtimes G_{n-1}$. Associated to this semidirect product, there are surjective homomorphisms $G_n \to G_{n-1}$ and inclusions $G_{n-1} \hookrightarrow G_n$. Let $\widehat{G} = \varprojlim G_n$ and $G = \varinjlim G_n$ be the associated inverse and direct limits.

Proposition 6.11 [Wilson 2010, (4.7)]. *The inverse limit $\widehat{G}$ is a hereditarily just infinite profinite group and the direct limit G is a hereditarily just infinite* (*discrete*) *group.*

We need the following additional properties of the groups G_n that are not recorded in Wilson's paper:

Lemma 6.12. (i) *For $n \geqslant 1$, the center of G_n is trivial.*

(ii) *If $n = 1$ and p_1 divides $q_0 - 1$, then the center of $D \rtimes G_0$ is cyclic generated by z.*

(iii) *If $n \geqslant 1$ and p_n divides $q_{n-1} - 1$, then the center of D is cyclic generated by z.*

Proof. (i) It is observed in [Wilson 2010, (4.6)(b)] that $C_{G_n}(W) = W$. Hence $Z(G_n) \leqslant W$. However, note $z \in D'$ by [Wilson 2010, (4.4)(a)] and $w^z = \zeta w$ for all $w \in W$ and so only the identity (that is, the zero vector in W) commutes with all elements of G_n.

(ii) Suppose that p_1 divides $q_0 - 1$ and recall that $U = G_0$ when $n = 0$. Consider first the action of G_0 on $X = \langle x_\gamma \mid \gamma \in \Gamma \rangle$. The group X is an elementary abelian p_n-group and so as an $\mathbb{F}_{p_n} G_0$-module is a direct sum $X = \bigoplus_{k=1}^{t_1} X_k$ where X_k is isomorphic to the group algebra $\mathbb{F}_{p_n} G_0$ (since G_0 acts regularly on U in this case). There is a unique 1-dimensional submodule of X_k upon which G_0 acts trivially, namely that generated by the product $v_k = \prod_{u \in U} x_{(u,k)}$, and an element of X is fixed by G_0 if and only if it belongs to $P = \langle v_1, v_2, \dots, v_{t_1} \rangle$.

Now if v_k were an element of $\widetilde{X}$, it could be written as $v_k = \prod_{u \neq u_0} \tilde{x}_{(u,k)}^{r_u}$ for some values r_u; that is, $v_k = x_{(u_0,k)}^{-s} \prod_{u \neq u_0} x_{(u,k)}^{r_u}$ where $s = \sum_{u \neq u_0} r_u$. Hence $r_u = 1$ for all $u \neq u_0$, but then $s = |U| - 1 \equiv 0 \pmod{p_1}$ since p_1 divides $q_0 - 1$. This is a contradiction and so we conclude $v_k \notin \widetilde{X}$ for all k. Since the set of $\tilde{x}_\gamma$ for $\gamma \in \widetilde{\Gamma}$ forms a basis for X, we deduce that $\widetilde{X} \cap P = \mathbf{1}$; that is, only the identity element of $\widetilde{X}$ is fixed under the action of G_0. Similarly, only the identity element is fixed under the action of G_0 on $\widetilde{Y}$. From these observations, we deduce that if $a = ghz^t$ is centralized by G_0 where $g \in \widetilde{X}$ and $h \in \widetilde{Y}$, then necessarily $g = h = 1$. The claim that $Z(D \rtimes G_0) = \langle z \rangle$ then follows.

(iii) Suppose that p_n divides $q_{n-1} - 1$. Let $a = ghz^t$ be in the center of D where $g \in \widetilde{X}$ and $h \in \widetilde{Y}$. From (2), it follows that, for $\gamma = (u, k)$ and $\delta = (v, \ell)$ with

$u, v \neq u_0$,

$$[\tilde{x}_\gamma, \tilde{y}_\delta] = [x_{(u_0,k)}^{-1} x_{(u,k)}, y_{(u_0,\ell)}^{-1} y_{(v,\ell)}] = \begin{cases} z^2 & \text{if } \gamma = \delta, \\ z & \text{if } k = \ell \text{ and } u \neq v, \\ 1 & \text{if } k \neq \ell. \end{cases}$$

Suppose $g = \prod_{\gamma \in \widetilde{\Gamma}} \tilde{x}_\gamma^{r_\gamma}$. Then, for $\delta = (v, \ell) \in \widetilde{\Gamma}$, $[g, \tilde{y}_\delta] = z^{N_\ell + r_\delta}$ where $N_\ell = \sum_{u \neq u_0} r_{(u,\ell)}$. It follows that $r_\delta \equiv -N_\ell \pmod{p_n}$ for all $\delta = (u, \ell) \in \widetilde{\Gamma}$. Hence $N_\ell \equiv -(|U| - 1)N_\ell \equiv 0 \pmod{p_n}$ for $1 \leqslant \ell \leqslant t_n$, using the fact that U is an elementary abelian q_{n-1}-group and p_n divides $q_{n-1}-1$. This shows $r_\delta \equiv 0 \pmod{p_n}$ for all $\delta \in \widetilde{\Gamma}$ and hence $g = 1$. It similarly follows that $h = 1$. We conclude that $a = z^t$ for some t and this establishes that $Z(D) = \langle z \rangle$. □

We shall now describe a method to construct some automorphisms of the groups G and $\widehat{G}$. For each $i \geqslant 0$, let λ_i be a nonzero scalar in the field $\mathbb{F}_{q_i}$. In particular, $\psi_0 : x \mapsto \lambda_0 x$ is an automorphism of the additive group $G_0 = \mathbb{F}_{q_0}$. Now suppose that we have constructed an automorphism ψ_{n-1} of G_{n-1}. Since U_{n-1} is the unique minimal normal subgroup of G_{n-1}, ψ_{n-1} induces an automorphism of $U = U_{n-1}$. Hence we induce a bijection $\psi_{n-1} : \Gamma \to \Gamma$ by $(u, k)\psi_{n-1} = (u\psi_{n-1}, k)$ and consequently determine an automorphism of A by $a_\gamma \mapsto a_{\gamma\psi_{n-1}}$ and this extends to an invertible linear map $\psi_{n-1} : V \to V$.

Lemma 6.13. *The induced linear map* $\psi_{n-1} \in \mathrm{GL}(V)$ *satisfies* $\psi_{n-1}^{-1} x_\delta \psi_{n-1} = x_{\delta\psi_{n-1}}$ *and* $\psi_{n-1}^{-1} y_\delta \psi_{n-1} = y_{\delta\psi_{n-1}}$ *for each* $\delta \in \Gamma$.

Proof. If $v \in V$, then $v\psi_{n-1}^{-1} x_\delta \psi_{n-1} = (v\psi_{n-1}^{-1} \cdot a_\delta)\psi_{n-1} = v \cdot a_{\delta\psi_{n-1}}$ and hence $\psi_{n-1}^{-1} x_\delta \psi_{n-1} = x_{\delta\psi_{n-1}}$. For an element $\prod a_\gamma^{r_\gamma} \in A$, we compute

$$\big(\prod a_\gamma^{r_\gamma}\big)\psi_{n-1}^{-1} y_\delta \psi_{n-1} = \big(\prod a_{\gamma\psi_{n-1}^{-1}}^{r_\gamma}\big) y_\delta \psi_{n-1} = \big(\zeta^{r_{\delta\psi_{n-1}}} \prod a_{\gamma\psi_{n-1}^{-1}}^{r_\gamma}\big)\psi_{n-1} = \zeta^{r_{\delta\psi_{n-1}}} \prod a_\gamma^{r_\gamma}$$

and hence $\psi_{n-1}^{-1} y_\delta \psi_{n-1} = y_{\delta\psi_{n-1}}$. □

As a consequence, we determine an automorphism ψ_{n-1}^* of the subgroup E of $\mathrm{GL}(V)$ given by conjugating by this linear map ψ_{n-1}. Notice furthermore that $D\psi_{n-1}^* = D$ since

$$\tilde{x}_{(u,k)}\psi_{n-1}^* = \psi_{n-1}^{-1} x_{(u_0,k)}^{-1} x_{(u,k)} \psi_{n-1} = x_{(u_0\psi_{n-1},k)}^{-1} x_{(u\psi_{n-1},k)} = \tilde{x}_{(u_0\psi_{n-1},k)}^{-1} \tilde{x}_{(u\psi_{n-1},k)}$$

and similarly for $\tilde{y}_{(u,k)}$. Finally, we determine a bijection $\psi_n : G_n \to G_n$ by applying ψ_{n-1}^* to elements in D and applying ψ_{n-1} to those in G_{n-1} and defining its effect on elements of $W = \mathbb{F}_{q_n}\tilde{A}$ by

$$\tilde{a}_{(u,k)}\psi_n = \lambda_n a_{(u_0\psi_{n-1},k)}^{-1} a_{(u\psi_{n-1},k)} = \lambda_n \tilde{a}_{(u_0\psi_{n-1},k)}^{-1} \tilde{a}_{(u\psi_{n-1},k)}$$

and extending by linearity. Thus, the effect of ψ_n on W is the composite of the linear map ψ_{n-1} defined above together with scalar multiplication by λ_n. Since each x_δ and y_δ is a linear map, it follows that ψ_n induces an automorphism of $W \rtimes D$. Also notice that, since the action of G_{n-1} on $U = U_{n-1}$ is given by the regular action of U_{n-1} on itself and the conjugation action of L_{n-1} on U_{n-1}, the automorphism ψ_{n-1} satisfies

$$(u^g)\psi_{n-1} = (u\psi_{n-1})^{g\psi_{n-1}}$$

for all $u \in U$ and $g \in G_{n-1}$ (and here exponentiation denotes the action). One determines, using Lemma 6.13, that $(x_\delta^g)\psi_{n-1}^* = (x_\delta\psi_{n-1}^*)^{g\psi_{n-1}}$ for $\delta \in \Gamma$ and $g \in G_{n-1}$. Similar formulae hold when we conjugate y_δ and a_δ by elements of G_{n-1} (in the latter case, we rely upon the fact that an element of G_{n-1} induces a linear map on V and so commutes with the operation of multiplying by the scalar λ_n). We conclude that ψ_n is indeed an automorphism of G_n that restricts to the previous one ψ_{n-1} on G_{n-1}. As a consequence, we determine an automorphism ψ of G whose restriction to each G_n equals ψ_n and an automorphism $\hat{\psi}$ of $\widehat{G}$ such that $\hat{\psi}\pi_n = \pi_n\psi_n$ for each n (where $\pi_n : \widehat{G} \to G_n$ is the surjective homomorphism associated to the inverse limit). The properties of this construction are analogous to those for the iterated wreath product and the first two parts of the following are established similarly to those of Lemma 6.8.

Lemma 6.14. *Let (λ_i), (μ_i) be sequences of scalars with $\lambda_i, \mu_i \in \mathbb{F}_{q_i}^*$. Define ψ and $\hat{\psi}$ to be the automorphisms of G and $\widehat{G}$ determined by the sequence (λ_i) and θ and $\hat{\theta}$ those determined by (μ_i). Then:*

(i) *$\hat{\psi}$ is a continuous automorphism of $\widehat{G}$.*

(ii) *$\psi\theta$ and $\hat{\psi}\hat{\theta}$ are the automorphisms of G and $\widehat{G}$ determined by the sequence $(\lambda_i\mu_i)$.*

(iii) *If p_i divides $q_{i-1} - 1$ for all $i \geqslant 1$ and, for some $n \geqslant 0$, $\lambda_i = 1$ in $\mathbb{F}_{q_i}$ for $0 \leqslant i \leqslant n-1$ and λ_n is not in the subgroup of order p_n in the multiplicative group of the field $\mathbb{F}_{q_n}$, then ψ is an outer automorphism of G and $\hat{\psi}$ is an outer automorphism of $\widehat{G}$.*

Proof. We prove part (iii). Suppose that p_i divides $q_{i-1} - 1$ for all $i \geqslant 1$ in addition to the original assumptions on the p_i and q_j. Suppose that $\lambda_i = 1$ for $0 \leqslant i \leqslant n-1$ and that λ_n is not a power of ζ where ζ is an element of order p_n in $\mathbb{F}_{q_n}^*$. Since $\lambda_i = 1$ for $0 \leqslant i \leqslant n-1$, the automorphism ψ_{n-1} of G_{n-1} is the identity map. We shall first show that $\psi_n \notin \operatorname{Inn} G_n$. We will need a different argument according to the value of n. If $n = 0$, then G_0 is abelian so ψ_0 cannot be an inner automorphism as it is not the identity.

Suppose that $n = 1$ and that ψ_1 is produced by conjugating by the element wdh where $w \in W$, $d \in D$ and $h \in G_0$. In this case, ψ_0 is the identity, so ψ_1 induces the identity on $D \rtimes G_0$ and hence $dh \in Z(D \rtimes G_0)$; that is, $h = 1$ and $d = z^k$ for some k by Lemma 6.12(ii). Now observe that w must normalize D since $D\psi_1 = D$ and it follows that $[w, g] = 1$ for all $g \in D$. Hence $\mathbb{F}_{q_n} w$ is a D-invariant subspace of W; so $w = 0$ as W is an irreducible D-module, by [Wilson 2010, (4.5)(c)]. In conclusion, ψ_1 is the inner automorphism of G_1 determined by conjugation by z^k. This means that $\lambda_1 = \zeta^k$, contrary to our assumption.

Suppose that $n \geqslant 2$ and that $\psi_n \in \operatorname{Inn} G_n$, and let conjugation by the element wdg (where $w \in W$, $d \in D$ and $g \in G_{n-1}$) achieve the same effect as applying ψ_n. In particular, wdg centralizes G_{n-1} and so wd normalizes G_{n-1}. It follows that $[wd, y] = 1$ for all $y \in G_{n-1}$ and hence w and d are both centralized by G_{n-1} and $g \in Z(G_{n-1})$. Therefore $g = 1$ by Lemma 6.12(i). Also necessarily $d \in Z(D)$, so $d = z^k$ for some k by Lemma 6.12(iii), while w spans a D-submodule of W and hence $w = 0$. We conclude, as in the previous case, that ψ_n is the inner automorphism of G_n determined by conjugation by z^k, which is impossible as $\lambda_n \notin \langle \zeta \rangle$ by assumption.

Now suppose that $\psi_m \notin \operatorname{Inn} G_m$ for some $m \geqslant n$. If it were the case that ψ_{m+1} is produced by conjugating by wdg where $w \in W$, $d \in D$ and $g \in G_m$, then ψ_m would coincide with conjugation by g, contrary to assumption. Hence $\psi_m \notin \operatorname{Inn} G_m$ for all $m \geqslant n$. It now follows that ψ is an outer automorphism of W and $\hat{\psi}$ is an outer automorphism of $\widehat{W}$. □

Theorem 6.15. *Let (p_n), for $n \geqslant 1$, and (q_n), for $n \geqslant 0$, be a sequence of prime numbers such that for every $n \geqslant 1$, $p_n \neq 2$, p_n divides both $q_{n-1} - 1$ and $q_n - 1$. Let (t_n) be any sequence of positive integers and define G to be the direct limit and $\widehat{G}$ to be the inverse limit of the semidirect products G_n built via Wilson's Construction B. Take $r_0 = q_0 - 1$ and, for each $i \geqslant 1$, write $q_i - 1 = r_i p_i^{m_i}$ where $p_i \nmid r_i$ and let C_{r_i} denote a cyclic group of order r_i. Then the group $A = \prod_{i=0}^{\infty} C_{r_i}$ embeds naturally*

(i) *as a subgroup of* $\operatorname{Aut} G$ *such that* $A \cap \operatorname{Inn} G = \mathbf{1}$;

(ii) *as a profinite subgroup of* $\operatorname{Aut_c} \widehat{G}$ *such that* $A \cap \operatorname{Inn} \widehat{G} = \mathbf{1}$.

Proof. The proof is similar to that of Theorem 6.9. For each i, let λ_i be an element of order r_i in the multiplicative group $\mathbb{F}_{q_i}^*$. Then, for $i \geqslant 1$, $\langle \lambda_i \rangle \cap \langle \zeta_i \rangle = \mathbf{1}$ where ζ_i denotes an element of order p_i in $\mathbb{F}_{q_i}^*$. Now if

$$g = (\lambda_i^{k_i}) \in \prod_{i=0}^{\infty} \langle \lambda_i \rangle \cong A,$$

write ψ_g for the automorphism ψ determined by the sequence $(\lambda_i^{k_i})$ as above. Lemma 6.14 ensures that $g \mapsto \psi_g$ is a homomorphism into $\operatorname{Aut} G$ whose image satisfies the conclusion of (i). The second part is established similarly: we determine an injective homomorphism $\theta : \prod_{i=0}^{\infty} \langle \lambda_i \rangle \to \operatorname{Aut_c} \widehat{G}$ and this is continuous since the inverse image under θ of the basic neighborhood of the identity comprising those automorphisms that act trivially on G_n is $\prod_{i \geqslant n+1} \langle \lambda_i \rangle$. □

Example 6.16. A specific example can be constructed as follows. Let (n_i) be any sequence of positive integers. Let (p_i), for $i \geqslant 1$, be any sequence of odd primes such that p_i does not divide n_i. When $i \geqslant 1$, take $a_i = \operatorname{lcm}(p_i n_i,\, p_{i+1})$ and $a_0 = \operatorname{lcm}(n_0,\, p_1)$. Now take, for $i \geqslant 0$, q_i to be any prime number of the form $a_i k+1$ for some $k \in \mathbb{N}$. (The existence of such a prime number is guaranteed by Dirichlet's theorem). These choices of sequences then fulfill the requirements of Theorem 6.15 and the integer r_i appearing in the statement is divisible by n_i by construction. Consequently, we deduce that the Cartesian product $\prod_{i=0}^{\infty} C_{n_i}$ embeds in the subgroup A appearing in Theorem 6.15. We may use any closed subgroup of this Cartesian product as the choice of A in Theorem 6.2. In particular, there are many choices of abelian profinite groups A such that $\widehat{G} \rtimes A$ is hereditarily JNAF including, as with the iterated wreath product, a hereditarily JNAF example of the form

$$(\varprojlim G_n) \rtimes \prod_{i=0}^{\infty} \hat{\mathbb{Z}}.$$

Hereditarily JNN$_c$F groups by use of the Nottingham group. The following construction brings together two facets of the study of pro-p groups. As a first ingredient, we make use of the work of Lubotzky–Shalev [1994] on R-analytic groups, in the specific case when R is the formal powers series ring $\mathbb{F}_p[\![T]\!]$, to identify a specific hereditarily just infinite pro-p group G. Secondly, we use the fact that every countably based pro-p group embeds in the automorphism group $\operatorname{Aut}(R)$ to obtain a wide range of groups of automorphisms of our group G.

Example 6.17. Let p be a prime number and let n be a positive integer with $n \geqslant 2$ such that p does not divide n. Take $R = \mathbb{F}_p[\![T]\!]$, the pro-$p$ ring of all formal power series over the field of p elements, which is a local ring with unique maximal ideal $\mathfrak{m} = T\mathbb{F}_p[\![T]\!]$ generated by the indeterminate T. Then take $G = \mathrm{SL}_n^1(R)$, the first principal congruence subgroup of the special linear group of all $n \times n$ matrices of determinant 1 over R; that is,

$$G = \{g \in \mathrm{SL}_n(R) \mid g \equiv I \pmod{\mathfrak{m}}\},$$

where I denotes the $n \times n$ identity matrix. Using the techniques of [Lubotzky and Shalev 1994], it is straightforward to observe that G is a hereditarily just infinite pro-p group. First, G is R-perfect and so the terms of its lower central series are the congruence subgroups

$$\gamma_k(G) = G_k = \{g \in \mathrm{SL}_n(R) \mid g \equiv I \pmod{\mathfrak{m}^k}\},$$

for each $k \geqslant 1$. Adapting slightly the notation used in [Lubotzky and Shalev 1994], we see that the (completed) graded Lie ring associated to the lower central series of

G satisfies

$$L(G) = L_G(G) = \overline{\bigoplus_{i=1}^{\infty} G_i/G_{i+1}} \cong \prod_{i=1}^{\infty} T^i \mathfrak{sl}_n(\mathbb{F}_p) \cong \mathfrak{sl}_n(\mathfrak{m}),$$

the latter being the Lie algebra over $\mathbb{F}_p$ of $n \times n$ matrices with entries in $\mathfrak{m}$ and trace 0. To every closed subgroup H of G we associate a closed Lie subalgebra of $L(G)$ that we denote by $L_G(H)$ and whose properties are described in [Lubotzky and Shalev 1994, Lemma 2.13]. Using the isomorphism above we view $L_G(H)$ as a Lie subalgebra of $\mathfrak{sl}_n(\mathfrak{m})$. In particular, $L_G(G_k)$ corresponds to the Lie subalgebra $\prod_{i=k}^{\infty} T^i \mathfrak{sl}_n(\mathbb{F}_p) \cong \mathfrak{sl}_n(\mathfrak{m}^k)$. If W is a nonzero $\mathbb{F}_p$-subspace of $\mathfrak{sl}_n(R)$ satisfying $[W, \mathfrak{sl}_n(\mathfrak{m}^k)]_{\mathrm{Lie}} \subseteq W$ for some $k \geqslant 1$, then a direct computation shows there exists r such that $\mathfrak{sl}_n(\mathfrak{m}^r) \subseteq W$. (It is this computation that uses the fact that $p \nmid n$).

Now let H and N be closed subgroups of G such that $\mathbf{1} \neq N \trianglelefteqslant H \trianglelefteqslant G$. Then $L_G(H)$ is an ideal of the Lie algebra $\mathfrak{sl}_n(\mathfrak{m})$ and hence there exists r such that $\mathfrak{sl}_n(\mathfrak{m}^r) \subseteq L_G(H)$. Consequently, $G_r \leqslant H$, so that $[N, G_r] \leqslant N$ and one deduces $[L_G(N), \mathfrak{sl}_n(\mathfrak{m}^r)]_{\mathrm{Lie}} \subseteq L_G(N)$. It follows that $\mathfrak{sl}_n(\mathfrak{m}^s) \subseteq L_G(N)$ for some s and hence $G_s \leqslant N$ and so $|G : N| < \infty$. This shows that G is hereditarily just infinite.

Next we exploit properties of the Nottingham group $\mathcal{N}$ over $\mathbb{F}_p$ to produce groups of automorphisms of the above group G. The group $\mathcal{N}$ is the Sylow pro-p subgroup of the profinite group $\mathrm{Aut_c}(R) = \mathrm{Aut}(R)$; it coincides with the group $\mathrm{Aut}^1(R)$ of all automorphisms of the ring R that act trivially modulo $\mathfrak{m}^2$. Any element α of $\mathcal{N}$ is then uniquely determined by its effect upon the indeterminate T and, conversely, for any $f \in R$ with $f \equiv T \pmod{\mathfrak{m}^2}$ there is a unique element of $\mathcal{N}$ mapping T to f. (Thus $\mathcal{N}$ could alternatively be defined as a group of power series $T + \mathfrak{m}^2$ with the binary operation given by substitution of power series. For our construction, however, the behavior as automorphisms is more relevant). We refer to [Camina 2000] for background material concerning the Nottingham group, which plays a role also in number theory and dynamics.

The action of the Nottingham group $\mathcal{N}$ on R induces a faithful action upon the group $G = \mathrm{SL}_n^1(R)$ and hence we construct a subgroup $\dot{\mathcal{N}} \leqslant \mathrm{Aut_c}\, G$ isomorphic to $\mathcal{N}$. Suppose $\alpha \in \mathcal{N}$ is an element that induces an inner automorphism $\dot{\alpha}$ of the group G, and put $f = T\alpha \in T + \mathfrak{m}^2$. Then there exists a matrix $h \in G$ such that $hx^{\dot{\alpha}} = xh$ for all $x \in G$. In particular, upon taking $x = I + Te_{ij}$ for $1 \leqslant i, j \leqslant n$ with $i \neq j$, we conclude that h must be a diagonal matrix such that every pair of distinct diagonal entries a and b are linked by the relation $Ta = fb$ in R. It follows that $f^2 = T^2$ and hence, since $f \equiv T \pmod{\mathfrak{m}^2}$, that $f = T$ and $\dot{\alpha} = \mathrm{id}_G$. In conclusion, the copy of the Nottingham group in $\mathrm{Aut_c}\, G$ satisfies $\dot{\mathcal{N}} \cap \mathrm{Inn}\, G = \mathbf{1}$.

As the final step in our construction, we use the result of Camina [1997] that every countably-based pro-p group can be embedded as a closed subgroup in $\mathcal{N}$. Hence if A is any finitely generated pro-p group that is virtually nilpotent (say, of class c), then it may be embedded in $\mathrm{Aut}_c\, G$ in such a way that $A \cap \mathrm{Inn}\, G = \mathbf{1}$. Hence we have satisfied the conditions of Theorem 6.2 and the semidirect product $G \rtimes A$ is an example of a hereditarily $\mathrm{JNN}_c\mathrm{F}$ pro-p group.

Acknowledgements

Some of this research was conducted while Quick was visiting Heinrich Heine University Düsseldorf. Quick thanks the university for its hospitality and we gratefully acknowledge partial support by the Humboldt Foundation. We also thank the referees for a careful reading of the paper and insightful suggestions, particularly that of using subprimitivity in Section 6.

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Received December 15, 2022. Revised August 1, 2023.

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