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ESTIMATE FOR THE FIRST FOURTH STEKLOV EIGENVALUE OF A MINIMAL HYPERSURFACE WITH FREE BOUNDARY

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We dedicate this paper to João Xavier da Cruz Neto on the occasion of his sixtieth birthday.

We explore the fourth-order Steklov problem of a compact embedded hypersurface Σ^n with free boundary in a $(n+1)$-dimensional compact manifold M^{n+1} which has nonnegative Ricci curvature and strictly convex boundary. If Σ is minimal we establish a lower bound for the first eigenvalue of this problem. When $M = B^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$, if Σ has constant mean curvature H^Σ we prove that the first eigenvalue satisfies $\sigma_1 \leq n + |H^\Sigma|$. In the minimal case ($H^\Sigma = 0$), we prove that $\sigma_1 = n$.

1. Introduction

Let Σ^n be an n-dimensional compact Riemannian manifold with nonempty boundary $\partial\Sigma \neq \varnothing$. Consider the fourth-order Steklov eigenvalue problem

$$\begin{cases} \Delta^2 \xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta\xi = \sigma \frac{\partial \xi}{\partial \nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{1}$$

where σ is a real number, Δ is the Laplacian operator on Σ and ν_Σ denotes the outward unit normal on $\partial\Sigma$. The first nonzero eigenvalue of the above problem will be denoted by $\sigma_1 = \sigma(\Sigma)$. The first eigenvalue of (1) has the following variational characterization:

$$\sigma_1 = \inf_{w_{|\partial\Sigma}=0} \frac{\int_\Sigma (\Delta w)^2}{\int_{\partial\Sigma} \left(\frac{\partial w}{\partial \nu_\Sigma}\right)^2}. \tag{2}$$

Wang and Xia [2009] proved that if Σ has nonnegative Ricci curvature and the mean curvature of $\partial\Sigma$ is bounded below by a positive constant c then $\sigma_1 \geq c \cdot n$. Furthermore, equality occurs if and only if Σ is isometric to an n-dimensional Euclidean ball of radius $\frac{1}{c}$.

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Since their first appearance in [Stekloff 1902], elliptic problems with parameters in the boundary conditions are called Steklov problems. Kuttler [1972] and Payne [1970] studied the isoperimetric properties of the first eigenvalue σ_1 of the fourth-order Steklov problem (1). Moreover, as already noticed in [Kuttler 1972; 1979; Kuttler and Sigillito 1985], σ_1 is the sharp constant for L^2 a priori estimates for solutions of the (second-order) Laplace equation under nonhomogeneous Dirichlet boundary conditions. In [Ferrero et al. 2005] the authors studied the spectrum of the biharmonic Steklov problem (1) and obtained a characterization of it, and presented a physical interpretation of σ_1. For comprehensive references on such Steklov problems, see [Berchio et al. 2006; Bucur et al. 2009; Wang and Xia 2009].

It should be pointed out that the problem

$$\begin{cases} \Delta^2\xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \frac{\partial^2\xi}{\partial\nu_\Sigma^2} = \lambda\frac{\partial\xi}{\partial\nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{3}$$

is a natural Steklov problem and one can check that when the mean curvature of $\partial\Sigma$ is constant, it is equivalent to (1).

Let M be a compact Riemannian manifold with nonempty boundary ∂M and $\Sigma \subset M$ a compact hypersurface (with boundary $\partial\Sigma$) properly embedded into M, that is, $\Sigma \cap \partial M = \partial\Sigma$. We say that Σ is a minimal hypersurface with free boundary if Σ is a minimal hypersurface and Σ meets ∂M orthogonally along $\partial\Sigma$. In this setting, Fraser and Li [2014] obtained a lower bound for the first eigenvalue of the second-order Steklov problem.

If $M = B^n$ is the unit ball in $\mathbb{R}^n$, it is known [Fraser and Schoen 2013] that the coordinate functions are eigenfunctions of the second-order Steklov problem with eigenvalue 1. Taking that into consideration, Fraser and Li [2014] conjectured that the first eigenvalue of the second-order Steklov problem of a compact properly embedded minimal hypersurface in B^n is 1 and proved that this is limited from below by $\frac{1}{2}$.

On the one hand, we did not find in the literature an extrinsic approach to the fourth-order Steklov eigenvalue problem. Motivated by the work of Fraser and Li, in this paper we consider the fourth-order Steklov problem of a compact properly embedded minimal hypersurface Σ with free boundary in a compact manifold M.

On the other hand, Ferrero, Gazzola and Weth [Ferrero et al. 2005] explored the fourth-order Steklov eigenvalue problem in a bounded domain Ω of $\mathbb{R}^n$ and proved that the first eigenvalue of this problem is equal to n when $\Omega = B^n$. It is known that the unit ball B^n is a minimal hypersurface with free boundary in B^{n+1}. In this setting, we have established an upper estimate for the first eigenvalue of the fourth-order Steklov problem of a compact properly embedded CMC hypersurface in B^{n+1} with free boundary on ∂B^{n+1}:

Proposition 1. *Let Σ^n be a compact properly embedded hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$. Assume that Σ has constant normalized mean curvature H^Σ. Then*

$$\sigma_1 \leq n + |H^\Sigma|.$$

It follows from Proposition 1 that if Σ is minimal ($H^\Sigma = 0$), then $\sigma_1 \leq n$. This, together with the result of Ferrero, Gazzola and Weth [Ferrero et al. 2005], naturally led us to formulate and prove the main result of this paper:

Theorem 2. *Let Σ^n be a compact properly embedded minimal hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$. Then the first eigenvalue of the fourth-order Steklov problem of Σ is equal to n.*

Wang and Xia [2009] proved that any compact connected Riemannian manifold Σ with boundary $\partial\Sigma$ satisfies

$$|\Sigma|\sigma_1 \leq |\partial\Sigma|, \tag{4}$$

where $|\partial\Sigma|$ and $|\Sigma|$ denote the area of $\partial\Sigma$ and the volume of Σ, respectively. If in addition the Ricci curvature of Σ is nonnegative and the equality holds, then Σ is isometric to an n-dimensional Euclidean ball. In our context, the equality always holds even for codimension greater than 1 (see Proposition 2.4 in [Li 2020]), i.e.,

$$k|\Sigma| = |\partial\Sigma|$$

for every k-dimensional immersed free boundary minimal submanifold Σ^k in the unit ball B^{n+1}. As a consequence of this equality and from (4) we get that $\sigma_1 \leq k$ for free boundary minimal submanifolds $\Sigma^k \subset B^{n+1}$.

Taking that into consideration, it is natural to consider the following question.

Problem 3. *Under what additional assumption is it possible to ensure that a compact properly embedded minimal hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$, such that $\sigma_1 = n$ is the unit ball B^n?*

In our next result, we prove a lower estimate for σ_1 when Σ^n is a compact properly embedded minimal hypersurface with free boundary in a compact manifold which has nonnegative Ricci curvature and strictly convex boundary. More precisely, we prove the following theorem.

Theorem 4. *Let M^{n+1} be an $(n+1)$-dimensional compact orientable Riemannian manifold with nonnegative Ricci curvature and nonempty boundary ∂M. Assume the second fundamental form of ∂M satisfies $A^{\partial M}(v, v) \geq k > 0$, for any unit vector v tangent to ∂M.*

Let Σ^n be a properly embedded minimal hypersurface in M with free boundary on ∂M. Assume $\partial\Sigma$ has constant mean curvature $H^{\partial\Sigma}$. If

(i) Σ *is orientable*, *or*

(ii) $\pi_1(M)$ *is finite*,

then we have the eigenvalue estimate $\sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}$, *where* σ_1 *is the first eigenvalue of the fourth-order Steklov problem on* Σ.

This estimate for σ_1 is analogous to the estimates of Fraser and Li [2014] for the first nonzero Steklov eigenvalue of the Dirichlet-to-Neumann map on Σ.

Remark 5. If $M = B^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$ and $\Sigma = B^n \subset B^{n+1}$ is the unit ball in $\mathbb{R}^n$ ("equatorial disk"), then $H^{\partial\Sigma} = n-1$ and $k = 1$, and we get that $\sigma_1 = H^{\partial\Sigma} + k$. For this reason, we believe that $\sigma_1 \geq H^{\partial\Sigma} + k$ is the sharp estimate. Consequently, the hypothesis in Theorem 4 that $\partial\Sigma$ has constant mean curvature becomes natural to assume.

Combining the inequality (4) with our Theorem 4 we deduce the following corollary.

Corollary 6. *Let* M^{n+1} *be an* $(n+1)$*-dimensional compact orientable Riemannian manifold with nonnegative Ricci curvature and nonempty boundary* ∂M. *Assume the second fundamental form of* ∂M *satisfies* $A^{\partial M}(v, v) \geq k > 0$, *for any unit vector* v *tangent to* ∂M.

Let Σ *be a properly embedded minimal hypersurface in* M *with free boundary on* ∂M. *Assume* $\partial\Sigma$ *has constant mean curvature* $H^{\partial\Sigma}$. *Then*

$$|\partial\Sigma| \geq \Big(H^{\partial\Sigma} + \frac{k}{2}\Big)|\Sigma|.$$

2. Preliminaries

In this section we will collect some basic results that are essential to deduce Theorem 4. Let M^{n+1} be a $(n+1)$-dimensional compact Riemannian manifold with nonempty boundary ∂M. Denote by $\langle \cdot, \cdot \rangle$ the metric on M and D the Riemannian connection on M. We define the second fundamental form of the boundary ∂M with respect to the outward unit normal μ by $A^{\partial M}(u, v) = \langle D_u\mu, v\rangle$, where u, v are tangent to ∂M. The mean curvature $H^{\partial M}$ of ∂M is then defined as the trace of $A^{\partial M}$, i.e.,

$$H^{\partial M} = \sum_{j=1}^{n} A^{\partial M}(e_j, e_j),$$

where $e_1, \dots, e_n$ is any orthonormal basis for $T\partial M$.

The following, known as Reilly's formula, was settled in [Fraser and Li 2014, Lemma 2.6]; see also [Choi and Wang 1983].

Proposition 7 [Fraser and Li 2014]. *Let* Ω *be a compact* $(n+1)$*-manifold with piecewise smooth boundary* $\partial\Omega = \bigcup \sum_{i=1}^{k} \Sigma_i$. *Suppose* f *is a continuous function* *on* Ω *where* $f \in C^\infty(\Omega \setminus S)$, $S = \bigcup \sum_{i=1}^k \partial \Sigma_i$, *and there exists some* $C > 0$ *such that* $\|f\|_{C^3(\Omega')} \leq C$ *for all* $\Omega' \subset \Omega \setminus S$. *Then, Reilly's formula holds*:

$$(5) \quad 0 = \int_\Omega \mathrm{Ric}^\Omega(Df, Df) - (\Delta_\Omega f)^2 + \|\mathrm{Hess}_\Omega f\|^2 + \sum_{i=1}^k \int_{\Sigma_i} \Big[\Big(\Delta_{\Sigma_i} f + H^{\Sigma_i} \frac{\partial f}{\partial \eta_i} \Big) \frac{\partial f}{\partial \eta_i} - \Big\langle \nabla^{\Sigma_i} f, \nabla^{\Sigma_i} \frac{\partial f}{\partial \eta_i} \Big\rangle + h^{\Sigma_i}(\nabla^{\Sigma_i} f, \nabla^{\Sigma_i} f) \Big].$$

Here, Ric^Ω *is the Ricci tensor of* Ω; Δ_Ω, Hess_Ω *and* ∇_Ω *are the Laplacian*, *Hessian and gradient operators on* Ω, *respectively*; Δ_{Σ_i} *and* ∇^{Σ_i} *are the Laplacian and gradient operators on each* Σ_i, *respectively*; η_i *is the outward unit normal of* Σ_i; H^{Σ_i} *and* h^{Σ_i} *are the mean curvature and second fundamental form of* Σ_i *in* Ω *with respect to the outward unit normal*, *respectively.*

To prove our main result we need a few considerations. Let $\varphi : \Sigma \to M$ be a properly embedded minimal hypersurface with free boundary in a compact orientable manifold M. Assume that ∂M is strictly convex and M has nonnegative Ricci curvature. Under these assumptions, ∂M is connected [Fraser and Li 2014, Proposition 2.8], and any properly embedded minimal hypersurface in M with free boundary is connected [Fraser and Li 2014, Lemma 2.5]. Furthermore, if both Σ and M are orientable then $M \setminus \varphi(\Sigma)$ consists of two components Ω_1 and Ω_2 (see [Fraser and Li 2014, Corollary 2.10]). Take $\Omega = \Omega_1$. Let $\partial\Omega = \Sigma \cup \Gamma$ where $\Gamma \subset \partial M$. Thus, $\partial\Sigma = \partial\Gamma$. Note that Γ is not necessarily connected, but each component of Γ must intersect Σ along some component of $\partial\Sigma$. Otherwise, ∂M would have more than one component, a contradiction.

Remark 8. From a result due to M. C. Li [2011, Theorem 1.1.8], any compact Riemannian 3-manifold M with nonempty boundary ∂M admits a nontrivial compact embedded minimal surface Σ with free boundary. Some examples of free boundary submanifolds in the unit ball are given in [Fraser and Schoen 2013].

3. Proof of the results

3.1. *Proof of Proposition 1.*

Proof. Let $\xi : B^{n+1} \to \mathbb{R}$ be defined by $\xi(x) = 1 - \|x\|^2$. As can be easily seen

$$\xi_{|\partial\Sigma} = 0 \quad \text{and} \quad \Delta_\Sigma \xi(x) = -2\big(n + H^\Sigma \langle x, N(x) \rangle\big),$$

where N is a unit vector field normal to Σ^n in B^{n+1}. Thus,

$$(\Delta_\Sigma \xi)^2 \leq 4n^2 \Big(1 + \frac{|H^\Sigma|}{n}\Big)^2.$$

On the other hand, if ν_Σ is the outward unit conormal along $\partial\Sigma$ and x_i are the coordinate functions, the condition

$$\frac{\partial x_i}{\partial \nu_\Sigma} = x_i$$

is equivalent to $\nu_\Sigma = x$, which is equivalent to the condition that Σ meets ∂B^n orthogonally. Then, Σ meets ∂B^n orthogonally if and only if

$$\frac{\partial \xi}{\partial \nu_\Sigma} = -2.$$

Now, using the variational characterization of σ_1 we get

$$\sigma_1 \cdot |\partial\Sigma| \le n^2\left(1 + \frac{|H^\Sigma|}{n}\right)^2 \cdot |\Sigma|,$$

and applying inequality (4) we conclude that

$$\sigma_1 \le n + |H^\Sigma|. \qquad \square$$

3.2. *Proof of Theorem 2.*

Proof. Again let us consider the function $\xi : B^{n+1} \to \mathbb{R}$ defined by $\xi(x) = 1 - \|x\|^2$. Since Σ is minimal, it follows from the proof of Proposition 1 that $\Delta_\Sigma \xi = -2n$. Thus

$$\begin{cases} \Delta^2_\Sigma \xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta_\Sigma \xi = n \frac{\partial \xi}{\partial \nu_\Sigma} & \text{on } \partial\Sigma, \end{cases}$$

which implies that n is an eigenvalue. Now we will show that $\sigma_1 = n$.

It is known (see Theorem 1 in [Berchio et al. 2006]) that the infimum in (2) is achieved and that, up to a multiplicative constant, the minimizer is unique, smooth up to the boundary, positive in Σ, and the normal derivative relative to the outward unit normal is negative on $\partial\Sigma$. Arguing as in the proof of Lemma 2.2 in [Ferrero et al. 2005] we conclude that $\sigma_1 = n$. $\square$

3.3. *Proof of Theorem 4.*

Proof. Firstly suppose that Σ is orientable. Since M is orientable we have Σ is connected and $M \setminus \varphi(\Sigma)$ consists of two components Ω_1 and Ω_2 (see [Fraser and Li 2014, Corollaries 2.5 and 2.10]). Let $\Omega = \Omega_1$ and $\partial\Omega = \Sigma \cup \Gamma$, where $\Gamma \subset \partial M$, so that $\partial\Sigma = \partial\Gamma$.

Let $\xi \in C^\infty(\Sigma)$ be an eigenfunction corresponding to the first eigenvalue σ_1 of the fourth-order Steklov problem, that is,

$$\begin{cases} \Delta^2_\Sigma \xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta_\Sigma \xi = \sigma_1 \frac{\partial \xi}{\partial \nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{6}$$

where ν_Σ is the outward conormal vector of $\partial\Sigma$ with respect to Σ. Next, we consider the Dirichlet–Neumann boundary value problem on the compact $(n+1)$-manifold Ω with piecewise smooth boundary $\partial\Omega = \Sigma \cup \Gamma$

$$\begin{cases} \Delta_\Omega f = 0 & \text{in } \Omega, \\ f = \Delta_\Sigma \xi & \text{on } \Sigma, \\ \frac{\partial f}{\partial \eta_\Gamma} = (\sigma_1 - H^{\partial\Sigma}) f & \text{on } \Gamma. \end{cases} \tag{7}$$

Analyzing the relationship between the first eigenvalues of problems (1) and (3) it is possible to conclude that $\sigma_1 > H^{\partial\Sigma}$. To ensure the existence of a solution for problem (7), we will consider the homogeneous problem

$$\begin{cases} \Delta_\Omega f = 0 & \text{in } \Omega, \\ f = 0 & \text{on } \Sigma, \\ \frac{\partial f}{\partial \eta_\Gamma} = \mu f & \text{on } \Gamma. \end{cases} \tag{8}$$

This mixed Steklov–Dirichlet problem has a discrete spectrum $\{\mu_i\}$ (see [Guo and Xia 2019, Section 2]) where

$$0 < \mu_1 \le \mu_2 \le \cdots \to +\infty.$$

Next, we will establish a lower bound for μ_1. Consider f_1 an eigenfunction associated with μ_1 and assume without loss of generality that $\int_\Sigma h^\Sigma(\nabla^\Sigma f_1, \nabla^\Sigma f_1) \ge 0$ (otherwise, we choose $\Omega = \Omega_2$ instead). We get by Reilly's formula (5) applied to f_1

$$0 \ge nk \int_\Gamma \left(\frac{\partial f_1}{\partial \eta_\Gamma}\right)^2 + \int_\Gamma \Delta_\Gamma f \frac{\partial f_1}{\partial \eta_\Gamma} - \int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle + k \int_\Gamma |\nabla^\Gamma f_1|^2,$$

where η_Σ and η_Γ are the outward unit normals of Σ and Γ, respectively, with respect to Ω. Integrating by parts we get

$$\int_\Gamma \Delta_\Gamma f_1 \frac{\partial f_1}{\partial \eta_\Gamma} = - \int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle + \int_{\partial\Gamma} \frac{\partial f_1}{\partial \nu_\Gamma} \frac{\partial f_1}{\partial \eta_\Gamma},$$

where ν_Σ and ν_Γ are the outward conormal vectors of $\partial\Sigma = \partial\Gamma$ with respect to Σ and Γ, respectively. Since Σ meets Γ orthogonally along $\partial\Sigma = \partial\Gamma$, we have $\nu_\Sigma = \eta_\Gamma$ and $\eta_\Sigma = \nu_\Gamma$ along the common boundary $\partial\Sigma$. Thereby

$$0 = \int_{\partial\Sigma} \frac{\partial f_1}{\partial \nu_\Sigma} \frac{\partial f_1}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f_1}{\partial \nu_\Sigma} \frac{\partial f_1}{\partial \nu_\Gamma} = \int_{\partial\Gamma} \frac{\partial f_1}{\partial \nu_\Gamma} \frac{\partial f_1}{\partial \eta_\Gamma},$$

which implies

$$2 \int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle \ge nk \int_\Gamma \left(\frac{\partial f_1}{\partial \eta_\Gamma}\right)^2 + k \int_\Gamma |\nabla^\Gamma f_1|^2.$$

We conclude that $\mu_1 \ge \frac{k}{2}$. Having proved this fact, we will make an analysis divided into two cases. Namely, if there is $i \in \mathbb{N}$ such that $\sigma_1 - H^{\partial\Sigma} = \mu_i \ge \mu_1$ we get $\sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}$. Otherwise, $\sigma_1 - H^{\partial\Sigma} \neq \mu_i$ for all $i \in \mathbb{N}$. So, the homogeneous problem (8) has only the trivial solution, and it follows from standard elliptic PDE theory, more specifically from the Fredholm alternative, that the problem (7) has a unique solution f. Note that $\Delta_\Sigma(f|_\Sigma) = \Delta_\Sigma^2 \xi = 0$ in Σ, and assuming without loss of generality that $\int_\Sigma h^\Sigma(\nabla^\Sigma f, \nabla^\Sigma f) \geq 0$, by substituting this function f in formula (5) we obtain

$$0 \geq -\int_\Sigma \left\langle \nabla^\Sigma f, \nabla^\Sigma \frac{\partial f}{\partial \eta_\Sigma} \right\rangle + nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + \int_\Gamma \Delta_\Gamma f \frac{\partial f}{\partial \eta_\Gamma} - \int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + k \int_\Gamma |\nabla^\Gamma f|^2.$$

Now, using that

$$\int_\Sigma \left\langle \nabla^\Sigma f, \nabla^\Sigma \frac{\partial f}{\partial \eta_\Sigma} \right\rangle = -\int_\Sigma \frac{\partial f}{\partial \eta_\Sigma} \Delta_\Sigma f + \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma}$$

and

$$\int_\Gamma \Delta_\Gamma f \frac{\partial f}{\partial \eta_\Gamma} = -\int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma},$$

we have

$$0 \geq -\int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} + \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma} - 2\int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + k \int_\Gamma |\nabla^\Gamma f|^2.$$

As we saw previously,

$$\int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \nu_\Gamma} = \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma}.$$

Therefore,

$$2\int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle \geq nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + k \int_\Gamma |\nabla^\Gamma f|^2.$$

Now, using the last equality in (7) we get

$$2(\sigma_1 - H^{\partial\Sigma}) \geq k \implies \sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}.$$

This proves the theorem when Σ is orientable. In the case when Σ nonorientable and $\pi_1(M)$ finite, we can argue as in [Fraser and Li 2014, Theorem 3.1]. □

3.4. *Proof of Corollary 6.*

Proof. The proof of Corollary 6 follows directly from (4) and Theorem 4. □

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