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For a pair of distinct non-CM newforms of weights at least 2 and having rational integral Fourier coefficients $a_1(n)$ and $a_2(n)$, under GRH, we obtain an estimate for the set of primes p such that

$$\omega(a_1(p)-a_2(p)) \le \left[7k+\tfrac{1}{2}+k^{1/5}\right],$$

where $\omega(n)$ denotes the number of distinct prime divisors of an integer n and k is the maximum of their weights. As an application, under GRH, we show that the number of primes giving congruences between two such newforms is bounded by $\left[7k+\frac{1}{2}+k^{1/5}\right]$. We also obtain a multiplicity-one result for newforms via congruences.

1. Introduction and statement of the results

For an elliptic curve $E/\mathbb{Q}$ and a prime p of good reduction, let $N_p(E) := p+1-a(p)$ be the number of points of the reduction of E modulo p. Assume that E is not $\mathbb{Q}$-isogenous to an elliptic curve with torsion. Then Koblitz's conjecture [7] says that the number of primes $p \le X$ for which $N_p(E)$ is prime is asymptotically equal to $C_E(X/(\log X)^2)$, where C_E is a positive constant depending on E. In particular, $N_p(E)$ is prime infinitely often when p runs over the set of primes. This conjecture is still open but there are many results towards this in the literature (see [17]). Indeed, Koblitz's conjecture can be seen as a variant of the twin prime conjecture (for more details, see [7]).

Inspired by Koblitz's conjecture, Kirti Joshi [6] studied the prime divisors of $N_p(f) := p^{k-1}+1-a(p)$, where $a(p)$ is the (integer) p-th Fourier coefficient of a newform $f \in S_k(N)$, the space of cusp forms of weight k and level N. Note that for the Ramanujan delta function $\Delta \in S_{12}(1)$, $\omega(N_p(\Delta)) \ge 3$ for any $p \ge 5$, where $\omega(n)$ is the number of distinct prime divisors of an integer n. This shows that, in general, the obvious variant of Koblitz's conjecture is not true for modular forms of higher weights. In fact, Joshi shows that there exist infinitely many cusp forms f_{k_i}

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(which need not be eigenforms) of increasing weight k_i and of level 1 such that $\omega(N_p(f_{k_i})) \geq 2$ for all primes p.

If f is a non-CM newform of weight $k \geq 4$ then in the same paper Joshi gives an estimate for the primes p for which $N_p(f)$ is an almost prime, i.e., has few prime divisors. More precisely, under GRH and Artin's holomorphy conjecture, he uses a suitably weighted sieve due to Richert to prove that

$$\left|\left\{p \leq X : \omega(N_p(f)) \leq \left[5k + 1 + \sqrt{\log k}\,\right]\right\}\right| \gg \frac{X}{(\log X)^2}, \tag{1-1}$$

where $[\,\cdot\,]$ is the greatest integer function. He also proves a similar result for the function $\Omega(N_p(f))$, where $\Omega(n)$ counts the number of prime divisors of n with multiplicity.

One can interpret $N_p(f)$ as the difference of p-th Fourier coefficients of the normalized Eisenstein series E_k and the newform f. This leads us to study the number of prime divisors of the difference between the p-th Fourier coefficients of any two distinct cuspidal newforms which allows us to deduce many interesting consequences about congruences between newforms, multiplicity-one results, etc. More precisely, we prove the following.

Theorem 1.1. *Let $f_1 \in S_{k_1}(N_1)$ and $f_2 \in S_{k_2}(N_2)$ be non-CM newforms with integer Fourier coefficients $a_1(n)$ and $a_2(n)$, respectively, of weights at least* 2. *We also assume that f_1 and f_2 are not character twists of each other if $k_1 = k_2$. Put*

$$k = \max\{k_1, k_2\}.$$

Then under GRH, we have

$$\left|\left\{p \leq X : a_1(p) \neq a_2(p),\ \omega(a_1(p) - a_2(p)) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right]\right\}\right| \gg \frac{X}{(\log X)^2}. \tag{1-2}$$

If $k \geq 6$ then the term $k^{1/5}$ appearing in (1-2) *can be replaced with the smaller term $\sqrt{\log k}$.*

We remark that because of Deligne's estimate of Fourier coefficients, for any p, $N_p(f)$ in (1-1) never vanishes, whereas $a_1(p) - a_2(p)$ may be zero. Therefore we remove such primes from (1-2).

Remark 1.2. In Theorem 1.1 and all the subsequent results in this section, by GRH, we mean the generalized Riemann hypothesis holds for all the number fields L_h, $h \geq 1$ (see Section 2B for the definition of L_h), i.e., the Dedekind zeta functions associated with L_h have no zeros in the complex region $\mathrm{Re}(s) > \frac{1}{2}$ for all h.

We now state a few applications of our main result. Unless stated otherwise, throughout the paper we shall work with forms f_1 and f_2 as in Theorem 1.1. We also assume that a newform is always normalized so that its first Fourier coefficient is 1. An immediate consequence of Theorem 1.1 is the following.

Corollary 1.3. *Let f_1 and f_2 be newforms as in Theorem 1.1. Then under GRH there exist infinitely many primes p such that $a_1(p) \neq a_2(p)$ and*

$$\omega(a_1(p) - a_2(p)) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right].$$

We now recall a multiplicity-one result which says that if $a_1(p) = a_2(p)$ for all but finitely many primes p, then $f_1 = f_2$. Rajan [14] has extensively generalized this result by proving that if $a_1(p) = a_2(p)$ for a set of primes p of positive upper density, then f_1 is a character twist of f_2. This is known as a strong multiplicity-one result. Recently, in [12], a variant of this result for normalized Fourier coefficients has been obtained. In this direction, we prove in Proposition 4.2 that, under GRH, if

$$\left|\{p \leq X : a_1(p) = a_2(p)\}\right| \gg X^{13/14+\epsilon} \tag{1-3}$$

for any $\epsilon > 0$, then f_1 is a character twist of f_2. As a consequence of Theorem 1.1, we obtain the following interesting result that can be seen as a variant of a multiplicity-one result in terms of congruences.

Corollary 1.4. *Let f_1 and f_2 be non-CM normalized newforms of weight k_1 and k_2 with integer Fourier coefficients $a_1(n)$ and $a_2(n)$, respectively. Put $k = \max\{k_1, k_2\}$ and assume GRH. If there exist primes $\ell_1, \ell_2, \dots, \ell_n$ such that $n > \left[7k + \frac{1}{2} + k^{1/5}\right]$ and, for each $1 \leq i \leq n$,*

$$a_1(p) \equiv a_2(p) \pmod{\ell_i}, \tag{1-4}$$

for all p except for a set of primes of order $o(X/(\log X)^2)$, then $k_1 = k_2$ and f_1 is a character twist of f_2.

Proof. On the contrary, assume that f_1 is not a character twist of f_2. For $1 \leq i \leq n$, let $B_i(X) = \{p \leq X : a_1(p) \not\equiv a_2(p) \pmod{\ell_i}\}$. Put $B(X) = \bigcup_{i=0}^{n} B_i(X)$. Then, for $p \notin B(X)$,

$$\ell_1\ell_2\cdots\ell_n \mid (a_1(p) - a_2(p)) \implies \omega(a_1(p) - a_2(p)) \geq n.$$

In particular,

$$\left\{p \leq X : a_1(p) \neq a_2(p) \text{ and } \omega(a_1(p) - a_2(p)) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right]\right\} \subset B(X).$$

But from our assumptions in (1-4) we have $|B(X)| = o(X/(\log X)^2)$ and this contradicts Theorem 1.1. □

We now mention the last application of Theorem 1.1 which is related to the number of congruence primes of a newform. Recall that for a newform $f_1 \in S_k(N_1)$ with integer Fourier coefficients $a_1(n)$, a positive integer D is called a congruence divisor if there exists another newform $f_2 \in S_k(N_2)$ with integer Fourier coefficients $a_2(n)$ which is not a character twist of f_1 such that f_1 and f_2 are congruent modulo D, that is, $a_1(n) \equiv a_2(n) \pmod{D}$ for all $(n, N_1N_2) = 1$. Indeed, this is equivalent to the condition that $a_1(p) \equiv a_2(p) \pmod D$ for all $(p, N_1N_2) = 1$. If D is a prime, then D is called a congruence prime and we refer to [3] for a nice overview of the subject. A congruence divisor of a newform is an important object to study as it is connected to many well-known problems. To name a few, a bound of the largest congruence divisor is related to the ABC conjecture, and if $k = 2$, then the congruence primes for f_1 are related to the prime divisors of the minimal degree of the modular parametrization to the elliptic curve attached to f_1 via the Eichler–Shimura mapping (see [11, p. 179–180]). It would be also of great interest to bound the number of congruence primes of a newform (see remark on page 180 of [11]). However, if we fix two newforms, then the following result gives a bound on the number of congruence primes which is immediate by Corollary 1.3.

Corollary 1.5. *Let f_1 and f_2 be newforms as in Theorem 1.1. Suppose there exists a positive integer D such that f_1 and f_2 are congruent modulo D. Then under GRH*

$$\omega(D) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right].$$

Each prime divisor of D gives a congruence between f_1 and f_2; therefore, Corollary 1.5 ensures that the number of primes giving congruences between two newforms is bounded uniformly in terms of their weights and not on the levels. This is the novelty of this result.

We now discuss some results about the function $\Omega(a_1(p) - a_2(p))$, where p varies over the set of primes. Using a similar idea as the proof of Theorem 1.1, we obtain the following.

Theorem 1.6. *Let f_1 and f_2 be as in Theorem 1.1. Then under GRH, we have*

$$\left|\left\{p \leq X : a_1(p) \neq a_2(p) \text{ and } \Omega(a_1(p) - a_2(p)) \leq \left[13k + \tfrac{1}{2} + \sqrt{\log k}\,\right]\right\}\right| \gg \frac{X}{(\log X)^2}.$$

It is clear that Theorem 1.6 also has applications of similar nature to that of Theorem 1.1 mentioned above and we would not repeat it here.

Remark 1.7. It is possible to obtain an upper bound of the right order of magnitude for the estimate in Theorem 1.6. In fact, we can do so by using Selberg's sieve and the ideas used in the proof of [6, Theorem 2.3.1]. More precisely, under GRH, one can obtain that if f_1 and f_2 are as in Theorem 1.1, then

$$\left|\left\{p \leq X : a_1(p) \neq a_2(p) \text{ and } \Omega(a_1(p) - a_2(p)) \leq \left[\tfrac{1}{2}(29k - 13)\right]\right\}\right| \ll \frac{X}{(\log X)^2}.$$

From the above estimate, it follows that under GRH

$$\left|\{p \leq X : a_1(p) - a_2(p) \text{ is prime}\}\right| \ll \frac{X}{(\log X)^2}.$$

In particular, the natural density of the set $\{p : a_1(p) - a_2(p) \text{ is prime}\}$ is zero. It would be interesting to obtain a suitable lower bound of this set or at least to know whether there are infinitely many primes p for which $a_1(p) - a_2(p)$ is a prime.

In fact, all the above results are valid even if we replace $a_1(p) - a_2(p)$ with $a_1(p) + a_2(p)$. Also, similar results but with better bounds hold in Theorems 1.1 and 1.6 if we assume Artin's holomorphicity conjecture in addition to GRH. It is also worth mentioning that the full strength of GRH is not essential to prove our theorems. Rather, a quasi-GRH, which assumes a zero-free region for the associated Dedekind zeta functions in the region $\mathrm{Re}(s) = 1 - \epsilon$ for some $\epsilon \in \left(0, \frac{1}{2}\right)$, is sufficient for our purpose (see the discussion and results proved in [13]). In this case, we obtain similar results to Propositions 2.1 and 4.1, with the only difference being that the exponent of x becomes $1 - \epsilon$ instead of $\frac{1}{2}$. However, it then requires a more careful analysis of handling the error terms in the subsequent part of the proof of our results, which will not be carried out here.

Contents and structure of the paper. The theorem of Deligne connecting the theory of ℓ-adic Galois representations to Fourier coefficients of newforms opens the door for obtaining many new results regarding the arithmetical nature of these coefficients. This connection and the Chebotarev density theorem play a prominent role in this paper. These are recalled in Section 2. To prove our results, we first establish Proposition 4.3 which gives an asymptotic formula for the number of primes p up to X for which $a_1(p) \neq a_2(p)$ and $a_1(p) - a_2(p)$ is divisible by a fixed positive integer. Proof of Proposition 4.3 requires computations of the image of the product Galois representations attached to f_1 and f_2 and this is obtained in Section 3. Finally, we apply a suitably weighted sieve due to Richert, recalled in Section 5, to prove our results. To establish the sieve conditions with the required uniformity of parameters, Proposition 4.3 plays a crucial role. We use the ideas employed in [6; 17] to prove our main results in Sections 6 and 7.

Notation. For any real number $X \geq 2$, $\pi(X)$ denotes the number of primes less than or equal to X. Along with the standard analytic notation $\ll, \gg, O, o, \sim$ (the implied constants will often depend on the pair of forms under consideration), we use the letters p, ℓ, q, ℓ_1, ℓ_2, etc. to denote prime numbers throughout the paper.

2. Preliminaries

We summarize some standard results without proofs which will be used throughout the paper. We closely follow [2] for our exposition.

2A. ***Chebotarev density theorem.*** We recall the Chebotarev density theorem which is one of the principal tools needed for proving the main theorems of this paper.

Let K be a finite Galois extension of $\mathbb{Q}$ with the Galois group G and degree n_K. For an unramified prime p, we denote by Frob_p, a Frobenius element of K at p in G. For a subset C of G, stable under conjugation, we define

$$\pi_C(X) := \{p \leq X : p \text{ unramified in } K \text{ and } \mathrm{Frob}_p \in C\}.$$

The Chebotarev density theorem states that

$$\pi_C(X) \sim \frac{|C|}{|G|}\pi(X).$$

We will use the following conditional effective version of this theorem which was first obtained by Lagarias and Odlyzko [8] and was subsequently refined by Serre [16]. To state this, let d_K be the absolute value of the discriminant of $K/\mathbb{Q}$ and $\zeta_K(s)$ be the Dedekind zeta function associated with K.

Proposition 2.1. *Suppose $\zeta_K(s)$ satisfies GRH. Then*

$$\pi_C(X) = \frac{|C|}{|G|}\pi(X) + O\left(\frac{|C|}{|G|}X^{1/2}(\log d_K + n_K \log X)\right).$$

In addition to GRH for $\zeta_K(s)$, by assuming Artin's holomorphy conjecture (which states that the Artin L-function associated to any nontrivial representation of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ has an analytic continuation on the whole complex plane) one can improve the error term in the above asymptotic formula for $\pi_C(X)$.

2B. ***mod-h Galois representations.*** Let $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Let $k \geq 2$, $N \geq 1$ and ℓ be a prime. Suppose $f \in S_k(N)$ is a newform with integer Fourier coefficients $a(n)$. The work of Eichler, Shimura and Deligne (see [1]) give the existence of a two-dimensional continuous, odd and irreducible Galois representation

$$\rho_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_\ell),$$

which is unramified at $p \nmid N\ell$. If Frob_p denotes a Frobenius element corresponding to such a prime, then the representation $\rho_{f,\ell}$ has the property that

$$\mathrm{tr}(\rho_{f,\ell}(\mathrm{Frob}_p)) = a(p), \quad \det(\rho_{f,\ell}(\mathrm{Frob}_p)) = p^{k-1}.$$

By reduction and semisimplification, we obtain a mod-ℓ Galois representation,

$$\bar{\rho}_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell),$$

where $\mathbb{F}_\ell := \mathbb{Z}/\ell\mathbb{Z}$.

Let $h = \prod_{j=1}^{t} \ell_j^{n_j}$ be a positive integer. Using the ℓ_j-adic representations attached to f, we consider an h-adic representation given by products of mod-ℓ_j's representations

$$\rho_{f,h} : G_{\mathbb{Q}} \to \mathrm{GL}_2\Big(\prod_{1\le j\le t} \mathbb{Z}_{\ell_j}\Big).$$

For each $1 \le j \le t$, we have the natural projection $\mathbb{Z}_{\ell_j} \twoheadrightarrow \mathbb{Z}/\ell_j^{n_j}\mathbb{Z}$, and hence we obtain a mod-h Galois representation given by

$$\bar{\rho}_{f,h} : G_{\mathbb{Q}} \to \mathrm{GL}_2\Big(\prod_{1\le j\le t} \mathbb{Z}/\ell_j^{n_j}\mathbb{Z}\Big) \xrightarrow{\cong} \mathrm{GL}_2(\mathbb{Z}/h\mathbb{Z}).$$

If $p \nmid Nh$ is a prime, then $\bar{\rho}_{f,h}$ is unramified at p and

$$\mathrm{tr}(\bar{\rho}_{f,h}(\mathrm{Frob}_p)) \equiv a(p) \pmod{h}, \qquad \det(\bar{\rho}_{f,h}(\mathrm{Frob}_p)) \equiv p^{k-1} \pmod{h}.$$

Let $f_1 \in S_{k_1}(N_1)$ and $f_2 \in S_{k_2}(N_2)$ be newforms having integer Fourier coefficients $a_1(n)$ and $a_2(n)$, respectively. Then one can consider the product representation $\bar{\rho}_h$ of $\bar{\rho}_{f_1,h}$ and $\bar{\rho}_{f_2,h}$, defined by

$$\begin{aligned}\bar{\rho}_h : G_{\mathbb{Q}} &\to \mathrm{GL}_2(\mathbb{Z}/h\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/h\mathbb{Z}),\\ \sigma &\mapsto (\bar{\rho}_{f_1,h}(\sigma), \bar{\rho}_{f_2,h}(\sigma)).\end{aligned}$$

Let $\mathscr{A}_h$ denote the image of $G_{\mathbb{Q}}$ under $\bar{\rho}_h$. By the fundamental theorem of Galois theory, the fixed field of $\ker(\bar{\rho}_h)$, say L_h, is a finite Galois extension of $\mathbb{Q}$ and

$$\mathrm{Gal}(L_h/\mathbb{Q}) \cong \mathscr{A}_h. \tag{2-1}$$

Let $\mathscr{C}_h$ be the subset of $\mathscr{A}_h$ defined by

$$\mathscr{C}_h = \{(A, B) \in \mathscr{A}_h : \mathrm{tr}(A) = \mathrm{tr}(B)\}.$$

We now define the following function on the set of positive integers which will play an important role throughout the paper. For an integer $h > 1$, define

$$\delta(h) := \frac{|\mathscr{C}_h|}{|\mathscr{A}_h|} \tag{2-2}$$

and $\delta(1) := 1$. Since the trace of the image of complex conjugation is always zero, $\mathscr{C}_h \neq \phi$, and hence $\delta(h) > 0$ for every integer h.

3. Technical results

Let f_1 and f_2 be newforms as before. The main aim of this section is to obtain an asymptotic size of $\delta(\ell^n)$ for $n = 1, 2$ and this requires the computation of the cardinalities of $\mathscr{A}_{\ell^n}$ and $\mathscr{C}_{\ell^n}$. Building on the work of Ribet [15] and Momose [10], Loeffler [9] has determined the image $\mathscr{A}_{\ell^n}$ of the product Galois representations.

More precisely, Loeffler [9, Theorem 3.2.2] has proved that that there exists a positive constant $M(f_1, f_2)$ such that, for all primes $\ell \geq M(f_1, f_2)$ and $n \geq 1$,

$$\mathscr{A}_{\ell^n} = \big\{(A, B) \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(A) = v^{k_1-1}, \det(B) = v^{k_2-1}, v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\big\}. \tag{3-1}$$

In other words, the mod-ℓ^n representations of two newforms (that are not character twists of each other) are as independent as possible. In the rest of the paper, we denote the constant $M(f_1, f_2)$ by M and without loss of generality, we assume that $M \geq 3$. Clearly, for $\ell \geq M$,

$$\mathscr{C}_{\ell^n} = \{(A, B) \in \mathscr{A}_{\ell^n} : \mathrm{tr}(A) = \mathrm{tr}(B)\}. \tag{3-2}$$

3A. *Combinatorial lemmas.* Here we obtain results about cardinalities of $\mathscr{A}_{\ell^n}$ and $\mathscr{C}_{\ell^n}$ for any $\ell \geq M$. We first assume that

$$\lambda_n = \gcd(\ell^n - \ell^{n-1}, k_1 - 1, k_2 - 1)$$

and

$$\Lambda_n = \{(v^{k_1-1}, v^{k_2-1}) : v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\}. \tag{3-3}$$

Recall that $\ell \geq 3$. We now consider the group homomorphism

$$\phi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to \Lambda_n \quad \text{defined by } \phi(v) = (v^{k_1-1}, v^{k_2-1}).$$

Since ϕ is surjective and its kernel $\{v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times : v^{\lambda_n} = 1\}$ is a cyclic subgroup of $(\mathbb{Z}/\ell^n\mathbb{Z})^\times$ of order λ_n, we obtain

$$|\Lambda_n| = \frac{|(\mathbb{Z}/\ell^n\mathbb{Z})^\times|}{\lambda_n} = \frac{\ell^n - \ell^{n-1}}{\lambda_n}. \tag{3-4}$$

We first recall the following result proved in [2, Lemma 3.3].

Lemma 3.1. *For any prime $\ell \geq M$,*

$$|\mathscr{A}_\ell| = \frac{1}{\lambda_1}(\ell - 1)^3(\ell^2 + \ell)^2.$$

Using Lemma 3.1, we now compute $|\mathscr{A}_{\ell^n}|$ for any $n \geq 1$.

Lemma 3.2. *For any prime $\ell \geq M$ and integer $n \geq 1$,*

$$|\mathscr{A}_{\ell^n}| = \frac{1}{\lambda_n}\ell^{7(n-1)}(\ell - 1)^3(\ell^2 + \ell)^2.$$

Proof. Let $\psi : \mathscr{A}_{\ell^n} \to \mathscr{A}_\ell$ be the natural map defined by

$$(A, B) \mapsto (A \pmod{\ell}, B \pmod{\ell}). \tag{3-5}$$

Since it is a surjective group homomorphism, we have

$$|\mathscr{A}_{\ell^n}| = |\ker(\psi)|\,|\mathscr{A}_\ell|.$$

Therefore, in view of Lemma 3.1, to evaluate $|\mathscr{A}_{\ell^n}|$ it is sufficient to compute $|\ker(\psi)|$. For that we first compute the cardinality of the set

$$\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(\gamma) = d,\ \gamma \equiv \mathrm{Id} \pmod{\ell}\},$$

where $d \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$ such that $d \equiv 1 \pmod{\ell}$ is fixed and Id is the identity element in $\mathrm{GL}_2(\mathbb{F}_\ell)$. Any general element of the above set will be of the form

$$\begin{pmatrix} 1+x\ell & y\ell \\ z\ell & 1+w\ell \end{pmatrix},$$

where $0 \le x, y, z, w < \ell^{n-1}$ with the condition that

$$(1+x\ell)(1+w\ell) - yz\ell^2 = d.$$

As $d \equiv 1 \pmod{\ell}$, the above equation reduces to

$$x(1+w\ell) = \frac{d-1}{\ell} + yz\ell - w.$$

Since $1 + w\ell \in (Z/\ell^n\mathbb{Z})^\times$ for such w, for any choices of $0 \le y, z, w < \ell^{n-1}$ the above equation gives a unique x. Therefore

$$\big|\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(\gamma) = d,\ \gamma \equiv \mathrm{Id} \pmod{\ell}\}\big| = \ell^{3(n-1)}. \tag{3-6}$$

Now, we note that

$$\ker(\psi) = \big|\{(A, B) \in \mathscr{A}_{\ell^n} : (A, B) \equiv (\mathrm{Id}, \mathrm{Id}) \pmod{\ell}\}\big|;$$

therefore from (3-1)

$$|\ker(\psi)| = \sum_{(d_1,d_2)\in\Lambda_n} \sum_{\substack{A\in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \det(A)=d_1 \\ A\equiv \mathrm{Id} \pmod{\ell}}} 1 \sum_{\substack{B\in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \det(b)=d_2 \\ b\equiv \mathrm{Id} \pmod{\ell}}} 1.$$

In the above, congruence conditions on A and B compel that $d_1 \equiv d_2 \equiv 1 \pmod{\ell}$, and hence using (3-6) gives

$$|\ker(\psi)| = \ell^{6(n-1)} \sum_{\substack{(d_1,d_2)\in\Lambda_n \\ d_1\equiv d_2\equiv 1 \pmod{\ell}}} 1.$$

Since the sum appearing on the right side of the above equation is the cardinality of the kernel of the natural (surjective) reduction map $\Lambda_n \to \Lambda_1$ given in (3-5), we have

$$|\ker(\psi)| = \frac{|\Lambda_n|}{|\Lambda_1|}\ell^{6(n-1)}.$$

Now using (3-4) in the above yields the desired result. □

Our next aim is to compute the cardinalities of $\mathscr{C}_\ell$ and $\mathscr{C}_{\ell^2}$. Though an explicit computation is possible, we only obtain asymptotic formulas here and that is enough for our purpose. To simplify our notation, we denote the set of quadratic and nonquadratic residue elements in $(\mathbb{Z}/\ell^n\mathbb{Z})^\times$ by Q_n and Q_n^c, respectively.

Lemma 3.3. *For any prime $\ell \geq M$,*

$$|\mathscr{C}_\ell| = \frac{\ell^6}{\lambda_1} + O(\ell^5).$$

Proof. From the definition of $\mathscr{C}_\ell$

$$\begin{aligned}|\mathscr{C}_\ell| &= \sum_{(d_1,d_2)\in\Lambda_1} \Big|\{(A,B)\in \mathrm{GL}_2(\mathbb{F}_\ell)\times\mathrm{GL}_2(\mathbb{F}_\ell) : \\ &\qquad\qquad \det(A)=d_1,\ \det(B)=d_2,\ \mathrm{tr}(A)=\mathrm{tr}(B)\}\Big| \\ &= \sum_{t\in\mathbb{F}_\ell}\ \sum_{(d_1,d_2)\in\Lambda_1}\ \sum_{\substack{A\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(A)=d_1\\ \mathrm{tr}(A)=t}} 1 \sum_{\substack{B\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1.\end{aligned}$$

Split the sum over Λ_1 into three parts, namely

$$|\mathscr{C}_\ell| = \sum_{t\in\mathbb{F}_\ell}\Bigg[\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2=4d_1}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1^c}}\Bigg] \sum_{\substack{A\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(A)=d_1\\ \mathrm{tr}(A)=t}} 1 \sum_{\substack{B\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1 \tag{3-7}$$

and we denote the corresponding sums by S_1, S_2 and S_3, respectively. Thus

$$S_1 = \sum_{t\in\mathbb{F}_\ell}\ \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}}\ \sum_{\substack{A\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(A)=d_1\\ \mathrm{tr}(A)=t}} 1 \sum_{\substack{B\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1.$$

To proceed further, note that for given $d\in\mathbb{F}_\ell^\times$ and $t\in\mathbb{F}_\ell$ one can obtain the following result by employing an elementary counting argument:

$$\Big|\{\gamma\in\mathrm{GL}_2(\mathbb{F}_\ell) : \det(\gamma)=d,\ \mathrm{tr}(\gamma)=t\}\Big| = \begin{cases}\ell^2+\ell & \text{if } t^2-4d\in Q_1,\\ \ell^2 & \text{if } t^2=4d,\\ \ell^2-\ell & \text{if } t^2-4d\in Q_1^c.\end{cases} \tag{3-8}$$

Using (3-8) gives

$$\begin{aligned}S_1 &= (\ell^2+\ell)\sum_{t\in\mathbb{F}_\ell}\ \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}}\ \sum_{\substack{B\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1 \\ &= (\ell^2+\ell)\sum_{t\in\mathbb{F}_\ell}\Bigg[\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2=4d_2}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1^c}}\Bigg] \sum_{\substack{B\in\mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=t_2\\ \mathrm{tr}(B)=t}} 1.\end{aligned}$$

Again using (3-8)

$$S_1 = (\ell^2+\ell)\sum_{t\in\mathbb{F}_\ell}\Bigg[(\ell^2+\ell)\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1}} 1 + \ell^2 \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2=4d_2}} 1 + (\ell^2-\ell)\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1^c}} 1\Bigg].$$

Collecting the terms containing ℓ^4 gives

$$S_1 = \ell^4 \sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}} 1 + O(\ell^5).$$

Similarly, we have

$$S_2 = \ell^4 \sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2=4d_1}} 1 + O(\ell^5), \quad S_3 = \ell^4 \sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1^c}} 1 + O(\ell^5).$$

Combining all together, we have, from (3-7),

$$|\mathscr{C}_\ell| = \ell^4 \sum_{t\in\mathbb{F}_\ell}\sum_{(d_1,d_2)\in\Lambda_1} 1 + O(\ell^5)$$

and now using (3-4) completes the proof. □

To compute $|\mathscr{C}_{\ell^2}|$, we first prove the following result which is a generalization of (3-8) for the ring $\mathbb{Z}/\ell^2\mathbb{Z}$.

Lemma 3.4. *For any $d \in (\mathbb{Z}/\ell^2\mathbb{Z})^\times$ and $t \in \mathbb{Z}/\ell^2\mathbb{Z}$, we have*

$$\text{(3-9)}\quad \big|\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) : \det(\gamma) = d,\ \mathrm{tr}(\gamma) = t\}\big| = \begin{cases} \ell^4+\ell^3-\ell^2 & \text{if } t^2-4d = 0,\\ \ell^4-\ell^2 & \text{if } 0 \neq t^2-4d \equiv 0 \pmod{\ell},\\ \ell^4+\ell^3 & \text{if } t^2-4d \in Q_2,\\ \ell^4-\ell^3 & \text{if } t^2-4d \in Q_2^c.\end{cases}$$

Proof. It is clear that

$$\big|\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) : \det(\gamma) = d,\ \mathrm{tr}(\gamma) = t\}\big| = |\mathscr{N}|,$$

where $\mathscr{N} := \{(a,b,c) \in (\mathbb{Z}/\ell^2\mathbb{Z})^3 : a^2-at+bc = -d\}$. To compute $|\mathscr{N}|$ we divide the set $\mathscr{N}$ into three disjoint subsets $\mathscr{N}_1$, $\mathscr{N}_2$ and $\mathscr{N}_3$ based on the following three cases, respectively. Hence

$$\text{(3-10)}\qquad |\mathscr{N}| = |\mathscr{N}_1| + |\mathscr{N}_2| + |\mathscr{N}_3|.$$

Case (i): $a = 0$. Then the condition $bc = -d$ forces that b and c both have to be units and for any b there exists a unique c. Hence

$$|\mathcal{N}_1| = \ell^2 - \ell.$$

Case (ii): $a \neq 0$ and $bc = 0$. The latter condition implies that either b or c is 0, or both are (nonzero) zero-divisors of $\mathbb{Z}/\ell^2\mathbb{Z}$. The total number of such pairs is $2\ell^2 - 1 + (\ell - 1)^2 = 3\ell^2 - 2\ell$. Therefore,

$$|\mathcal{N}_2| = |\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d = 0\}| \times (3\ell^2 - 2\ell). \tag{3-11}$$

We now claim that

$$|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d = 0\}| = \begin{cases} \ell & \text{if } t^2 - 4d = 0, \\ 0 & \text{if } 0 \neq t^2 - 4d \equiv 0 \pmod{\ell}, \\ 2 & \text{if } t^2 - 4d \in Q_2, \\ 0 & \text{if } t^2 - 4d \in Q_2^c. \end{cases} \tag{3-12}$$

To prove this, we see that if $t^2 - 4d = 0$, then any $a \equiv \frac{t}{2} \pmod{\ell}$ is a solution of $a^2 - at + d = 0$ and there are ℓ such choices for a. Next, assume that $0 \neq t^2 - 4d \equiv 0 \pmod{\ell}$. If $a^2 - at + d = 0$ has solutions, say x and y, then

$$(x - y)^2 = (x + y)^2 - 4xy = t^2 - 4d \equiv 0 \pmod{\ell}.$$

Therefore, $x - y \equiv 0 \pmod{\ell} \implies t^2 - 4d = (x - y)^2 = 0$, which is a contradiction. The last two cases are clear.

Thus using (3-12) in (3-11) gives the cardinality of $\mathcal{N}_2$.

Case (iii): $a \neq 0$ and $bc \neq 0$. In this case, bc can be either a (nonzero) zero-divisor or a unit. Clearly, the number of choices for b and c such that bc is a given nonzero zero-divisor is $2\ell(\ell - 1)$ and for a given unit the number of such choices is $\ell^2 - \ell$. Therefore, we have

$$\begin{aligned} |\mathcal{N}_3| = \big|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : 0 \neq a^2 - at + d \equiv 0 \pmod{\ell}\}\big| \times 2\ell(\ell - 1) \\ + \big|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d \in (\mathbb{Z}/\ell^2\mathbb{Z})^\times\}\big| \times (\ell^2 - \ell). \end{aligned} \tag{3-13}$$

If $a^2 - at + d = m\ell$ for some $m \in \mathbb{F}_\ell^\times$, then from (3-12)

$$|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d = m\ell\}| = \begin{cases} \ell & \text{if } t^2 - 4(d - m\ell) = 0, \\ 0 & \text{if } 0 \neq t^2 - 4(d - m\ell) \equiv 0 \pmod{\ell}, \\ 2 & \text{if } t^2 - 4(d - m\ell) \in Q_2 \iff t^2 - 4d \in Q_2, \\ 0 & \text{if } t^2 - 4(d - m\ell) \in Q_2^c \iff t^2 - 4d \in Q_2^c. \end{cases}$$

Note that there exists a unique $m \in \mathbb{F}_\ell^\times$ such that $t^2 - 4(d - m\ell) = 0$, and in that case $0 \neq t^2 - 4d \equiv 0 \pmod{\ell}$. Therefore

$$\left|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : 0 \neq a^2 - at + d \equiv 0 \pmod{\ell}\}\right| = \begin{cases} 0 & \text{if } t^2 - 4d = 0, \\ \ell & \text{if } 0 \neq t^2 - 4d \equiv 0 \pmod{\ell}, \\ 2(\ell - 1) & \text{if } t^2 - 4d \in Q_2, \\ 0 & \text{if } t^2 - 4d \in Q_2^c. \end{cases} \tag{3-14}$$

As we have $\ell^2 - 1$ choices of a in this case, (3-12) and (3-14) immediately gives

$$\left|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d \in (\mathbb{Z}/\ell^2\mathbb{Z})^\times\}\right| = \begin{cases} \ell^2 - \ell - 1 & \text{if } t^2 - 4d = 0, \\ \ell^2 - \ell - 1 & \text{if } 0 \neq t^2 - 4d \equiv 0 \pmod{\ell}, \\ \ell^2 - 2\ell - 1 & \text{if } t^2 - 4d \in Q_2, \\ \ell^2 - 1 & \text{if } t^2 - 4d \in Q_2^c. \end{cases} \tag{3-15}$$

Substituting (3-14) and (3-15) in (3-13) and then combining all the above three cases in (3-10) gives the desired result. □

We are now ready to give a desirable estimate for $|\mathscr{C}_{\ell^2}|$.

Lemma 3.5. *For any prime $\ell \geq M$,*

$$|\mathscr{C}_{\ell^2}| = \frac{\ell^{12}}{\lambda_2} + O(\ell^{11}).$$

Proof. We use similar arguments as in the proof of Lemma 3.3, and hence we will only give an outline of the proof here. We write

$$|\mathscr{C}_{\ell^2}| = \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \sum_{(d_1, d_2) \in \Lambda_2} \sum_{\substack{A \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(A) = d_1 \\ \mathrm{tr}(A) = t}} 1 \sum_{\substack{B \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(B) = d_2 \\ \mathrm{tr}(B) = t}} 1.$$

We split the sum over Λ_2 into four parts, namely

$$|\mathscr{C}_{\ell^2}| = \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \Bigg[\sum_{\substack{(d_1, d_2) \in \Lambda_2 \\ t^2 - 4d_1 \in Q_2}} + \sum_{\substack{(d_1, d_2) \in \Lambda_1 \\ t^2 = 4d_1}} + \sum_{\substack{(d_1, d_2) \in \Lambda_1 \\ 0 \neq t^2 - 4d_1 \equiv 0 \pmod{\ell}}} + \sum_{\substack{(d_1, d_2) \in \Lambda_1 \\ t^2 - 4d_1 \in Q_2^c}} \Bigg] \sum_{\substack{A \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(A) = d_1 \\ \mathrm{tr}(A) = t}} 1 \sum_{\substack{B \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(B) = d_2 \\ \mathrm{tr}(B) = t}} 1$$

and denote the corresponding sums by S_1', S_2', S_3' and S_4' so that

$$|\mathscr{C}_{\ell^2}| = S_1' + S_2' + S_3' + S_4'. \tag{3-16}$$

Now applying Lemma 3.4, we obtain

$$S_1' = (\ell^4+\ell^3) \sum_{t\in\mathbb{Z}/\ell^2\mathbb{Z}} \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2}} \sum_{\substack{B\in\mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(B)=d_2 \\ \mathrm{tr}(B)=t}} 1.$$

As before, splitting the middle sum into four parts and applying Lemma 3.4 yields

$$S_1' = (\ell^4+\ell^3) \sum_{t\in\mathbb{Z}/\ell^2\mathbb{Z}} \Bigg[(\ell^4+\ell^3) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ t^2-4d_2\in Q_2}} + (\ell^4+\ell^3-\ell^2) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ t^2=4d_2}} + (\ell^4-\ell^2) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ 0\equiv t^2-4d_2\equiv 0 \pmod{\ell}}} + (\ell^4-\ell^3) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ t^2-4d_2\in Q_2^c}} \Bigg] 1$$

and then collecting the terms containing ℓ^8 gives

$$S_1' = \ell^8 \sum_{t\in\mathbb{Z}/\ell^2\mathbb{Z}} \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2}} 1 + O(\ell^{11}).$$

Computing S_2', S_3' and S_4' in a similar manner and substituting in (3-16), we have

$$|\mathscr{C}_{\ell^2}| = \ell^8 \sum_{t\in\mathbb{Z}/\ell^2\mathbb{Z}} \sum_{(d_1,d_2)\in\Lambda_2} 1 + O(\ell^{11})$$

and finally using (3-4) completes the proof. □

Let $h = \ell_1^{n_1}\ell_2^{n_2}\cdots\ell_t^{n_t}$. Since the fixed field of $\ker(\bar{\rho}_h)$ is contained in the compositum of fixed fields of $\ker(\bar{\rho}_{\ell_i^{n_i}})$, from (2-1)

$$|\mathscr{A}_h| \le |\mathscr{A}_{\ell_1^{n_1}}||\mathscr{A}_{\ell_2^{n_2}}|\cdots|\mathscr{A}_{\ell_t^{n_t}}| \quad\text{and}\quad |\mathscr{C}_h| \le |\mathscr{C}_{\ell_1^{n_1}}||\mathscr{C}_{\ell_2^{n_2}}|\cdots|\mathscr{C}_{\ell_t^{n_t}}|.$$

For any prime ℓ and integer $n \ge 1$, $\mathscr{A}_{\ell^n}$ is contained in the set

$$\big\{(A, B) \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(A) = v^{k_1-1}, \det(B) = v^{k_2-1}, v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\big\},$$

and hence a simple counting argument gives

$$|\mathscr{A}_{\ell^n}| \ll \ell^{7n} \quad\text{and}\quad |\mathscr{C}_{\ell^n}| \ll \ell^{6n}.$$

Therefore now it is clear that, for any integer $h \ge 1$,

$$|\mathscr{A}_h| \ll h^7 \quad\text{and}\quad |\mathscr{C}_h| \ll h^6. \tag{3-17}$$

3B. ***Asymptotic size of*** $\boldsymbol{\delta(\ell)}$**.** Recall that, for any positive integer $h > 1$,

$$\delta(h) = \frac{|\mathscr{C}_h|}{|\mathscr{A}_h|}.$$

An immediate consequence of the results in the previous section is the following.

Proposition 3.6. *If ℓ varies over primes then, for $n = 1, 2$,*

$$\delta(\ell^n) \sim \frac{1}{\ell^n} \quad \text{as } \ell \to \infty.$$

Finally, we state the following multiplicative property which plays an important role to prove our main result.

Proposition 3.7 [2, Proposition 3.6]. *For primes $\ell_1, \ell_2 > M$ with $\ell_1 \neq \ell_2$, we have*

$$\delta(\ell_1 \ell_2) = \delta(\ell_1)\delta(\ell_2).$$

4. Analytic results on primes

Recall that f_1 and f_2 are non-CM newforms with integer Fourier coefficients which are not character twists of each other. For a positive integer $h \geq 1$ and a real number $X \geq 2$, consider the function

$$\pi_{f_1,f_2}(X, h) := \sum_{\substack{p \leq X, (p,hN)=1 \\ h|(a_1(p)-a_2(p))}} 1. \tag{4-1}$$

The representation $\bar{\rho}_h$, defined in Section 2, is unramified outside hN. Also, it is ramified at all the primes $\ell \mid h$ because its determinant contains a nontrivial power of the mod-ℓ cyclotomic character which is ramified at ℓ. However, there may exist some primes dividing N at which $\bar{\rho}_h$ is unramified. It follows that a prime p is unramified in L_h only if either $(p, hN) = 1$ or $p \mid N$. Since the image of Frobenius elements under $\bar{\rho}_h$ generate $\mathscr{A}_h$, we can write

$$\pi_{f_1,f_2}(X, h) = \big|\{p \leq X : p \text{ unramified in } L_h,\ \bar{\rho}_h(\mathrm{Frob}_p) \in \mathscr{C}_h\}\big| + O(1),$$

where the error term is due to the possible primes divisors of N which are unramified in L_h. Since the trace and the determinant maps are stable under conjugation, the group $\mathscr{A}_h$ and the set $\mathscr{C}_h$ are also stable under conjugation. Now applying the Chebotarev density theorem (see Proposition 2.1) for the field L_h, we obtain the following.

Proposition 4.1. *Let $f_1 \in S_{k_1}(N_1)$ and $f_2 \in S_{k_2}(N_2)$ be non-CM newforms with rational integral coefficients $a_1(n)$ and $a_2(n)$, respectively. Assume that f_1 and f_2*

are not character twists of each other. Let $N = \mathrm{lcm}(N_1, N_2)$ *and* $h \geq 1$ *be an integer. If GRH holds for the field* L_h, *then*

$$\pi_{f_1,f_2}(X, h) = \delta(h)\pi(X) + O(h^6 X^{1/2} \log(hNX)). \tag{4-2}$$

We remark that to establish Proposition 4.1, we need to use (3-17) and the following variation of a result of Hensel (see [16, Proposition 5, p. 129]):

$$\log d_{L_h} \leq \mathscr{A}_h \log(hN\mathscr{A}_h). \tag{4-3}$$

For our purpose, we now use Proposition 4.1 to obtain the following result giving an upper bound for the set of primes p with $a_1(p) = a_2(p)$. This may be also of independent interest.

Proposition 4.2. *Let* f_1 *and* f_2 *be newforms as before. Then under GRH*

$$\bigl|\{p \leq X : a_1(p) = a_2(p)\}\bigr| = O(X^{13/14}).$$

Proof. Clearly, for any prime ℓ,

$$\bigl|\{p \leq X : a_1(p) = a_2(p)\}\bigr| \leq \pi_{f_1,f_2}(X, \ell) + O(1).$$

Hence using Proposition 4.1, for a large prime ℓ,

$$\bigl|\{p \leq X : a_1(p) = a_2(p)\}\bigr| = O\left(\frac{\pi(X)}{\ell}\right) + O(\ell^6 X^{1/2} \log(\ell NX)).$$

Now by Bertrand's postulate, we choose a prime ℓ between $X^{1/14}/\log X$ and $2(X^{1/14}/\log X)$ and this proves the result. □

Note that in Proposition 4.2, GRH is used for the field L_ℓ, for all but finitely many primes ℓ.

We remark that for newforms of weight 2 and by making use of various abelian extensions, in [13, Theorem 10], a better estimate in Proposition 4.2 is obtained.

We now define

$$\pi^*_{f_1,f_2}(X, h) = \sum_{\substack{p \leq X \\ h \mid (a_1(p) - a_2(p)) \\ a_1(p) \neq a_2(p)}} 1. \tag{4-4}$$

Using Propositions 4.1 and 4.2 we deduce the following.

Proposition 4.3. *Let* f_1 *and* f_2 *be newforms as in Proposition 4.1 and* $h \geq 1$ *be an integer. If GRH holds for the field* L_h, *then*

$$\pi^*_{f_1,f_2}(X, h) = \delta(h)\pi(X) + O(h^6 X^{1/2} \log(hNX)) + O(X^{13/14}). \tag{4-5}$$

Remark 4.4. Indeed, the estimates given in Propositions 4.1 and 4.3 are also valid for the set of primes $p \le X$ with $h \mid (a_1(p) + a_2(p))$. This can be achieved by considering the set $\mathscr{C}'_h = \{(A, B) \in \mathscr{A}_h : \operatorname{tr}(A) = -\operatorname{tr}(B)\}$ instead of $\mathscr{C}_h$ in Section 3 and following the same arguments.

Remark 4.5. In the above propositions, if one assumes Artin's holomorphy conjecture in addition to GRH, then an improved error term can be obtained. More precisely, in Propositions 4.1 and 4.3, we have $O(h^3 X^{1/2} \log(hNX))$ instead of $O(h^6 X^{1/2} \log(hNX))$ which gives the estimate for Proposition 4.2

$$\bigl|\{p \le X : a_1(p) = a_2(p)\}\bigr| = O(X^{7/8}). \tag{4-6}$$

5. Sieving tool: Richert's weighted one-dimensional sieve form

We will prove Theorem 1.1 by using a suitably weighted sieve due to Richert [4]. The sieve problem we encounter here is a one-dimensional sieve problem in the parlance of "sieve methods". We will use notation and conventions from [4].

Let $\mathcal{A}$ be a finite set of integers not necessarily positive or distinct. Let $\mathcal{P}$ be an infinite set of prime numbers. For each prime $\ell \in \mathcal{P}$, let $\mathcal{A}_\ell := \{a \in \mathcal{A} : a \equiv 0 \pmod{\ell}\}$. We write

$$|\mathcal{A}| = X + r_1 \quad \text{and} \quad |\mathcal{A}_\ell| = \delta(\ell) X + r_\ell, \tag{5-1}$$

where X (resp. $\delta(\ell) X$) and r_1 (resp. r_ℓ) are a close approximation and remainder to $\mathcal{A}$ (resp. $\mathcal{A}_\ell$), respectively. For a square free integer d composed of primes of $\mathcal{P}$, let

$$\mathcal{A}_d = \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}, \quad \delta(d) = \prod_{\ell \mid d} \delta(\ell) \quad \text{and} \quad r_d = |\mathcal{A}_d| - \delta(d) X.$$

Notice that the function δ depends on both $\mathcal{A}$ and $\mathcal{P}$. For a real number $z > 0$, let

$$P(z) = \prod_{\ell \in \mathcal{P}, \ell < z} \ell \quad \text{and} \quad W(z) = \prod_{\ell \in \mathcal{P}, \ell < z} (1 - \delta(\ell)).$$

Hypothesis 5.1 [4, p. 29, 142, 219]. *For the above setup, we now state a series of hypotheses.*

Ω_1: *There exists a constant* $A_1 > 0$ *such that*

$$0 \le \delta(\ell) \le 1 - \frac{1}{A_1} \quad \textit{for all } \ell \in \mathcal{P}.$$

$\Omega_2(1, L)$: *If* $2 \le w \le z$, *then*

$$-L \le \sum_{w \le \ell \le z} \delta(\ell) \log \ell - \log \frac{z}{w} \le A_2,$$

where $A_2 \ge 1$ *and* $L \ge 1$ *are some constants independent of* z *and* w.

$R(1,\alpha)$: *There exist* $0<\alpha<1$ *and* $A_3, A_4 \geq 1$ *such that, for* $X \geq 2$,

$$\sum_{d \leq \frac{X^{\alpha}}{(\log X)^{A_3}}} \mu(d)^2\, 3^{\omega(d)}\, |r_d| \leq A_4 \frac{X}{(\log X)^2}.$$

For $\mathcal{A}$ and $\mathcal{P}$ as above and for real numbers u, v and λ with $u \leq v$, define the weighted sum

$$\mathcal{W}(\mathcal{A},\mathcal{P},v,u,\lambda) = \sum_{\substack{a\in\mathcal{A}\\(a,P(X^{1/v}))=1}} \Bigg(1 - \sum_{\substack{X^{1/v}\leq q<X^{1/u}\\ q\mid a,\, q\in\mathcal{P}}} \lambda\bigg(1-u\frac{\log q}{\log X}\bigg)\Bigg). \tag{5-2}$$

We now state the following form of Richert's weighted one-dimensional sieve.

Theorem 5.2 [4, Theorem 9.1, Lemma 9.1]. *With notation as above, assume that Hypothesis 5.1 for* Ω_1, $\Omega_2(1,L)$ *and* $R(1,\alpha)$ *hold for suitable constants L and* α. *Suppose further that there exists* $u, v, \lambda \in \mathbb{R}$ *and* $A_5 \geq 1$ *such that*

$$\frac{1}{\alpha} < u < v, \quad \frac{2}{\alpha} \leq v \leq \frac{4}{\alpha}, \quad 0<\lambda<A_5.$$

Then

$$\mathcal{W}(\mathcal{A},\mathcal{P},v,u,\lambda) \geq XW(X^{1/v})\bigg(F(\alpha,v,u,\lambda) - \frac{cL}{(\log X)^{1/14}}\bigg),$$

where c is a constant depends at most on u and v (*as well as on the* A_i*'s and* α) *and*

$$F(\alpha,v,u,\lambda) = \frac{2e^{\gamma}}{\alpha v}\bigg(\log(\alpha v-1) - \lambda\alpha u\log\frac{v}{u} + \lambda(\alpha u-1)\log\frac{\alpha v-1}{\alpha u-1}\bigg). \tag{5-3}$$

Here γ *is Euler's constant and X is the approximation of* $\mathscr{A}$ *given in* (5-1).

6. Proof of Theorem 1.1

We shall closely follow the arguments of [6]. The idea is to apply Theorem 5.2 to the situation

$$\mathcal{A} := \big\{|a_1(p)-a_2(p)| : p \leq X,\, a_1(p)\neq a_2(p)\big\} \quad \text{and} \quad \mathcal{P} := \{\ell : \ell \geq M\},$$

where $M = M(f_1, f_2)$ is the constant in Section 3. It is clear that, for any $\ell \in \mathcal{P}$,

$$|\mathcal{A}_\ell| = \big|\big\{p \leq X : a_1(p)\neq a_2(p),\, \ell \mid (a_1(p)-a_2(p))\big\}\big| = \pi^*_{f_1,f_2}(X,\ell).$$

Applying Proposition 4.3, under GRH, we obtain

$$|\mathcal{A}_\ell| = \delta(\ell)\frac{X}{\log X} + r_\ell,$$

where $r_\ell = O(\ell^6 X^{1/2}\log(\ell N X)) + O(X^{13/14})$. If d is a square free integer composed of primes from $\mathcal{P}$, then from Propositions 3.7 and 4.3 we have

$$\delta(d) = \prod_{\ell \mid d, \ell \in \mathcal{P}} \delta(\ell) \quad \text{and} \quad r_d = O(d^6 X^{1/2}\log(dNX)) + O(X^{13/14}). \tag{6-1}$$

To apply Theorem 5.2, we now verify that hypotheses Ω_1, $\Omega_2(L, 1)$ and $R(1, \alpha)$, given in Hypothesis 5.1, hold for our choice of $\mathcal{A}$ and $\mathcal{P}$.

Lemma 6.1. *Let f_1 and f_2 be newforms as before. Then we have the following*:

(1) *Hypothesis Ω_1 holds with a suitable A_1.*

(2) *Hypothesis $\Omega_2(1, L)$ holds with a suitable L.*

(3) *Under GRH, the hypothesis $R(1, \alpha)$ holds with any $\alpha < \frac{1}{14}$.*

Proof. By Proposition 3.6 the validity of hypotheses Ω_1 and $\Omega_2(1, L)$ are immediate because if $\ell \in P$ then $\delta(\ell) \sim \frac{1}{\ell}$ and this proves hypothesis Ω_1 while the latter one can be achieved by using Mertens's theorem (see [6, Lemmas 4.6.1, 4.6.2, 4.6.3]). So we only give a proof of part (3). From [5, p. 260], we know that $3^{\omega(n)} \le d(n)^{3\log 3/\log 2} \ll n^{\epsilon}$. Therefore, for any positive constant A_3, from (6-1), we have

$$\sum_{d \le \frac{X^\alpha}{(\log X)^{A_3}}} \mu(d)^2\, 3^{\omega(d)}\, |r_d| \ll \sum_{d \le \frac{X^\alpha}{(\log X)^{A_3}}} (d^{6+\epsilon} X^{1/2}\log(dNX) + X^{13/14}).$$

We now see that, for any $\alpha < \frac{1}{14}$,

$$\sum_{d \le \frac{X^\alpha}{(\log X)^{A_3}}} \mu(d)^2\, 3^{\omega(d)}\, |r_d| \ll \frac{X}{(\log X)^2}$$

and this completes the proof. □

Next we need to choose sieve parameters α, u, v, λ satisfying conditions in Theorem 5.2. For $k \ge 2$ we take

$$\alpha = \frac{k-1}{14k}, \quad u = \frac{14k+1}{k-1}, \quad v = \frac{56k}{k-1}, \quad \lambda = \frac{1}{k^{1/5}}. \tag{6-2}$$

Clearly, $\frac{1}{\alpha} < u < v$, $\frac{2}{\alpha} \le v \le \frac{4}{\alpha}$ and $0 < \lambda < 1$. This shows that these parameters satisfy the conditions required for applying Theorem 5.2, and hence for our choices of $\mathcal{A}$ and $\mathcal{P}$, we obtain

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) \gg \frac{X}{(\log X)^2}\left(F(\alpha, v, u, \lambda) - \frac{cL}{(\log X)^{1/14}}\right).$$

Note that here we have used the fact that $|\mathcal{A}| \gg X/\log X$ and $W(X) \gg 1/\log X$ for $X \gg 0$ which follows immediately by using Proposition 3.6. Also for the choices of sieve parameters α, u, v, λ given in (6-2), the function $F(\alpha, v, u, \lambda)$, defined by (5-3), can be computed explicitly and is given by

$$F\Big(\frac{k-1}{14k}, \frac{56k}{k-1}, \frac{14k+1}{k-1}, \frac{1}{k^{1/5}}\Big) = \frac{e^{\gamma}\big(14k^{6/5}\log 3+\log 42k-(1+14k)\log\big(\frac{56k}{14k+1}\big)\big)}{28k^{6/5}}.$$

Also, $F(\alpha, v, u, \lambda) > 0$ for $k > 1.71\ldots$. Therefore for a fixed weight $k \geq 2$ one can choose X, sufficiently large, such that $F(\alpha, v, u, \lambda) - (cL)/(\log X)^{1/14} > 0$. In other words, we have

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) \gg \frac{X}{(\log X)^2}. \tag{6-3}$$

There are at least $X/(\log X)^2$ many primes $p \leq X$ which make a positive contribution to the left-hand side of (6-3). Therefore to complete the proof of the first part of Theorem 1.1 it is sufficient to show that, for any such prime p,

$$\omega(a_1(p) - a_2(p)) \leq \big[7k + \tfrac{1}{2} + k^{1/5}\big].$$

Let p be such a prime. Then $(a_1(p) - a_2(p), X^{1/v}) = 1$ and

$$1 - \sum_{\substack{X^{1/v} \leq q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} \lambda\bigg(1 - u\frac{\log q}{\log X}\bigg) > 0. \tag{6-4}$$

Therefore, we write

$$\omega(a_1(p) - a_2(p)) = \sum_{q \mid (a_1(p)-a_2(p))} 1 = \sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1 + \sum_{\substack{q \geq X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1. \tag{6-5}$$

Now to estimate the first sum on the right of (6-5) we use (6-4) and obtain

$$\sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1 < \frac{1}{\lambda} + u \sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} \frac{\log q}{\log X}.$$

For the second sum we observe that if $q \geq X^{1/u}$ then $\log q/\log X \geq \frac{1}{u}$ that gives

$$\sum_{\substack{q \geq X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1 \leq u \sum_{\substack{q \geq X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} \frac{\log q}{\log X}.$$

Substituting the last two inequalities in (6-5) yields

$$\omega(a_1(p) - a_2(p)) \leq \frac{1}{\lambda} + u \sum_{q \mid (a_1(p)-a_2(p))} \frac{\log q}{\log X} \leq \frac{1}{\lambda} + u\frac{\log |a_1(p) - a_2(p)|}{\log X}.$$

Using Deligne's estimate we know $|a_1(p) - a_2(p)| \le 4p^{(k-1)/2}$. Therefore for any $p \le X$ as above, we have

$$\omega(a_1(p) - a_2(p)) \le \frac{1}{\lambda} + u\frac{k-1}{2} + u\frac{\log 4}{\log X}.$$

Substituting the values of u and λ from (6-2) and choosing X large enough completes the proof of the first part of the theorem.

Finally, the last assertion of the theorem when $k \ge 6$ can be achieved by taking $\lambda = 1/\sqrt{\log k}$ instead of $\lambda = 1/k^{1/5}$ in the above proof and then following the same arguments.

7. Proof of Theorem 1.6

The idea of the proof is similar to the proof of Theorem 1.1 with minor modifications. We shall apply Theorem 5.2 with the same setting as in Section 6. For $k \ge 2$, we choose the sieve parameters as

$$\alpha = \frac{k-1}{14k}, \quad u = \frac{26k+1}{k-1}, \quad v = \frac{30k}{k-1}, \quad \lambda = \frac{1}{\sqrt{\log k}}.$$

Again, these parameters satisfy the conditions required for Theorem 5.2 and the corresponding function $F(\alpha, v, u, \lambda) > 0$ for $k > 1.006$. Hence as in the proof of Theorem 1.1, the corresponding weighted sum satisfies

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) \gg \frac{X}{(\log X)^2}. \tag{7-1}$$

Next we observe that

$$\left|\left\{p \le X : \ell^2 \mid (a_1(p) - a_2(p)),\, X^{1/v} \le \ell \le X^{1/u}\right\}\right| = \sum_{X^{1/v} \le \ell \le X^{1/u}} (\pi_{f_1, f_2}(X, \ell^2) + O(1)),$$

where the error term is due to the presence of those primes p such that $p \mid \ell N$ and $\ell^2 \mid (a_1(p) - a_2(p))$. Applying Proposition 4.1 gives that the left side of the above equality is equal to

$$\pi(X) \sum_{X^{1/v} \le \ell \le X^{1/u}} \frac{1}{\ell^2} + O\left(X^{1/2+\epsilon} \sum_{X^{1/v} \le \ell \le X^{1/u}} \ell^{12}\right).$$

Since $u > 26$, we have

$$\left|\left\{p \le X : \ell^2 \mid (a_1(p) - a_2(p)),\, X^{1/v} \le \ell \le X^{1/u}\right\}\right| = o\left(\frac{X}{(\log X)^2}\right). \tag{7-2}$$

We conclude, by combining (7-1) and (7-2), that there are at least $X/(\log X)^2$ many primes $p \le X$ such that

(a) $a_1(p) - a_2(p)$ does not have any prime divisors less than $X^{1/v}$,

(b) for primes $\ell \mid (a_1(p) - a_2(p))$ with $X^{1/v} < \ell < X^{1/u}$, $\ell^2 \nmid (a_1(p) - a_2(p))$,

(c) the contribution of p to the sifting function $\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda)$ is positive, i.e.,

$$1 - \sum_{\substack{X^{1/v} \le q < X^{1/u} \\ q \| (a_1(p) - a_2(p))}} \lambda \left(1 - u \frac{\log q}{\log X}\right) > 0.$$

In order to complete the proof, we will show that if $p \le X$ is a prime satisfying the three conditions above then

$$\Omega(a_1(p) - a_2(p)) \le \left[13k + \tfrac{1}{2} + \sqrt{\log k}\,\right].$$

Let $p \le X$ be a prime satisfying (a), (b) and (c). Then, as in the proof of Theorem 1.1,

$$\Omega(a_1(p) - a_2(p)) = \sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \| (a_1(p) - a_2(p))}} 1 + \sum_{\substack{q \ge X^{1/u} \\ q^m | (a_1(p) - a_2(p))}} 1 < \frac{1}{\lambda} + u \sum_{q^m | (a_1(p) - a_2(p))} \frac{\log q}{\log X}$$

which gives

$$\Omega(a_1(p) - a_2(p)) \le \frac{1}{\lambda} + u \frac{\log |a_1(p) - a_2(p)|}{\log X}.$$

Now applying Deligne's estimate and arguing as in the proof of Theorem 1.1, we get the desired result.

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