Pacific Journal of Mathematics

A NOTE ON THE DISTINCT DISTANCES PROBLEM IN THE HYPERBOLIC PLANE

ZHIPENG LU AND XIANCHANG MENG

Volume 327 No. 1 November 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 327, No. 1, 2023
https://doi.org/10.2140/pjm.2023.327.129

A NOTE ON THE DISTINCT DISTANCES PROBLEM IN THE HYPERBOLIC PLANE

ZHIPENG LU AND XIANCHANG MENG

We provide a proof of a Guth–Katz-type lower bound for the distinct distances problem in the hyperbolic plane. Our construction follows the framework of Guth and Katz to deal with $\mathrm{PSL}_2(\mathbb{R})$ and the corresponding incidence structure in projective geometry. In addition, we deduce a new sum-product estimate in the form of a hyperbolic metric formula based on this lower bound.

1. Introduction

The distinct distances problem was first proposed by Erdős [3] in the Euclidean plane. He conjectured the lower bound $\gtrsim N/\sqrt{\log N}$ for the number of distinct distances between pairs of points among N points in the plane. (Here $A \gtrsim B$ means $A \geq cB$ for some absolute constant $c > 0$.) After a half-century of progression with partial results, there came the major breakthrough by Guth and Katz [4] who proved the nearly optimal bound $\gtrsim N/\log N$. Foremostly they invented the tool of polynomial partitioning and promoted profound applications in incidence geometry and other areas, later developed by themselves and many other authors; for instances, see [1; 6].

In this paper, we deal with the distinct distances problem in the hyperbolic plane $\mathbb{H}^2$ and prove the nearly optimal bound in equivalent strength with [4]. Following an idea of Tao's blog [11], Rudnev and Selig [9] described a proof using the Klein quadric in Plüker coordinates without exploiting symmetries in the hyperbolic plane. By contrast, following the framework of Elekes and Sharir, as in [4], we give an independent proof by carefully studying isometries of $\mathbb{H}^2$ in a more Guth–Katz ethnic language. More specifically, we prove:

Theorem 1.1. *For any set $P \subset \mathbb{H}^2$ of N points, we have*

$$|\{d_{\mathbb{H}^2}(p, q),\, p, q \in P\}| \gtrsim N/\log N,$$

where $|A|$ denotes the cardinality of a set A and $d_{\mathbb{H}^2}$ denotes the hyperbolic metric on $\mathbb{H}^2$.

MSC2020: 11P70, 52C10.

Keywords: Erdős distinct distances, hyperbolic geometry, sum-product.

In the case of the Euclidean plane, Guth and Katz [4] used the framework of Elekes and Sharir [2] to reduce the distinct distances problem to an incidence problems of lines, then derived the lower bound resorting to ruled surface theory and polynomial partitioning. Elekes and Sharir's framework serves as a realization of the Erlangen program (see [7] for historical background) for the distinct distances problem in the Euclidean plane. However, this framework cannot apply directly to the case of the hyperbolic plane. For the hyperbolic plane $\mathbb{H}^2$, we consider its isometry group $\mathrm{PSL}_2(\mathbb{R})$. Distinguished from Guth and Katz's coordinate of lines, our lines lie in $\mathbb{P}^3$ rather than $\mathbb{R}^3$. We need further linearizations to reduce our coordinate of lines to $\mathbb{R}^3$. Subsequently we need to overcome the difficulty of constructing vector fields in order to use ruled surface theory. See Section 2 for details.

In addition, we deduce a new sum-product-type result using Theorem 1.1. For any finite sets $A \subset \mathbb{R}\backslash\{0\}$, $B \subset \mathbb{R}$, define $P = \{b + i|a| : a \in A,\ b \in B\}$, and $P' = \{-b + i|a| : a \in A,\ b \in B\}$. Note that explicitly we have the hyperbolic distance formula

$$2\cosh d_{\mathbb{H}^2}(x_1 + iy_1, x_2 + iy_2) = \frac{(x_1 - x_2)^2 + y_1^2 + y_2^2}{y_1 y_2}$$

and $|\{|x| : x \in E\}| \geq \frac{1}{2}|E|$ for any finite set $E \subset \mathbb{R}$. By applying Theorem 1.1 to P and P', we get:

Theorem 1.2. *Let $A \subset \mathbb{R}\backslash\{0\}$, $B \subset \mathbb{R}$ be finite sets. Then we have*

$$\left|\left\{\frac{a_1^2 + a_2^2 + (b_1 - b_2)^2}{a_1 a_2} : a_1, a_2 \in A,\ b_1, b_2 \in B\right\}\right| \gtrsim \frac{|A|\,|B|}{\log(|A|) + \log(|B|)},$$

and

$$\left|\left\{\frac{a_1^2 + a_2^2 + (b_1 + b_2)^2}{a_1 a_2} : a_1, a_2 \in A,\ b_1, b_2 \in B\right\}\right| \gtrsim \frac{|A|\,|B|}{\log(|A|) + \log(|B|)}.$$

By adding or subtracting 2 on the elements in the above sets, the factor $a_1^2 + a_2^2$ can be replaced by $(a_1 + a_2)^2$ or $(a_1 - a_2)^2$.

Remark 1. In particular, if $|A|$ and $|B|$ are all about the size $\asymp N$, the above lower bounds become $\gtrsim N^2/\log N$.

A variant of the distinct distances problem has been previously used by Roche-Newton and Rudnev [8] to study sum-product-type estimates. See also the work of Jones [5] for estimates of other sum-product-types using incidence geometry. Very recently, Sheffer and Zahl [10] derived a sum-product-type estimate for complex numbers.

2. Proof of Theorem 1.1

We use Elekes and Sharir's framework to reduce the counting of distinct distances to an incidence problem of lines in the real projective space $\mathbb{P}^3$. To overcome the difficulty of linearizing projective lines in $\mathbb{P}^3$, we turn the incidence of lines in $\mathbb{P}^3$ into that of lines in $\mathbb{R}^3$ by certain conjugation. Then fulfilling the requirements for our lines in $\mathbb{R}^3$ as Guth and Katz in Proposition 2.8 of [4] amounts to a more concrete proof of the lower bound $\gtrsim N/\log N$ of distinct distances among N points in $\mathbb{H}^2$.

Framework. Let $\mathbb{H}^2$ be the hyperbolic plane and $G = \mathrm{PSL}_2(\mathbb{R})$ be its isometry group which acts on $\mathbb{H}^2$ by Möbius transformation:

$$z \mapsto \gamma \cdot z = \frac{az+b}{cz+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}),\ z \in \mathbb{H}^2.$$

Let $P \subset \mathbb{H}^2$ be a set of N points and define the set of *distance quadruples*

$$Q(P) := \{(p_1, p_2, p_3, p_4) \in P^4 : d(p_1, p_2) = d(p_3, p_4) \neq 0\}, \tag{1}$$

where $d(\cdot\,,\cdot)$ denotes the hyperbolic metric. Denote the distance set by

$$d(P) := \{d(p_1, p_2) : p_1 \neq p_2 \in P\}.$$

Then we have a close relation between $d(P)$ and $Q(P)$ as follows. Suppose $d(P) = \{d_i : 1 \le i \le m\}$ and n_i is the number of pairs of points in P with distance d_i. So $|Q(P)| = \sum_{i=1}^m n_i^2$. Since $\sum_{i=1}^m n_i = 2\binom{N}{2} = N^2 - N$, by Cauchy–Schwarz inequality we get

$$(N^2 - N)^2 = \left(\sum_{i=1}^m n_i\right)^2 \le \left(\sum_{i=1}^m n_i^2\right) m = |Q(P)|\,|d(P)|.$$

Rearranging the inequality gives

$$|d(P)| \ge \frac{N^4 - 2N^3}{|Q(P)|}. \tag{2}$$

Any quadruple $(p_1, p_2, p_3, p_4) \in Q(P)$ uniquely determines an isometry $g \in G$ such that $g(p_1) = p_3$, $g(p_2) = p_4$. Suppose $p_1 = x + iy$, $p_3 = x' + iy' \in \mathbb{H}^2$ $(y, y' > 0)$ and there is some $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ such that

$$A \cdot (x + iy) = \frac{a(x+iy)+b}{c(x+iy)+d} = x' + iy',$$

for $i = \sqrt{-1}$. Rearranging terms we get

$$ax + b + iay = cxx' + dx' - cyy' + i(cxy' + dy' + cx'y),$$

or equivalently the system of linear equations

$$
\begin{aligned}
xa+b+(yy'-xx')c-x'd&=0,\\
ya-(xy'+x'y)c-y'd&=0.
\end{aligned}
\tag{3}
$$

Its solution set in $\mathbb{R}^4$ is the intersection of two distinct hyperplanes, which turns out to be a two-dimensional plane passing through the origin. If, in addition, $A\cdot p_2=p_4$, the point (a,b,c,d) also lies in another distinct two-dimensional plane intersecting the above plane at a line since $p_1\neq p_2$, $p_3\neq p_4$ as follows.

Lemma 2.1. *The equations of (3) determine a unique dimension-2 hyperplane in $\mathbb{R}^4$ for each distinct pair of points in $\mathbb{H}^2$. In particular, any quadruple $(p_1,p_2,p_3,p_4)\in Q(P)$ determines a unique isometry.*

Proof. A fairly complicated elementary computation on 4×4 matrices derived from (3) allows us to see this, but here we prove it by geometric arguments.

First, a nonidentity real Möbius transformation can have at most one fixed point in $\mathbb{H}^2$, since $\frac{az+b}{cz+d}=z$ implies $cz^2+(d-a)z-b=0$ which has 1 or no roots in $\mathbb{H}^2$ for real coefficients. If two isometries $\gamma_1,\gamma_2\in \mathrm{PSL}_2(\mathbb{R})$ satisfy $\gamma_i\cdot p_1=p_3$ and $\gamma_i\cdot p_2=p_4$, then $\gamma_1^{-1}\gamma_2$ fixes both p_1 and p_2, a contradiction ($p_1\neq p_2$). This is to say a quadruple in $Q(P)$ determines at most one isometry, or equivalently, two systems of equations for two pairs of points as in (3) define different planes that intersect on at most one line.

Then we verify the existence of solution. Since $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2$ transitively (which can also be seen from (3)), let $\gamma_j\cdot i=p_j$, $j=1,\dots,4$. Then

$$
\gamma\cdot p_1=p_3,\quad \gamma\cdot p_2=p_4 \quad\iff\quad \gamma_3^{-1}\gamma\gamma_1\cdot i=i,\quad \gamma_4^{-1}\gamma\gamma_2\cdot i=i.
$$

For $i=(0,1)$, (3) simply becomes

$$
\begin{aligned}
b+c&=0,\\
a-d&=0.
\end{aligned}
$$

Let its solution plane be π; then the desired solution set of γ is $\gamma_3\pi\gamma_1^{-1}\cap\gamma_4\pi\gamma_2^{-1}=\gamma_3(\pi\cap\gamma_3^{-1}\gamma_4\pi\gamma_2^{-1}\gamma_1)\gamma_1^{-1}$. Note that $d(i,\gamma_2^{-1}\gamma_1\cdot i)=d(\gamma_2\cdot i,\gamma_1\cdot i)=d(p_2,p_1)=d(p_4,p_3)=d(\gamma_4\cdot i,\gamma_3\cdot i)=d(i,\gamma_4^{-1}\gamma_3\cdot i)$. Hence there exists a rotation $\gamma\in\pi$ about i that transfers $\gamma_2^{-1}\gamma_1\cdot i$ to $\gamma_4^{-1}\gamma_3\cdot i$, that is, $\gamma\gamma_2^{-1}\gamma_1\cdot i=\gamma_4^{-1}\gamma_3\cdot i$, or $\gamma_3^{-1}\gamma_4\gamma\gamma_2^{-1}\gamma_1\cdot i=i$. This is to say

$$
\gamma\in\pi\cap\gamma_3^{-1}\gamma_4\pi\gamma_2^{-1}\gamma_1,
$$

so that $\gamma_3^{-1}\gamma_4\pi\gamma_2^{-1}\gamma_1\neq\varnothing$ and then the desired solution set $\gamma_3\pi\gamma_1^{-1}\cap\gamma_4\pi\gamma_2^{-1}$ is not empty. $\square$

Thus all (a, b, c, d) lying in the intersection line of two planes defined by (3) in $\mathbb{R}^4$ project to a single point as $[a : b : c : d] \in \mathbb{P}^3$. This gives a map $E : Q(P) \to G$. Define, for any $p, q \in \mathbb{H}^2$,

$$S_{pq} := \{g \in G : g(p) = q\},$$

which are one-dimensional curves in G. Similar to [4, Lemmas 2.4 and 2.6], we have

(i) if $|P \cap gP| = k$, then $|E^{-1}(g)| = 2\binom{k}{2}$;

(ii) and $|P \cap gP| \geq k$ if and only if g lies in at least k of the curves $\{S_{pq}\}_{p,q\in P}$.

Thus we derive that

$$|Q(P)| = \sum_{k=2}^{N} 2\binom{k}{2} |\{g : |P \cap gP| = k\}| \lesssim \sum_{k=2}^{N} k\, |G_k(P)|, \tag{4}$$

where $G_k(P) \subset G$ consists of $g \in G$ with $|P \cap gP| \geq k$. Henceforth we focus on estimating $|G_k(P)|$ for $k = 2$ and $k \geq 3$ as in Sections 3 and 4 of [4].

Incidence of projective lines in $\mathbb{P}^3$**.** For any $g \in G$, we have $d(gp, gq) = d(p, q)$ so that shifting P to gP does not affect counting of distinct distances. Now for a quadruple $(p_1, p_2, p_3, p_4) \in Q(P)$, suppose $E((p_1, p_2, p_3, p_4)) = h$, i.e., $hp_1 = p_3$, $hp_3 = p_4$. After shifting we get

$$E((gp_1, gp_2, gp_3, gp_4)) = ghg^{-1}.$$

In the matrix form of G, we manage to reshape the distance quadruples as follows.

Proposition 2.2. *For any finite set of points* $P \subset \mathbb{H}^2$, *there is an isometry* $g \in \mathrm{PSL}_2(\mathbb{R})$ *such that all matrices in* $E(Q(gP))$ *have nonvanishing upper-left corners.*

Proof. We use translations $T_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in \mathbb{R}$. For any $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ we calculate that

$$T_x h T_x^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+cx & -cx^2+(d-a)x+b \\ c & d-cx \end{pmatrix}.$$

Suppose $E(Q(P))$ consists of $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$, $1 \leq i \leq K$. Note that a_i and c_i cannot be both zero, we choose nonzero x such that $a_i + c_i x \neq 0$ for all $i = 1, \ldots, K$. For such x we have $E(Q(T_x P)) = T_x E(Q(P)) T_x^{-1}$ consisting of matrices with nonvanishing upper-left corners. $\square$

Remark 2. For any finite set of points in the upper-half plane, we may also dilate points by hyperbolic isometries so that they all have sufficiently large absolute values. Note that a Möbius transformation $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \cdot z = \frac{b}{cz+d}$ basically inverts the absolute value of z, so that it cannot map z with large absolute values to points with large absolute values. Thus after dilation, Möbius transformations with vanishing upper-left corners do not occur as isometries in consideration.

Hence without loss of generality, we assume $p_x,\ p_y,\ q_x,\ q_y \gg 1$ for points $p = px + ip_y$, $q = q_x + q_y$ in consideration, that is, far away in the first quadrant. We have the following observation through (3). First, each S_{pq} is a projective line in $\mathbb{P}^3 \supset G = \mathrm{PSL}_2(\mathbb{R})$. We use the natural manifold atlas

$$\mathbb{P}^3 = \mathbb{R}^3_1 \cup \mathbb{R}^3_2 \cup \mathbb{R}^3_3 \cup \mathbb{R}^3_4,$$

with $\mathbb{R}^3_1 = \{[1 : b : c : d] \mid b, c, d \in \mathbb{R}\} \simeq \mathbb{R}^3$ and $\mathbb{R}^3_i \simeq \mathbb{R}^3$, $i = 2, 3, 4$, similarly defined with i-th entry equal to 1 in the projective coordinate. Analogously we use

$$G = \bigcup_{i=1}^{4} G_i, \quad G_i = \mathrm{PSL}_2(\mathbb{R}) \cap \mathbb{R}^3_i.$$

In particular, G_1 consists of matrices with nonvanishing upper-left corners. Then the restriction $S_{pq} \cap G_i$ becomes a real line in $\mathbb{R}^3_i$, and by Proposition 2.2, there exists $g \in G$ such that $G_k(gP) \subset G_1$ for each $k \geq 2$. Abusing notation, we always denote by L_{pq} the real line $S_{(gp)(gq)} \cap \mathbb{R}^3_1$ in the manifold atlas of $\mathbb{P}^3$. The incidences among curves S_{pq} are now equivalent to that of lines L_{pq} in $\mathbb{R}^3$ ($\mathbb{R}^3_1$). Explicitly L_{pq} has the following linear parametrization.

Proposition 2.3. *For any $p = p_x + ip_y$, $q = q_x + iq_y \in \mathbb{H}^2$, the line L_{pq} can be parametrized as*

$$\left(-\frac{q_y(p_x^2 + p_y^2) + p_y(q_x^2 + q_y^2)}{p_x q_y + q_x p_y}, \frac{p_y + q_y}{p_x q_y + q_x p_y}, 0\right) + t\left(\frac{p_y(q_x^2 + q_y^2)}{p_x q_y + q_x p_y}, -\frac{q_y}{p_x q_y + q_x p_y}, 1\right), \tag{5}$$

for $t \in \mathbb{R}$.

Proof. For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot p = q$ with $a = 1$ and $t = d + 1$ as parameter, we get from (3),

$$\begin{aligned} b &= -\frac{q_y(p_x^2 + p_y^2) + p_y(q_x^2 + q_y^2)}{p_x q_y + q_x p_y} + \frac{p_y(q_x^2 + q_y^2)}{p_x q_y + q_x p_y} t, \\ c &= \frac{p_y + q_y}{p_x q_y + q_x p_y} - \frac{q_y}{p_x q_y + q_x p_y} t, \end{aligned} \tag{6}$$

which gives us the parametrization of points $(b, c, t) \in L_{pq}$. □

Remark 3. There are other parametrizations of L_{pq}, say for $b = t$ as the parameter. Here the roles of p and q are symmetric in that the intersection of L_{pq} and L_{qp} is on the plane $t = 0$.

Since there are nonlinear terms in our parametrization, which is not a problem for Guth and Katz [4], we have to consider different families of lines that rule surfaces and the vector fields on reguli to get the following.

Proposition 2.4. *For any set of N points $P \subset \mathbb{H}^2_{>0} := \{x + iy : x, y > 0\}$ and $\mathcal{L} = \{L_{pq} : p, q \in P\}$, no more than N lines of $\mathcal{L}$ lie in a common plane and no more than $O(N)$ lines of $\mathcal{L}$ lie in a common regulus.*

Proof. We consider the families $L_q := \{L_{pq}\}_{p\in\mathbb{H}^2_{>0}}$ of lines targeting at q. First, for any $p' \neq p$, the line $L_{p'q}$ does not intersect L_{pq}. Note that $L_{pq} \subset S_{pq}$, and suppose $L_{pq} \cap L_{p'q} \neq \varnothing$. Then there would be some $g \in G$ such that $gp' = gp = q$, a contradiction. Moreover by (5), the directions of L_{pq} and $L_{p'q}$ are different:

$$\left(\frac{p_y(q_x^2 + q_y^2)}{p_x q_y + q_x p_y}, -\frac{q_y}{p_x q_y + q_x p_y}, 1\right) = (\xi_1, \xi_2, 1)$$

has a unique solution for fixed q and ξ_1, ξ_2. Thus different L_q's have no lines in common and belong to different rulings of a ruled surface if any. Note that ξ_1, ξ_2 cannot be zero since $p_x, p_y, q_x, q_y > 0$. Indeed, equivalently we have

$$\begin{pmatrix} -\xi_1 q_y & q_x^2 + q_y^2 - q_x\xi_1 \\ \xi_2 q_y & \xi_2 q_x \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} 0 \\ -q_y \end{pmatrix},$$

whose associate matrix has determinant $-(q_x^2 + q_y^2)\xi_2 q_y \neq 0$. Hence lines of L_q are pairwise skew and no two of its lines lie in a common plane. Therefore any plane intersects each L_q at most one line and intersects $\mathcal{L}$ at most N lines.

To prove the second part, we construct a vector field $V = (V_1, V_2, V_3)$ on $\mathbb{R}^3$ tangent to lines of L_q for any fixed $q = q_x + iq_y \in \mathbb{H}^2_{>0}$. By (3) we locate p such that L_{pq} passes through any given $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ as follows ($a = 1$, $x_1 = b$, $x_2 = c$, $x_3 = d$):

$$\begin{aligned} p_x + x_1 + (p_y q_y - p_x q_x)x_2 - q_x x_3 &= 0, \\ p_y - (p_x q_y + q_x p_y)x_2 - q_y x_3 &= 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} (1 - q_x x_2)p_x + (q_y x_2)p_y &= q_x x_3 - x_1, \\ (-q_y x_2)p_x + (1 - q_x x_2)p_y &= q_y x_3, \end{aligned}$$

which has solution

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \frac{1}{(1 - q_x x_2)^2 + q_y^2 x_2^2} \begin{pmatrix} q_x x_1 x_2 - (q_x^2 + q_y^2)x_2 x_3 - x_1 + q_x x_3 \\ -q_y x_1 x_2 + q_y x_3 \end{pmatrix}.$$

By (5), we set the direction of L_{pq} as

$$((q_x^2 + q_y^2)p_y, -q_y, q_y p_x + q_x p_y) = \frac{1}{(1 - q_x x_2)^2 + q_y^2 x_2^2}(V_1, V_2, V_3),$$

where

$$\begin{aligned} V_1 &= -q_y(q_x^2+q_y^2)(x_1x_2-x_3),\\ V_2 &= -q_y[(1-q_xx_2)^2+q_y^2x_2^2],\\ V_3 &= -q_y(q_x^2+q_y^2)x_2x_3-q_yx_1+2\,q_xq_yx_3. \end{aligned}$$

Let $V=(V_1, V_2, V_3)$; then V has degree 2. Note that $p\in\mathbb{H}^2_{>0}$, the vector field is defined over the open subset

$$U_q := \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid q_xx_1x_2-(q_x^2+q_y^2)x_2x_3-x_1+q_xx_3>0,\ -q_yx_1x_2+q_yx_3>0\},$$

and we always consider the pieces of reguli restricted in U_q.

Now suppose a line L_{pq} lies in a regulus R defined by a degree-2 irreducible polynomial f in $\mathbb{R}^3$. Then at any point $x\in L_{pq}$ we have the Taylor expansion

$$f(x+tV(x)) = f(x)+\nabla(f)\cdot V(x)t+\tfrac{1}{2}V^TH(f)Vt^2,$$

where $\nabla(f)$ is the gradient of f and $H(f)$ is the Hessian matrix of f.

By Bezout's lemma (Lemma 3.1 of [4]), if more than 9 lines of L_q are contained in R, f would have a common factor with both $\nabla(f)\cdot V$ and $V^TH(f)V$, which have degree 3 and 4, respectively. By irreducibility, f must be the common factor so that f vanishes on each line of L_q with direction $V(x)$ for any $x\in R$ by the Taylor expansion above, that is, L_q is a ruling of R. Since a regulus has only two rulings, R can only contain at most 8 lines from $N-2$ families L_q which are not rulings of R and $2N$ lines of L_{q_1}, L_{q_2} if they are rulings of R. In total, there are at most $2N+8(N-1)=10N-8$ lines of $\mathcal{L}$ lying in R. □

Now we already reduced the problem to incidence geometry in the Euclidean space. Applying ruled surface theory and polynomial partitioning to reproduce Guth and Katz's Theorem 2.10 and 2.11 of [4], we get the following lower bound for the distinct distances problem in the hyperbolic plane. It has the same strength as the result of Guth and Katz for the Euclidean plane.

Theorem 2.5. *For $P\subset\mathbb{H}^2$ any set of N points and $\mathcal{L}=\{L_{pq}\mid p,q\in P\}$, let G_k be the set of points where at least k lines of $\mathcal{L}$ meet for $2\le k\le N$. Then*

$$|G_k|\lesssim N^3k^{-2}.$$

Consequently, by (4), $|Q(P)|\lesssim N^3\log N$, and by (2), we have $|d(p)|\gtrsim N/\log N$.

Acknowledgements

We are thankful for helpful discussions with Harald Helfgott. We are also thankful for the comments from Misha Rudnev. Both authors were partially supported by the Humboldt Professorship of Professor Helfgott while completing the work.

Currently, Lu is supported by Guangdong Basic and Applied Basic Research Foundation (No. 2021A1515110206) and 2023 Shenzhen National Science Foundation (No. 20231128220938001). Meng is supported by the National Natural Science Foundation of China (Grant No. 12201346) and Shandong Provincial Foundation (No. 2022HWYQ-046 & No. ZR2022QA001).

References

[1] X. Du, L. Guth, and X. Li, "A sharp Schrödinger maximal estimate in $\mathbb{R}^2$", *Ann. of Math.* (2) **186**:2 (2017), 607–640. MR Zbl

[2] G. Elekes and M. Sharir, "Incidences in three dimensions and distinct distances in the plane", *Combin. Probab. Comput.* **20**:4 (2011), 571–608. MR Zbl

[3] P. Erdős, "On sets of distances of n points", *Amer. Math. Monthly* **53**:5 (1946), 248–250. Zbl

[4] L. Guth and N. H. Katz, "On the Erdős distinct distances problem in the plane", *Ann. of Math.* (2) **181**:1 (2015), 155–190. MR Zbl

[5] T. G. F. Jones, "New results for the growth of sets of real numbers", *Discrete Comput. Geom.* **54**:4 (2015), 759–770. MR Zbl

[6] H. Kaplan, J. Matoušek, and M. Sharir, "Simple proofs of classical theorems in discrete geometry via the Guth–Katz polynomial partitioning technique", *Discrete Comput. Geom.* **48**:3 (2012), 499–517. MR Zbl

[7] F. Klein, "Vergleichende betrachtungen über neuere geometrische forschungen", *Math. Ann.* **43**:1 (1893), 63–100. Translated in *Bull. Amer. Math. Soc.* **2**:10 (1893), 215–249. MR Zbl

[8] O. Roche-Newton and M. Rudnev, "On the Minkowski distances and products of sum sets", *Israel J. Math.* **209**:2 (2015), 507–526. MR Zbl

[9] M. Rudnev and J. M. Selig, "On the use of the Klein quadric for geometric incidence problems in two dimensions", *SIAM J. Discrete Math.* **30**:2 (2016), 934–954. MR Zbl

[10] A. Sheffer and J. Zahl, "Distinct distances in the complex plane", *Trans. Am. Math. Soc.* **374**:9 (2021), 6691–6725. MR Zbl

[11] T. Tao, "Lines in the Euclidean group SE(2)", blog post, 2011, http://terrytao.wordpress.com/2011/03/05/lines-in-the-euclidean-group-se2/.

Received September 4, 2023. Revised January 19, 2024.

ZHIPENG LU
GUANGDONG LABORATORY OF MACHINE PERCEPTION AND INTELLIGENT COMPUTING
DEPARTMENT OF ENGINEERING
SHENZHEN MSU-BIT UNIVERSITY
SHENZHEN
CHINA
zhipeng.lu@hotmail.com

XIANCHANG MENG
SCHOOL OF MATHEMATICS
SHANDONG UNIVERSITY
JINAN
CHINA
xianchang.meng@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2023 is US $605/year for the electronic version, and $820/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 327 No. 1 November 2023
