Adem relations in the Dyer-Lashof algebra and modular invariants

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Dedicated to the memory of Professor F. P. Peterson

April 21, 2004

Abstract

This work deals with Adem relations in the Dyer-Lashof algebra from a modular invariant point of view. Our purpose is to give a moderate explanation of the complexity of Adem relations. An algorithm is provided which has two effects. Firstly, to calculate the hom-dual of an element in the Dyer-Lashof algebra; and secondly, to find the image of a non-admissible element after applying Adem relations. The advantage of our method is that one has to deal with polynomials instead of homology operations.

Category: Primary: 55S10, 13F20; Secondary: 55P10.
Keywords: Adem relations, Dyer-Lashof algebra, Dickson algebra, Borel invariants.

1 Introduction

The relationship between the (canonical sub-co-algebras) Dyer-Lashof algebra, $R[k]$ and the Dickson invariants $D[k]$ is well-known, see May's paper in [3], relevant parts of which will be quoted here. We provide an algorithm for calculating Adem relations in the Dyer-Lashof algebra using modular co-invariants. Much of our work involves the calculation of the hom-duals of elements of R in terms of the generators of the polynomial algebra $D[k]$. The results described here will be applied to give an invariant theoretic description of the mod $-p$ cohomology of a finite loop space in [6].

We note that the idea for our algorithm was inspired by May's theorem 3.7, page 29, in [3]. The key ingredient for relating homology operations and polynomial invariants is the relation between the map which imposes Adem relations and the decomposition map between certain rings of invariants. This relation was studied by Mui for $p = 2$ in [8], and we extend it here for any prime. Recall $R[n]$ is obtained as the quotient of a free associative algebra by imposing two conditions, one that elements of negative excess are 0 and the other that the Adem relations hold. $T[n]$ is the analogue obtained by imposing only the condition that elements of negative excess are 0.

Theorem 43 *Let $\rho : T[n] \to R[n]$ be the map which imposes Adem relations. Let $\hat{\imath} : S(E(n))^{GL_n} \otimes D[n] \hookrightarrow S(E(n))^{B_n} \otimes B[n]$ be the natural inclusion. Then $\rho^* \equiv \hat{\imath}$, i.e. for any $e_I \in T[n]$ and $d^m M^\varepsilon \in S(E(n))^{GL_n} \otimes D[n]$,*

$$< d^m M^\varepsilon, \rho(e_I) >=< \hat{\imath}(d^m M^\varepsilon), e_I > .$$

Campbell, Peterson and Selick studied self maps f of $\Omega_0^{m+1} S^{m+1}$ and proved that if f induces an isomorphism on $H_{2p-3}(\Omega_0^{m+1} S^{m+1}, Z/pZ)$, then $f_{(p)}$ is a homotopy equivalence for p odd and m even [2]. A key ingredient for their proof was the calculation of

$$AnnPH^*(\Omega_0^{m+1} S^{m+1}, Z/pZ)$$

They gave a convenient method for calculating the hom-dual of elements of $H_*(\Omega_0^{m+1} S^{m+1}, /p)$ which do not involve Bockstein operations. Our algorithm computes the hom-duals of elements of $R[n]$ in terms of the generators of the polynomial algebra $D[n]$. Please see Theorem 45 and a short description before Example 44.

A direct application of the last two theorems is the computation of Adem relations. The main difference between the classical and our approach is that we consider Adem relations globally instead of consecutive elements and it requires fewer calculations. This algorithm is described in Proposition 56 and a short description before Example 54.

The paper is purely algebraic and its applications are deferred to [6]. There are three sections in this paper beyond this introduction, sections 2, 3 and 4. Section 2 recalls well known facts about the Dyer-Lashof algebra from May's article, cited above. In section 3, the Dickson algebra and its relation with the ring of invariants of the Borel subgroup is examined. That relation is studied using a certain family of matrices which suitably summarizes the expressions for Dickson invariants in terms of the invariants of the Borel subgroup. In the view of the author, the complexity of Adem relations is reflected in the different ways in which the same monomial in the generators of the Borel subgroup can show up as a term in a Dickson invariant. The ways in which this can happen can be understood using these matrices. For p odd, the dual of the Dyer-Lashof algebra is a subalgebra of the full ring of invariants. This subalgebra is also discussed in full details. In the last section a great amount of work is devoted to the proof of the analog of Mui's result mentioned above. Then our algorithms more or less naturally follows. A number of examples are included.

This paper has been written for odd primes with minor modifications needed when $p = 2$ provided in statements in square brackets following the odd primary statements.

We must note that the first draft of this work did not concern with Bockstein operations and Campbell, Peterson and Selick's paper was not mentioned because we were not aware of their method.

We thank Eddy Campbell very much for his great effort regarding the presentation and organization of the present work.

2 The Dyer-Lashof algebra

Let us briefly recall the construction of the Dyer-Lashof algebra. Let F be the free graded associative algebra on $\{f^i,\ i \geq 0\}$ and $\{\beta f^i,\ i > 0\}$ over $K := Z/pZ$ with $|f^i| = 2(p-1)i$, $[|f^i| = i]$ and $|\beta f^i| = 2i(p-1)-1$. F becomes a co-algebra equipped with coproduct $\psi :\ F \longrightarrow F \otimes F$ given by

$$\psi f^i = \sum f^{i-j} \otimes f^j \text{ and } \psi\beta f^i = \sum \beta f^{i-j} \otimes f^j \ + \ \sum f^{i-j} \otimes \beta f^j.$$

Elements of F are of the form $f^{I,\varepsilon} = \beta^{\epsilon_1} f^{i_1} \ldots \beta^{\epsilon_n} f^{i_n}$ where $(I,\varepsilon) = ((i_1,\ldots,i_n),(\epsilon_1,...,\epsilon_n))$ with $\epsilon_j = 0$ or 1 and i_j a non-negative integer for $j = 1,\ \ldots\ ,n$, $|f^{I,\varepsilon}| = 2(p-1)\left(\sum\limits_{t=1}^{n} i_t\right) - \left(\sum\limits_{t=1}^{n} e_t\right)$ $[|f^{I,\varepsilon}| = \left(\sum\limits_{t=1}^{n} i_t\right)]$. Let $l(I,\varepsilon) = n$ denote the length of I,ε or $f^{I,\varepsilon}$ and let the excess of (I,ε) or $f^{I,\varepsilon}$, denoted $exc(f^{I,\varepsilon}) = i_1 - \epsilon_1 - |f^{I_2}|$, where $(I_t,\varepsilon_t) = ((i_t,\ldots,i_n),(\epsilon_t,\ldots,\epsilon_n))$.

$$exc(f^{I,\varepsilon}) = i_1 - \epsilon_1 - 2(p-1)\sum_{2}^{n} i_t,\ [exc(f^I) = i_1 - \sum_{2}^{n} i_t]$$

The excess is defined ∞, if $I = \emptyset$ and we omit the sequence $(\epsilon_1,...,\epsilon_n)$ if all $e_i = 0$. We refer to elements f^I as having non-negative excess if $exc(f^{I_t})$ is non-negative for all t.

It is sometimes convenient to use lower notation for elements of F and its quotients. We define $f^i x = f_{\frac{1}{2}(2i-|x|)}x$ $[f^i x = f_{i-|x|}x]$. Let $I = (i_1,...,i_n)$ and $\varepsilon = (\epsilon_1,...,\epsilon_n)$, then the degree of $Q_{I,\varepsilon}$ is

$$|f_{I,\varepsilon}| = 2(p-1)\left(\sum_{t=1}^{n} i_t p^{t-1}\right) - \left(\sum_{t=1}^{n} e_t p^{t-1}\right),\ [|f_{I,\varepsilon}| = \left(\sum_{t=1}^{n} i_t 2^{t-1}\right)]$$

In lower notation we see immediately that $f_{I,\varepsilon}$ has non-negative excess if and only if (I,ε) is a sequence of non-negative integers: $exc(I,\varepsilon) = 2i_1 - e_1$

Given sequences I and I' we call the direct sum of I and I' the sequence $I \oplus I' = (i_1,...,i_n,i'_1,...,i'_m)$. Using a sequence I we use the above idea for the appropriate decomposition. Let 0_k denote the zero sequence of length k.

Remark 1 *Let us make a remark at this point concerning the sequences (I,ε) in upper and lower terms.*

For upper notation: We consider N as a monoid of the rationals, then (I,ε) is an element of $N^n \times (Z/2Z)^n$. $[I \in N^n]$

For lower notation: Let $< N, \frac{1}{2} >$ be the monoid generated by N and $\frac{1}{2}$ in the rationals. Let $< N, \frac{1}{2} >^n$ be the monoid which is the n-th Cartesian product of $< N, \frac{1}{2} >$. Then $(I,\varepsilon) \in < N, \frac{1}{2} >^n \times (Z/2Z)^n$. $[I \in N^n]$

F admits a Hopf algebra structure with unit $\eta : K \longrightarrow F$ and augmentation $\epsilon : F \longrightarrow K$ given by:

$$\epsilon(f^i) = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2 *There is a natural order on the elements $f^{(I,\varepsilon)}$ or $f_{(I,\varepsilon)}$ defined as follows: for (I,ε) and (I',ε') we say that $(I,\varepsilon) < (I',\varepsilon')$ if $exc(I_l,\varepsilon_l) = exc(I'_l,\varepsilon'_l)$ for $1 \leq l \leq t$ and $exc(I_t,\varepsilon_t) < exc(I'_t,\varepsilon'_t)$ for some $1 \leq t \leq n$.*

We define $T = F/\mathcal{I}_{exc}$, where $\mathcal{I}_{exc}$ is the two sided ideal generated by elements of negative excess. T inherits the structure of a Hopf algebra and if we let $T[n]$ denote the set of all elements of T with length n, then $T[n]$ is a co-algebra of finite type. We denote the image of $f^{I,\varepsilon}$ ($f_{I,\varepsilon}$) by $e^{I,\varepsilon}$ ($e_{I,\varepsilon}$). Degree, excess and ordering for upper or lower notation described above passes to T and $T[n]$.

The Adem relations are given by:

$$e^r e^s = \sum_i (-1)^{\mathbf{r}+i} \binom{(p-1)(i-s)-1}{pi-r} e^{r+s-i} e^i, \qquad \text{if } r > ps; \qquad (1)$$

$$e_r e_s = \sum_i (-1)^{\mathbf{r}-i} \binom{(p-1)(i-s)-1}{r-i-1} e_{r+ps-pi} e_i, \text{ if } r > s$$

and if $p > 2$ and $r \geq ps$,

$$e^r \beta e^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta e^{r+s-i} e^i - \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} e^{r+s-i} \beta e^i$$

$$e_r \beta e_s = \sum_i (-1)^{r+i+1/2} \binom{(p-1)(i-s)}{r-1/2-i} \beta e_{r+ps-pi-1/2} e_i +$$

$$\sum_i (-1)^{r+i-1/2} \binom{(p-1)(i-s)-1}{r-1/2-i} e_{r+ps-pi} \beta e_i, \qquad \text{if } r \geq s.$$

Let $\mathcal{I}_{Adem}$ be the two sided ideal of T generated by the Adem relations. We denote R the quotient $T/\mathcal{I}_{Adem}$ and this quotient algebra is called **the Dyer-Lashof algebra.** R is a Hopf algebra and $R[n]$ is again a co-algebra of finite type. We will denote the obvious epimorphism above which imposes Adem relations by

$$\rho : T \to R$$

If (I,ε) is admissible then $Q^{I,\varepsilon}$ is the image of $e^{I,\varepsilon}$. Respectively, $Q_{I,\varepsilon}$ of $e_{I,\varepsilon}$.

The following lemma will be applied in section 4.

Lemma 3 *a) $\rho(e_{p^k} e_0) = Q_0 Q_{p^{k-1}}$; $\rho(e_1 e_0) = 0$.*

b) $\rho(e_{p^k+1/2} e_{1/2}) = Q_{1/2} Q_{p^{k-1}+1/2}$; $\rho(e_{3/2} e_{1/2}) = 0$.

c) $\rho(e_{p^k+1} e_1) = Q_1 Q_{p^{k-1}+1}$; $\rho(e_2 e_1) = 0$.

d) $\rho(e_{p^k} e_1) = Q_0 Q_{p^{k-1}+1}$; $\rho(e_p e_1) = 2 Q_0 Q_2$.

e) $\rho(e_{p^k} \beta e_{1/2}) = Q_0 \beta Q_{p^{k-1}+1/2}$; $\rho(e_1 \beta e_{1/2}) = \beta Q_{1/2} Q_{1/2}$.

f) $\rho(e_{p^k+1/2} e_{1/2}) = 0$; $\rho(e_{3/2} \beta e_1) = \beta Q_1 Q_1$.

g) $\rho(e_{p^k+1} \beta e_{1/2}) = \beta Q_{1/2} Q_{p^{k-1}+1/2}$; $\rho(e_2 \beta e_{1/2}) = 0$.

The passage from lower to upper notation between elements of R is given as follows. Let by $Jx\varepsilon$ and $Ix\varepsilon$ as defined above. Then,

$$\beta^{\epsilon_1}Q_{j_1}...\beta^{\epsilon_n}Q_{j_n} \equiv \beta^{\epsilon_1}Q^{i_1}...\beta^{\epsilon_n}Q^{i_n}$$

up to a unit in Z/pZ, where $i_n = j_n$, and

$$i_{n-t} = \frac{1}{2}(2j_{n-t} + |I_{n-t+1}x\varepsilon_{n-t+1}|)$$
$$j_{n-t} = \frac{1}{2}(2i_{n-t} - |J_{n-t+1}x\varepsilon_{n-t+1}|)$$

Definition 4 *We say that $Q^{I,\varepsilon}$ or (I,ε) itself is admissible if there are no Adem relations between its factors: if $i_t \leq pi_{t+1} - \epsilon_{t+1}$ for all t. Note that in lower notation an element $Q_{I,\varepsilon}$ is admissible, if $0 \leq 2i_t - 2i_{t-1} + e_{t-1}$ for $2 \leq t \leq n-1$.*

The ordering described above passes to R and $R[n]$.

Since $R[n]$ and $T[n]$ are of finite type, they are isomorphic to their duals as vector spaces and these duals become algebras. We shall describe these duals giving an invariant theoretic description, namely: they are isomorphic to subalgebras of rings of invariants over the appropriate subgroup of $GL(n, K)$ in section 4.

3 The Dickson algebra and a special family of matrices

Let V^k denote a K-dimensional vector space generated by $\{e_1, ..., e_k\}$ for $1 \leq k \leq n$. Let the dual basis of V^n be $\{x_1, ..., x_n\}$ and the contragradient representation of $W_{\Sigma_{p^n}}(V^n) \longrightarrow Aut(V^n) \equiv GL_n$ induces an action of GL_n on the graded algebra $E(x_1, ..., x_n) \otimes P[y_1, ..., y_n]$, $[P[y_1, ..., y_n]]$, where $\beta x_i = y_i$. Let $E(n) = E(x_1, ..., x_n)$ and $S[n] = K[y_1, \cdots, y_n]$. The degree is given by $|x_i| = 1$ and $|y_i| = 2$ (if $p = 2$, then $|y_i| = 1$).

The following theorems are well known:

Theorem 5 *[4] $S[n]^{GL_n} := D[n] = K[d_{n,0}, \cdots, d_{n,n-1}]$ is a polynomial algebra where the degrees are $|d_{n,i}| = 2(p^n - p^i)$, $[2^n - 2^i]$.*

$D[n]$ is called the Dickson algebra.

Theorem 6 *[7] $S[n] := B[n] = K[h_1, \cdots, h_n]$ is a polynomial algebra where the degrees are $|h_i| = 2p^{i-1}(p-1)$, $[2^{i-1}]$.*

The generators above are related as follows:

Let $f_{k-1}(x) = \prod_{u \in V^{k-1}} (x - u)$, then $f_{k-1}(x) = \sum_{i=0}^{k-1} (-1)^{n-i} x^{p^i} d_{k-1,i}$ and $h_k = \prod_{u \in V^{k-1}} (y_k - u)$. Moreover, (see [5]),

$$d_{n,n-i} = \sum_{1 \leq j_1 < \cdots < j_i \leq n} \prod_{s=1}^{i} (h_{j_s})^{p^{n-i+s-j_s}} \tag{2}$$

Let us consider an example.

Example 7 *Let* $n = 3$.

$d_{3,2} = h_1^{p^2} + h_2^p + h_3$. *We associate the following family of matrices to the last decomposition.*

$$A_{3,2} = \left\{ \begin{pmatrix} & h_1 & h_2 & h_3 \\ d_{3,0} & 0 & 0 & 0 \\ d_{3,1} & 0 & 0 & 0 \\ d_{3,2} & p^2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} & h_1 & h_2 & h_3 \\ d_{3,0} & 0 & 0 & 0 \\ d_{3,1} & 0 & 0 & 0 \\ d_{3,2} & 0 & p & 0 \end{pmatrix}, \begin{pmatrix} & h_1 & h_2 & h_3 \\ d_{3,0} & 0 & 0 & 0 \\ d_{3,1} & 0 & 0 & 0 \\ d_{3,2} & 0 & 0 & 1 \end{pmatrix} \right\}$$

Respectively: $d_{3,1} = h_1^p h_2^p + h_1^p h_3 + h_2 h_3$.

$$A_{3,1} = \left\{ \begin{pmatrix} & h_1 & h_2 & h_3 \\ d_{3,0} & 0 & 0 & 0 \\ d_{3,1} & p & p & 0 \\ d_{3,2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} & h_1 & h_2 & h_3 \\ d_{3,0} & 0 & 0 & 0 \\ d_{3,1} & p & 0 & 1 \\ d_{3,2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} & h_1 & h_2 & h_3 \\ d_{3,0} & 0 & 0 & 0 \\ d_{3,1} & 0 & 1 & 1 \\ d_{3,2} & 0 & 0 & 0 \end{pmatrix} \right\}$$

And $d_{3,0} = h_1 h_2 h_3$.

$$A_{3,0} = \left\{ \begin{pmatrix} & h_1 & h_2 & h_3 \\ d_{3,0} & 1 & 1 & 1 \\ d_{3,1} & 0 & 0 & 0 \\ d_{3,2} & 0 & 0 & 0 \end{pmatrix} \right\}$$

Rows are associated with Dickson generators and columns with certain powers of $B[n]$ *generators.*

Next we shall give an interpretation of formulae 2 using matrices. We will use this interpretation to examine relations between Dickson generators of different height. Those relations will be used for the proof of the main theorem in section 4.

Let $m = (m_0, ..., m_{n-1})$ and $k = (k_1, ..., k_n)$ be sequences of non-negative integers. Let d^m denote an element of $D[n]$ given by $\prod_{t=0}^{n-1} d_{n,t}^{m_t}$ and h^k denote an element of $B[n]$ given by $\prod_{t=1}^{n} h_t^{k_t}$. Let $I_{(t)}$ denote the t-th element of the sequence $I = (i_{l_1}, ..., i_{l_n})$ from the left: i.e. $I_{(t)} := i_{l_t}$.

For any non-negative matrix C with integral entries and $\mathbf{1} = (1, ..., 1)$, the matrix product $\mathbf{1}{\cdot}C$ is a sequence of non-negative integers, then $h^{\mathbf{1}\cdot C}$ stands for $\prod_{t=1}^{n} h_t^{(\mathbf{1}\cdot C)_{(t)}}$ Let $C(d_{n,j}) = \{h^I \in B[n]$ and h^I is a non-trivial summand in $d_{n,j}\}$, then $C(d_{n,i}) \cap C(d_{n,j}) = \emptyset$ for $j \neq i$.

Remark 8 *1) Before we start considering sets of matrices, we would like to stress the point that the zero matrix is excluded from our sets, unless otherwise stated.*

2) Until the end of this section, we number matrices beginning with $(0,0)$ *in the upper left corner. In this case* $h^{\mathbf{1}\cdot C}$ *stands for* $\prod_{t=1}^{n} h_t^{(\mathbf{1}\cdot C)_{(t-1)}}$.

Let $0 \leq j \leq n-1$. Here j corresponds to the value $n-i$ in formula 2.

Definition 9 *For each matrix $A = (a_{it})$ such that a_{it} is a non-negative integer (N), $\sum_{t=0}^{n-1} a_{jt} = n - j$ and $\sum_{t=0}^{n-1} a_{it} = 0$ for $i \neq j$, we define an $n \times n$ matrix $C(A) = (b_{ij}) = \big(b_{(0)}, \cdots, b_{(n-1)}\big)$ such that $b_{it} = a_{it} p^{i-1-t+a_{i0}+\cdots+a_{it}}$. Let us call this collection $A_{n,j}$.*

For $C \in A_{n,j}$, $\mathbf{1}{\cdot}C$ is the j-th row of C which is the only non-zero row of that matrix.

Let us also note that there is an obvious bijection between $A_{n,j}$ and $C(d_{n,j})$.

Lemma 10 $d_{n,j} = \sum_{C \in A_{n,j}} h^{\mathbf{1}\cdot C}$.

Definition 11 *Let $m = (m_0, \cdots, m_{n-1})$ be a sequence of zeros or powers of p. Let $A^m_{n,j}$ stand for*

$$A^m_{n,j} = \{m \cdot C_j = (m_0 b_{(0)}, \cdots, m_{n-1} b_{(n-1)}) \mid C_j = (b_{(0)}, \cdots, b_{(n-1)}) \in A_{n,j}\} \tag{3}$$

and A^m_n for

$$A^m_n = \{\sum_{j=0}^{n-1} m \cdot C_j \mid C_j \in A_{n,j}\}$$

Note that different elements of A^m_n may provide the same element of $B[n]$ as the next example demonstrates:

Example 12 *Let $m = (0,1,1,0,0)$. Then* $C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ p & p & 0 & 1 & 1 \\ p^2 & 0 & p & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *and* $C' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ p & p & p & 0 & 1 \\ p^2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *are different elements of A^m_n but $h^{\mathbf{1}\cdot C} = h^{\mathbf{1}\cdot C'}$.*

Note that the remark made above indicates why Adem relations are complicated as we shall examine more in Proposition 18. We shall also note that the motivation of this section was exactly to demonstrate this difficulty using an elementary method.

The following lemma is easily deduced from formulae 2

Lemma 13 *Let $m = (m_0, \cdots, m_{n-1})$ such that $m_i = 0$ or p^{k_i}, then*

$$d^m = \prod_0^{n-1} d^{m_i}_{n,i} = \sum_{C \in A^m_n} \prod_{t=1}^{n} (h_t)^{(C(\mathbf{1}))_{t-1}}.$$

Example 12 implies that coefficients might appear in the last summation. Hence one needs to partition the set A_n^m as the following lemma suggests.

Lemma 14 *Let $m = (m_0, \cdots, m_{n-1})$ be a sequence of zeros or powers of p. Let $A = (a_{it})$ and $A' = (a'_{it})$ such that $a_{it}, a'_{it} \in N$, $\sum_{t=0}^{n-1} a_{jt} = \sum_{t=0}^{n-1} a'_{jt} = n - j$ if $m_i \neq 0$, otherwise the last sums are zero. Suppose that $\mathbf{1} \cdot A = \mathbf{1} \cdot A'$ and let $\{i_1, ..., i_q\}$ denote their different columns. Consider only their different rows and for each column i_r partition them according to where 1's appear: $\{j_1, ..., j_s\}$ and $\{j'_1, ..., j'_s\}$. If for each j_t there exists a j'_ℓ such that the number of zeros next to a_{i_r,j_t} and a'_{i_r,j'_t} are equal and this is true for all i_r, then $\mathbf{1}{\cdot}C(A) = \mathbf{1}{\cdot}C(A')$.*

Proof. We use the definition of $C(A)$ in 9. ■

The next example seeds some light to the required partition of A_n^m in order to control coefficients.

Example 15 *Let $p = 2$ and* $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $A' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
Then $\mathbf{1}{\cdot}A \neq \mathbf{1}{\cdot}A'$ but $\mathbf{1}{\cdot}C(A) = \mathbf{1}{\cdot}C(A')$.

On $1xn$ or $nx1$ matrices we give the left or upper lexicographical ordering respectively.

Definition 16 *Let m be a non-negative integer, we denote by ${}_{|A_{n,j}|}(m)$ the set of partitions of m in $|A_{n,j}|$ terms. A typical element of ${}_{|A_{n,j}|}(m)$ is of the form $\pi = (\pi_1, ..., \pi_{|A_{n,j}|})$.*

For $\pi = (\pi_1, ..., \pi_{|A_{n,j}|}) \in_{|A_{n,j}|}(m)$, let (π) denote the integer $\frac{m!}{\prod \pi_t!}$.

Lemma 17 *Let $m_j = \sum_{\alpha=0}^{\ell_j} m_{j,\alpha} p^\alpha$. Then*

$$d_{nj}^{m_j} = \sum_{\substack{0 \leq \alpha \leq \ell_j \\ \pi^{(j,\alpha)} \in_{|A_{n,j}|}(m_{j,\alpha})}} \prod_{\alpha=0}^{\ell_j} \left(\pi^{(j,\alpha)}\right) h^{\sum_\alpha \pi_i^{(j,\alpha)} {}_{C_{j,i} \in A_{n,j}} p^\alpha \mathbf{1} \cdot C_{j,i}}$$

Proof. First, we show the formulae above for $m_{j,\alpha}$:

$$\sum_{\pi^{(j,\alpha)} \in_{|A_{n,j}|}(m_{j,\alpha})} \left(\pi^{(j,\alpha)}\right) h^{\pi_i^{(j,\alpha)} {}_{C_{j,i} \in A_{n,j}} p^\alpha \mathbf{1} \cdot C_{j,i}}$$

and then we extend by direct multiplication. ■

Proposition 18 *Let $m = (m_0, ..., m_{n-1})$ be a sequence of non-negative integers, then*

$$d^m = \sum_{\substack{0 \le j \le n-1, 0 \le \alpha \le \ell_j \\ \pi^{(j,\alpha)} \in |A_{n,j}|^{(m_{j,\alpha})}}} \prod_{j=0}^{n-1} \prod_{\alpha=0}^{\ell_j} \left(\pi^{(j,\alpha)}\right) h^{\sum_j \sum_\alpha \pi_i^{(j,\alpha)} \sum_{\substack{C_{j,i} \in A_{n,j} \\ 0 \le j \le n-1}} p^\alpha \mathbf{1} \cdot C_{j,i}}$$

The following lemma which is of great importance for dealing with Adem relations involving Bockstein operations is an application.

Lemma 19 *Each term of $d_{k+t,s}$ is also a term of $d_{k,s}d_{k+t,k}$. Here $0 \le s < k$ and $1 \le t$. Moreover, no term of $d_{k,s}d_{k+t,k} - d_{k+t,s}$ is divisible by $\prod_{k+1}^{k+t} h_i$.*

Proof. Let us consider a non-zero row of a typical matrix of $A_{k+t,s}$. Such a row is of the form

$$b_{sl} = a_{sl} p^{s-1-l+a_{s0}+\cdots+a_{sl}}$$

Let j_r such that $a_{sj_r} = 1$ for $1 \le r \le k-s+t$. Let $b'_{sl} = b_{sl}$ for $j_{k-s+1} \le l \le k+t$ and $b'_{sl} = 0$ for $1 \le l < j_{k-s+1}$. Let $b''_{sl} = b_{sl}$ for $1 \le l \le j_{k-s}$ and $b''_{sl} = 0$ for $j_{k-s} + 1 \le l \le k+t$. Then $b_{sl} = b'_{sl} + b''_{sl}$. Now (b'_{sl}) is an element of $A_{k+t,s}$ and (b''_{sl}) is an element of $A_{k,s}$ under the obvious assumption. For the second statement: there is only one term in $d_{k,s}d_{k+t,k}$ which is divisible by $\prod_{k+1}^{k+t} h_i$, namely $d_{k,s} \prod_{k+1}^{k+t} h_i$, and this is also a term in $d_{k+t,s}$. The lemma follows. ■

In order to prove the main theorem in the next section, the following formula for decomposing Dickson generators will be needed. This formula is a special case of the lemma above. Formulas of this kind might be of interest for other circumstances involving the Dickson algebra. One of them may be the transfer between the Dickson algebra and the ring of invariants of parabolic subgroups.

Lemma 20 *Let $0 \le s < k$. Then $d_{k,s}d_{k+1,k} - d_{k+1,s} =$*
$\sum_{t=0}^{s-1} d_{k-t-1,s-t}^{p^t} d_{k-t-1,k-t-2}^{p^{t+2}} h_{k-t}^{p^t} + d_{k-s,0}^{p^s} d_{k-s,k-s-1}^{p^{s+1}} + d_{k-s,1}^{p^{s-1}} h_{k-s+1}^{p^{s-1}}$.

Proof. We shall use the formula $d_{k,s} = d_{k-1,s-1}^p + d_{k-1,s} h_k$. Firstly: $d_{k,s}d_{k+1,k} - d_{k+1,s} = d_{k,s}d_{k,k-1}^p - d_{k,s-1}^p$. We use reverse induction starting from $d_{k-s+1,1}^{p^{s-1}} d_{k-s+1,k-s}^{p^s} - d_{k-s+1,0}^{p^s} =$
$d_{k-s,1}^{p^{s-1}} d_{k-s,k-s-1}^{p^{s+1}} h_{k-s+1}^{p^{s-1}} + d_{k-s,0}^{p^s} d_{k-s,k-s-1}^{p^{s+1}} + d_{k-s,1}^{p^{s-1}} h_{k-s+1}^{p^{s-1}}$. For the inductive step: $d_{k-t,s-t}^{p^t} d_{k-t,k-t-1}^{p^{t+1}} - d_{k-t,s-t-1}^{p^{t+1}} =$
$d_{k-t-1,s-t-1}^{p^{t+1}} d_{k-t-1,k-t-2}^{p^{t+2}} - d_{k-t-1,s-t-2}^{p^{t+2}} + d_{k-t-1,s-t}^{p^t} d_{k-t-1,k-t-2}^{p^{t+2}} h_{k-t}^{p^t}$. ■

Lemma 21 *Each term of* $d_{k+q,k}d_{k+t,s}$ *is also a term of* $d_{k+q,s}d_{k+t,k}$. *Here* $0 \leq s < k$ *and* $0 \leq q < t$. *Moreover, no term of* $d_{k+q,s}d_{k+t,k} - d_{k+q,k}d_{k+t,s}$ *is divisible by* $\prod_{k+q+1}^{k+t} h_i$.

Proof. We consider $(k+t)x(k+t)$ matrices of the following form:

$$\begin{array}{c} \\ s\text{-th} \\ \\ k\text{-th} \\ \\ \end{array} \left[\begin{array}{cccc} & & \overset{k+q}{|} & \\ \leftarrow & k+q-s \rightarrow & | & \\ & & | & \\ \leftarrow & t & | & \rightarrow \\ & & | & \end{array}\right] - \begin{array}{c} \\ s\text{-th} \\ \\ k\text{-th} \\ \\ \end{array} \left[\begin{array}{cccc} & & \overset{k+q}{|} & \\ \leftarrow & k+t-s & | & \rightarrow \\ & & | & \\ \leftarrow & q \rightarrow & | & \\ & & | & \end{array}\right]$$

The last column of the matrices above is of size $t-q$. If this column is full of non-zero elements in the last matrix, we require the same in the k-th row of the first matrix. Then our matrices under consideration become:

$$\left[\begin{array}{cccc} & & \overset{k+q}{|} & \\ \leftarrow & k+q-s \rightarrow & | & \\ & & | & \\ \leftarrow & q & | & \overset{t-q}{\rightarrow} \\ & & | & \end{array}\right] - \left[\begin{array}{cccc} & & \overset{k+q}{|} & \\ \leftarrow & k+q-s & | & \overset{t-q}{\rightarrow} \\ & & | & \\ \leftarrow & q \rightarrow & | & \\ & & | & \end{array}\right]$$

Now the assertion follows because there is no other choice for the first matrix of this kind. For the general case, let the non-zero elements in the last column of the second matrix be $l < t-q$. Then the situation is as follows:

$$\left[\begin{array}{cccc} & & \overset{k+q}{|} & \\ \leftarrow & k+q-s \rightarrow & | & \\ & & | & \\ \leftarrow & t-l & | & \overset{l}{\rightarrow} \\ & & | & \end{array}\right] - \left[\begin{array}{cccc} & & \overset{k+q}{|} & \\ \leftarrow & k+t-s-l & | & \overset{l}{\rightarrow} \\ & & | & \\ \leftarrow & q \rightarrow & | & \\ & & | & \end{array}\right]$$

Hence we have to consider the following $(k+q)x(k+q)$ matrices:

$$\left[\begin{array}{ccc} \leftarrow & k+q-s & \rightarrow \\ \leftarrow & t-l & \rightarrow \end{array}\right] - \left[\begin{array}{ccc} \leftarrow & k+t-s-l & \rightarrow \\ \leftarrow & q & \rightarrow \end{array}\right]$$

Here the s-th column of the second matrix and the k-th column of the first one have been raised to the power p^{t-q-l}. Because the exponents are of the right form the assertion follows. ■

For the rest of this section we recall the ring of invariants $(E(x_1, ..., x_n) \otimes P[y_1, ..., y_n])^{GL_n}$ from [7]. Here $p > 2$.

Theorem 22 *[7]1) The algebra* $(E(n)\otimes S[n])^{B_n}$ *is a tensor product between the polynomial algebra* $B[n]$ *and the* Z/pZ *-module spanned by the set of elements consisting of the following monomials:*

$$M_{s;s_1,\ldots,s_m}L_s^{p-2};\ 1\le m\le n,\ m\le s\le n,\ and\ 0\le s_1<\cdots<s_m=s-1.$$

Its algebra structure is determined by the following relations:
a) $(M_{s;s_1}L_s^{p-2})^2=0$, *for* $1\le s\le n, 0\le s_1\le s-1$.
b) $M_{s;s_1,\ldots,s_m}L_s^{p-2}(L_s^{p-1})^{m-1}=$
$(-1)^{m(m-1)/2}\prod_{q=1}^{m}(\sum\limits_{r=s_q+1}^{s}M_{r;r-1}L_r^{p-2}h_{r+1}\ldots h_s d_{r-1,s_q})$
Here $1\le m\le n,\ m\le s\le n,\ and\ 0\le s_1<\cdots<s_m=s-1$.

2) The algebra $(E(n)\otimes S[n])^{GL_n}$ *is a tensor product between the polynomial algebra* $D[n]$ *and the* Z/pZ *-module spanned by the set of elements consisting of the following monomials:*

$$M_{n;s_1,\ldots,s_m}L_n^{p-2};\ 1\le m\le n,\ and\ 0\le s_1<\cdots<s_m\le n-1.$$

Its algebra structure is determined by the following relations:
a) $(M_{n;s_1,\ldots,s_m}L_n^{p-2})^2=0$ *for* $1\le m\le n,\ and\ 0\le s_1<\cdots<s_m\le n-1$.
b) $M_{n;s_1,\ldots,s_m}L_n^{(p-2)}d_{n,n-1}^{m-1}=(-1)^{m(m-1)/2}M_{n;s_1}L_n^{p-2}\ldots M_{n;s_m}L_n^{p-2}$.
Here $1\le m\le n,\ and\ 0\le s_1<\cdots<s_m\le n-1$.

The elements $M_{n;s_1,\ldots,s_m}$ above have been defined by Mui in [7]. The degree of elements above are $|M_{n;s_1,\ldots,s_m}|=m+2((1+\cdots+p^{n-1})-(p^{s_1}+\cdots+p^{s_m}))$ and $|L_n^{p-2}|=2(p-2)(1+\cdots+p^{n-1})$.

Definition 23 *Let* $S(E(n))^{B_n}$ *be the subspace of* $(E(n)\otimes S[n])^{B_n}$ *generated by:*
i) $M_{s;s-1}(L_s)^{p-2}$ *for* $1\le s\le n$,
ii) $\prod\limits_{t=1}^{\ell}\left(M_{s_{2t-1}+1;s_{2t-1}}(L_{s_{2t-1}+1})^{p-2}M_{s_{2t}+1;s_{2t}}(L_{s_{2t}+1})^{p-2}\right)/d_{s_{2t-1}+1,0}$ *for* $0\le s_1<\ldots<s_{2\ell}\le n-1$,
iii) $M_{s_1+1;s_1}(L_s)^{p-2}\prod\limits_{t=1}^{\ell}\left(M_{s_{2t}+1;s_{2t}}(L_{s_{2t}+1})^{p-2}M_{s_{2t+1}+1;s_{2t+1}}(L_{s_{2t+1}+1})^{p-2}\right)/d_{s_{2t}+1,0}$ *for* $0\le s_1<\ldots<s_{2\ell+1}\le n$;
and $S(E(n))^{GL_n}$ *be the subspace of* $(E(n)\otimes S[n])^{GL_n}$ *generated by:*
$M_{n;s}(L_n)^{p-2}$ *for* $0\le s\le n-1$,
$\prod\limits_{t=1}^{\ell}M_{n;s_{2t-1},s_{2t}}(L_n)^{p-2}$ *for* $0\le s_1<\ldots<s_{2\ell}\le n-1$,
$M_{n;s_1-1}(L_n)^{p-2}\prod\limits_{t=1}^{\ell}M_{n;s_{2t},s_{2t+1}}(L_n)^{p-2}$ *for* $0\le s_1<\ldots<s_{2\ell+1}<n$.

The following lemmata provide the decomposition of $M_{n;s,m}(L_n)^{p-2}$ in $S(E(n))^{B_n}\otimes B[n]$ and relations between them.

Lemma 24 *Let* $s<\ell$, *then* $M_{s;s-1}L_s^{p-2}M_{\ell;\ell-1}L_\ell^{p-2}$ *can be written with respect to basis elements of* $B[k]\otimes S(E_k)^{B_k}$.

Proof. We use induction on $s = \ell - 1, ..., 1$.
$M_{\ell-1;\ell-2}L_{\ell-1}^{p-2}M_{\ell;\ell-1}L_{\ell}^{p-2}h_{\ell} = -M_{\ell;l-2\ell-1}L_{\ell}^{p-2}L_{\ell}^{p-1} = M_{\ell;\ell-2}L_{\ell}^{p-2}M_{\ell;\ell-1}L_{\ell}^{p-2} =$
$(M_{\ell-1;\ell-2}L_{\ell-1}^{p-2}M_{\ell;\ell-1}L_{\ell}^{p-2}h_{\ell}+M_{\ell;\ell-1}L_{\ell-1}^{p-2}d_{\ell-1,\ell-2}M_{\ell;\ell-1}L_{\ell}^{p-2}) = M_{\ell-1;\ell-2}L_{\ell-1}^{p-2}M_{\ell;\ell-1}L_{\ell}^{p-2}h_{\ell}$.
Now the general step:
$M_{s;s-1}L_{s}^{p-2}M_{\ell;\ell-1}L_{\ell}^{p-2}\left(\frac{L_\ell}{L_s}\right)^{p-1} =$
$-M_{\ell;s,\ell-1}L_{\ell}^{p-2}L_{\ell}^{p-1} - M_{s+1;s}L_{s+1}^{p-2}d_{s,s-1}M_{\ell;\ell-1}L_{\ell}^{p-2}\left(\frac{L_\ell}{L_{s+1}}\right)^{p-1} -$
$M_{\ell-2;\ell-3}L_{\ell-2}^{p-2}d_{\ell-3,s-1}M_{\ell;\ell-1}L_{\ell}^{p-2}\left(\frac{L_\ell}{L_{\ell-2}}\right)^{p-1} + M_{\ell;\ell-2,\ell-1}L_{\ell}^{p-2}d_{\ell-2,s-1}L_{\ell}^{p-1}$. Now the claim follows by induction hypothesis. ∎

Lemma 25 *Let $m < s-1$, then $M_{s;\ell,s-1}L_s^{p-2}M_{m;m-1}L_m^{p-2}$ can be written with respect to basis elements of $B[k] \otimes S(E_k)^{B_k}$.*

Proof. Let $\ell < m < s-1$, then
$M_{s;\ell,s-1}L_s^{p-2}M_{m;m-1}L_m^{p-2}L_s^{p-1} = -M_{s;\ell}L_s^{p-2}M_{s;s-1}L_s^{p-2}M_{m;m-1}L_m^{p-2} =$
$M_{s;\ell}L_s^{p-2}M_{m;m-1}L_m^{p-2}M_{s;s-1}L_s^{p-2} =$
$(M_{\ell+1;\ell}L_{\ell+1}^{p-2}\left(\frac{L_\ell}{L_{\ell+1}}\right)^{p-1} + M_{\ell+2;\ell+1}L_{\ell+2}^{p-2}d_{\ell+1,\ell}\left(\frac{L_\ell}{L_{\ell+2}}\right)^{p-1} + ...+$
$M_{s;s-1}L_s^{p-2}d_{s-1,\ell})M_{m;m-1}L_m^{p-2}M_{s;s-1}L_s^{p-2}$.
Now the claim follows by the previous lemma. ∎

Lemma 26 $M_{n;s,m}(L_n)^{p-2} =$
$$\sum_{\substack{s\le q<t\\ m\le t\le n-1}} M_{q+1;q}(L_{q+1})^{p-2}M_{t+1;t}(L_{t+1})^{p-2}h_{t+2}...h_n(d_{q,s}d_{t,m}-d_{q,m}d_{t,s})/d_{q+1,0}.$$
Here $d_{i,i}=1$ and $d_{i,j}=0$ if $i<j$.

Corollary 27 *Let $\kappa = [\frac{n+1}{2}]$ and $\varepsilon = (\epsilon_1, ..., \epsilon_n) \in (Z/2Z)^n$, then $S(E_n)^{GL_n}$ is spanned by at most κ monomials:*

$$M^{\varepsilon} := \begin{cases} M_{n;s_1,s_2}^{[\frac{\epsilon_1+\epsilon_2}{2}]}L_n^{p-2}...M_{n;s_{n-1},s_n}^{[\frac{\epsilon_{n-1}+\epsilon_n}{2}]}L_n^{p-2}, & \text{if } n \text{ is even}\\ M_{n;s_1}^{\epsilon_1}L_n^{p-2}M_{n;s_2,s_3}^{[\frac{\epsilon_2+\epsilon_3}{2}]}L_n^{p-2}...M_{n;s_{n-1},s_n}^{[\frac{\epsilon_{n-1}+\epsilon_n}{2}]}L_n^{p-2}, & \text{if } n \text{ is odd}\end{cases}$$

The analogue corollary holds for $S(E_n)^{B_n}$.

The Steenrod algebra acts naturally on $S(E(n))^{GL_n} \otimes D[n]$ and $S(E(n))^{B_n} \otimes B[n]$.

Let $\hat{\imath} : S(E(n))^{GL_n} \otimes D[n] \hookrightarrow S(E(n))^{B_n} \otimes B[n]$ be the inclusion, then $\hat{\imath}(d^m M^{\varepsilon})$ means the decomposition of $d^m M^{\varepsilon}$ in $S(E(n))^{B_n} \otimes B[n]$.

Lemma 28 *Let $0 \le s_1(s_1') < k_1(k_1') < ... < s_{l'}(s_{l'}') < k_{l'}(k_{l'}') \le n-1$. If $\sum_0^{n-1} m_i(p^n-p^i) + \sum_1^{l'}(p^n-p^{s_i}-p^{k_i}) = \sum_0^{n-1} m_i'(p^n-p^i) + \sum_1^{l'}(p^n-p^{s_i'}-p^{k_i'})$, then $s_i = s_i'$ and $k_i = k_i'$. Moreover, if in addition $0 \le k_0(k_0') < s_1(s_1')$ and $\sum_0^{n-1} m_i(p^n-p^i) + (p^n-p^{k_0}) + \sum_1^{l'}(p^n-p^{s_i}-p^{k_i}) = \sum_0^{n-1} m_i'(p^n-p^i) + (p^n-p^{k_0'}) + \sum_1^{l'}(p^n-p^{s_i'}-p^{k_i'})$, then $s_i = s_i'$ and $k_j = k_j'$.*

Proof. We prove the first statement, the second is completely analogous.

Let $k'_{l'} > k_{l'}$, then $k_{l'} \leq k'_{l'} - 1$. Thus $p^{k'_{l'}} > 1 + ... + p^{k_{l'}-1} \geq \sum(p^{k_i} + p^{s_i})$. Hence, $\sum(p^{k'_i} + p^{s'_i}) > \sum(p^{k_i} + p^{s_i}) \Rightarrow \sum(p^n - p^{k'_i} - p^{s'_i}) < \sum(p^n - p^{k_i} - p^{s_i})$.

Now, if $k'_{l'} < n-1$, then $\sum(p^n - p^{k'_i} - p^{s'_i}) - \sum(p^n - p^{k_i} - p^{s_i}) < p^{n-1}$. Otherwise $(k'_{l'} = n-1)$, $\sum(p^n - p^{k'_i} - p^{s'_i}) - \sum(p^n - p^{k_i} - p^{s_i}) \leq p^{n-1} - 1$. But $p^{n-1} - 1 < \sum\limits_{0}^{n-1} (m'_i - m_i)(p^n - p^i)$ and this is a contradiction. Finally, $k'_{l'} = k_{l'}$. ■

4 Calculating the hom-duals and Adem relations

We start this section by recalling the description of $R[n]^*$ as an algebra, for p odd please see May [3] Theorem 3.7 page 29. The analogue Theorem for $p = 2$ was given by Madsen who expressed the connection between $R[n]^*$ and Dickson invariants back in 1975, [9].

For convenience we shall write I instead of (I, ε).

Let $I_{n,i} = (\underbrace{0, ..., 0}_{i}, \underbrace{1, ..., 1}_{n-i})$. Here $0 \leq i \leq n-1$ and $n-i$ denotes the number of p-th powers. The degree $|Q_{I_{n,i}}| = 2p^i(p^{n-i} - 1)$ $[2^n - 2^i]$ and the $exc(Q_{I_{n,i}}) = 0$, if $i < n$, and 1 if $i = 0$.

Let $J_{n;i} = (\underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{i}, \underbrace{1, ..., 1}_{n-i})x(\underbrace{0, ..., 0, 1}_{i+1}, \underbrace{0, ..., 0}_{n-i-1})$. Here $\varepsilon = (\underbrace{0, ..., 0}_{i}, 1, \underbrace{0, ..., 0}_{n-i-1})$ and $0 \leq i \leq n-1$. The degree $|Q_{J_{n;i}}| = 2p^i(p^{n-i}-1)-1$ and the $exc(Q_{J_{n;i}}) = 1$.

Let $K_{n;s,i} = (\underbrace{0, ..., 0}_{s}, \underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{i-s}, \underbrace{1, ..., 1}_{n-i})x(\underbrace{0, ..., 0}_{s}, \underbrace{1, 0, ..., 0, 1}_{i-s+1}, \underbrace{0, ..., 0}_{n-i})$. Here $\varepsilon = (\underbrace{0, ..., 0}_{s}, \underbrace{1, 0, ..., 0, 1}_{i-s}, \underbrace{0, ..., 0}_{n-i})$ and $0 \leq s < i \leq n-1$. There are two Bockstein operations in this element: at the s-th and i-th position from the left. The degree $|Q_{K_{n;s,i}}| = 2(p^i(p^{n-i} - 1) - p^s)$ and the $exc(Q_{K_{n;s,i}}) = 0$.

Let $O_{n,i} = (0, ..., 0, 1, 0, ..., 0)$, where there are $n-i$ zeros. Its degree is $|e_{O_{n,i}}| = 2p^{i-1}(p-1)$ $[2^{i-1}]$ and $exc(e_{O_{n,i}}) = 0$. Here $1 \leq i \leq n$.

Let $J_{n,i;i-1} = (\underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{i}, \underbrace{0, ..., 0}_{n-i})x(\underbrace{0, ..., 0, 1}_{i}, \underbrace{0, ..., 0}_{n-i})$. Here $\varepsilon = (\underbrace{0, ..., 0}_{i-1}, 1, \underbrace{0, ..., 0}_{n-i})$ and $1 \leq i \leq n$. Its degree $|Q_{J_{n,i;i-1}}| = 2p^{i-1}(p-1)-1$ and the $exc(Q_{J_{n,i;i-1}}) = 1$.

Let $K_{n,i;s,i-1} = (\underbrace{0, ..., 0}_{s}, \underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{i-s}, 1, \underbrace{0, ..., 0}_{n-i})x(\underbrace{0, ..., 0}_{s}, \underbrace{1, 0, ..., 0, 1}_{i-s}, \underbrace{0, ..., 0}_{n-i})$. Here $\varepsilon = (\underbrace{0, ..., 0}_{s}, \underbrace{1, 0, ..., 0, 1}_{i-s}, \underbrace{0, ..., 0}_{n-i})$ and $0 \leq s < i-1 \leq n-1$. Its degree $|Q_{K_{n,i;s,i-1}}| = 2(p^i - p^s - p^{i-1})$ and the $exc(Q_{K_{n,i;s,i-1}}) = 0$.

Let $\xi_{n,0} = ((Q_0)^n)^* = ((Q^0)^n)^*$;

$\xi_{n,i} = (Q_{I_{n,i}})^* = (Q^{(p^{i-1}(p^{n-i}-1),\dots,(p^{n-i}-1),p^{n-i-1},\dots,p,1)})^*$, $0 \le i \le n-1$;
$\tau_{n;i} = (Q_{J_{n;i}})^* = (Q^{(p^{i-1}(p^{n-i}-1),\dots,(p^{n-i}-1),p^{n-i-1},\dots,p,1,)x\varepsilon})^*$, $0 \le i \le n-1$;
$\sigma_{n;s,i} = (Q_{K_{n;s,i}})^* = (Q^{(p^{i-1}(p^{n-i}-1)-p^{s-1},\dots,p^{i-s-1}(p^{n-i}-1),\dots,\ p^{n-i}-1,p^{n-i-1},\dots,p,1)x\varepsilon})^*$, $0 \le s < i \le n-1$;
$\zeta_{n,i} = \left(e_{O_{n,i}}\right)^* = (e^{(p^{i-2}(p-1),\dots,(p-1),1,0,\dots,0)})^*$, $1 \le i \le n$;
$\nu_{n,i;i-1} = (e_{J_{n,i;i-1}})^* = (e^{(p^{i-2}(p-1),\dots,(p-1),1,0,\dots,0)x\varepsilon})^*$, $1 \le i \le n$;
$\upsilon_{n,i;s,i-1} = (e_{K_{n,i;s,i-1}})^* =$
$(e^{(p^{i-1}(p^{n-i}-1)-p^{s-1},\dots,p^{i-s-1}(p^{n-i}-1),\dots,p^{n-i}-1,p^{n-i-1},\dots,p,1,0,\dots,0)x\varepsilon})^*$, $0 \le s < i-1 \le n-1$.

Theorem 29 (Madsen $p = 2$, May $p > 2$) . *As an A algebra $R[n]^* \cong$ free associative commutative algebra generated by $\{\xi_{n,i}, \tau_{n;i}$, and $\sigma_{n;s,i} \mid 0 \le i \le n-1$, and $0 \le s < i\}$, [$\{\xi_{n,i} \mid 0 \le i \le n-1\}$], modulo the following relations:*
a) $\tau_{n;i}\ \tau_{n;i} = 0$.
b) $\tau_{n;s}\tau_{n;i} = \sigma_{n;s,i}\xi_{n,0}$. *Here* $0 \le s < i \le n-1$.
c) $\tau_{n;s}\tau_{n;i}\tau_{n;j} = \tau_{n;s}\sigma_{n;i,j}\xi_{n,0}$. *Here* $0 \le s < i < j \le n-1$.
d) $\tau_{n;s}\tau_{n;i}\tau_{n;j}\tau_{n;k} = \sigma_{n;s,i}\sigma_{n;j,k}\xi^2_{n,0}$. *Here* $0 \le s < i < j < k \le n-1$.

Theorem 30 *[5] $R[n]^* \approxeq S(E(n))^{GL_n} \otimes D[n]$ [$R[n]^* \approxeq D[n]$] and $T[n]^* \approxeq S(E(n))^{B_n} \otimes B[n]$ [$T[n]^* \approxeq B[n]$] as algebras over the Steenrod algebra and the isomorphism Φ is given by $\Phi(\xi_{n,i} = (Q_{I_{n,i}})^*) = d_{n,n-i}$, $\Phi(\tau_{n;i} = (Q_{J_{n;i}})^*) = M_{n;i}(L_n)^{p-2}$, $\Phi(\sigma_{n;s,i} = (Q_{K_{n;s,i}})^*) = M_{n;s,i}(L_n)^{p-2}$. Here $0 \le i \le n-1$ and $0 \le s < i$.*

$\Phi(\zeta_{n,i} = \left(e_{O_{n,i}}\right)^*) = h_i$, $\Phi(\nu_{n,i;i-1} = (e_{J_{n,i;i-1}})^*) = M_{i;i-1}(L_i)^{p-2}$, $\Phi(\upsilon_{n,i;s,i-1} = (e_{K_{n,i;s,i-1}})^*) = (M_{s+1;s}(L_{s+1})^{p-2}M_{i;i-1}(L_i)^{p-2})/d_{s+1,0}$. Here $1 \le i \le n$ and $0 \le s < i-1$.

Under isomorphism Φ in Theorem 30 we identify $R[n]^*$ with $S(E(n))^{GL_n} \otimes D[n]$ and $B[n]^*$ with $S(E(n))^{B_n} \otimes B[n]$.

Let $\rho : T[n] \to R[n]$ be the induced map which imposes Adem relations between the respected coalgebras of length n.

$$\rho(e_I) = \sum a_{I,J} Q_J$$

The set Ţ$[n]$ and Ŗ$[n]$ of admissible monomials in $T[n]$ and $R[n]$ provide vector space bases respectively. Let θ : Ŗ$[n] \rightarrowtail$Ţ$[n]$ be the map given by

$$\theta(Q_I) = e_I$$

The image of the dual of these bases are denoted by Ţ$[n]^*$ in $\Phi(T[n])^* = S(E(n))^{B_n} \otimes B[n]$ and Ŗ$[n]^*$ in $\Phi(R[n])^* = S(E(n))^{GL_n} \otimes D[n]$. Of course there are also the bases of monomials which are denoted by ß$_n(S(E(n))^{B_n} \otimes B[n])$ and ß$_n(S(E(n))^{GL_n} \otimes D[n])$ respectively. We shall note that Ţ$[n]^*$ =ß$_n(S(E(n))^{B_n} \otimes B[n])$.

The decomposition relations between the other two bases are not obvious and this is the first topic of this section. Campbell, Peterson and Selick provided a method to pass from ß$_n(S(E(n))^{GL_n} \otimes D[n])$ to Ŗ$[n]^*$ in [2]. We shall describe and compare their algorithm with ours. Firstly we shall establish some machinery to work with those bases.

Definition 31 *Let $\chi_{\min}$ and $\chi_{\max}$ be the set functions from $\beta(D[n]\otimes S(E_n)^{GL_n})$ $(\beta(B[n] \otimes S(E_n)^{B_n})$) to the monoid $< N, \frac{1}{2} >^n \times (Z/2Z)^n$ given by*

$$1)\ \chi_{\min}(d_{n,i}) = I_{n,i} \text{ and}$$
$$\chi_{\max}(d_{n,i}) = (p^{n-i}, ..., p^{n-i}, 0, ..., 0)x(0, ..., 0).$$
$$2)\ \chi_{\min}(M_{n;s}L_n^{(p-2)}) = J_{n;s} \text{ and}$$
$$\chi_{\max}(M_{n;s}L_n^{(p-2)}) = (\underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{s}, \underbrace{1\frac{1}{2}, ..., 1\frac{1}{2}}_{n-s-1}, 1)x(0, ..., 0, 1).$$
$$3)\ \chi_{\min}(M_{n;s,m}L_n^{(p-2)}) = K_{n;s,m} \text{ and}$$
$$\chi_{\max}(M_{n;s,m}L_n^{(p-2)}) = (\underbrace{0, ..., 0}_{m}, \underbrace{1\frac{1}{2}, ..., 1\frac{1}{2}}_{n-m-1}, 1)x(\underbrace{0, ..., 0}_{m}, \underbrace{1, 0, ..., 0, 1}_{n-m}).$$

and the rule $\chi_{\min}(dd'MM') = \chi_{\min}(d) + \chi_{\min}(d') + \chi_{\min}(M) + \chi_{\min}(M')$. Here d, d' $\in\beta(D[n])$ and M, M' $\in\beta(S(E_n)^{GL_n})$. The same holds for $\chi_{\max}$.

Note that the function $\chi_{\min}$ always provides an admissible element and $\hat{\imath}(d_{n,i})$ contains a monomial with a unique admissible sequence, namely $h^{\chi_{\min}(d_{n,i})}$, and a monomial with a unique maximal sequence, namely $h^{\chi_{\max}(d_{n,i})}$. The same is true for elements $M_{n;s-1}L_n^{p-2}$ and $M_{n;s,m}L_n^{p-2}$. Moreover, $\hat{\imath}(d_n^m M)$ might contain a number of monomials with admissible sequences and this is the main point of investigation because of its applications in [6]. Namely, those monomials provide possible candidates for $(d_n^m M)^*$. Primitives in R are well known and so are their duals as generators in R^*. But it is not the case for their expression with respect to the Dickson algebra. On the other hand, the action on the Dickson algebra is well known on $S(E(n))^{GL_n} \otimes D[n]$ and hence it is easier to compute the annihilator ideal in the mod $-p$ cohomology of a certain finite loop space.

Definition 32 *Let Ψ be the correspondence between $\beta_n(S(E(n))^{GL_n} \otimes D[n])$ and $\underset{\sim}{R}[n]$ given by $d \longmapsto \Psi(d) = Q_{\chi_{\min}(d)}$ and the corresponding one between $\beta_n(S(E(n))^{B_n} \otimes B[n])$ and $\underset{\sim}{T}[n]$ denoted by Ψ_T where $\Psi_T(h^J M^\varepsilon) = e_{J+\epsilon_1 J_{n;s_1} + \sum_t [\frac{\epsilon_{2t}+\epsilon_{2t+1}}{2}] K_{n;s_{2t},s_{2t=1}}}$. The maps Ψ and Ψ_T are set bijections.*

Let ι be the map

$$\iota : \beta_n(S(E(n))^{GL_n} \otimes D[n]) \rightarrowtail \beta_n(S(E(n))^{B_n} \otimes B[n]) \tag{4}$$

defined by $\iota(d) = h^{\chi_{\min}(d)}$.

Note that $e_{\chi_{\max}(d)}, e_{\chi_{\min}(d)} \in \c{T}[n]$. The following diagram is commutative.

$$\begin{array}{ccccc} \text{ß}_n(S(E(n))^{B_n} \otimes B[n]) & \stackrel{\iota}{\hookleftarrow} & \text{ß}_n(S(E(n))^{GL_n} \otimes D[n]) & \stackrel{\chi_{\min}}{\rightarrow} & < N, \frac{1}{2} >^n \times (Z/2Z)^n \\ \downarrow \ \Psi_T & & \downarrow \ \Psi & \swarrow & \\ \c{T}[n] & \stackrel{\theta}{\hookleftarrow} & \c{R}[n] & & \end{array}$$

Definition 33 *A monomial in* $\text{ß}_n(S(E(n))^{B_n} \otimes B[n])$ *is called admissible if it is an element of* $\iota\left(\text{ß}_n(S(E(n))^{GL_n} \otimes D[n])\right)$.

Lemma 34 *Let* $h^J M^\varepsilon \in S(E(n))^{B_n} \otimes B[n]$. *The following are equivalent:*
i) $h^J M^\varepsilon$ *is admissible;*
ii) $j_t \leq j_{t+1}$ *for* $t = 1, ..., n-1$ *and* h^J *is divisible by* $\prod_{t=0}^{l} (h_{s_{2t+1}+2} ... h_n)^{\epsilon_{2t+1}}$ *for* κ *odd (see 27); or* $\prod_{t=1}^{l} (h_{s_{2t}+2} ... h_n)^{\epsilon_{2t}}$, *otherwise. If* $s_{2t+1} + 2$ *or* $s_{2t+2} + 2 = n + 1$, *then the corresponding product must be* 1.
iii) $\rho(\Psi_T(h^J M^\varepsilon))$ *is admissible in* $R[n]$.

Proof. This follows from the following relation: $M_{k;s} L_k^{p-2} = M_{s+1;s} L_{s+1}^{p-2} h_{s+1} ... h_k + \sum_{t=2}^{k-s} M_{s+t;s+t-1} L_{s+t}^{p-2} d_{s+t-1,s} h_{s+t+1} ... h_k$. Explicitly, if $h^{J'} = h^J / \prod_{t=0}^{l} (h_{s_{2t+1}+2} ... h_n)^{\epsilon_{2t+1}}$, then $\chi_{\min}(d^m M^\varepsilon) = (J', \varepsilon)$. Here $d^m M^\varepsilon = \prod_{i=0}^{n-1} d_{n,i}^{m_i} M_{n;s_1}^{\epsilon_1} L_n^{p-2} M_{n;s_2,s_3}^{[\frac{\epsilon_2+\epsilon_3}{2}]} L_n^{p-2} ... M_{n;s_{n-1},s_n}^{[\frac{\epsilon_{n-1}+\epsilon_n}{2}]} L_n^{p-2}$ and $m_t = j'_t - j'_{t-1}$, $m_0 = j'_0$. ■

Firstly, we shall show that $\rho^* \equiv \hat{\imath}$, $\hat{\imath}$ as in 3, i.e. for any $e_I \in T[n]$ and $d^m M^\varepsilon$,

$$< d^m M^\varepsilon, \rho(e_I) > = < \hat{\imath}(d^m M^\varepsilon), e_I > .$$

Here, $< -, - >$ is the Kronecker product. This is done by studying all monomials in $T[n]$ which map to primitives in $R[n]$ after applying Adem relations.

Let $n(mx\varepsilon) = \sum m_i + \kappa$. Let $\psi_{n(mx\varepsilon)} : R[n] \to \overset{n(mx\varepsilon)}{\otimes} R[n]$ be the iterated coproduct $n(mx\varepsilon)$ times. Let J admissible, $\rho e_J = Q_J$, then

$$\psi Q_J = \psi \rho e_J = \rho \psi e_J = \rho(\Sigma \pm e_{J_1} \otimes \cdots \otimes e_{J_{n(mx\varepsilon)}}), \quad \Sigma J_i = J$$
$$\psi_{n(mx\varepsilon)} Q_J = \Sigma a_{J_1, ..., J_{n(mx\varepsilon)}} Q_{J'_1} \otimes \cdots \otimes Q_{J'_{n(mx\varepsilon)}}.$$

Since J_i may not be in admissible form, after applying Adem relations we have $J'_i \leq J_i$.

$$< d^m M^\varepsilon, \rho e_I > = < \prod_{i=0}^{n-1} d_{n,i}^{m_i} M^\varepsilon, \psi_{n(mx\varepsilon)} \rho e_I > = < \prod_{i=0}^{n-1} d_{n,i}^{m_i} M^\varepsilon, \rho \psi_{n(mx\varepsilon)} e_I > =$$
$$< \prod_{i=0}^{n-1} d_{n,i}^{m_i} M^\varepsilon, \sum_{I_j} \bigotimes_{j}^{n(mx\varepsilon)} \rho e_{I_j} > = \sum_{I_j} \prod_{j}^{n(m)} < d_{n,i}^j, \rho e_{I_j} > \prod_{j}^{n(\varepsilon)} < M_{n;s_{j-1},s_j}^{[\frac{\epsilon_{j-1}+\epsilon_j}{2}]} L_n^{p-2}, \rho e_{I_j} >.$$

Lemma 35 *Let* $d^m = \prod_{i=1}^{n} d_{n,i}^{m_i}$. *Then* $\iota(d^m) = \prod_{t=1}^{n} h_t^{\sum_{i=0}^{t-1} m_i}$ *and* $(\iota(d^m))^* = e_{m_0} e_{m_0+m_1} ... e_{m_0+...m_{n-1}}$.

Lemma 36 *Let* $I = \chi_{\max}(d_{n,n-i})$, *then* $\rho(e_I) = Q_{I_{n,n-i}} = \Psi\left(d_{n,n-i}\right)$ *in* $R[n]$.

Proof. For the sake of simplicity, we write I instead of Q_I or e_I. By hypothesis $\chi_{\max}(d_{n,n-i}) = I = (p^{n-i}, ..., p^{n-i}, 0, ..., 0)$. Let us apply Adem relations between the last n-i+1 elements of $I : ((p^{n-i}, 0, ..., 0))$. The last sequence becomes $(0, p^{n-i-1}, 0, ..., 0)$, because of excess and the binomial coefficients in the Adem relations: $pk - p^{n-i} \geq 0$ and $\binom{(p-1)k-1}{pk-p^{n-i}-1} \neq 0 \bmod p \Rightarrow k = p^{n-t-1}$. For the same reason $(0, p^{n-i-1}, 0, ..., 0)$ becomes $(0, ..., 0, 1)$. Next we consider the first i elements of the new sequence: $(p^{n-i+1}, 0, ..., 0, 1)$ and we continue on the same pattern. ■

Lemma 37 *Let* $e_I \in T[n]$ *be such that* $e_I = \Phi^{-1}\left(\prod_{s=1}^{i} h_{j_s}^{p^{n-i+s-j_s}}\right)$ *in (2). Here* $1 \leq j_1 < ... < j_i \leq n$. *Then* $\rho(e_I) = Q_{n,n-i} = \Psi\left(d_{n,n-i}\right)$ *in* $R[n]$.

Proof. The sequence I is given by:

$$\left(\underbrace{0,\cdots,0,p^{n-i+1-j_1}}_{j_1},\underbrace{0,\cdots,0,p^{n-i+2-j_2}}_{j_{i-2}-j_1},\cdots,\underbrace{0,\cdots,0,p^{n-j_i}}_{j_i-j_{i-1}},\underbrace{0,\cdots,0}_{n-j_i}\right)$$

Please note the analogy between I above and the corresponding row of a matrix in $A_{n,n-i}$ in section 3. Here $p^m := 0$, whenever $m < 0$. We shall work out the first steps to describe the idea of the proof. First, we consider the last $n-i+1$ elements of $\chi_{\max}(d_{n,n-i})$: $(p^{n-i}, 0, ..., 0)$ which becomes $(\underbrace{0,\cdots,0,p^{n-j_i}}_{j_i-j_{i-1}},\underbrace{0,\cdots,0}_{n-j_i})$.

Thus applying Adem relations on certain positions on $Q_{\chi_{\max}(d_{n,n-i})}$, Q_I is obtained and the lemma follows. ■

Proposition 38 *Let* $e_I \in T[n]$ *be the* hom-*dual of a monomial* $h^J \in T[n]$ *such that* $|h^J| = 2\left(p^n - p^{n-i}\right)$ *and* h^J *is not a summand in (2). Then* $\rho(e_I) = 0$ *in* $R[n]$.

Proof. Let j_{i-t} be the biggest index such that ${h_{j_{i-t}}}^{p^{n-t-j_{i-t}}}$ does not divide h^J or ${h_{j_{i-t}}}^{p^{n-t-j_{i-t}}+1}$ divides h^J. In other words, exponents do not have the right form. Two cases should be considered, namely: i) $p^{n-j_{i-t}}$ has been replaced by $p^{n-t-j_{i-t}} - m(t)$ and ii) by $p^{n-t-j_{i-t}} + m(t)$.

Let us start with i) and recall that $I_{n-t-j_{i-t}+1}$ of $(h^J)^*$ has the form

$$\left(p^{n-t-j_{i-t}} - m(t),\ \underbrace{0,\cdots,0}_{j_{i-t+1}-j_{i-t}}\ ,p^{n-t+1-j_{i-t+1}},\ \underbrace{0,\cdots,0}_{j_{i-t+2}-j_{i-t+1}}\ ,p^{n-t+2-j_{i-t+2}},...,p^{n-j_i},\underbrace{0,\cdots,0}_{n-j_i}\right)$$

i) Let us start with $m(t) > 0$. The last $n - j_{i-t} + 1$ elements of the sequence becomes $(p^{n-t-j_{i-t}} - m(t), 0, \cdots, 0, 1, \cdots, 1)$ after applying Adem relations. Because of excess and Adem relations, $m(t)$ must be divisible by $p^{n-t-j_{i-t}}$. Hence $m(t) = 0$.

ii) Now $m(t)p^{j_{i-t}-1} \leq p^{n-i} + ... + p^{n-1} - p^{n-t-1} - ... - p^{n-1}$ because of the degree. Thus $m(t)$ can be divisible at most by $p^{n-t-j_{i-t}-1}$. Because of excess and Adem relations, $m(t)$ must be divisible by $p^{n-t-j_{i-t}-1}$. We obtain a contradiction.

Now the following theorem is easily deduced because $R[n]$ is a coalgebra, the map ρ is a coalgebra map, and primitives which do not involve Bockstein operations have been considered. ■

Theorem 39 *Let ρ' be the restriction of ρ between the subcoalgebras $T'[n]$ and $R'[n]$ where no Bockstein operations are allowed. Let $\hat{\imath}' : D[n] \hookrightarrow B[n]$ be the natural inclusion. Then $(\rho')^* \equiv \hat{\imath}'$, i.e. for any $e_I \in T[n]$ and $d^m = \prod_{i=0}^{n-1} d_{n,i}^{m_i} \in D[n]$,*

$$< d^m, \rho'(e_I) >=< \hat{\imath}'(d^m), e_I > .$$

We shall extend last Theorem to cases including Bockstein operations as well.

$\chi_{\min}(M_{n;s}L_n^{(p-2)}) = J_{n;s}$.

Proposition 40 *a) Let $J = J_{n,t;t-1}+(I'_{t-1,s}\oplus 0_{n-t+1})+I_{n,t}$ such that $\rho'(e_{I'_{t-1,s}}) = Q_{I_{t-1,s}}$ for $s+1 \leq t \leq n$. Then $\rho(e_J) = Q_{J_{n;s}} = \Psi(M_{n;s}L_n^{p-2})$.*

b) Let J be a sequence of length n such that $|J| = 2(p^n - p^s) - 1$ and J is not of the form described in a), then $\rho(e_J) = 0$.

c) $\hat{\imath}(M_{n;s}L_n^{p-2}) = \rho^*(M_{n;s}L_n^{p-2})$.

Proof. a) If $J = J_{n,t;t-1} + (I_{t-1,s} \oplus 0_{n-t+1}) + I_{n,t}$, then $\rho(e_J) = Q_{J_{n;s}}$ by direct computation. Let $J = J_{n,t;t-1} + (I'_{t-1,s} \oplus 0_{n-t+1}) + I_{n,t}$, then $J_{n-t+1} = J_{1;0} \oplus I_{n-t,0} = J_{n-t+1;0}$. Let us call J'_{t-1} such that $J = J'_{t-1} \oplus J_{n-t+1}$, then $J'_{t-1} = (p-1)I_{t-1,t-2}+I'_{t-1,s}$. Now using theorem 39 the conclusion is received.

b) The proof of this part is divided into two steps. i) If J contains $J_{n,t;t-1}$, then J also contains $I_{n,t}$. Suppose that is not the case, then using theorem 39 $J_{n-t+1} = (J_{1;0} + p^q I_{1,0}) \oplus I'_{n-t}$ and I'_{n-t} contains a zero. Adem relations between the first two elements of J_{n-t+1} give a zero. ii) We proceed by reverse induction on the position where the Bockstein operation appears since it can be moved to the right only. Let $s = n-2$, then there are two candidates for this case: J contains $J_{n;n-1}$ and $J_{n,n-1;n-2}$. The second case has been considered in i). Because of degree, J contains a sequence I'_{n-1} of degree $2(p^{n-1} - p^{n-2})$. Thus $J = ((p-1)I_{n-1,n-2} + I'_{n-1}) \oplus J_{1;0}$. Because of theorem 39, it follows that $\rho(e_J) = 0$ unless I'_{n-1} is of the form described in a). For the general step. In order to apply induction hypothesis, we consider the two extreme cases: $J_{n,n-1;s-1}+I_{n,n-1}$ and $J_{n;n-1}+I_{n-1}$ such that $|I_{n-1}| = 2(p^{n-1} - p^{s-1})$.

- $J_{n,n-1;s-1}+I_{n,n-1}$: Using step i) and induction hypothesis, all possible candidates are as follows. $J_{n,t;t-1}+(I'_{t-1,s-1}\oplus 0_{n-t+1})+(I_{n-1,t}\oplus 0_1)+I_{n,n-1}$ such that $\rho'(e_{I'_{t-1,s-1}}) = Q_{I_{t-1,s-1}}$ for $s \leq t \leq n-1$.
- $J_{n;n-1}+I_{n-1}$: Our sequence can be decomposed as follows, $((p-1)I_{n-1,n-2}+$

$I_{n-1}) \oplus J_{1;0}$. We note that we are interested in finding sequences I_{n-1} such that $\rho(e_{((p-1)I_{n-1,n-2}+I_{n-1})}) = Q_{((p-1)I_{n-1,n-2}+I_{n-1,s-1})}$. Using theorem 39, we conclude that all possible candidates I_{n-1} are such that $\rho(e_{(I_{n-1})}) = Q_{(I_{n-1,s-1})}$.

c) Now part c) follows from parts a) and b). ■

Lemma 41 *a) Let the sequences $K_{n,t+1;q,t}$ and $I_{n,t+1}$, then $\rho e_{(K_{n,t+1;q,t}+I_{n,t+1})} = Q_{K_{n;q,t}}$.*

b) Let the sequence $K = K_{n;q,t} + (I''_q \oplus 0_{n-q}) + (I'_t \oplus 0_{n-t})$ such that $I'_t = I'_q \oplus I'_{t-q}$, $\rho(e_{I''_q}) = Q_{I''_{q,s}}$ and $\rho(e_{I'_t}) = Q_{I_{t,m}}$. If we allow Adem relations everywhere in the first t positions except at positions between q and $q+1$ from left, then $\rho'(e_K) = e_{K'}$ where $K' = K_{n;q,t} + (I''_{q,s} \oplus 0_{n-q}) + p^{t-q-m_2}(I'_{q,m_1} \oplus 0_{n-q}) + (0_q \oplus I'_{t-q,m_2} \oplus 0_{n-t})$ or $K' = K_{n;q,t} + p^{t-q-m_2}(I'_{q,s+m_1-q} \oplus 0_{n-q}) + (0_q \oplus I'_{t-q,m_2} \oplus 0_{n-t})$. For the first case $\rho(e_{I'_t}) = Q_{I'_{t-q,m_2}}$, $\rho(e_{I'_q}) = Q_{p^{t-q-m_2}I'_{q,m_1}}$ and $m = m_1+m_2$, and for the second $s+m_1 \geq q$ and $\rho(e_{I''_q+I'_q}) = Q_{p^{t-q-m_2}I'_{q,s+m_1-q}}$.

Proof. This is an application of theorem 39. ■

Proposition 42 *a) Let $K = K_{n,t+1;s,t} + (I'_t \oplus 0_{n-t}) + I_{n,t+1}$ such that $\rho'(e_{I'_t}) = Q_{I_{t,m}}$ for $m \leq t \leq n-1$. Then $\rho(e_K) = Q_{K_{n;s,m}} = \Psi(M_{n;s,m}L_n^{p-2})$.*

b) Let $K = K_{n,m+1;t,m} + (I'_t \oplus 0_{n-t}) + I_{n,m+1}$ such that $\rho'(e_{I'_t}) = Q_{I_{t,s}}$ for $s \leq t \leq m-1$. Then $\rho(e_K) = Q_{K_{n;s,m}} = \Psi(M_{n;s,m}L_n^{p-2})$.

c) Let $K = K_{n,t+1;q,t} + I + I_{n,t+1}$ for $m \leq q < t \leq n-1$ with $I = I' + I''$, $I' = (I'_q \oplus 0_{n-q})$, $I'' = (I''_t \oplus 0_{n-t})$ such that: $\rho'(e_{I''_t}) = Q_{I_{t,m}}$ and $\rho'(e_{I'_q}) = Q_{I_{q,s}}$ and not of the form $\rho'(e_{I''_t}) = Q_{I_{t,s}}$ and $\rho'(e_{I'_q}) = Q_{I_{q,m}}$. Then $\rho(e_K) = Q_{K_{n;s,m}} = \Psi(M_{n;s,m}L_n^{p-2})$.

d) Let K be a sequence of length n such that $|K| = 2(p^n - p^s - p^m)$ and K is not of the form described in a), b) and c) above, then $\rho(e_K) = 0$.

e) $\hat{\imath}(M_{n;s,m}L_n^{p-2}) = \rho^*(M_{n;s,m}L_n^{p-2})$.

Proof. Since $K_{n;s,m}$ contains two Bockstein operations, there are three choices for moving Bockstein operations from right to left by applying Adem relations and each choice provides the degrees of the additional sequences. Let us call first Bockstein operation the first one from left.

a) If the first Bockstein operation is fixed, the second one can be in any position to the left of its m-th position: $(-, ..., -, \underset{s}{\beta}, -..., -, \underset{m}{\bullet}, \underset{\leftarrow}{-}, \underset{-}{...}\beta, \underset{-}{...}, \underset{\rightarrow}{-})$. Thus we have the sequence $K_{n,t+1;s,t}$ plus two more sequences different in the sense that one is needed to fill the zeros to the left of $K_{n,t+1;s,t}$ and the other one to force the second Bockstein to be moved to its final position after Adem relations. Now the assertion follows by direct computation.

b) If the second one is fixed, the first one can be in any position to the left of it: $(-, ..., -, \underset{s}{\bullet}, \underset{\leftarrow}{-}, \underset{-}{...}\beta, \underset{-}{...}, \underset{\rightarrow}{-}, \underset{m}{\beta}, -..., -)$. Thus we have the sequence $K_{n,m+1;t,m}$ plus two more sequences different in the sense that one is needed to fill the zeros to the left of $K_{n,m+1;t,m}$ and the other one to force the first Bockstein to be moved to its final position after Adem relations. Again the assertion follows by direct computation.

c) Of course, there is also the choice of both first and second Bockstein operations being to the right of the m-th position: $(-,...,-,\underset{s}{\bullet},-...,-,\underset{m}{\bullet}\underset{\leftarrow}{-},\underset{-}{...}\beta,\underset{-}{...},\beta,\underset{-}{...},\underset{\rightarrow}{-})$. This is the difficult case. Three sequences are needed in this case: one $(I_{n,t+1})$ to fill the zeros to the left of $K_{n,t+1;q,t}$, one $(I_{q,s} \oplus 0_{n-q})$ to force the first Bockstein to be moved to its final position, and the third one $(I_{t,m} \oplus 0_{n-t})$ to force the second Bockstein to be moved to its final position. Sequences in brackets are the final ones after Adem relations. But we must exclude those sequences which will move the second Bockstein on the first one. Those will provide a zero, because two Bockstein operations will be on the same position after applying Adem relations and this is not allowed. They are all sequences which will provide $K_{n,t+1;q,t} + (I_{q,m} \oplus 0_{n-q}) + (I_{t,s} \oplus 0_{n-t})$ after suitable Adem relations. Please see lemmata 19 and 21 to get an idea of those sequences.

In order to avoid the bad cases we consider for example sequences of the form

$$(0,...,0,\underbrace{p^{t-q-l},...,p^{t-q-l}}_{t-m-l-q+s},\underbrace{p^{t-q-l}+1,...,p^{t-q-l}+1}_{q-s},\beta\frac{1}{2},...,\frac{1}{2},\underbrace{\frac{3}{2},...,\frac{3}{2}}_{l},\beta 1,0...,0) \tag{5}$$

or

$$(0,...,0,\underbrace{1,...,1,\overbrace{p^{t-q-l}+1,...,p^{t-q-l}+1}^{t-m-l}}_{q-s},\beta\frac{1}{2},...,\frac{1}{2},\underbrace{\frac{3}{2},...,\frac{3}{2}}_{l},\beta 1,0...,0) \tag{6}$$

and not of the form $(0,...,0,\underbrace{1,...,1}_{m-s},\underbrace{2,...,2}_{q-m},\beta\frac{3}{2},\frac{3}{2},...,\frac{3}{2},\beta 1,0...,0)$. All the above are expressed in the assertion of c).

Now we prove the statements above. We consider only $q > m$. If $q = m$, it is completely analogous. Let us start with the second one. The relevant sequences are of the following form.

$$\begin{array}{cccccccccccc}
 & & & s & & m & & q & q+1 & & t & t+1 \\
 & & & \downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow \\
1\bullet & 0 & & 0 & \dots & 0 & \dots & 0 & \beta 1/2 & \dots & 1/2 & \beta 1 \\
2\bullet & 0 & p^{\gamma} & & & \dots & & p^{\delta} & p^{\delta} & \dots & 1 & 0 \\
 & & \leftarrow & - & - & t-s-l_{\text{non zero}} & - & \rightarrow & \leftarrow & l_{\text{non zero}} & \rightarrow & \\
3\bullet & 0 & \dots & 0 & p^{\zeta} & \dots & & 1 & 0 & \dots & \dots & 0 \\
 & & & & \leftarrow & q-m_{\text{non zero}} & - & \rightarrow & & & &
\end{array}$$

The three rows marked with $\bullet$ should be added. We apply Adem relations between columns $q+1$ and t. Note that row 1$\bullet$ do not effect the result and row 2$\bullet$ becomes $(0,...,0,\underbrace{1,...,1})$, because of Theorem 39. We also note that $\delta = t-q-l$. Next we consider the first q columns. For the same reason rows 2$\bullet$ and 3$\bullet$ become $(0,...,0,\underbrace{p^{\delta},...,p^{\delta}}_{t-s-l})$ and $(0,...,0,\underbrace{1,...,1}_{q-m})$.

[*]We consider columns q, $q+1$, and $q+2$ and apply Adem relations between the first two: [+] $(1+p^\delta, \beta 1/2, 1/2)$. Lemma 3 is used repeatedly. We get $(\beta 1/2, p^{\delta-1}+1/2, 1/2)$.

[**]Applying Adem relations repeatedly to the right of β, the second β is moved one position to the left because $\delta = t-q-l$ equals the number of positions between the β's.

[***]Now we start again from column $q-1$ and finally the first β is moved to its m-th position: $(p^\delta, p^\delta, \beta\underbrace{1/2, ..., 1/2}_{t-l-q}, \beta 1)$.

[****]Now we get $(p^\delta, 0, \beta\underbrace{p^{\delta-1}+1/2, ..., 1/2}_{t-l-q}, \beta 1)$ and the second β will moved one position to its left. We repeat, $(0, p^{\delta-1}, \beta\underbrace{1/2, ..., 1/2}_{t-l-q-1}, \beta 1, 1)$. And finally the two β's will be at the same position because the number of p-th powers to the right of the first β equals the exponent of p i.e. $t-l-q$. The case $(0, ..., 0, \underbrace{p^{\delta'}, ..., p^{\delta'}}_{\theta}, 0, ..., 0, \underbrace{1, ..., 1}_{q-m-\theta}) + (0, ..., 0, \underbrace{p^\delta, ..., p^\delta}_{t-s-l})$ is excluded because $\delta' = t-s-l-q+m+\theta > t-l-q$ is not of the right form.

For the other case, after suitable Adem relations, rows 2 and 3 are of the form described in 5 and 6. Let us start with 5: We suppose that $t-q-l \geq 1$, otherwise it is reduced to the previous case. As in [*], [**], and [***] the first β will be moved to the s-th position and the second one to the $t-q-l+s$-th position. As in [****], the second β will be moved to the m-th position. The case $(0, ..., 0, \underbrace{p^{\delta'}, ..., p^{\delta'}}_{\theta}, 0, ..., 0, \underbrace{1, ..., 1}_{q-s-\theta}) + (0, ..., 0, \underbrace{p^\delta, ..., p^\delta}_{t-m-l})$ has been considered above because we must have $\delta' = t-m-l-q+s+\theta = t-l-q$ or $m = s+\theta$.

For 6: Applying [*] and [**], the first β is moved to $q-t+m+l$ and the second one to the m-th position: $(0, ..., 0, \underbrace{1, ..., 1}_{q-s-t+m+l}, \beta\frac{1}{2}, ..., \frac{1}{2}, \beta 1, ..., 1, 0..., 0)$. Using lemma 19, the first β is moved to the s-th position. The case $(0, ..., 0, \underbrace{p^{\delta'}, ..., p^{\delta'}}_{q-s}, 0, ..., 0)+$ $(0, ..., 0, \underbrace{p^\delta, ..., p^\delta}_{t-m-l})$ provides $q = m$, because we must have $t-m-l = t-l-q$ or $m = q$.

d) follows from a), b), and c).

e) follows from a), b), c), d) and lemmata 19, 21, and 26. ∎

It would be a nice exercise to find for example all $e_I \in T[n]$ such that $\rho e_I = Q_{K_{7;1,3}}$ (please see example 55). That would provide the reader a good feeling of the computations involved.

Theorem 43 *Let $\rho : T[n] \to R[n]$ be the map which imposes Adem relations. Let $\hat{\imath} : S(E(n))^{GL_n} \otimes D[n] \hookrightarrow S(E(n))^{B_n} \otimes B[n]$ be the natural inclusion. Then*

$\rho^* \equiv \hat{\imath}$, *i.e. for any* $e_I \in T[n]$ *and* $d^m M^\varepsilon \in S(E(n))^{GL_n} \otimes D[n]$,

$$< d^m M^\varepsilon, \rho(e_I) > = < \hat{\imath}(d^m M^\varepsilon), e_I > .$$

Next, an example is worked out to demonstrate the idea of our algorithm. We would like to express elements of the monomial basis $\text{ß}_n(S(E(n))^{GL_n} \otimes D[n])$ with respect to the dual basis $\text{Ŗ}[n]$. Roughly speaking: Given d^m, define $\chi_{\min}(d^m)$ and evaluate all greater sequences I with the same degree. For each such a sequence consider $\chi_{\min}(\frac{\psi Q_I}{d_{n,0}^{m_0}})$ as an element in $T[n]$. Start with the greatest sequence I and evaluate the coefficient of $(\chi_{\min}(\frac{\psi Q_I}{d_{n,0}^{m_0}}))^*$ in $\hat{\imath}(\frac{d^m}{d_{n,0}^{m_0}})$. This is the coefficient of $(Q_I)^*$ in d^m. Then we continue with the next sequence.

Example 44 *Let* $p = 3$ *and* $d^I = d_{2,0}^2 d_{2,1}^{19}$, *then* $I = (2,19)$ *and* $\chi_{\min}(d^I) = (2,21)$. *To calculate its expression with respect* $\text{Ŗ}[n]$ *the following elements given by greater sequences of the same degree should be considered:* $Q_{(14,17)}$, $Q_{(11,18)}$, $Q_{(8,19)}$, $Q_{(5,20)}$, *and* $Q_{(2,21)}$. *And their images under* Ψ^{-1} *should also be considered:* $d_{2,0}^{14} d_{2,1}^3$, $d_{2,0}^{11} d_{2,1}^7$, $d_{2,0}^8 d_{2,1}^{11}$, $d_{2,0}^5 d_{2,1}^{15}$, *and* $d_{2,0}^2 d_{2,1}^{19}$. *Here we examine the Kronecker product* $< d_{2,0}^2 d_{2,1}^{19}, Q_J >$ *for* Q_J *one of the elements defined above.*

i) Start with the greatest sequence $(14,17)$. *Calculate the coefficient of* $(Q_{(14,17)})^*$ *in* d^I. *Divide both* $d_{2,0}^{14} d_{2,1}^3$ *and* d^I *by* $d_{2,0}^2$ *since sequences of the form* $lI_{n,0}$ *do not effect Adem relations. Let* $I^{(1)} = (0,19)$ *and* $J^{(1)} = (12,3)$. *Let* $'I^{(1)} = (0,19)$ *and* $'J^{(1)} = (12,15)$ *the corresponding sequences. Let* $(Q_{'J^{(1)}}) = Q_{12} Q_{15}$ *be considered as* $e_{12} e_{15}$ *an element of* $T[n]$. *Decompose* $d^{I^{(1)}} = (h_1^6 + h_2^2)^{19}$ *in* $B[n]$ *and find the coefficient of* $\Phi(e_{12} e_{15})^* = h_1^{24} h_2^{30}$ *in the last decomposition:* $\binom{19}{15} \equiv 0 \bmod 3$. *Zero is the coefficient of* $(Q_{(14,17)})^*$ *in* d^I.

ii) Check for $(Q_{(11,18)})^* \longleftrightarrow d_{2,0}^{11} d_{2,1}^7$. *Let* $I^{(1)} = (0,19)$ *and* $J^{(1)} = (9,7)$. *Let* $'I^{(1)} = (0,19)$ *and* $'J^{(1)} = (9,16)$ *the corresponding sequences. Let* $e_{(9,16)}$ *be the corresponding element of* $T[n]$. *Decompose* $d^{I^{(1)}} = (h_1^6 + h_2^2)^{19}$ *in* $B[n]$ *and find the coefficient of* $\Phi(e_{(9,16)})^* = h_1^{18} h_2^{32}$ *as an element of* $B[n]$ *in the last decomposition:* $\binom{19}{16} \equiv 0 \bmod 3$. *That is the coefficient of* $(Q_{(11,18)})^*$ *in* d^I.

iii) Check for $(Q_{(8,19)})^* \longleftrightarrow d_{2,0}^8 d_{2,1}^{11}$. *By repeating steps described above, we obtain:* $\binom{19}{17} \equiv 0 \bmod 3$.

iv) $(Q_{(5,20)})^* \longleftrightarrow d_{2,0}^5 d_{2,1}^{15} \cdot \binom{19}{18} \equiv 1 \bmod 3$.

Thus $d_{2,0}^2 d_{2,1}^{19} = (Q_{(2,21)})^* + (Q_{(5,20)})^*$.

Let us make some remarks on the last remarks. Firstly, the exponent 19 of $d_{2,1}$ and its p-adic analysis plays an important role. The maximal involved element is $d_{2,0}^{14} d_{2,1}^3$ and hence the least exponent for h_2 is 15. Thus we shall start counting from 15. On the other hand, only 18 and 19 contribute a non-zero binomial coefficient mod 3 in this range.

Theorem 45 *Let* $d^m M^\varepsilon$ *be an element of* $\text{ß}_n(S(E(n))^{GL_n} \otimes D[n])$, *then the following algorithm calculates its image in* $R[n]^*$:

$$d^m M^\varepsilon = \sum_{J \geq \chi_{\min}(d^m)} < d^m, Q_J > (Q_{(J + \chi_{\min}(M^\varepsilon))})^*$$

1) Find all elements Q_J in $R[n]$ such that $|d^m| = |Q_J|$ and $J > \chi_{\min}(d^m)$, i.e. solve the Diophantine equation $\sum_0^{n-1} k_i(p^n - p^i) = |d^m|$ for $(k_0, ..., k_{n-1}) > (m_0, ..., m_{n-1})$. For each such a sequence J, let $J(1) = J - m_0(1, ..., 1)$ and consider $\Psi^{-1}(Q_{J(1)}) = d^{J'(1)}$ in $D[n]$.

2) Let $d^m M^\varepsilon = (Q_{\chi_{\min}(d^m M^\varepsilon)})^*$.

3) Let $d^{m(1)} = \frac{\mathrm{d^m}}{\mathrm{d_{n,0}^{m_0}}}$ and d^K be an element in step 1) corresponding to the biggest sequence among those which have not been considered yet. If $d^{K(1)} = d^{m(1)}$, then $\alpha_{(K)} =< d^m, Q_K >= 1$. Otherwise, proceed as follows: find the coefficient, $\alpha_{(K)}$, of $\iota(d^{K(1)})$ in $\hat{\imath}(d^{m(1)})$, $\alpha_{(K)} =< d^m, Q_K >$. Then add $\alpha_{(K)}(Q_{K+\chi_{\min}(M^\varepsilon)})^*$ in $d^m M^\varepsilon$.

4) Repeat step 3).

Proof. Since $R[n]^*$ is a free module over $D[n]$ with basis all elements which involve Bockstein operations, the computation of $d^m M^\varepsilon$ reduces to that of d^m, i.e.

$$d^m = \sum_{J \geq \chi_{\min}(d^m)} < d^m, Q_J > (Q_{(J)})^* \Rightarrow d^m M^\varepsilon = \sum_{J \geq \chi_{\min}(d^m)} < d^m, Q_J > (Q_{(J+\chi_{\min}(M^\varepsilon))})^*$$

Let $d^m = \sum \alpha_{(I)}(Q_I)^*$ and $n(m) = \sum_{t=0}^{n-1} m_t$. Because of the definition of the hom-dual, we have : $< d^m, Q_{\chi_{\min}(d^m)} >= 1$ and $< d^m, Q_I >= a_{(I)} \neq 0$ for a sequence I such that in the $n(m)$-times iterated coproduct:

$$\psi Q^I = \sum_{\Sigma J_t = I} e_{J_1} \otimes ... \otimes e_{J_{n(m)}} \overset{Adem}{=} \sum a_{I_1,...,I_{n(m)}} Q_{I_1} \otimes ... \otimes Q_{I_{n(m)}}$$

$a_{(I)} \bigotimes_{t=0}^{n-1} \bigotimes_1^{m_t} (Q_{I_{n,t}})$ is a summand. Thus $I \geq \chi_{\min}(d^m)$. Let $I_1 > \cdots > I_l > \chi_{\min}(d^m)$ be all sequences such that $|Q_{I_t}| = |Q_{\chi_{\min}(d^m)}|$.

We quote from May page 20: if for each $d^m M^\varepsilon$ we associate its coefficients $a_{(I)}$ as a matrix $(a_{\chi_{\min}(d^m M^\varepsilon),(I)})$, then this matrix is upper triangular with ones along the main diagonal. This allows us to express one basis element $d^m M^\varepsilon$ with respect to the dual basis of admissible monomials.

We consider the first sequence I_1. Our task is to evaluate $\alpha_{(I_1)}$. Let ψQ_{I_1} be the iterated coproduct applied $n(m)$-times. We shall write I_1 as a sum of $n(m)$ sequences such that each of them is a primitive element of $R[n]$ equals to one of those involved in $\chi_{\min}(d^m)$. This is possible, since $n(m) \geq n(\chi_{\min}(\Psi^{-1}(Q_{I_1})))$. The common element $d_{n,0}^{m_0}$ between $\Psi^{-1}(Q_{(I_1)})$ and d^m does not change the coefficient $\alpha_{(I_1)}$, because no Adem relation can reduce $Q_{I_{n,0}}$ to a smaller sequence. Instead, we consider $Q_{I_1 - m_0 I_{n,0}}$ $(d^{J_1} = \Psi^{-1}(Q_{(I_1)})/d_{n,0}^{m_0})$ and $Q_{(\chi_{\min}(d^m) - m_0 I_{n,0})}$ $(d^{m(1)} = d^m/d_{n,0}^{m_0})$. Now the iterated coproduct is applied $n(m(1))$-times.

For the second part of step 3), we use $\psi\rho = \rho\psi$, lemma 37 and proposition 38. All elements $e_I \in T[n]$, which have the property $\rho e_I = Q_{I_{n,n-i}}$, are known. Moreover, the dual of those elements, $(e_I)^* \in B[n]$, are summands in $\hat{\imath}(d_{n,n-i})$.

Using commutativity in $D[n]$ induced by symmetry in coproduct, we deduce that the required coefficient is the coefficient of $\iota(d^{J_1})$ in $\hat{\imath}(d^{m(1)})$. ■

Remark 46 *Suppose that $(Q_I)^*$ is to be expressed with respect to $\beta_n(S(E(n))^{GL_n}\otimes D[n])$, then one starts with the biggest sequence, say $K(1)$, $\Psi^{-1}(Q_{K(1)}) = (Q_{K(1)})^*$, then substitutes in the next element $\Psi^{-1}(Q_{K(2)}) = (Q_{K(2)})^* + a_{K(2),K(1)}(Q_{K(1)})^*$ $\Rightarrow (Q_{K(2)})^* = \Psi^{-1}(Q_{K(2)}) - a_{K(2),K(1)}\Psi^{-1}(Q_{K(2)})$ and so on.*

Let us make some comments. If the degree m of a monomial d^m is quite high, then there exist many elements of the same degree such that the dual of their images under Φ do not appear in d^m for a variety of reasons. We shall give a refinement of the algorithm described above through the next lemmas.

Example 47 *Let $d = d_{3,0}d_{3,1}^{p}d_{3,2}^{1+p^3}$. We will use relation ??, 37 and 4 in order to improve the algorithm described above. The idea is to use $d_{3,1}^{p}$ and or $d_{3,2}^{1+p^3}$ to create d' such that $(\Psi(d'))^*$ will be a term in the image of d in $R[n]^*$. Please note that we abuse notation for simplicity here. For $i(d_{3,1}) = h_2h_3$ we need a suitable product of $d_{3,2}$. Since $d_{3,2} = h_1^{p^2} + h_2^{p} + h_3$, our candidate $d_{3,1}$ must have exponent at least p (because of h_2) and $d_{3,2}^{p}$ also is needed to get h_3^{p}. On the other hand we have $\binom{1+p^3}{1} \equiv \binom{1}{1}$ choices for h_2^p. Thus $\Psi(d' = d_{3,0}d_{3,1}^{2p}d_{3,2}^{p^3-p}) = Q_1Q_{2p+1}Q_{p^3+p+1}$ is a term in d. Next we consider $i(d_{3,0}) = h_1h_2h_3$ to a suitable power. Let us try to use $d_{3,2}$ only. We need such a product $d_{3,2}^{\gamma(3)}d_{3,2}^{\gamma(2)}d_{3,2}^{\gamma(1)}$ and we shall define the exponents and its binomial coefficient. If $\gamma(3) = 1$, then $d_{3,2}^{\gamma(3)}$ provides $h_1^{p^2}$ and forces $\gamma(2) = p$ and $\gamma(1) = p^2$ for h_2 and h_3 respectively. Now its binomial coefficient: $\binom{1+p^3}{1}\binom{p^3}{p} \equiv 0$. Hence $i(d_{3,0})$ can not be created using only powers of $d_{3,2}$. Now let us consider $d_{3,0}^{\ell}$. Using $d_{3,1}d_{3,2}^{p}$ with two different ways $((h_1h_2)^p(h_3)^p$ or $h_1^p h_3 (h_2)^p (h_3)^{p-1})$ we get $d_{3,0}^{p}$ with coefficient $[\binom{p}{1} + \binom{p}{1}\binom{1+p^3}{1}] \equiv 0 \bmod p$. Using $d_{3,1}^{p}d_{3,2}^{p^2}$ we get $d_{3,0}^{p^2}$ with two different ways $((h_1h_2)^{p^2}(h_3)^{p^2}$ or $h_1^{p^2}h_3^{p}(h_2)^{p^2}(h_3)^{p^2-p})$, i.e. $d' = d_{3,0}^{p^2+1}d_{3,2}^{p^3-p^2+1}$. Here the coefficient is $[\binom{p}{p} + \binom{p}{p}\binom{i+p^3}{p^2}] \equiv 1 \bmod p$. Moreover, repeating this process we also get $d'' = d_{3,0}^{p^2+1}d_{3,1}^{p}d_{3,2}^{p^3-p^2-p}$. Finally,*

$$d = (Q_1Q_{p+1}Q_{p^3+p+2})^* + \left(Q_1Q_{2p+1}Q_{p^3+p+1}\right)^* + \left(Q_{p^2+1}Q_{p^3+2}Q_{p^3+2}\right)^* + \left(Q_{p^2+1}Q_{p^2+p+1}Q_{p^3+1}\right)^*$$

Definition 48 *Let $d^m = \prod_{i=0}^{n-1} d_{n,i}^{m_i}$ be a monomial in the Dickson algebra and $m_i = \sum_{t=0}^{\ell_i} a_{i,t}p^t$. Let $i_0 = \max\{i \mid m_i \neq 0\}$ and $0 \leq t < i_0$. Let $\delta(t)$ be a positive integer such that $t \leq \gamma(s) \leq n-1$ for $s = 1, ..., \delta(t)$ and $\sum_{1}^{\gamma(\delta(t))}(n-\gamma(s)) = n-t$.*

Let also $\ell_{(t,\gamma(1),...,\gamma(\delta(t)))} = \max\{\gamma(s) - (\sum_1^s \gamma(j)) + (s-1)n \mid s = 1, ..., \gamma(\delta(t))\}$ *and* $0 \le c \le \min\{\ell_{\gamma(s)} - \ell_{(t,\gamma(1),...,\gamma(\delta(t)))} + \sum_{j=1}^{s-1}(n-\gamma(j))\}$. *We define*

$$\zeta(t,\gamma(1),...,\gamma(\delta(t)),c,\mu) = \begin{cases} \prod_{s=2}^{\gamma(\delta(t))} \binom{a}{\gamma(s),c+\ell_{(t,\gamma(1),...,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))}, \\ \text{if } 0 \le m_{\gamma(1)} - \mu p^{c+\ell_{(t,\gamma(1),...,\gamma(\delta(t)))}} \\ 0, \text{ otherwise} \end{cases}.$$

Here $1 \le \mu \le \min\{a_{\gamma(s),c+\ell_{(t,\gamma(1),...,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))} \mid s = 2, ..., \delta(t)\}$.

Proposition 49 *Let* $d^m = \prod_{i=0}^{n-1} d_{n,i}^{m_i}$ *be a monomial in the Dickson algebra as above. Then* d^m *contains* $\left(\Psi(d^m d_{n,t}^{p^{\ell_{(t,\gamma(1),...,\gamma(\delta(t)))}}} / \prod_1^{\gamma(\delta(t))} d_{n,\gamma(s)}^{p^{\ell_{(t,\gamma(1),...,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))}})\right)^*$ *with coefficient*

$$\sum_{\gamma(1),...,\gamma(\delta(t))} \zeta(t,\gamma(1),...,\gamma(\delta(t)),0,1)+$$
$$\sum_{\gamma'(1),...,\gamma'(\delta'(t))} \prod_{i\in I_{(t,\gamma'(1),...,\gamma'(\delta'(t)))}} \binom{m_i}{\sigma_i(t,\gamma'(1),...,\gamma'(\delta'(t)))}$$

such that $\ell_{(t,\gamma(1),...,\gamma(\delta(t)))} = \ell'_{(t,\gamma'(1),...,\gamma'(\delta'(t)))}$ *and* $\prod_1^{\gamma(\delta(t))} d_{n,\gamma(s)}^{p^{\ell_{(t,\gamma(1),...,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))}} = \prod_{i\in I_{(t,\gamma'(1),...,\gamma'(\delta'(t)))}} d_{n,i}^{\sigma_i(t,\gamma'(1),...,\gamma'(\delta'(t)))}$. *Here* $\{\gamma(1), ..., \gamma(\delta(t))\}$ *and* $\{\gamma'(1), ..., \gamma'(\delta'(t))\}$ *are partitions of* $\{t+1, ..., n\}$ *of consecutive and non-consecutive elements respectively. For the definition of* $I_{(t,\gamma'(1),...,\gamma'(\delta'(t)))}$ *and* $\sigma_i(t,\gamma'(1),...,\gamma'(\delta'(t)))$, *please see the second case in the proof bellow because they strongly depend on the particular partition.*

Proof. Since $i(d_{n,t}) = (h_{t+1}...h_n)^{p-1}$, we are interested in partitions of $(h_{t+1}...h_n)^{(p-1)\ell}$ such that each part of each partition is a term of an appropriate power of $d_{n,\gamma(s)}$ for a suitable $\gamma(s)$.

1) Suppose that each of these terms consists of consecutive h_j's. In other words we consider all partitions of the set $\{t+1, ..., n\}$ such that

$$\{t+1, ..., t+n-\gamma(\delta(t)), t+n-\gamma(\delta(t))+1, ..., t+2n-\gamma(\delta(t))-\gamma(\delta(t)-1), ...,$$
$$\gamma(1)+...+\gamma(s)-(s-1)n+1, ..., \gamma(1)+...+\gamma(s-1)-(s-2)n, ...,$$
$$\gamma(1)-n+\gamma(2)+1, ..., \gamma(1), \gamma(1)+1, ..., n\}$$

where

$$\left(h_{\gamma(1)+...+\gamma(s)-(s-1)n+1}...h_{\gamma(1)+...+\gamma(s-1)-(s-2)n}\right)^{(p-1)p^{k_{\gamma(s)}}}$$

is a term of a suitable power of $d_{n,\gamma(s)}$. Firstly, we must have $\sum_{1}^{\gamma(\delta(t))}(n-\gamma(s))=n-t$. Secondly, the appropriate powers must be considered. Let $\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}=\max\{\gamma(s)-(\sum_{1}^{s}\gamma(j))+(s-1)n \mid s=1,\ldots,\gamma(\delta(t))\}$. Here we considered the p-th power of $h_{\gamma(1)+\ldots+\gamma(s)-(s-1)n+1}$. Let that entry be t_0'. Actually, in this case $t_0'=\gamma(\delta(t))$. We shall use the general description in the next case. We start with $d_{n,\gamma(t_0')}$ having exponent 1. This forces the rest of the exponents to be as follows:

$$d_{n,\gamma(\delta(t))}^{p^{\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}-\gamma(\delta(t))-t}} \ldots d_{n,\gamma(s)}^{p^{\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))}} \ldots d_{n,\gamma(1)}^{p^{\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}}}$$

and we are expecting to get $d_{n,t}^{p^{\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}}}$ with an appropriate coefficient because of the different choices which there exist for each particular term. The last term $(h_{\gamma(1)+1}\ldots h_n)^{p-1}$ has no choice and each of the other terms has $\binom{a_{\gamma(s),\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))}}{1}$ choices. In general, if ℓ is the exponent of $d_{n,t}$, then ℓ can be at most $\min\{m_{\gamma(s)}-\left(p^{\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))}\right) \mid s=1,\ldots,\gamma(\delta(t))\}$. Since we are considering only multiples of p-th powers we can increase each p-th power by c where

$$0\leq c\leq \min\{\ell_{\gamma(s)}-\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}+\sum_{j=1}^{s-1}(n-\gamma(j))\}$$

and each multiple μ can be between 1 and $\min\{a_{\gamma(s),c+\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}-\sum_{j=1}^{s-1}(n-\gamma(j))} \mid s=2,\ldots,\delta(t)\}$. In that case, $d_{n,t}^{\mu p^{c+\gamma(\delta(t))-t}}$ is expected with coefficient which contains $\sum_{\gamma(1),\ldots,\gamma(\delta(t)),c,\mu}\zeta(t,\gamma(1),\ldots,\gamma(\delta(t)),c,\mu)$ as a summand, because of choices of the second case.

2) Now we consider the case where partitions contain at least a part with no consecutive entries. The only difference is that extra elements $d_{n,\gamma(s)}$'s are needed in order to bring the exponents to the right number but this creates technical difficulties. So we will skip the definition of the analogue of c and μ in this case. Let such a partition be

$$(i_{\gamma(s),1},\ldots,i_{\gamma(s),n-\gamma(s)})$$

Let $(i_{\gamma(s),\alpha_1},\ldots,i_{\gamma(s),\alpha_q})$ be a subset of the set above such that $\alpha_1>1$, $i_{\gamma(s),\alpha_r}+1=i_{\gamma(s),\alpha_{r+1}}$ for $r=1,\ldots,q-1$ and $i_{\gamma(s),\alpha_1-1}+1<i_{\gamma(s),\alpha_1}$, $i_{\gamma(s),\alpha_q}+1=i_{\gamma(s),\alpha_{q+1}}$. For each such a subset we consider:

$$d_{n,n-q}^{p^{\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}-n+q-1+\alpha_1}-p^{\ell_{(t,\gamma(1),\ldots,\gamma(\delta(t)))}-n+q-1+\alpha_1+i_{\gamma(s),1}+\alpha_1-1-i_{\gamma(s),\alpha_1}}}$$

with the appropriate coefficient:

$$\binom{m_{n-q}}{p^{\ell(t,\gamma(1),...,\gamma(\delta(t)))-n+q-1+\alpha_1} - p^{\ell(t,\gamma(1),...,\gamma(\delta(t)))-n+q-1+\alpha_1+i_{\gamma(s),1}+\alpha_1-1-i_{\gamma(s),\alpha_1}}}$$

Here $\ell_{(t,\gamma(1),...,\gamma(\delta(t)))}$ is as in case 1). Note that many different parts of a particular partition may need powers of $d_{n,n-q}$. We also need the set of indices for such $d_{n,n-q}$. Let this set be $\{q_1, ..., q_{(t,\gamma(1),...,\gamma(\delta(t)))}\}$. The set of indices given by $I_{(t,\gamma(1),...,\gamma(\delta(t)))} := \{\gamma(1), ..., \gamma(\delta(t))\} \cup \{n-q_1, ..., n-q_{(t,\gamma(1),...,\gamma(\delta(t)))}\}$ must be considered. Let $\sigma_i(t,\gamma(1),...,\gamma(\delta(t)))$ denote the total number of $d_{n,i}$ involved in the particular partition for $i \in I_{(t,\gamma(1),...,\gamma(\delta(t)))}$, then the coefficient for $d_{n,i}$ is $\binom{m_i}{\sigma_i(t,\gamma(1),...,\gamma(\delta(t)))}$.

Finally, we combine both cases such that only partitions $\{\gamma(1), ..., \gamma(\delta(t))\}$ (for the first case) and $\{\gamma'(1), ..., \gamma'(\delta'(t))\}$ (for the second case) which provide $d_{n,t}^{p^{\ell(t,\gamma(1),...,\gamma(\delta(t)))}}$ and

$\prod_1^{\gamma(\delta(t))} d_{n,\gamma(s)}^{p^{\ell(t,\gamma(1),...,\gamma(\delta(t)))-\sum_{j=1}^{s-1}(n-\gamma(j))}} = \prod_{i\in I_{(t,\gamma'(1),...,\gamma'(\delta'(t)))}} d_{n,i}^{\sigma_i(t,\gamma'(1),...,\gamma'(\delta'(t)))}$ are considered. And the coefficient for

$\left(\Psi(d^m d_{n,t}^{p^{\ell(t,\gamma(1),...,\gamma(\delta(t)))}} / \prod_1^{\gamma(\delta(t))} d_{n,\gamma(s)}^{p^{\ell(t,\gamma(1),...,\gamma(\delta(t)))-\sum_{j=1}^{s-1}(n-\gamma(j))}})\right)^*$ in d^m is

$\sum_{\gamma(1),...,\gamma(\delta(t))} \zeta(t,\gamma(1),...,\gamma(\delta(t)),0,1) + \sum_{\gamma'(1),...,\gamma'(\delta'(t))} \prod_{i\in I_{(t,\gamma'(1),...,\gamma'(\delta'(t)))}} \binom{m_i}{\sigma_i(t,\gamma'(1),...,\gamma'(\delta'(t)))}$.

■

Next we consider a lemma in the opposite direction of last proposition.

Lemma 50 *Let $k \le n-i$ and $i < n$, then*
$d_{n,n-i}^{\alpha_k p^k+\alpha_0} = \left(\Psi(d_{n,n-i}^{\alpha_k p^k+\alpha_0})\right)^* +$
$\binom{\alpha_k}{\min(\alpha_k,\alpha_0)}\binom{\alpha_0}{\min(\alpha_k,\alpha_0)}\left(\Psi(d_{n,n-i-k}^{\min(\alpha_k,\alpha_0)p^k} d_{n,n-i}^{(\alpha_k-\min(\alpha_k,\alpha_0))p^k+(\alpha_0-\min(\alpha_k,\alpha_0))} d_{n,n-i+k}^{\min(\alpha_k,\alpha_0)})\right)^*$

Proof. We consider all admissible sequences in $\left(i(d_{n,n-i}^{p^k})\right)^{\alpha_k}(i(d_{n,n-i}))^{\alpha_0}$.

■

Note that $d_{n,n-i}^{\alpha_k p^k+...+\alpha_0}$ can be computed by repeated use of the formulae in the last lemma for all possible choices.

Remark 51 *We must admit that if $m(n) >> 0$, then there exist many candidates for m' and the bookkeeping described above can not be done by hand. We believe that it is harder but safer to consider all possible choices.*

Let us briefly discuss and describe Campbell, Peterson and Selick's method which calculates

$$d^m = \sum < d^m, Q_I > (Q_I)^*$$

Recall that $Q_{I_{n,i}} = \Psi(\iota)^{-1}(h^{J_{n,i}})$. For $d^m = \underbrace{d_{n,0}\ldots d_{n,0}}_{m_0}\ldots\underbrace{d_{n,n-1}\ldots d_{n,n-1}}_{m_{n-1}} = \prod_{i=1}^{n(m)} d(i)$, let $\Im(d^m)$ be the set of all $n(m) \times n$ matrices C whose i-th row C_i has the property that $\rho(e_{C_i}) = \Psi_C(\iota(d(i)))$.

For example, if $d(i) = d_{n,t}$ then $\rho(e_{C_i}) = Q_{\chi_{\min}(d(i))}$.

For each $C \in \Im(d^m)$, let $S(C) = \mathbf{1} \cdot C$. Here $\mathbf{1} = (1, \ldots, 1)$. Then

$$d^m = \sum_{\{C \in \Im(d^m) \mid S(C) \text{ is admissible}\}} Q^*_{S(C)}$$

Please note the analogy between matrices of exponents defined in section 3 and elements of $\Im(d^m)$.

We copy the next example from [2] and compare the two methods.

Example 52 *Let $p = 2$ and $n = 3$. Let $d = d_{3,0}d_{3,1}^2d_{3,2}^9$. The matrix associated with d is*

$$\begin{array}{l}
1\{\begin{matrix}0&0&1\end{matrix}\\
8\left\{\begin{matrix}0&0&1\\ &\vdots& \\0&0&1\end{matrix}\right.\\
2\left\{\begin{matrix}0&1&1\\0&1&1\end{matrix}\right.\\
1\{\begin{matrix}1&1&1\end{matrix}\\
S(C)=(1\ 3\ 12)
\end{array}$$

Since $p = 2$, in computing $\Im(d)$ it suffices to consider only matrices C whose braced rows are identical, since for other C, $S(C)$ will occur an even number of times by symmetry. There are $3 \cdot 3 \cdot 3 = 9$ choices. Most of them will be excluded by admissibility. This refinement can easily be done if the exponents involved are relatively small. Since the rest of them appear just once:

$$\begin{array}{l}
1\{\begin{matrix}0&2&0\end{matrix}\\
8\left\{\begin{matrix}0&0&1\\ &\vdots& \\0&0&1\end{matrix}\right.\\
2\left\{\begin{matrix}0&1&1\\0&1&1\end{matrix}\right.\\
1\{\begin{matrix}1&1&1\end{matrix}\\
S(C)=(1\ 5\ 11)
\end{array}
\qquad
\begin{array}{l}
1\{\begin{matrix}0&0&1\end{matrix}\\
8\left\{\begin{matrix}0&0&1\\ &\vdots& \\0&0&1\end{matrix}\right.\\
2\left\{\begin{matrix}2&2&0\\2&2&0\end{matrix}\right.\\
1\{\begin{matrix}1&1&1\end{matrix}\\
S(C)=(5\ 5\ 10)
\end{array}
\qquad
\begin{array}{l}
1\{\begin{matrix}0&2&0\end{matrix}\\
8\left\{\begin{matrix}0&0&1\\ &\vdots& \\0&0&1\end{matrix}\right.\\
2\left\{\begin{matrix}2&2&0\\2&2&0\end{matrix}\right.\\
1\{\begin{matrix}1&1&1\end{matrix}\\
S(C)=(5\ 7\ 9)
\end{array}$$

So $d = Q^*_{(1,3,12)} + Q^*_{(1,5,11)} + Q^*_{(5,5,10)} + Q^*_{(5,7,9)}$.

According to our method, $d_{3,0}$ is not considered and $9 = 8 + 1$. There are $3 \cdot 3 \cdot 3 = 9$ monomials in $d_{3,1}^2 d_{3,2}^9$ and only 4 of them are admissible elements of $B[n]$. Namely: $h_2^2h_3^{11}$ $h_2^4h_3^{10}$, $h_1^4h_2^4h_3^9$ and $h_1^4h_2^6h_3^8$. So $d = Q^*_{(1,3,12)} + Q^*_{(1,5,11)} + Q^*_{(5,5,10)} + Q^*_{(5,7,9)}$.

Remark 53 *C-P-S's method is more or less the same with our method of decomposing d^m using matrices in section 3.*

One advantage of the method of monomials is the use of computational packages and a theoretic advantage is the description of the inclusion map

$$i : \Sigma_p \int \cdots \int \Sigma_p \to \Sigma_{p^n}$$

in mod $-p$ *(co)homology.*

In principal both methods are equivalent.

Next, the algorithm which calculates Adem relations using modular invariants is demonstrated. Roughly speaking: For the given e_I, let $h^{I'} = (e_I)^*$ and compute all admissible sequences K with the same degree and smaller than the given one. For each such a K, let $d^{K'} = \psi(Q_K)$ and evaluate i) the coefficient of $h^{I'}$ in $\hat{\imath}(d^{K'})$ and ii) $d^{K'}$ in $R[n]^*$. Starting with the greatest sequence K whose coefficient in $\rho(e_I)$ has not been defined yet, using the Kronecker product, and i) and ii) above, we compute its coefficient in $\rho(e_I)$. Then we proceed to the next sequence.

Example 54 *Let $p = 3$ and $I = (20, 15)$. Then $\rho(e_I) = 2Q_{(11,18)} + 2Q_{(8,19)}$ using Adem relations in the Dyer-Lashof algebra. We shall also evaluate $\rho(e_I)$ using the following algorithm.*

1) Solve the Diophantic equation: $m_0(p^2 - 1) + m_1(p^2 - p) = (p - 1)(20 + p15)$ and for each solution define d^m or find all sequences K, $Q_K \in R[n]$ and $\Psi^{-1}(Q_K) = d^{K'} \in D[n]$ such that $|d^{K'}| = |e_I| = 2(p-1)(20+p15)$. Using their decomposition in $B[n]$ check those which contain $(e_I)^* = h_1^{20}h_2^{15}$ as a summand. Those are:

$d_{2,0}^{14}d_{2,1}^{3}$, $Q_{(14,17)}$, *no;*
$d_{2,0}^{11}d_{2,1}^{7}$, $Q_{(11,18)}$, *yes with coefficient* 2*;*
$d_{2,0}^{8}d_{2,1}^{11}$, $Q_{(8,19)}$, *no;*
$d_{2,0}^{5}d_{2,1}^{15}$, $Q_{(5,20)}$, *no;*
$d_{2,0}^{2}d_{2,1}^{19}$, $Q_{(2,21)}$, *no.*

2) Calculate their duals:

$d_{2,0}^{14}d_{2,1}^{3} = (Q_{(14,17)})^*$ *(there is only one choice).*

$d_{2,0}^{11}d_{2,1}^{7}$. *Divide both $d_{2,0}^{14}d_{2,1}^{3}$ and $d_{2,0}^{11}d_{2,1}^{7}$ by $d_{2,0}^{11}$. Consider $d_{2,0}^{3}d_{2,1}^{3}$ and $d_{2,1}^{7}$. $7 = 2\cdot 3+1$. Hence we can take only $d_{2,1}$ and replace it by h_1^3. This gives $h_1^3h_2^{2\cdot 3} = h_1^3\iota(d_{2,1}^3) = \iota(d_{2,0}^3d_{2,1}^3)$. Thus $d_{2,0}^{11}d_{2,1}^{7} = (Q_{(11,18)})^* + (Q_{(14,17)})^*$.*

$d_{2,0}^{8}d_{2,1}^{11}$. *Repeating steps described above, we obtain: $d_{2,0}^{8}d_{2,1}^{11} = (Q_{(8,19)})^* + 2(Q_{(11,18)})^* + (Q_{(14,17)})^*$.*

$d_{2,0}^{5}d_{2,1}^{15} = (Q_{(5,20)})^* + 2(Q_{(14,17)})^*$.

$d_{2,0}^{2}d_{2,1}^{19} = (Q_{(2,21)})^* + (Q_{(5,20)})^*$.

3) Use the Kronecker product to evaluate $\rho(e_I)$.

Start with $d^{K'}$ such that K' is the biggest sequence where the first non-zero coefficient of $(e_I)^* = h_1^{20}h_2^{15}$ appears in $d^{K'}$.

$< d_{2,0}^{11}d_{2,1}^{7}, \rho(e_I) > = < \hat{\imath}(d_{2,0}^{11}d_{2,1}^{7}), e_I > \equiv 2 \Rightarrow < (Q_{(11,18)})^* + (Q_{(14,17)})^*, \rho(e_I) > \equiv 2 \Rightarrow Q_{(11,18)}$ *has coefficient* 2 *in* $\rho(e_I)$.

$< d_{2,0}^{8}d_{2,1}^{11}, \rho(e_I) >=< \hat{\imath}(d_{2,0}^{8}d_{2,1}^{11}), e_I >\equiv 0 \Rightarrow < (Q_{(8,19)})^* + 2(Q_{(11,18)})^* + (Q_{(14,17)})^*, \rho(e_I) >\equiv 0 \Rightarrow < (Q_{(8,19)})^*, \rho(e_I) > +1 \equiv 0 \Rightarrow Q_{(8,19)}$ *has coefficient* 2 *in* $\rho(e_I)$.

$< d_{2,0}^{5}d_{2,1}^{15}, \rho(e_I) >=< \hat{\imath}(d_{2,0}^{5}d_{2,1}^{15}), e_I >\equiv 0 \Rightarrow Q_{(5,20)}$ *has coefficient* 0 *in* $\rho(e_I)$.

$< d_{2,0}^{2}d_{2,1}^{19}, \rho(e_I) >=< \hat{\imath}(d_{2,0}^{2}d_{2,1}^{19}), e_I >\equiv 0 \Rightarrow Q_{(2,21)}$ *has coefficient* 0 *in* $\rho(e_I)$.

Hence $\rho(e_I) = 2Q_{(11,18)} + 2Q_{(8,19)}$.

Example 55 *Let* $I = (p^3+p, p^3+p, p^3+p, 0, \beta\frac{1}{2}, \frac{1}{2}, \beta 1)$*, then* $\Psi_T^{-1}(e_I) = (h_1h_2h_3)^{p^3+p}(M_{5;4}L_5^{p-2}M_{7;6}L_7^{p-2}/d_{5,0})$.

We apply Adem relations on e_I*:* $I \to (p^3+p, p^3+p, 0, p^2+1, \beta\frac{1}{2}, \frac{1}{2}, \beta 1) \to (p^3+p, 0, p^2+1, p^2+1, \beta\frac{1}{2}, \frac{1}{2}, \beta 1) \to (0, p^2+1, p^2+1, p^2+1, \beta\frac{1}{2}, \frac{1}{2}, \beta 1) \to (0, p^2+1, p^2+1, \beta\frac{1}{2}, p+\frac{1}{2}, \frac{1}{2}, \beta 1) \to (0, p^2+1, p^2+1, \beta\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \beta 1) \to (0, p^2+1, \beta\frac{1}{2}, p+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \beta 1) \to (0, p^2+1, \beta\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \beta 1) \to (0, \beta\frac{1}{2}, p+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \beta 1) \to (0, \beta\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \beta 1) \to (0, \beta\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \beta 1, 1) \to (0, \beta\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \beta 1, 1, 1) \to (0, \beta\frac{1}{2}, \frac{1}{2}, \beta 1, 1, 1, 1) = K_{7;1,3}$. *Thus* $\rho(e_I) = Q_{K_{7;1,3}}$.

We use the monomial method. For degree reasons $Q_{K_{7;1,3}}$ *is the only possible candidate. We must check whether* $\hat{\imath}(\Psi_T^{-1}(Q_{K_{7;1,3}})) = \hat{\imath}(M_{7;1,3}L_7^{p-2})$ *contains* $(h_1h_2h_3)^{p^3+p}(M_{5;4}L_5^{p-2}M_{7;6}L_7^{p-2}/d_{5,0})$. $\hat{\imath}(M_{7;1,3}L_7^{p-2})$ *contains (lemma 26)* $(M_{5;4}L_5^{p-2}M_{7;6}L_7^{p-2}/d_{5,0})(d_{4,1}d_{6,3} - d_{4,3}d_{6,1})$ *and* $(h_1h_2h_3)^{p^3+p}$ *is a summand in (application of formula 2)* $(d_{4,1}d_{6,3} - d_{4,3}d_{6,1})$*. It follows that* $\rho(e_I) = Q_{K_{7;1,3}}$.

Now, the following proposition is obvious.

Proposition 56 *Let* $e_I \in T[n]$*. The following algorithm computes* $\rho(e_I)$ *in* $R[n]$.

i) Let $\Re = \{m = (m_0, ..., m_{n-1})\}$ *be all solutions of* $|I| = \sum_0^{n-1} m_i(p^n - p^i) + \sum_1^{l'}(p^n - p^{s_i} - p^{k_i})$*. Note that* s_i *and* k_i *are uniquely defined by lemma 28. Let* $\c{K}$ *be the set of all admissible sequences* K *such that* $|K| = |I|$ *and* $K \leq I$*. Moreover,* $Q_K \in R[n]$ *and* $Q_K = \Psi^{-1}(d^m M^\epsilon)$ *for* $m \in \Re$.

ii) Let $h^{I'} = \Psi_T^{-1}(e_I)$ *and find* $b_{I,K}$ *the coefficient of* $h^{I'}$ *in* $\hat{\imath}(d^m M^\epsilon)$ *for all elements of* $\Re$.

iii) Compute the image of $d^m M^\epsilon$ *in* $(R[n])^*$.

iii) Use the Kronecker product to evaluate $\rho(e_I)$ *:*

Start with the first non-zero b_{I,K_1}*,* $\rho(e_I)$ *contains* $a_{I,K_1}Q_{K_1}$*; i.e.* $< d^{K_1'}, \rho(e_I) >= a_{I,K_1} = b_{I,K_1}$*. Proceed to the next sequence* K_2 *and use* b_{I,K_2} *(whether or not is zero) and the image of* $d^{K_2'}$ *to compute the coefficient* a_{I,K_2} *of* Q^{K_2} *in* $\rho(e_I)$*. Repeat last step for all remaining sequences.*

We close this work by making some remarks about evaluating $\rho(e_I)$ using matrices introduced in section 4. Since $(e_I)^* = h^{I'}$ is an element of $B[n]$, one has to find all sequences $m = (m_0, \cdots, m_{n-1})$ such that d^m contains $(e_I)^*$ as a summand. This is equivalent to find all matrices C such that $(e_I)^* = \prod_{t=1}^{n} h_t^{(\mathbf{1}\cdot C)_{t-1}}$ and then group them in different sets such that each set corresponds to an m. The coefficient $\alpha_{'m}$ of $Q^{'m}$ in $\rho(e_I)$ is a function of the order of the set corresponding to m. Given $h^{I'}$, there is a great number of choices for C depending on I' as the interested reader can easily check and this is the reason for the high complexity of Adem relations.

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