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Regular geodesic languages and the falsification by fellow traveler property

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Abstract We furnish an example of a finite generating set for a group that does not enjoy the falsification by fellow traveler property, while the full language of geodesics is regular.

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Keywords Regular language, falsification by fellow traveler property

1 Introduction

In this short note we answer the following question of Neumann and Shapiro from [5]:

Question Can one find a monoid generating set A of a group G so that the language of geodesics is regular but A does not have the falsification by fellow traveler property?

The converse to this statement is Proposition 4.1 of their paper, which states that if A has the falsification by fellow traveler property then the full language of geodesics on A is regular, and this fact is the reason for the property's existence. Several authors have used the falsification by fellow traveler property as a route to finding other (geometric) properties of groups; Rebbechi uses a version of the property to prove that relatively hyperbolic groups are biautomatic [6], and the author exploits the property to prove that a certain class of groups is almost convex [4]. The author discusses various attributes and extensions of the property in [1, 2, 3].

In this article we answer the question via an example first given by Cannon to demonstrate that a group may have a regular language of geodesics with respect to one generating set but not another. Neumann and Shapiro include it in [5] to prove that the falsification by fellow traveler property is generating set dependent.

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2 Definitions

Definition 2.1 (Finite state automaton; regular language) Let A be a finite set of letters, and let A^* be the set of all finite strings, including the empty string, that can be formed from the letters of A. A *finite state automaton* is a quintuple (S, A, τ, Y, s_0), where S is a finite set of *states*, τ is a map $\tau : S \times A \to S$, $Y \subseteq S$ are the *accept states*, and $s_0 \in S$ is the *start state*. A finite string $w \in A^*$ is *accepted* by the finite state automaton if starting in the state s_0 and changing states according to the letters of w and the map τ, the final state is in Y. The set of all finite strings that are accepted by a finite state automaton is called the *language* of the automaton. A language $L \subseteq A^*$ is *regular* if it is the language of a finite state automaton.

Suppose G is a group with finite generating set A. A word in A^* represents a path in the Cayley graph based at any vertex. Define $d(a, b)$ to be the distance between two points a and b in the Cayley graph with respect to the path metric. Paths can be parameterized by non-negative $t \in \mathbb{R}$ by defining $w(t)$ as the point at distance t along the path w if t is between 0 and the length of w, and the endpoint of w otherwise.

Definition 2.2 (The (asynchronous) fellow traveler property) Paths u and v are said to *k-fellow travel* if $d(u(t), v(t)) \leq k$ for all $t \geq 0$. They *asynchronously k-fellow travel* if there is a non-decreasing proper continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $d(u(t), v(\phi(t))) \leq k$. A language $L \subseteq A^*$ enjoys the *(asynchronous) fellow traveler property* if there is a constant k such that for each $u, v \in L$ that start at the identity and end at distance 0 or 1 apart in the Cayley graph, u and v (asynchronously) k-fellow travel.

Definition 2.3 (The (asynchronous) falsification by fellow traveler property) A finite generating set A for a group G has the *(asynchronous) falsification by fellow traveler property* if there is a constant k such that every non-geodesic word in the Cayley graph of G with respect to A is (asynchronously) k-fellow traveled by a shorter word.

The property arises naturally in the context of geodesic regular languages, and the proof of Proposition 4.1 in [5] uses the property to build an appropriate finite state automaton. The author proves in [1] that the synchronous and asynchronous versions of the falsification by fellow traveler property are equivalent.

3 The Example

Let G be the split extension of $\mathbb{Z}^2$, generated by $\{a, b\}$, by $\mathbb{Z}_2$, generated by $\{t\}$, such that t conjugates a to b and b to a, with presentation

$$\langle a, b, t \mid t^2 = 1, ab = ba, tat = b \rangle.$$

Performing one Tietze transformation (removing $b = tat$) we obtain

$$\langle a, t \mid t^2 = 1, atat = tata \rangle.$$

Let $A = \{a^{\pm 1}, t^{\pm 1}\}$ be the inverse-closed generating set corresponding to this presentation. The Cayley graph for G with respect to A is shown in Figure 1. We can consider the vertices of the graph as either being in the top or the

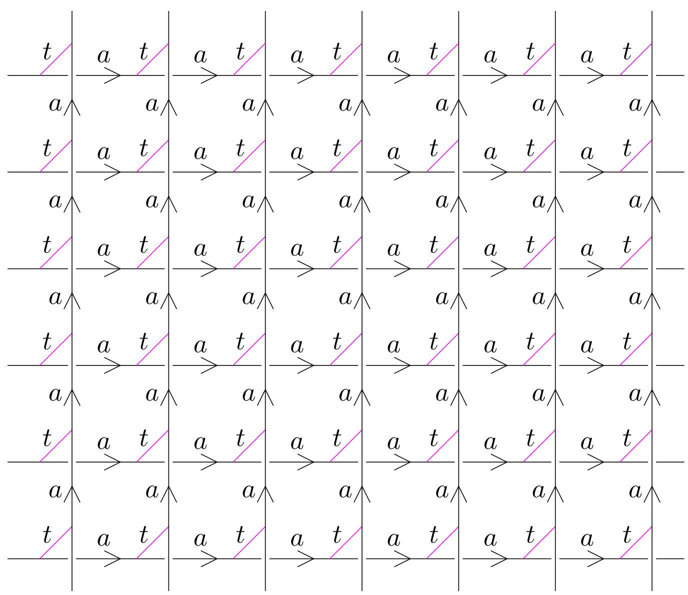

Figure 1: The Cayley graph for $G = \langle a, t \mid t^2 = 1, atat = tata \rangle$

bottom layer, where edges labeled t link top and bottom layers. We declare the identity vertex to lie in the bottom. Each vertex can also be given a coordinate (x, y) where x is the distance in the East-West direction from the identity, and

y is the distance in the North-South direction. In this way each vertex (group element) is uniquely specified by the triple (x, y, bottom) or (x, y, top).

For example, the identity has the coordinate $(0, 0, \text{bottom})$, the word a^3ta^4 has the coordinate $(3, 4, \text{top})$, the word a^3ta^4t has the coordinate $(3, 4, \text{bottom})$, and the word ta^3ta^4 has the coordinate $(4, 3, \text{bottom})$.

Lemma 3.1 *Each word from 1 to a vertex in the top layer has an odd number of t letters, and each word from 1 to a vertex in the bottom layer has an even number of t letters.*

Proof Suppose a vertex has the coordinate (x, y, top). Then a^xta^y is a word to this vertex. Suppose w is any other word to this vertex. Then $wa^{-y}ta^{-x} =_G 1$ so under the map which sends t to t and a to 1 this word must be sent to an even power of t, so w has an odd number of t letters. Similarly if a vertex lies in the bottom layer there is a word a^xta^yt to it from 1. If w is any other word to this vertex then $wta^{-y}ta^{-x} =_G 1$ gets sent to an even power of t so w has an even number of t letters. □

Lemma 3.2 *The word ta^nta^mt for any $m, n \in \mathbb{Z}$ is not geodesic.*

Proof The word ta^nta^mt can be written as a^mta^n which is shorter. □

Lemma 3.3 *Each vertex in the top layer has a unique geodesic to it from the identity of the form a^xta^y, where (x, y, top) is the coordinate of the vertex.*

Proof By Lemma 3.1 any geodesic to the vertex with coordinate (x, y, top) has an odd number of t letters. If a word has three or more t letters then it has a subword of the form ta^nta^mt which is not geodesic by Lemma 3.2. So any geodesic to this vertex has exactly one t letter, so is of the form a^ita^j. This path has coordinate (i, j, top) so it must be that $i = x$ and $j = y$. □

Proposition 3.4 *A does not have the falsification by fellow traveler property.*

Proof Suppose by way of contradiction that A has the falsification by fellow traveler property with positive constant k, and choose $n >> k$. Consider the word $w = ta^nta^nt$ which ends at the coordinate (n, n, top). Any word that ends at this vertex must move East at least n units and North at least n units, and by Lemma 3.1 must have an odd number of t letters. If it has just one t letter then it must be the unique geodesic a^nta^n which clearly does not k-fellow travel w. Otherwise it has three or more t letters, so has length at least $2n+3$ so is not shorter than w, so we are done. □

Theorem 3.5 *There is a group and finite generating set such that the language of all geodesics is regular but fails to have the falsification by fellow traveler property.*

Proof By Proposition 3.4 the group G with generating set A fails the falsification by fellow traveler property.

Consider the language $L = \{a^x, a^x t a^y, a^{x_1} t a^y t a^{x_2} : x, x_1, x_2, y \in \mathbb{Z}, x_1 \cdot x_2 \geq 0\}$. L is the language of the finite state automaton in Figure 2. All states are accept states.

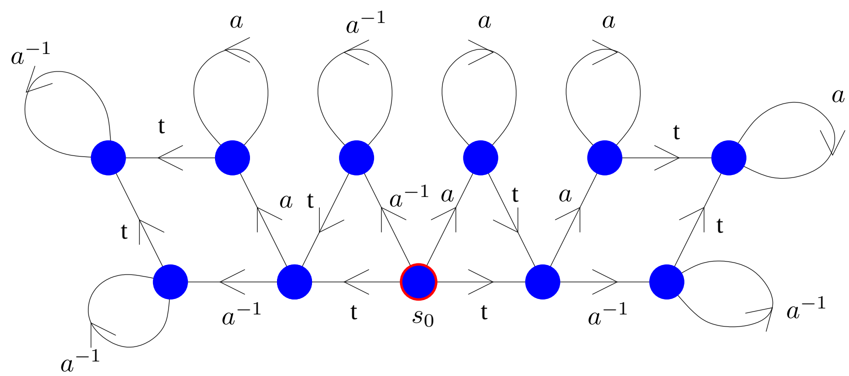

Figure 2: A finite state automaton accepting the language L

We now show that L is the language of all geodesics on A for G. By Lemma 3.3 every group element corresponding to a vertex in the top layer has a unique geodesic representative of the form $a^x t a^y$. Otherwise the group element corresponds to a vertex in the bottom, so has an even number of t letters. If a word has more than two t letters then it has a subword of the form $ta^n ta^m t$ which is not geodesic by Lemma 3.2, so a geodesic word for a bottom element has either zero t letters, so is of the form a^x, or has two t letters, so is of the form $a^{x_1} t a^y t a^{x_2}$. If x_1 and x_2 don't have the same sign then we can find a shorter word $a^{(x_1+x_2)} t a^y t$. Otherwise $a^{x_1} t a^y t a^{x_2}$ is a geodesic to a vertex with coordinate $(x_1 + x_2, y, \text{bottom})$. Notice that this gives a family of $x_1 + x_2 + 1$ geodesics to this vertex. □

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