ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 5 (2005) 183–205
Published: 23 March 2005

The 3–cocycles of the Alexander quandles $\mathbb{F}_q[T]/(T-\omega)$

Takuro Mochizuki

Abstract We determine the third cohomology of Alexander quandles of the form $\mathbb{F}_q[T]/(T-\omega)$, where $\mathbb{F}_q$ denotes the finite field of order q and ω is an element of $\mathbb{F}_q$ which is neither 0 nor 1. As a result, we obtain many concrete examples of non-trivial 3–cocycles.

AMS Classification 18H40; 55A25, 57Q45

Keywords Quandle, cohomology, knot

1 Introduction

1.1 Aims

Interest in quandles has been growing recently, particularly because of their applications to the study of classical knots and 2–knots. A *quandle* is a set X equipped with a binary operation $*\colon X \times X \to X$ satisfying the following conditions:

(Idempotency) For any $a \in X$, $a * a = a$.

(Right-Invertibility) For any $a, b \in X$, there exists a unique $c \in X$ such that $a = c * b$.

(Self-Distributivity) The identity $(a * b) * c = (a * c) * (b * c)$ holds for all $a, b, c \in X$.

Quandle cohomology $H^*(X, A)$ is defined for any quandle X and any abelian group A, and may be used (see [1, 2, 7] for details) to construct isotopy invariants of classical knots and links, and also of higher-dimensional embeddings. In particular, such invariants obtained from 3–cocycles play an interesting rôle in the study of 2–knots (see [8, 9] for example). However, there are not very many concrete examples of nontrivial 3–cocycles, and so it would be useful to find a systematic method of constructing nontrivial 3–cocycles with calculable forms for some classes of quandles.

In this paper, we discuss the third cohomology group of Alexander quandles. Let R be a commutative ring with a unit element, and let M be an R–module. For any invertible element ω of R, a binary operation $*\colon M \times M \to M$ may be defined by

$$a * b = \omega \cdot a + (1-\omega) \cdot b.$$

It is easy to check that $(M, *)$ satisfies the three quandle axioms, and we call a quandle of this type an *Alexander quandle*. In this paper, we restrict ourselves to the case $R = M = \mathbb{F}_q$, where q is a power of a prime p and $\mathbb{F}_q$ denotes a finite field of order q, and denote the resulting quandle $\mathbb{F}_q[T]/(T-\omega)$. We ignore the case $\omega = 1$, which yields a trivial quandle, and the case $\omega = 0$, which is forbidden by the right-invertibility axiom.

Quandle cohomology groups are well understood in the case where A is a field of characteristic 0 (see [4, 6]), and do not typically give rise to interesting cocycles. However, we may expect interesting examples to arise from the case where A is a field of positive characteristic. For example, the third quandle cohomology group $H^3\big(\mathbb{F}_p[T]/(T-\omega), \mathbb{F}_p\big)$ was calculated in [6], and the case $\omega = -1$ gives rise to a particular nontrivial 3–cocycle which has been used by Satoh and Shima in their study of 2–knots.

By generalizing our previous work, we will determine the third quandle cohomology group $H^3\big(\mathbb{F}_q[T]/(T-\omega), A\big)$ where A is an algebraic closure k of $\mathbb{F}_q$ (this includes the case where A is a field of characteristic p). In doing so (see subsections 2.3 and 2.4), we obtain many examples of nontrivial 3–cocycles, which we hope will be useful in the study of 2–knots.

1.2 Outline

In subsection 2.1, we recall the definition of the quandle cohomology groups and explain our description of the cocycles, which is slightly different to the description in [6]. In subsection 2.2, we give some concrete examples. Subsection 2.3 contains Theorem 2.11, the main result of this paper, which we apply to certain quandles in subsection 2.4. The proof of the theorem is given in Section 3.

1.3 Acknowledgements

The author is grateful to the referee for his comments, and for drawing attention to a paper of Rourke and Sanderson [7]. The author thanks Mikiya Masuda, Yoshifumi Tsuchimoto and Akira Ishii for their encouragement, Nicholas

Jackson for polishing the English translation, the Japan Society for the Promotion of Science, and the Sumitomo Foundation, for financial support, and the Institute of Advanced Study (where this paper was written) for their kind hospitality. This material is based upon work supported by National Science Foundation grant DMS–9729992, although any opinions, findings, conclusions or recommendations expressed herein are the responsibility of the author.

2 The main result

2.1 Preliminaries

2.1.1 Quandle cohomology groups

Let $(X,*)$ be a quandle, and let A be an abelian group. We define a complex $C^*(X,A)$ with cochain groups

$$C^n\big(X,A\big) := \big\{ f\colon\ X^n \to A \,\big|\, f(x_1,\dots,x_n) = 0 \text{ when } x_i = x_{i+1} \ \text{ for some } i \big\}$$

and differential $\delta\colon\ C^n(X,A) \longrightarrow C^{n+1}(X,A)$ defined as follows:

$$\delta(f)(x_1,\dots,x_{n+1}) := \sum_{i=1}^{n+1}(-1)^{i-1} f\big(x_1 * x_i,\dots,x_{i-1} * x_i, x_{i+1},\dots,x_{n+1}\big) \\ - \sum_{i=1}^{n+1}(-1)^{i-1} f\big(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}\big)$$

The cohomology of this complex is denoted $H^*(X,A)$, and called the *quandle cohomology* of X (with coefficient group A).

Let q be a power of a prime p, let $\mathbb{F}_q$ denote a finite field of order q, and let ω be a non-zero element of $\mathbb{F}_q$. We wish to calculate $H^3\big(\mathbb{F}_q[T]/(T-\omega),k\big)$, where $\mathbb{F}_q[T]/(T-\omega)$ is the Alexander quandle discussed in section 1, and k is an algebraic closure of $\mathbb{F}_q$. It is obvious that the differential map is trivial in the case $\omega = 1$, so we will consider only the cases $\omega \neq 0,1$.

2.1.2 The quandle complex

Let k be a field, and ω an element of k which is neither 0 nor 1. (Later k will denote an algebraic closure of $\mathbb{F}_q$, but for the moment we consider arbitrary fields.) Let $k[U_1,\dots,U_n]$ be the polynomial ring over k with n variables $U_1,\dots,U_n$, and set $\Omega_{n-1} := \prod_{i=1}^{n-1} U_i$ and $C^n := \Omega_{n-1}\cdot k[U_1,\dots,U_n]$.

For any element $f \in C^n$, we define $\delta(f) \in C^{n+1}$ as follows:

$$\delta(f)\bigl(U_1, \dots, U_{n+1}\bigr) := \sum_{i=1}^{n} (-1)^{i-1} f\bigl(\omega \cdot U_1, \dots, \omega \cdot U_{i-1}, \omega \cdot U_i + U_{i+1}, U_{i+2}, \dots, U_{n+1}\bigr) - \sum_{i=1}^{n-1} (-1)^{i-1} f\bigl(U_1, \dots, U_{i-1}, U_i + U_{i+1}, U_{i+2}, \dots, U_{n+1}\bigr)$$

We thus obtain a homomorphism $\delta\colon\ C^n \to C^{n+1}$. A routine calculation verifies that $\delta \circ \delta = 0$, and so we have a complex $C^* = \bigl(\bigoplus_{n=1}^{\infty} C^n, \delta\bigr)$, which we call the *quandle complex associated with k and ω*.

Remark 2.1 The complex C^* was discussed in [6] in the case where k is a field of characteristic 0, and was shown to be acyclic. The above definition looks slightly different, due to a different choice of coordinates in k^n.

2.1.3 A convenient description of the complex $C^*\bigl(\mathbb{F}_q[T]/(T-\omega), k\bigr)$

Let $\mathbb{F}_q$ denote a finite field of order q. In the following, k is an algebraic closure of $\mathbb{F}_q$, and ω an element of $\mathbb{F}_q$ such that $\omega \neq 0, 1$. Let C^* be the complex described in subsection 2.1.2. Then an element $f = f(U_1, \dots, U_n)$ of $C^n = k[U_1, \dots, U_n]$ induces a k–valued function on $\mathbb{F}_q^n$, given by

$$\mathbb{F}_q^n \ni \bigl(x_1, \dots, x_n\bigr) \mapsto f\bigl(x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n, x_n\bigr) \in k,$$

and so we obtain a map $\varphi\colon\ C^n \to C^n\bigl(\mathbb{F}_q[T]/(T-\omega), k\bigr)$ for any n. The following lemma can be checked by a direct calculation.

Lemma 2.2 *The homomorphism φ is compatible with the differentials (that is, $\varphi \circ \delta = \delta \circ \varphi$) and so we obtain a morphism of the cochain complexes $C^* \to C^*\bigl(\mathbb{F}_q[T]/(T-\omega), k\bigr)$.*

We now define

$$C^n(q) := \Bigl\{ \sum a_{i_1, \dots, i_n} \cdot U_1^{i_1} \cdots U_n^{i_n} \in C^n \Bigm| 0 \leq i_j \leq q-1 \Bigr\}.$$

It is easy to check that $\delta\bigl(C^n(q)\bigr) \subset C^{n+1}(q)$. Thus, we obtain the subcomplex $C^*(q) := \bigl(\bigoplus C^n(q), \delta\bigr)$. The following lemma can be checked easily.

Lemma 2.3 *The induced morphism $C^*(q) \to C^*\bigl(\mathbb{F}_q[T]/(T-\omega), k\bigr)$ is an isomorphism.*

2.2 Some 3–cocycles of the complex C^*

We now give some concrete examples of 3–cocycles in the complex C^*. The last variable U_n for any element $f(U_1,\dots,U_n)$ of C^n is denoted by T_n in the following argument, and will be useful for specific calculations.

2.2.1 The cocycles Ψ, E_0 and E_1

For positive integers a and b set $\mu_a(x,y) := (x+y)^a - x^a - y^a$ and define the polynomial $\Psi(a,b) \in k[U_1,U_2,T_3]$ as follows:

$$\begin{aligned}\Psi(a,b) &:= \Big(\mu_a\big(\omega\cdot U_1,U_2\big) - \mu_a\big(U_1,U_2\big)\Big)\cdot T_3^b \\ &= \Big(\big(\omega\cdot U_1+U_2\big)^a - \big(U_1+U_2\big)^a + \big(1-\omega^a\big)\cdot U_1^a\Big)\cdot T_3^b\end{aligned}$$

Lemma 2.4 *If $\omega^{a+p^s}=1$, then $\Psi(a,p^s)$ is a quandle 3–cocycle.*

Proof Set $h(U_1,T_2) := \mu_a(\omega\cdot U_1,T_2) - \mu_a(U_1,T_2)$. Then from the relation $\delta(T_1^a) = (\omega\cdot U_1+T_2)^a - (U_1+T_2)^a$, we see that $h(U_1,T_2) = \delta(T_1^a) + (1-\omega^a)\cdot U_1^a$, and so

$$\delta(h)(U_1,U_2,T_3) = (1-\omega^a)\cdot\delta(U_1^a) = (1-\omega^a)\cdot h(U_1,U_2).$$

Then

$$\begin{aligned}\delta\big(\Psi(a,p^s)\big) &= (1-\omega^a)\cdot h(U_1,U_2)\cdot T_4^{p^s} \\ &\quad + \Big(\omega^a\cdot h(U_1,U_2)\cdot(\omega\cdot U_3+T_4)^{p^s} - h(U_1,U_2)\cdot(U_3+T_4)^{p^s}\Big) = 0,\end{aligned}$$

so $\Psi(a,p^s)$ is a quandle cocycle. □

We now introduce the following polynomials with $\mathbb{Z}/p\mathbb{Z}$ coefficients:

$$\chi(x,y) := \sum_{i=1}^{p-1}(-1)^{i-1}\cdot i^{-1}\cdot x^{p-i}\cdot y^i \equiv \frac{1}{p}\Big((x+y)^p - x^p - y^p\Big) \pmod p$$

For positive integers a and b, we define the polynomial $E_0\big(a\cdot p,b\big)(U_1,U_2,T_3)$ to be:

$$E_0\big(a\cdot p,b\big) := \Big(\chi(\omega\cdot U_1,U_2) - \chi(U_1,U_2)\Big)^a\cdot T_3^b$$

Lemma 2.5 *If $\omega^{p^s+p^h}=1$ and $s>0$, then $E_0(p^s,p^h)$ is a quandle 3–cocycle.*

Proof This can be verified by an argument similar to that used in the proof of Lemma 2.4. □

For positive integers a and b, we define the polynomial $E_1\big(a, b{\cdot}p\big) \in k[U_1, U_2, T_3]$ as

$$E_1\big(a, b \cdot p\big) := U_1^a \cdot \Big(\chi\big(U_2, T_3\big) - \chi\big(U_2, \omega^{-1} \cdot T_3\big)\Big)^b.$$

Lemma 2.6 *If $\omega^{p^t+p^s} = 1$ and $s > 0$, then $E_1(p^t, p^s)$ is a quandle 3–cocycle.*

Proof Set $h(U_2, T_3) := \chi(U_2, T_3) - \chi(U_2, \omega^{-1} \cdot T_3)$. Then

$$\begin{aligned}\delta\big(E_1(p^t, p^s)\big) = \Big(\big(\omega \cdot U_1 + U_2\big) - \big(U_1 + U_2\big)\Big)^{p^t} \cdot h(U_3, T_4)^{p^{s-1}} \\ - U_1^{p^t} \cdot \Big(\omega^{p^t} h\big(\omega \cdot U_2 + U_3, T_4\big)^{p^{s-1}} - h\big(U_2 + U_3, T_4\big)^{p^{s-1}}\Big) \\ + U_1^{p^t} \cdot \Big(\omega^{p^t} h\big(\omega \cdot U_2,\ \omega \cdot U_3 + T_4\big)^{p^{s-1}} - h\big(U_2, U_3 + T_4\big)^{p^{s-1}}\Big).\end{aligned}$$

By using the relation $\omega^{p^t} = \omega^{-p^s}$, the right hand side can be rewritten

$$\begin{aligned}U_1^{p^t} \cdot \Big(-\big(1 - \omega^{-p}\big) \cdot h\big(U_3, T_4\big) - \omega^{-p} \cdot h\big(\omega U_2 + U_3, T_4\big) + h\big(U_2 + U_3, T_4\big) \\ + \omega^{-p} \cdot h\big(\omega U_2, \omega U_3 + T_4\big) - h\big(U_2, U_3 + T_4\big)\Big)^{p^{s-1}}.\end{aligned}$$

It can be directly shown that this expression is zero, and so $E_1(p^t, p^s)$ is a quandle 3–cocycle. □

2.2.2 The set $\mathcal{Q}$ and the cocycles F and Γ

In the following, let q_i be powers of the prime p. For any non-negative integers a, b, c and d, we define polynomials $F(a, b, c) \in k[U_1, U_2, T_3]$ and $G(a, b, c, d) \in k[U_1, U_2, U_3, T_4]$ as

$$F(a, b, c) := U_1^a \cdot U_2^b \cdot T_3^c,$$
$$G(a, b, c, d) := U_1^a \cdot U_2^b \cdot U_3^c \cdot T_4^d.$$

The following lemma is helpful for our later calculations.

Lemma 2.7 *We have the following identities.*

$$\begin{aligned}
\delta\big(F(q_1,q_2,q_3)\big) &= \big(\omega^{q_1+q_2+q_3}-1\big)\cdot G(q_1,q_2,q_3,0)\\
\delta\big(F(q_1+q_2,q_3,q_4)\big) &= \big(\omega^{q_1}-1\big)\cdot G(q_1,q_2,q_3,q_4)\\
&\quad+\big(\omega^{q_2}-1\big)\cdot G(q_2,q_1,q_3,q_4)\\
&\quad+\big(\omega^{q_1+q_2+q_3+q_4}-1\big)\cdot G(q_1+q_2,q_3,q_4,0)\\
\delta\big(F(q_1,q_2+q_3,q_4)\big) &= -\big(\omega^{q_1+q_2}-1\big)\cdot G(q_1,q_2,q_3,q_4)\\
&\quad-\big(\omega^{q_1+q_3}-1\big)\cdot G(q_1,q_3,q_2,q_4)\\
&\quad+\big(\omega^{q_1+q_2+q_3+q_4}-1\big)\cdot G(q_1,q_2+q_3,q_4,0)\\
\delta\big(F(q_1,q_2,q_3+q_4)\big) &= \big(\omega^{q_1+q_2+q_3}-1\big)\cdot G(q_1,q_2,q_3,q_4)\\
&\quad+\big(\omega^{q_1+q_2+q_4}-1\big)\cdot G(q_1,q_2,q_4,q_3)\\
&\quad+\big(\omega^{q_1+q_2+q_3+q_4}-1\big)\cdot G(q_1,q_2,q_3+q_4)
\end{aligned}$$

Proof These identities may be verified by direct calculation. □

Corollary 2.8 *Let q_1, q_2 and q_3 be powers of a prime p. Then*

(1) $F(q_1,q_2,q_3)$ *is a quandle 3–cocycle if* $\omega^{q_1+q_2+q_3}=1$*, and*

(2) $F(q_1,q_2,0)$ *is a quandle 3–cocycle if* $\omega^{q_1+q_2}=1$.

Let $\mathcal{Q}$ denote the set of quadruples (q_1,q_2,q_3,q_4) satisfying the following conditions:

Condition 2.9

- $q_2 \le q_3$, $q_1 < q_3$, $q_2 < q_4$, and $\omega^{q_1+q_3}=\omega^{q_2+q_4}=1$.
- One of the following holds:

  Case 1 $\omega^{q_1+q_2}=1$.

  Case 2 $\omega^{q_1+q_2}\neq 1$ and $q_3>q_4$.

  Case 3 $(p\neq 2)$ $\omega^{q_1+q_2}\neq 1$ and $q_3=q_4$.

  Case 4 $(p\neq 2)$ $\omega^{q_1+q_2}\neq 1$, $q_2\le q_1<q_3<q_4$, and $\omega^{q_1}=\omega^{q_2}$.

  Case 5 $(p=2)$ $\omega^{q_1+q_2}\neq 1$, $q_2<q_1<q_3<q_4$, and $\omega^{q_1}=\omega^{q_2}$.

The polynomial $\Gamma(q_1,q_2,q_3,q_4)$ is defined for any element (q_1,q_2,q_3,q_4) of $\mathcal{Q}$ as follows: **Case 1**

$$\Gamma(q_1,q_2,q_3,q_4) := F(q_1,q_2+q_3,q_4).$$

Case 2

$$\Gamma(q_1,q_2,q_3,q_4) := F(q_1,q_2+q_3,q_4) - F(q_2,q_1+q_4,q_3) \\ - \big(\omega^{q_2}-1\big)^{-1}\cdot\big(1-\omega^{q_1+q_2}\big)\cdot\Big(F(q_1,q_2,q_3+q_4) - F(q_1+q_2,q_4,q_3)\Big).$$

Case 3

$$\Gamma(q_1,q_2,q_3,q_4) := F(q_1,q_2+q_3,q_4) - 2^{-1}\cdot(1-\omega^{-q_3})\cdot F(q_1,q_2,q_3+q_4).$$

Case 4 and Case 5

$$\Gamma(q_1,q_2,q_3,q_4) := F(q_1,q_2+q_3,q_4) + F(q_2,q_1+q_3,q_4) \\ - (\omega^{q_1}-1)^{-1}\cdot(1-\omega^{2q_1})\cdot F(q_1+q_2,q_3,q_4).$$

The next lemma follows from Lemma 2.7 together with a direct calculation.

Lemma 2.10 *The polynomials $\Gamma(q_1,q_2,q_3,q_4)$ are quandle 3–cocycles for any quadruple $(q_1,q_2,q_3,q_4)\in\mathcal{Q}$.*

We define $\mathcal{Q}(q) := \{(q_1,q_2,q_3,q_4)\in\mathcal{Q} \mid q_i<q\}$ and, for d a positive integer, $\mathcal{Q}_d(q) := \{(q_1,q_2,q_3,q_4)\in\mathcal{Q}(q) \mid \sum q_i = d\}$.

2.3 Statement of the main theorem

As before let q_i denote a power of a prime p, and define:

$$\begin{aligned} I(q) := &\big\{F(q_1,q_2,q_3) \mid \omega^{q_1+q_2+q_3}=1,\ q_1<q_2<q_3<q\big\} \\ &\cup\big\{F(q_1,q_2,0) \mid \omega^{q_1+q_2}=1,\ q_1<q_2<q\big\} \\ &\cup\left\{\Psi(a,q_1)\ \middle|\ \begin{array}{l}\omega^{a+q_1}=1,\ 0<a<q,\ 1<q_1<q,\\ a\not\equiv 0 \pmod{q_1},\ a \text{ is not a power of } p\end{array}\right\} \\ &\cup\big\{E_0(p\cdot q_1,q_2) \mid \omega^{p\cdot q_1+q_2}=1,\ q_1<q_2<q\big\} \\ &\cup\big\{E_1(q_1,p\cdot q_2) \mid \omega^{q_1+p\cdot q_2}=1,\ q_1\le q_2<q\big\} \\ &\cup\{\Gamma(q_1,q_2,q_3,q_4) \mid (q_1,q_2,q_3,q_4)\in\mathcal{Q}(q)\} \end{aligned} \tag{1}$$

Let $H^3(q)$ denote the subspace of $C^3(q)$ generated by $I(q)$. The following theorem is the main result of this paper, and will be proved in Section 3.

Theorem 2.11 *The natural map $H^3(q)\to H^3\big(C^*(q)\big)$ is an isomorphism.*

Remark 2.12 It will also turn out that the cocycles given in (1) are linearly independent, and hence form a basis for $H^3(q)$.

2.4 Examples

2.4.1 The case $\omega = -1$

Let p be an odd prime, and let $\omega = -1$. Then:

- $\omega^{q_1+q_2+q_3} \neq 1$, so we have no quandle 3–cocycles of the form $F(q_1, q_2, q_3)$.
- $\omega^{a+q_1} = -\omega^a$, and so the identity $\omega^{a+q_1} = 1$ implies that a is odd.
- $\omega^{q_1+q_2} = 1$ for any powers q_i of p. Hence the polynomials $F(q_1, q_2, 0)$, $E_0\big(p \cdot q_1, q_2\big)$ and $E_1\big(q_1, p \cdot q_2\big)$ are quandle 3–cocycles. In addition, $\mathcal{Q}(q) = \{(q_1, q_2, q_3, q_4) \mid q_2 \leq q_3,\ q_1 < q_3,\ q_2 < q_4\}$, and $\omega^{q_1+q_2} = 1$ for any $(q_1, q_2, q_3, q_4) \in \mathcal{Q}(q)$.

Thus we obtain the following 3–cocycles, which form a basis for the cohomology group $H^3\big(\mathbb{F}_q[T]/(T+1), k\big)$:

$$\begin{aligned} &\big\{F(q_1, q_2, 0) \,\big|\, 0 < q_1 < q_2 < q\big\} \\ \cup &\big\{E_0\big(p \cdot q_1, q_2\big) \mid q_1 < q_2 < q\big\} \\ \cup &\big\{E_1\big(q_1, p \cdot q_2\big) \mid q_1 < q_2 < q\big\} \\ \cup &\left\{\Psi(a, q_1) \,\middle|\, \begin{array}{l} a \text{ odd},\ 0 < a < q,\ q_1 < q, \\ a \not\equiv 0 \ (\mathrm{mod}\, q_1),\ a \text{ is not a power of } p \end{array}\right\} \\ \cup &\{F(q_1, q_2 + q_3, q_4) \mid q_2 \leq q_3,\ q_1 < q_3,\ q_2 < q_4,\ q_i < q\} \end{aligned} \tag{2}$$

If $q = p^2$ then the basis in (2) is:

$$\begin{aligned} &\big\{F(1,p),\ E_0(p,p),\ E_1(1,p),\ E_1(1,p^2),\ E_1(p,p^2),\ F(1,p+1,p)\big\} \\ \cup\ &\big\{\Psi(a,p) \mid a \text{ odd},\ a < p^2,\ a \not\equiv 0 \ (\mathrm{mod}\, p),\ a \neq 1\big\} \end{aligned}$$

If $q = p$ then the basis in (2) is simply $\{E_1(1,p)\}$, and so $H^3\big(\mathbb{F}_p[T]/(T+1), k\big)$ is 1–dimensional, as previously noted in [6].

2.4.2 Some other examples

Example 2.13 If $\mathbb{F}_q = \mathbb{Z}_2[\omega]/(1+\omega+\omega^2)$, then $q = 4 = 2^2$, and the order of ω is $3 = 2+1$. Then:

- We have no triples (q_1, q_2, q_3) of powers of 2 satisfying $q_1 < q_2 < q_3 < 2^2$.
- If a pair (q_1, q_2) of powers of 2 satisfies $q_1 < q_2 < 2^2$, then we have $q_1 = 1$ and $q_2 = 2$. In this case, $\omega^{q_1+q_2} = \omega^3 = 1$, and so we have a cocycle $F(1,2,0)$.

- The identity $\omega^{a+2} = 1$ implies $a \equiv 1 \pmod 3$, and a cannot be a power of 2 because of the definition of $I(q)$ in (1), hence cocycles of the form $\Psi(b \cdot p^t, p^s)$ do not occur.
- If $(q_1, q_2) = (1, 2)$, then $\omega^{2 \cdot q_1 + q_2} = \omega^4 \neq 1$.
- If $(q_1, q_2) = (1, 2)$, then $\omega^{q_1 + 2 \cdot q_2} = \omega^5 \neq 1$. On the other hand, if $(q_1, q_2) = (1, 1)$ or $(2, 2)$, then $\omega^{q_1 + 2 \cdot q_2} = \omega^{3 q_1} = 1$, and so we have the cocycles $E_1(1, 2)$ and $E_1(2, 4)$.
- The set $\mathcal{Q}(q)$ is empty.

Thus we have cocycles

$$\bigl\{F(1,2),\ E_1(1,2),\ E_1(2,4)\bigr\}$$

which form a basis for the cohomology group $H^3\bigl(\mathbb{F}_q[T]/(T+1), k\bigr)$.

Example 2.14 If $\mathbb{F}_q = \mathbb{Z}_3[\omega]/(\omega^2+1)$, then $q = 3^2$, ω has order 8, and $\mathcal{Q}(q) = \{(1,1,3,3)\}$, so we have a cocycle $\Gamma(1,1,3,3) = F(1,4,3) - 2^{-1}(1-\omega) \cdot F(1,1,6)$. If $\omega^{a+3} = 1$ and $0 < a < 9$, then we have $a = 1, 5$. Hence we have the cocycles

$$\bigl\{F(1,3,0),\ \Psi(5,3),\ \Gamma(1,1,3,3),\ E_1(1,3),\ E_1(3,9)\bigr\}$$

which form a basis for the third quandle cohomology group.

Example 2.15 If $\mathbb{F}_q = \mathbb{Z}_3[\omega]/(\omega^2 + \omega - 1)$, then $q = 3^2$, ω has order 8, and so $H^3\bigl(\mathbb{F}_q[T]/(T - \omega)\bigr)$ is generated by the cocycle $\Psi(5,3)$.

Example 2.16 If $\mathbb{F}_q = \mathbb{Z}_2[\omega]/(\omega^3 + \omega^2 + 1)$, then $q = 2^3$ and ω has order 7. We have the triple $(1,2,4)$ of powers of 2 satisfying $1 < 2 < 4$, and thus have a cocycle $F(1,2,4)$. Note that $\omega^{q_1 + q_2} \neq 1$ in the case where q_i are powers of 2 satisfying $q_i < 8$. Hence the cocycles

$$\bigl\{F(1,2,4),\ \Psi(5,2),\ \Psi(3,4)\bigr\}$$

form a basis for $H^3\bigl(\mathbb{F}_q[T]/(T - \omega)\bigr)$.

3 Proof of Theorem 2.11

3.1 Preliminaries

3.1.1 A decomposition of the complex C^n

We may decompose C^n by the total degree, as follows:

$$C^n_d := \Big\{ \sum a_{i_1,\dots,i_n} \cdot \prod_{h=1}^{n-1} U_h^{i_h} \cdot T_n^{i_n} \in C^n \ \Big|\ \sum i_h = d \Big\}$$
$$C^n_d(q) := C^n_d \cap C^n(q)$$

Then $C^n(q) = \bigoplus_d C^n_d(q)$, and it is easy to see that $\delta(C^n_d(q)) \subset C^{n+1}_d(q)$. We denote the complex $\big(\bigoplus C^n_d(q), \delta\big)$ by $C^*_d(q)$.

The following easy lemma follows by a standard argument (see [6]).

Lemma 3.1 *In the case $\omega^d \neq 1$, the complex $C^*_d(q)$ is acyclic.*

The next lemma shows the relationship between this decomposition and the differential δ.

Lemma 3.2 *Let $f = \sum_a f_a(U_1,\dots,U_{n-1}) \cdot T_n^a$ be an element of $C^n_d(q)$. Then*

$$\delta(f)\big(U_1,\dots,U_n,T_{n+1}\big) = \sum_a \delta(f_a)\big(U_1,\dots,U_n\big) \cdot T_{n+1}^a$$
$$+ (-1)^{n-1} \sum_a f_a(U_1,\dots,U_{n-1}) \cdot \Big(\omega^d \cdot \big(U_n + \omega^{-1} T_{n+1}\big)^a - \big(U_n + T_{n+1}\big)^a\Big).$$

Example 3.3 Let $\lambda_d(T_1) := T_1^d \in C^1_d$. Then

$$\delta(\lambda_d)(U_1,T_2) = (\omega \cdot U_1 + T_2)^d - (U_1 + T_2)^d$$
$$= \omega^d \cdot (U_1 + \omega^{-1} \cdot T_2)^d - (U_1 + T_2)^d.$$

Example 3.4 For an element $f(U_1,T_2) = \sum f_a(U_1) \cdot T_2^a \in C^2_d$,

$$\delta(f)(U_1,U_2,T_3) = \sum_a \Big(f_a(\omega U_1 + U_2) - f_a(U_1 + U_2) \Big) \cdot T_3^a$$
$$- \sum_a f_a(U_1) \cdot \Big(\omega^d (U_2 + \omega^{-1} T_3)^a - (U_2 + T_3)^a \Big)$$
$$= \sum_a \delta(f_a)(U_1,U_2) \cdot T_3^a$$
$$- \sum_a f_a(U_1) \cdot \Big(\omega^d (U_2 + \omega^{-1} T_3)^a - (U_2 + T_3)^a \Big).$$

Example 3.5 If $f = \sum f_a(U_1, U_2) \cdot T_3^a$, then

$$\delta(f)(U_1,U_2,U_3,T_4) = \sum \delta(f_a)(U_1,U_2,U_3) \cdot T_3^a + \sum f_a(U_1,U_2) \cdot \Big(\omega^d (U_3 + \omega^{-1}T_4)^a - (U_3+T_4)^a\Big).$$

3.1.2 The filtration and the derivatives

Let

$$C_d^{n\,(s)} := \Big\{ \sum_a f_a(U_1,\dots,U_{n-1}) \cdot T_n^{ap^s} \in C_d^n \Big\},$$
$$C_d^{n\,(s)}(q) := C_d^{n\,(s)} \cap C_d^n(q),$$

and let

$$C_d^{n\,(\infty)} := \Big\{ f_0(U_1,\dots,U_{n-1}) \in C_d^n(q) \Big\} = \bigcap_{s\geq 0} C_d^{n\,(s)}(q),$$
$$C_d^{n\,(\infty)}(q) := C_d^{n\,(\infty)} \cap C_d^n(q).$$

It is easy to see that $\delta\big(C_d^{n\,(s)}(q)\big)$ is contained in $C_d^{n+1\,(s)}(q)$, and we define

$$Z_d^n(q) := \mathrm{Ker}(\delta) \cap C_d^n(q), \qquad B_d^n(q) := \delta\big(C_d^{n-1}(q)\big),$$
$$Z_d^{n\,(s)}(q) := \mathrm{Ker}(\delta) \cap C_d^{n\,(s)}(q), \qquad B_d^{n\,(s)}(q) := \delta\big(C_d^{n-1\,(s)}(q)\big).$$

There is a homomorphism $D_n^{(s)}\colon C_d^{n\,(s)}(q) \to C_{d-p^s}^{n\,(s)}(q)$ defined as follows:

$$D_n^{(s)}\Big(\sum_a f_a \cdot T_n^{ap^s}\Big) = \sum_a (a \cdot f_a) \cdot T_n^{(a-1)p^s}$$

The kernel $\ker D_n^{(s)} = C^{n\,(s+1)}(q)$, and the relation $\delta \circ D_n^{(s)} = D_{n+1}^{(s)} \circ \delta$ can be checked easily. Where the meaning is clear, we may omit the subscript n.

Let s be a positive integer such that $p^s < q$, and define

$$\mathcal{P}(s,q) := \big\{p^t \mid 0 \leq t < s\big\} \cup \big\{b \cdot p^s \mid 0 < b \cdot p^s < q,\ b \not\equiv -1 \pmod p \text{ or } b = p-1\big\}.$$

For any positive integer $d < q$, set $\lambda_d(T_1) := T_1^d \in C_d^1$.

Lemma 3.6 *Let s and d be integers such that $p^s < q$ and $0 < d < q$. If $\delta(\lambda_d) \in C_d^2(q)$ is contained in the subset $\mathrm{Im}(D^{(s)}) \subset C_d^{2\,(s)}(q)$, then $d \in \mathcal{P}(s,q)$.*

Proof We note that $\delta(\lambda_d) = (\omega U_1+T_2)^d-(U_1+T_2)^d$, and consider the integers d_t such that $d=\sum d_t p^t$ and $0 \le d_t \le p-1$ for any non-negative integer t. Set $i := \min\{t \mid d_t > 0\}$, then $(\omega U_1+T_2)^{d-p^i} - (U_1+T_2)^{d-p^i} = 0$ for $i < s$, and hence $d-p^i = 0$.

If $i \ge s$, then d is of the form $b \cdot p^s$ for some positive integer b. We will suppose that $b = ap-1$ for some $a > 1$, and show that this leads to a contradiction. Consider the partition $a = \sum_{t \ge 0} a_t \cdot p^t$ such that $0 \le a_t \le p-1$. If $\sum a_t = 1$, then $ap = p^h$ for some $h > 1$, and so

$$\begin{pmatrix} a\cdot p-1 \\ p \end{pmatrix} \not\equiv 0 \pmod p \text{ for } \omega^p - 1 \neq 0.$$

Thus the coefficient of $(U_1^p \cdot T_2^{ap-p-1})^{p^s}$ in $\delta(\lambda_d)$ is nonzero.

Now consider the case $\sum a_t > 1$, and set $j := \max\{t \mid a_t > 0\}$. Then

$$\begin{pmatrix} a\cdot p-1 \\ p^{j+1} \end{pmatrix} \not\equiv 0 \pmod p \text{ for } \omega^{p^{j+1}} - 1 \neq 0,$$

and so the coefficient of $\left(U_1^{p^{j+1}} T_2^{ap-1-p^{j+1}}\right)^{p^s}$ is nonzero, hence $\delta(\lambda_d)$ cannot be contained in $\mathrm{Im}(D^{(s)})$. □

On the other hand, if $d \in \mathcal{P}(s,q)$, then $\delta(\lambda_d)$ is contained in $\mathrm{Im}(D_2^{(s)})$. For example,

$$\delta(\lambda_{(p-1)p^s}) = D_2^{(s)}\left[p^{-1}\cdot\left((\omega U_1+T_2)^p - (U_1+T_2)^p + (1-\omega^p)\cdot U_1^p\right)\right]^{p^s}$$

when $b = (p-1)p^s$. Note that $\lambda_{(p-1)p^s} \in \mathrm{Im}(D_2^{(s)})$ even in the case $p^{s+1} = q$. The other cases can be checked more easily.

3.1.3 The 2–cocycles

A routine calculation proves the following lemma.

Lemma 3.7 *If s and t are non-negative integers such that $\omega^{p^s+p^t} = 1$, then $U_1^{p^t} \cdot T_2^{p^s}$ is a quandle 2–cocycle.*

Let d be a positive integer such that $\omega^d = 1$, then for any s such that $p^s < q$, we consider the following sets of 2–cocycles:

$$J_d^{(s)}(q) := \left\{U_1^{p^t} \cdot T_2^{p^s} \,\middle|\, t < s,\ p^s + p^t = d,\ p^s < q\right\}$$

Remark 3.8 Clearly the order of $J_d^{(s)}(q)$ is at most 1.

Let $H_d^{2\,(s)}(q)$ denote the subspace of $C_d^{2\,(s)}(q)$ generated by $J_d^{(s)}(q)$.

Lemma 3.9 *If* $\omega^d = 1$, *then*

$$Z_d^{2\,(s)}(q) = H_d^{2\,(s)}(q) \oplus \big(B_d^2(q) \cap C_d^{2\,(s)}(q) + Z_d^{2\,(s+1)}(q)\big)$$

and $Z_d^{2\,(\infty)}(q) = 0$.

Proof By an argument similar to the proof of Lemma 3.11 below,

$$Z_d^{2\,(s)}(q) = H_d^{2\,(s)}(q) \oplus \big(B_d^2(q) \cap C_d^{2\,(s)}(q) + Z_d^{2\,(s+1)}(q)\big).$$

If $\delta(U_1^d) = 0$ for $U_1^d \in C_d^{2\,(\infty)}(q)$, then $d = p^s$ for some s, and so $\omega^d \neq 1$. □

Note that $B_d^2(q) \cap C_d^{2\,(s)}(q) \cap Z_d^{2\,(s+1)}(q)$ is contained in $B_d^2(q) \cap C_d^{2\,(s+1)}(q)$. Then we obtain, as one of the simplest special cases, the following proposition, originally stated and proved in [6].

Proposition 3.10 *We have a decomposition* $Z_d^2(q) = \bigoplus_{s \geq 0} H_d^{2\,(s)}(q) \oplus B_d^2(q)$. *In particular, the natural map* $\bigoplus H_d^{2\,(s)}(q) \to H^2\big(C_d^*(q)\big)$ *is an isomorphism.*

3.1.4 Preliminaries for 3–coboundaries

Lemma 3.11 *If* $\omega^d = 1$, *then* $B_d^{3\,(s)}(q) = B_d^3(q) \cap C_d^{3\,(s)}(q)$ *and* $B_d^{3\,(\infty)}(q) = B_d^3(q) \cap C_d^{3\,(\infty)}(q)$.

Proof Let f be an element of $C_d^{2\,(s)}$ such that $f = \sum_a f_{ap^s}(U_1) \cdot T_2^{ap^s}$. Then

$$\delta(f) = \sum_a \delta(f_{ap^s})(U_1, U_2) \cdot T_3^{ap^s} - \sum_a f_{ap^s}(U_1) \cdot \Big((U_2 + \omega^{-1} T_3)^{ap^s} - (U_2 + T_3)^{ap^s}\Big).$$

Assume that $\delta(f) \in C_d^{3\,(s+1)}(q)$. By comparing coefficients of $T_3^{p^s}$, we find that

$$\delta(f_{p^s})(U_1, U_2) - \sum_a f_{ap^s}(U_1) \cdot a \cdot U_2^{(a-1)p^s} \cdot (\omega^{-p^s} - 1) = 0,$$

so $\delta(f_{p^s}) + (1 - \omega^{-p^s}) \cdot D^{(s)}(f) = 0$, and hence $(d - p^s) \in \mathcal{P}(s, q)$.

If $d - p^s = p^t$ for some $t \geq 0$, then $f(U_1, T_2) = A \cdot U_1^{p^t} \cdot T_2^{p^s} + h$ for some $A \in k$ and $h \in C_d^{2\,(s+1)}(q)$. If $d - p^s = b \cdot p^s$ for some $b \not\equiv -1 \mod p$, then

$f = A\cdot\delta(\lambda_d)+h$ for some $A\in k$ and some $h\in C_d^{2\,(s+1)}(q)$. We can exclude the possibility $b=p-1$, since in that case $d=p^s+(p-1)\cdot p^s=p^{s+1}$, and so $\omega^d\neq 1$.

Hence we see that $\delta(f)=\delta(h)$, and so $B_d^{3\,(s)}(q)\cap C_d^{3\,(s+1)}(q)=B_d^{3\,(s+1)}(q)$, which implies that $B_d^{3\,(s)}(q)=B_d^3(q)\cap C_d^{3\,(s)}(q)$ for any finite s, and also that $B_d^{3\,(\infty)}(q)=B_d^3(q)\cap C_d^{3\,(\infty)}(q)$. □

3.2 Reductions

3.2.1 Subsets of cocycles

Let s and t be non-negative integers such that $p^t<p^s<q$. Then the subsets $I_d^{(s,t)}(q)$ and $I_d^{(s,s)}(q)$ of $I_d(q)$ are defined as follows:

$$\begin{aligned}
I_d^{(s,t)}(q) :=\ & \left\{F(q_1,p^t,p^s)\mid q_1+p^t+p^s=d,\ q_1<p^t\right\}\\
&\cup\left\{\Psi(b\cdot p^t,p^s)\mid b\cdot p^t+p^s=d,\ b\not\equiv 0\ (\mathrm{mod}\,p),\ b\neq 1\right\}\\
&\cup\left\{E_0(p\cdot p^t,p^s)\mid p^{t+1}+p^s=d\right\}\\
&\cup\left\{\Gamma(q_1,p^t,q_3,p^s)\mid (q_1,p^t,q_3,p^s)\in\mathcal{Q}_d(q)\right\}\\
I_d^{(s,s)}(q) :=\ & \left\{E_1(q_1,p\cdot p^s)\mid q_1\le p^s,\ q_1+p^{s+1}=d\right\}
\end{aligned}$$

Set $I_d^{(s)}(q):=\bigcup_{t\le s}I_d^{(s,t)}(q)$, and

$$I_d^{(\infty)}(q):=\left\{F(q_1,q_2,0)\,\middle|\,q_1+q_2=d,\ 0<q_1<q_2<q\right\}.$$

Let $H_d^{(s,t)}(q)$ denote the subspace of $C_d^{3\,(s)}(q)$ generated by $I_d^{(s,t)}(q)$, let $H_d^{3\,(s)}(q)$ denote $\bigoplus_{t\le s}H_d^{(s,t)}(q)$, and let $H_d^{3\,(\infty)}(q)$ denote the subspace of $C_d^{3\,(\infty)}(q)$ generated by $I_d^{(\infty)}(q)$.

It is easy to see that $I(q)$ and $H^3(q)$ decompose as follows:

$$I(q)=\coprod_d\Big(I_d^{(\infty)}\sqcup\coprod_s I_d^{(s)}\Big)$$

$$H^3(q)=\bigoplus_d\Big(H_d^{3\,(\infty)}(q)\oplus\bigoplus_s H_d^{3\,(s)}(q)\Big)$$

3.2.2 First reduction

The following theorem implies Theorem 2.11.

Theorem 3.12 *We have the following decomposition:*

$$Z_d^3(q)=\Big(\bigoplus_s H_d^{3\,(s)}(q)\oplus H_d^{3\,(\infty)}(q)\Big)\oplus B_d^3(q)$$

In particular, the natural homomorphism

$$\Big(\bigoplus_s H_d^{3\,(s)}(q)\Big)\oplus H_d^{3\,(\infty)}(q)\to H^3(C^*(q))$$

is an isomorphism.

Proof This follows directly from Lemma 3.13 and Lemma 3.14 below. □

3.2.3 Second reduction

Lemma 3.13 *If* $\omega^d=1$*, then*

$$Z_d^{3\,(\infty)}(q)=H_d^{3\,(\infty)}(q)\oplus B_d^{3\,(\infty)}(q).$$

Proof Let $f(U_1,U_2,T_3)=f_0(U_1,U_2)$ be an element of $Z_d^{3\,(\infty)}(q)$. Then the polynomial f_0 is an element of $Z_d^2(q)$. We have a decomposition $f_0=g+\delta(\lambda_d)$, where g is an element of $\bigoplus_s H_d^{2\,(s)}(q)$, which gives the required decomposition of $Z_d^{3\,(\infty)}(q)$. □

Lemma 3.14 *There is a decomposition*

$$Z_d^{3\,(s)}(q)=H_d^{3\,(s)}(q)\oplus\big(B_d^{3\,(s)}(q)+Z_d^{3\,(s+1)}(q)\big).$$

Proof This follows from Lemma 3.16 and Lemma 3.17 below. □

3.2.4 Third reduction

Let s denote a non-negative integer such that $p^s<q$, and let $f\in Z_d^{3\,(s)}$ such that $f=\sum_a f_{ap^s}(U_1,U_2)T_3^{ap^s}$. Then

$$\sum_a\delta(f_{ap^s})(U_1,U_2,U_3)T_4^{ap^s}$$
$$+\sum_a f_{ap^s}(U_1,U_2)\Big((U_3+\omega^{-1}T_4)^{ap^s}-(U_3+T_4)^{ap^s}\Big)=0.$$

By considering the coefficients of $T_4^{p^s}$, we find that

$$\delta(f_{p^s})(U_1,U_2,U_3) + (\omega^{-p^s}-1)\cdot\sum_a f_{ap^s}(U_1,U_2)\cdot a\cdot U_3^{(a-1)p^s} = 0. \qquad (3)$$

Hence $\delta(f_{p^s}) + (\omega^{-p^s}-1)\cdot D_3^{(s)} f = 0$, and so f_{p^s} is contained in $\delta^{-1}(\operatorname{Im} D_3^{(s)})$, where we denote $D_n^{(s)}\big(C_d^{n\,(s)}(q)\big)$ by $\operatorname{Im} D_n^{(s)}$ for simplicity. We thus obtain a map $\phi\colon Z_d^{3\,(s)}(q) \to \delta^{-1}(\operatorname{Im} D_3^{(s)})$ given by $\phi(f) = f_{p^s}$. It is clear that $\phi\big(Z_d^{3\,(s+1)}(q)\big) = 0$.

Lemma 3.15 *If $g \in C_d^{2\,(s)}(q)$ such that $g = \sum g_{ap^s}(U_1)\cdot T_2^{ap^s}$, then*

$$\phi\big(\delta(g)\big) = \delta(g_{p^s}) + (1-\omega^{-p^s})\cdot D_2^{(s)}(g).$$

Proof This follows by direct calculation. □

We thus obtain a homomorphism:

$$\overline{\phi}\colon\ \frac{Z_d^{3\,(s)}(q)}{B_d^{3\,(s)}(q) + Z_d^{3\,(s+1)}(q)} \longrightarrow \frac{\delta^{-1}(\operatorname{Im} D_3^{(s)})}{\operatorname{Im} D_2^{(s)} + B_{d-p^s}^{2\,(s)}(q)}$$

Lemma 3.16 *The homomorphism $\overline{\phi}$ is injective.*

Proof Let $f \in Z_d^{3\,(s)}(q)$. If $\overline{\phi}(f) = 0$, then $f_{p^s} = \delta(h) + D^{(s)}(g)$ for some $\delta(h) \in B_{d-p^s}^{2\,(s)}(q)$ and $g \in C_d^{2\,(s)}(q)$. Let $\overline{f}$ denote $f - (1-\omega^{-p^s})^{-1}\cdot\delta(g)$. Then $D^{(s)}(\overline{f}) = 0$, and so $\overline{f} \in Z_d^{3\,(s+1)}(q)$. □

We denote by ψ_s the composition:

$$H_d^{3\,(s)}(q) \longrightarrow \frac{Z_d^{3\,(s)}(q)}{B_d^{3\,(s)}(q) + Z_d^{3\,(s+1)}(q)} \xrightarrow{\bar{\phi}} \frac{\delta^{-1}(\operatorname{Im} D_3^{(s)})}{\operatorname{Im} D_2^{(s)} + B_{d-p^s}^{2\,(s)}(q)}$$

Lemma 3.17 *The map ψ_s is an isomorphism.*

Proof This follows immediately from Proposition 3.18 below. □

3.2.5 Fourth reduction

We define

$$\begin{aligned} K^{(t)} &:= \delta^{-1}\big(\mathrm{Im}(D_3^{(s)})\big) \cap C^{2\,(t)}_{d-p^s}(q) \quad \text{for } t \le s, \\ K^{(s+1)} &:= \mathrm{Im}\, D_2^{(s)} + B^{2\,(s)}_{d-p^s}(q). \end{aligned}$$

Then

$$K^{(s+1)} \subset K^{(s)} \subset \cdots \subset K^{(0)}.$$

It can be easily checked that the image $\psi_s(H_d^{3\,(s,t)})$ is contained in $K^{(t)}$ and that $\psi_s(H_d^{3\,(s,s)})$ is contained in $K^{(s)}$. There is an induced homomorphism

$$\psi_{(s,t)}\colon\ H_d^{(s,t)} \to K^{(t)}/K^{(t+1)}$$

for all $0 \le t \le s$.

Proposition 3.18 *The homomorphisms $\psi_{(s,t)}$ are isomorphisms.*

We now define

$$\mathcal{A}(s,t) := \begin{cases} \left\{ (q_1,q_2,q_3) \;\middle|\; \begin{array}{l} q_1 < q_3,\ q_2 < p^t \le q_3 < p^s,\ \omega^{q_1+q_3} = 1, \\ \omega^{q_1+q_2} \ne 1,\ \text{if } \omega^{q_1} = \omega^{q_2} \text{ then } q_1 < q_2 \end{array} \right\} & \text{if } p \ne 2, \\[2ex] \left\{ (q_1,q_2,q_3) \;\middle|\; \begin{array}{l} q_1 < q_3,\ q_2 < p^t \le q_3 \le p^s,\ \omega^{q_1+q_3} = 1, \\ \omega^{q_1+q_2} \ne 1,\ \text{if } \omega^{q_1} = \omega^{q_2} \text{ then } q_1 \le q_2 \end{array} \right\} & \text{if } p = 2 \end{cases}$$

and consider the condition

$$\delta(g) - \sum_{(q_1,q_2,q_3)\in\mathcal{A}(s,t)} a_{q_1,q_2,q_3} \cdot U_1^{q_1} \cdot U_2^{q_2} \cdot T_3^{q_3} \in \mathrm{Im}(D_3^{(s)}) \tag{4}$$

for any element $g \in C^{2\,(t)}_{d-p^s}(q)$.

Proposition 3.19 *If there exists an element $g \in C^{2\,(t)}_{d-p^s}(q)$ satisfying (4), then all of the coefficients a_{q_1,q_2,q_3} are zero.*

We will prove propositions 3.18 and 3.19 later, by descending induction on t. Before going into the proof, we give some remarks:

- If $p > 2$ and $s = t$, then Proposition 3.19 is trivial.
- If $p > 2$, then either $a_{q_1,q_2,q_3} = 0$ or $a_{q_2,q_1,q_3} = 0$.
- If $p = 2$ and $q_1 \ne q_2$, then either $a_{q_1,q_2,q_3} = 0$ or $a_{q_2,q_1,q_3} = 0$.

3.2.6 The case $t = s$

Let us consider Proposition 3.19 in the case $p = 2$ and $s = t$. By taking the coefficients of $T_3^{p^s}$ in (4), we see that

$$\delta(g_{p^s})(U_1, U_2) + (1 - \omega^{-p^s}) \cdot D_2^{(s)}(g)(U_1, U_2) - \sum a_{q_1,q_2,p^s} \cdot U_1^{q_1} \cdot U_2^{q_2} = 0. \quad (5)$$

We have $a_{q_1,q_2,p^s} = 0$ unless $q_i < p^s$. We also know that either a_{q_1,q_2,p^s} or a_{q_2,q_1,p^s} vanishes if $q_1 \neq q_2$, and so (5) has no solution if one of a_{q_1,q_2,p^s} is nonzero.

Let us consider Proposition 3.18 for general p. We remark that $h(U_1) \cdot T_2^{ap^s}$ is contained in the image of $D_2^{(s)}$ if $a \not\equiv -1 \mod p$, and thus we consider an element $g \in C_{d-p^s}^{2\,(s)}$ of the form

$$g(U_1, T_2) = \sum g_{(ap-1)p^s}(U_1) \cdot T_2^{(ap-1)p^s},$$

to obtain

$$\delta(g)(U_1, U_2, T_3) = \sum \delta(g_{(ap-1)p^s})(U_1, U_2) \cdot T_3^{(ap-1)p^s}$$
$$- \sum g_{(ap-1)p^s}(U_1) \cdot \Big(\omega^{-p^s}(U_2 + \omega^{-1}T_3)^{(ap-1)p^s} - (U_2 + T_3)^{(ap-1)p^s}\Big). \quad (6)$$

If $\delta(g) \in \mathrm{Im}(D_3^{(s)})$, then the coefficients in the right hand side of (6) sum to zero. Taking the terms in $T_3^{(ap-1)p^s}$ and dividing by $T_3^{(p-1)p^s}$, we see that

$$\sum \delta(g_{(ap-1)p^s})(U_1, U_2) \cdot T_3^{(a-1)p^{s+1}}$$
$$- \sum g_{(ap-1)p^s}(U_1) \cdot \Big(\omega^{-p^{s+1}}(U_2 + \omega^{-1}T_3)^{(a-1)p^{s+1}} - (U_2 + T_3)^{(a-1)p^{s+1}}\Big) = 0. \quad (7)$$

Substituting $T_3 = 0$ gives

$$\delta(g_{(p-1)p^s})(U_1, U_2) + (1 - \omega^{-p^{s+1}}) \cdot \sum g_{(ap-1)p^s}(U_1) \cdot U_2^{(a-1)p^{s+1}} = 0,$$

which shows that $g(U_1, T_2) = (\omega^{-p^s} - 1)^{-1} \cdot \delta g_{(p-1)p^s}(U_1, T_2) \cdot T_2^{(p-1)p^s}$, and also that $\delta g_{(p-1)p^s} \in C_{d-p^{s+1}}^{3\,(s+1)}(q)$. Thus the degree $\deg(g_{(p-1)p^s}) = d - p^{s+1}$ is either p^h (for $0 \le h \le s$) or $b \cdot p^{s+1}$ (see the first half of the proof of Lemma 3.6). In the case $\deg(g_{p^s(p-1)}) = b \cdot p^{s+1}$, the polynomial g is of the form $\big(\omega^{-p^s} - 1\big)^{-1} \cdot \delta(\lambda_{bp^{s+1}})(U_1, T_2) \cdot T_2^{(p-1)p^s}$. Then

$$\delta\big(\lambda_{(b+1)p^{s+1}-p^s}\big) - \delta\big(\lambda_{bp^{s+1}}\big) \cdot T_2^{(p-1)p^s} \in \mathrm{Im}\big(D_2^{(s)}\big),$$

and so the term $\delta\big(\lambda_{bp^{s+1}}\big) \cdot T_2^{(p-1)p^s}$ can be killed.

On the other hand,

$$\psi_s\big(E_1(p^h, p \cdot p^s)\big) = (1 - \omega^{-p^s}) \cdot U_1^{p^h} \cdot T_2^{(p-1)\cdot p^s},$$

and so the term in $\delta\big(\lambda_{p^h}\big) \cdot T_2^{(p-1)p^s}$ can be killed by $E_1(p^h, p \cdot p^s)$. Thus we conclude that $\psi_{(s,s)}$ is surjective.

We remark that $p^h + p^s(p-1) \not\equiv 0 \pmod{p^s}$ if $h < s$, and thus the injectivity of $\psi_{(s,s)}$ can be checked easily.

3.2.7 The case $t < s$

We assume that the claims of the propositions 3.18 and 3.19 hold for larger than $t+1$, and we will prove the claims for t.

Let $g \in C^{2\,(t)}_{d-p^s}$ be an element satisfying (4). Then, comparing coefficients of $T_3^{p^t}$, we find that

$$\delta(D^{(t)}g)(U_1, U_2, T_3)_{|T_3=0} = \sum a_{q_1,q_2,p^t} \cdot U_1^{q_1} \cdot U_2^{q_2}. \tag{8}$$

Here "$|T_3 = 0$" means the substitution $T_3 = 0$. If we decompose g as

$$g = \sum g_{ap^t}(U_1) \cdot T_2^{ap^t},$$

then it is easy to see from (8) that

$$\delta(g_{p^t})(U_1, U_2) - \sum a_{q_1,q_2,p^t} \cdot U_1^{q_1} \cdot U_2^{q_2}$$

is contained in $\mathrm{Im}(D_2^{(t)})$. Recall that we have $a_{q_1,q_2,p^t} = 0$ unless $q_i < p^t$. If $p > 2$, or if $p = 2$ and $q_1 \neq q_2$, then either a_{q_1,q_2,p^t} or a_{q_2,q_1,p^t} is zero, and so we can conclude that all of the a_{q_1,q_2,p^t} are zero in both cases, by using

$$\delta\big(\lambda_{q_1+q_2}\big)(U_1, U_2) = \big(\omega^{q_1+q_2} - 1\big) \cdot U_1^{q_1+q_2} + \big(\omega^{q_1} - 1\big) \cdot U_1^{q_1} \cdot U_2^{q_2} + \big(\omega^{q_2} - 1\big) \cdot U_1^{q_2} \cdot U_2^{q_1}.$$

Hence $\delta(D^{(t)}g) = 0$, and so $D^{(t)}g$ is of the form

$$D^{(t)}g(U_1, T_2) = A \cdot \delta(\lambda_{d-p^s-p^t})(U_1, T_2) + B \cdot U_1^{p_1} \cdot T_2^{p_2}. \tag{9}$$

Here we have $p_1 < p_2$ and $p^t \leq p_2$. If $B \neq 0$, then $\omega^{p_1+p_2} = 1$.

Lemma 3.20 *If $p = 2$ and $p^t = p_2$, then $B = 0$.*

Proof Assume that $B \neq 0$. Since the terms in $D^{(t)}g$ are of the form $a \cdot U_1^b \cdot T_2^{c \cdot 2^{t+1}}$, it follows that $A \neq 0$. We have

$$\begin{aligned}\delta\big(\lambda_{p_1+p_2}\big) &= \big(\omega \cdot U_1 + T_2\big)^{p_1+p_2} - \big(U_1 + T_2\big)^{p_1+p_2} \\ &= \big(\omega^{p_1} - 1\big) \cdot U_1^{p_1} \cdot T_2^{p_2} + \big(\omega^{p_2} - 1\big) \cdot U_1^{p_2} \cdot T_2^{p_1}.\end{aligned}$$

Note that $p_1 < p_2 = p^t$, and so the right hand side of (9) cannot be contained in $\mathrm{Im}(D^{(t)})$, a contradiction. □

Thus $B \cdot U_1^{p_1} \cdot T_2^{p_2} \in \mathrm{Im}(D^{(t)})$. It follows from Lemma 3.6 that $d - p^s - p^t \in \mathcal{P}(t, q)$ if $A \cdot \delta(\lambda_{d-p^s-p^t}) \neq 0$.

Lemma 3.21 *We can kill $A\delta(\lambda_{d-p^s-p^t})$ by using one of $D^{(t)}\psi_s(F(p^h, p^t, p^s))$, $D^{(t)}\psi_s(E_0(p \cdot p^t, p^s))$ or $D^{(t)}\psi_s\big(\Psi\big(b \cdot p^t, p^s\big)\big)$.*

Proof The following may be easily verified by routine calculation.

$$\begin{aligned}\psi_s\big(F(p^h, p^t, p^s)\big) &= U_1^{p^h} \cdot T_2^{p^t}, \\ \psi_s\big(E_0(p \cdot p^t, p^s)\big) &= \left(\frac{1}{p}\Big(\big(\omega U_1 + T_2\big)^p - \big(U_1 + T_2\big)^p - \big(\omega^p - 1\big) \cdot U_1^p\Big)\right)^{p^t}, \\ \psi_s\big(\Psi(b \cdot p^t, p^s)\big) &= \big(\omega U_1 + T_2\big)^{b \cdot p^t} - \big(U_1 + T_2\big)^{b \cdot p^t} - \big(\omega^{b \cdot p^t} - 1\big) \cdot U_1^{b \cdot p^t}.\end{aligned}$$

As an immediate consequence, we obtain the following.

$$\begin{aligned}D^{(t)}\psi_s\big(F(p^h, p^t, p^s)\big) &= U_1^{p^h} = (\omega^{p^h} - 1)^{-1} \cdot \delta\big(\lambda_{p^h}\big), \\ D^{(t)}\psi_s\big(E_0(p \cdot p^t, p^s)\big) &= \big(\omega \cdot U_1 + T_2\big)^{(p-1) \cdot p^t} - \big(U_1 + T_2\big)^{(p-1) \cdot p^t} \\ &= \delta\big(\lambda_{p^t \cdot (p-1)}\big)(U_1, T_2), \\ D^{(t)}\psi_s\big(\Psi(b \cdot p^t, p^s)\big) &= b \cdot \Big(\big(\omega \cdot U_1 + T_2\big)^{(b-1) \cdot p^t} - \big(U_1 + T_2\big)^{(b-1) \cdot p^t}\Big) \\ &= b \cdot \delta\big(\lambda_{(b-1)p^t}\big).\end{aligned} \tag{10}$$

Then the claim of the lemma follows immediately. □

We now consider the case $B \neq 0$.

Lemma 3.22 *If $(p_1, p^t, p_2, p^s) \in \mathcal{Q}_d(q)$, then we can kill the term $B \cdot U_1^{p_1} T_2^{p_2}$ by using $D^{(t)}\psi_s(\Gamma(p_1, p^t, p_2, p^s))$.*

Proof In cases 1–3 of Condition 2.9, $\psi_s\big(\Gamma(p_1,p^t,p_2,p^s)\big)=U_1^{p_1}\cdot T_2^{p^t+p_2}$, and so $D^{(t)}\psi_s\big(\Gamma(p_1,p^t,p_2,p^s)\big)=U_1^{p_1}\cdot T_2^{p_2}$. In cases 4 and 5 of Condition 2.9,

$$\psi_s\big(\Gamma(p_1,p^t,p_2,p^s)\big)=U_1^{p_1}\cdot T_2^{p^t+p_2}+U_1^{p^t}\cdot T_2^{p_1+p_2} \\ -\big(\omega^{q_1}-1\big)^{-1}\cdot\big(1-\omega^{2q_1}\big)\cdot U_1^{p_1+p^t}\cdot T_2^{p_2},$$

and so

$$D^{(t)}\psi_s\big(\Gamma(p_1,p^t,p_2,p^s)\big)=\begin{cases} U_1^{p_1}\cdot T_2^{p_2}, & \text{if } (p^t<p_1),\\ 2\cdot U_1^{p_1}\cdot T_2^{p_2}, & \text{if } (p^t=p_1,\ p\neq 2),\end{cases}$$

and the claim of the lemma follows immediately. □

Lemma 3.23 *$g-C\cdot\psi_s(f)$ satisfies (4) for any $f\in H_d^{(s,t)}$ and any $C\in k$.*

Proof Since $\delta(f)=0$ for $f\in H_d^{(s,t)}$, we have $\delta\psi_s(f)\in\operatorname{Im}D_2^{(s)}$ due to (3), and the lemma follows immediately. □

Lemma 3.24 *If $(p_1,p^t,p_2,p^s)\notin\mathcal{Q}_d(q)$, then $(p_1,p^t,p_2)\in\mathcal{A}(s,t+1)$.*

Proof We remark that $p_1<p_2$, $p^t<p^{t+1}$, $p^t\leq p_2$ and $\omega^{p_1+p_2}=1$.

The possibility $\omega^{p_1+p^t}=1$ is removed by case 1 of Condition 2.9. We remark that $\omega^{p_1+p^t}\neq 1$ and $\omega^{p_1+p_2}=1$ imply that $p^t\neq p_2$, and so $p^{t+1}\leq p_2$. If $p\neq 2$, then cases 2 and 3 remove the possibility $p^s\leq p_2$. Case 4 removes the possibility that both $p^t\leq p_1$ and $\omega^{p^t}=\omega^{p_1}$ simultaneously, and so we see that $(p_1,p^t,p_2)\in\mathcal{A}(s,t+1)$ if $p\neq 2$. If $p=2$, then the claim follows by a similar argument. □

We can choose a constant $C\in k$ such that we can kill the term $B\cdot U_1^{p_1}\cdot T_2^{p_2}$ by using $D^{(t)}\big(C\cdot U_1^{p_1}\cdot T_2^{p_2+p^t}\big)$. Then $g'=g-C\cdot U_1^{p_1}\cdot T_2^{p_2+p^t}$ satisfies $D^{(t)}(g'-g)=0$, ie $g'\in C^{2\,(t+1)}$. Then we obtain

$$\delta(g')(U_1,U_2,T_3)=\delta(g)(U_1,U_2,T_3)-C\cdot\delta(U_1^{p_1}\cdot T_2^{p_2+p^t}) \\ =\sum_{q_3>p^t}a_{q_1,q_2,q_3}\cdot U_1^{q_1}U_2^{q_2}T_3^{q_3}-C\cdot(1-\omega^{p_1+p^t})\cdot U_1^{p_1}U_2^{p^t}T_3^{p_2}+h$$

for some $h\in\operatorname{Im}(D_3^{(s)})$. We conclude that all of the a_{q_1,q_2,q_3} and C are zero, by Proposition 3.19 for $t+1$ and Lemma 3.24, and so Proposition 3.19 is true for t as well.

Let g be an element of $K^{(t)}$. Then $\delta(g) \in \operatorname{Im}\bigl(D_3^{(s)}\bigr)$. By the argument above, it can be shown that $g - \psi_s(f) \in K^{(t+1)}$ for some suitable $f \in H_d^{(s,t)}(q)$. The linear independence of $\bigl\{\psi_s(f) \bigm| f \in I^{(s,t)}(q)\bigr\}$ in $K^{(t)}/K^{(t+1)}$ follows by the argument above, and (10). Thus Proposition 3.18 holds for t, and so the proof of propositions 3.18 and 3.19 is complete.

References

[1] **J S Carter**, **D Jelsovsky**, **S Kamada**, **L Langford**, **M Saito**, *Quandle cohomology and State-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003) 3947–3989

[2] **J S Carter**, **D Jelsovsky**, **S Kamada**, **M Saito**, *Computations of quandle cocycle invariants of knotted curves and surfaces*, Adv. Math. 157 (2001) 36–94

[3] **J S Carter**, **D Jelsovsky**, **S Kamada**, **M Saito**, *Quandle homology groups, their Betti numbers, and virtual knots*, J. Pure Appl. Algebra 157 (2001) 135–155

[4] **P Etingof**, **M Graña**, *On rack cohomology*, J. Pure Appl. Algebra 177 (2003) 49–59

[5] **A Kawauchi** (editor), *A survey of knot theory*, Birkhäuser, Basel (1996)

[6] **T Mochizuki**, *Some calculations of cohomology groups of finite Alexander quandles*, J. Pure Appl. Algebra 179, (2003) 287–330

[7] **C Rourke**, **B Sanderson**, *A new classification of links and some calculations using it*, `arXiv:math.GT/0006062`

[8] **S Satoh**, **A Shima**, *The 2–twist spun trefoil has the triple number four*, Trans. Amer. Math. Soc. 356 (2004) 1007–1024

[9] **S Satoh**, **A Shima**, *Triple point numbers of surface-knots and colorings by quandles*, preprint

Department of Mathematics, Kyoto University
Kyoto, 606–8502, Japan

Email: `takuro@math.kyoto-u.ac.jp`

Received: 31 May 2004