



## All integral slopes can be Seifert fibered slopes for hyperbolic knots

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**Abstract** Which slopes can or cannot appear as Seifert fibered slopes for hyperbolic knots in the 3-sphere  $S^3$ ? It is conjectured that if  $r$ -surgery on a hyperbolic knot in  $S^3$  yields a Seifert fiber space, then  $r$  is an integer. We show that for each integer  $n \in \mathbb{Z}$ , there exists a tunnel number one, hyperbolic knot  $K_n$  in  $S^3$  such that  $n$ -surgery on  $K_n$  produces a small Seifert fiber space.

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**Keywords** Dehn surgery, hyperbolic knot, Seifert fiber space, surgery slopes

*This paper is dedicated to Donald M. Davis on the occasion of his 60th birthday.*

### 1 Introduction

Let  $K$  be a knot in the 3-sphere  $S^3$  with a tubular neighborhood  $N(K)$ . Then the set of *slopes* for  $K$  (i.e.,  $\partial N(K)$ -isotopy classes of simple loops on  $\partial N(K)$ ) is identified with  $\mathbb{Q} \cup \{\infty\}$  using preferred meridian-longitude pair so that a meridian corresponds to  $\infty$ . A slope  $\gamma$  is said to be *integral* if a representative of  $\gamma$  intersects a meridian exactly once, in other words,  $\gamma$  corresponds to an integer under the above identification. In the following, we denote by  $(K; \gamma)$  the 3-manifold obtained from  $S^3$  by Dehn surgery on a knot  $K$  with slope  $\gamma$ , i.e., by attaching a solid torus to  $S^3 - \text{int} N(K)$  in such a way that  $\gamma$  bounds a meridian disk of the filled solid torus. If  $\gamma$  corresponds to  $r \in \mathbb{Q} \cup \{\infty\}$ , then we identify  $\gamma$  and  $r$  and write  $(K; r)$  for  $(K; \gamma)$ .

We denote by  $\mathcal{L}$  the *set of lens slopes*  $\{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is a lens space}\}$ , where  $S^3$  and  $S^2 \times S^1$  are also considered as lens spaces. Then the cyclic surgery theorem [7] implies that  $\mathcal{L} \subset \mathbb{Z}$ . A

result of Gabai [10, Corollary 8.3] shows that  $0 \notin \mathcal{L}$ , a result of Gordon and Luecke [14] shows that  $\pm 1 \notin \mathcal{L}$ . In [19] Kronheimer and Mrowka prove that  $\pm 2 \notin \mathcal{L}$ . Furthermore, a result of Kronheimer, Mrowka, Ozsváth and Szabó [20] implies that  $\pm 3, \pm 4 \notin \mathcal{L}$ . Besides, Berge [4, Table of Lens Spaces] suggests that if  $n \in \mathcal{L}$ , then  $|n| \geq 18$  and not every integer  $n$  with  $|n| \geq 18$  appears in  $\mathcal{L}$ . Fintushel and Stern [9] had shown that 18-surgery on the  $(-2, 3, 7)$  pretzel knot yields a lens space.

*Which slope (rational number) can or cannot appear in the set of Seifert fibered slopes  $\mathcal{S} = \{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is Seifert fibered}\}$ ? It is conjectured that  $\mathcal{S} \subset \mathbb{Z}$  [12].*

The purpose of this paper is to prove:

**Theorem 1.1** *For each integer  $n \in \mathbb{Z}$ , there exists a tunnel number one, hyperbolic knot  $K_n$  in  $S^3$  such that  $(K_n; n)$  is a small Seifert fiber space (i.e., a Seifert fiber space over  $S^2$  with exactly three exceptional fibers).*

**Remark** Since  $K_n$  has tunnel number one, it is embedded in a genus two Heegaard surface of  $S^3$  and strongly invertible [26, Lemma 5]. See [22, Question 3.1].

Theorem 1.1, together with the previous known results, shows:

**Corollary 1.2**  $\mathcal{L} \subsetneq \mathbb{Z} \subset \mathcal{S}$ .

### Remarks

(1) For the set of reducing slopes  $\mathcal{R} = \{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is reducible}\}$ , Gordon and Luecke [13] have shown that  $\mathcal{R} \subset \mathbb{Z}$ . In fact, the cabling conjecture [11] asserts that  $\mathcal{R} = \emptyset$ .

(2) For the set of toroidal slopes  $\mathcal{T} = \{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is toroidal}\}$ , Gordon and Luecke [15] have shown that  $\mathcal{T} \subset \mathbb{Z}/2$  (integers or half integers). In [28], Teragaito shows that  $\mathbb{Z} \subset \mathcal{T}$  and conjectures that  $\mathcal{T} \subsetneq \mathbb{Z}/2$ .

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## 2 Hyperbolic knots with Seifert fibered surgeries

Our construction is based on an example of a longitudinal Seifert fibered surgery given in [17].

Let  $k \cup c$  be a 2-bridge link given in Figure 1, and let  $K_n$  be a knot obtained from  $k$  by  $\frac{1}{-n+4}$ -surgery along  $c$ .

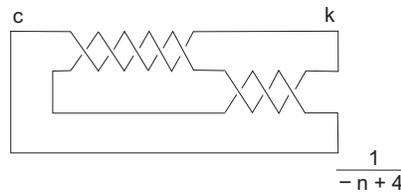


Figure 1:  $K_n$

We shall say that a Seifert fiber space is of *type*  $S^2(n_1, n_2, n_3)$  if it has a Seifert fibration over  $S^2$  with three exceptional fibers of indices  $n_1, n_2$  and  $n_3$  ( $n_i \geq 2$ ). Since  $K_4$  is unknotted,  $(K_4; 4)$  is a lens space  $L(4, 1)$ . For the other  $n$ 's, we have:

**Lemma 2.1**  $(K_n; n)$  is a small Seifert fiber space of type  $S^2(3, 5, |4n - 15|)$  for any integer  $n \neq 4$ .

**Proof** Since the linking number of  $k$  and  $c$  is one (with suitable orientations),  $(K_n; n)$  has surgery descriptions as in Figure 2.

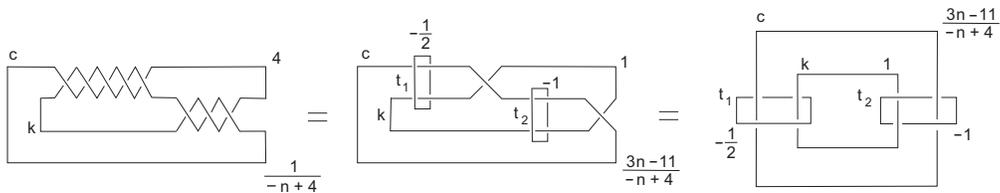


Figure 2: Surgery descriptions of  $(K_n; n)$

Let us take the quotient by the strong inversion of  $S^3$  with an axis  $L$  as shown in Figure 3.

Then we obtain a branch knot  $b'$  which is the image of the axis  $L$ . The Montesinos trick ([25], [6]) shows that  $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$  and 1-surgery on  $t_1, t_2, c$  and

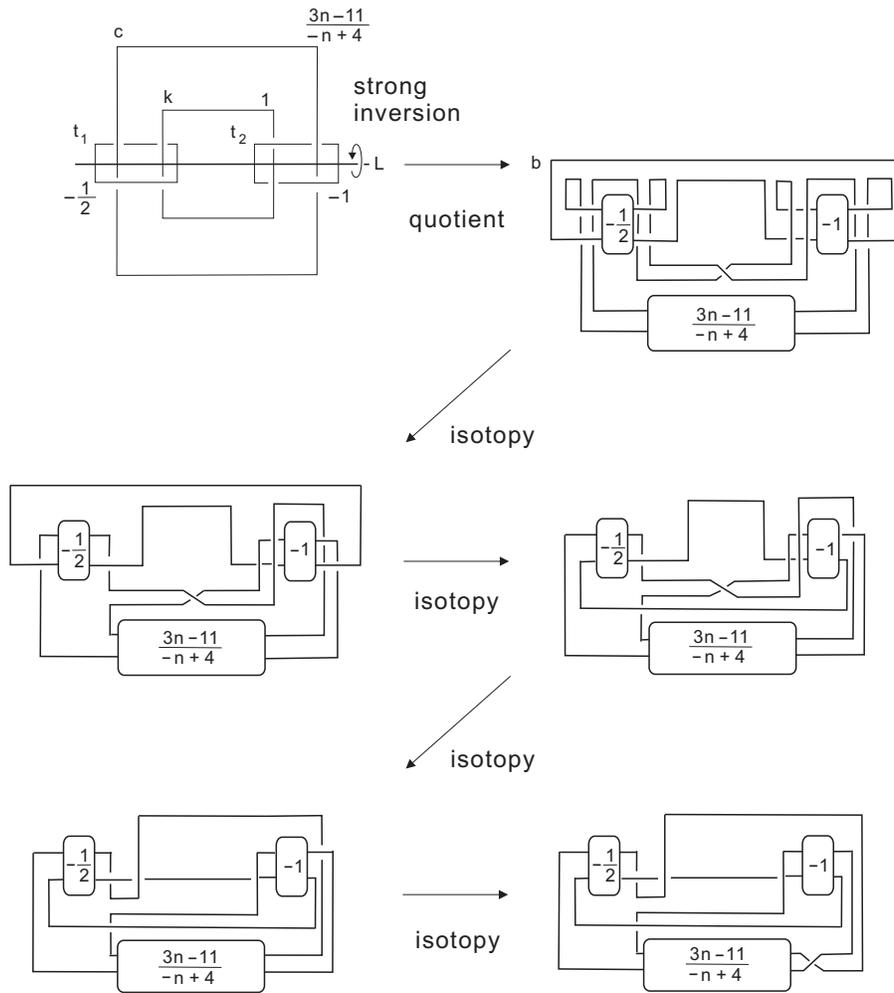


Figure 3

$k$  in the upstairs correspond to  $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$  and 1-untangle surgery on  $b'$  in the downstairs, where an  $r$ -untangle surgery is a replacement of  $\frac{1}{0}$ -untangle by  $r$ -untangle. (We adopt Bleiler's convention [5] on the parametrization of rational tangles.) These untangle surgeries convert  $b'$  into a link  $b$  (Figure 3).

Following the sequence of isotopies in Figures 3 and 4, we obtain a Montesinos link  $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$ .

Since  $(K_n; n)$  is the double branched cover of  $S^3$  branched over the Montesinos

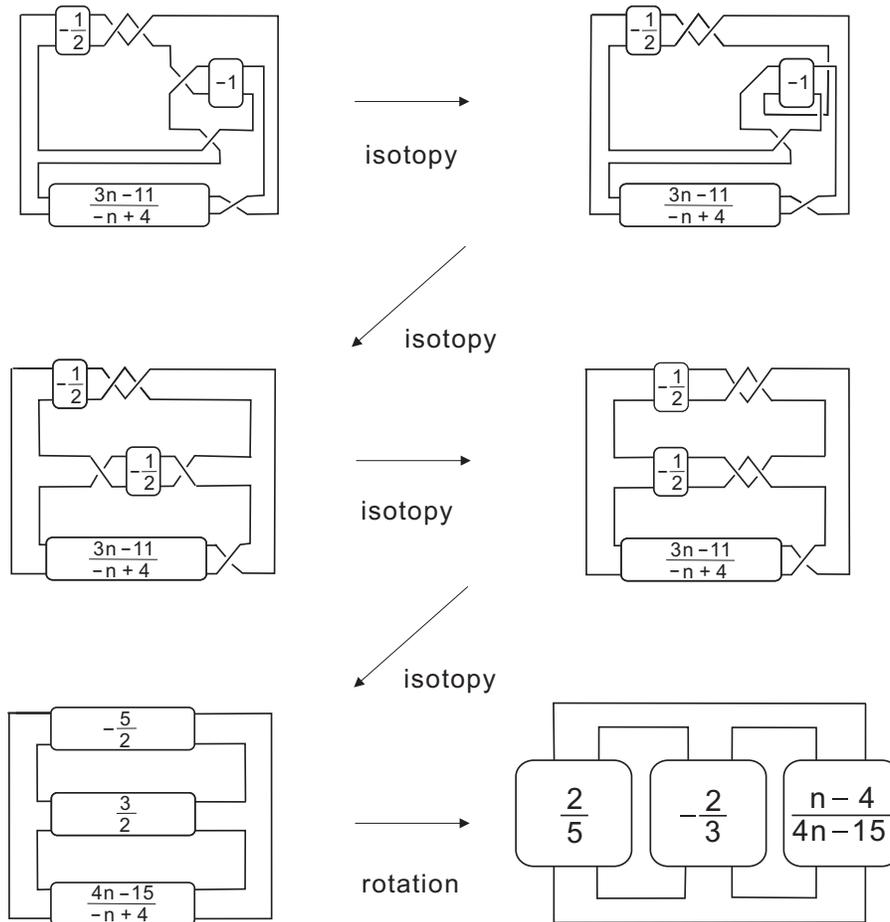


Figure 4: Continued from Figure 3

link  $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$ ,  $(K_n; n)$  is a Seifert fiber space of type  $S^2(3, 5, |4n - 15|)$  as desired.  $\square$

**Lemma 2.2** *The knot  $K_n$  is hyperbolic if  $n \neq 3, 4, 5$ .*

**Proof** Note that the 2-bridge link given in Figure 1 is not a  $(2, p)$ -torus link, and hence by [23] it is a hyperbolic link. If  $n \neq 3, 4, 5$ , then  $|-n + 4| > 1$  and it follows from [1, Theorem 1] (also [3, Theorem 1.2]) that  $K_n$  is a hyperbolic knot. See also [16, Corollary A.2], [24, Theorem 1.2] and [2, Theorem 1.1].  $\square$

**Remark** It follows from [21], [18] that  $K_n$  is a nontrivial knot except when  $n = 4$ . An experiment using Weeks' computer program "SnapPea" [31] suggests that  $K_3$  and  $K_5$  are hyperbolic, but we will not use this experimental results.

**Lemma 2.3** *The knot  $K_n$  has tunnel number one for any integer  $n \neq 4$ .*

**Proof** Since the link  $k \cup c$  is a two-bridge link, the tunnel number of  $k \cup c$  is one with unknotting tunnel  $\tau$ ; A regular neighborhood  $N(k \cup c \cup \tau)$  is a genus two handlebody and  $S^3 - \text{int}N(k \cup c \cup \tau)$  is also a genus two handlebody, see Figure 5.

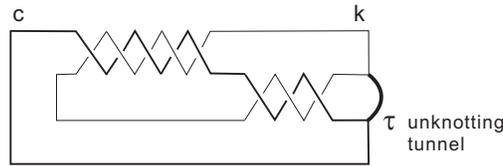


Figure 5

Then the general fact below (in which  $k \cup c$  is not necessarily a two-bridge link) shows that the tunnel number of  $K_n$  is less than or equal to one. Since our knot  $K_n$  ( $n \neq 4$ ) is knotted in  $S^3$ , the tunnel number of  $K_n$  is one.  $\square$

**Claim 2.4** *Let  $k \cup c$  be a two component link in  $S^3$  which has tunnel number one. Assume that  $c$  is unknotted in  $S^3$ . Then every knot obtained from  $k$  by twisting along  $c$  has tunnel number at most one.*

**Proof** Let  $\tau$  be an unknotting tunnel and  $V$  a regular neighborhood of  $k \cup c \cup \tau$  in  $S^3$ ;  $V$  is a genus two handlebody. Since  $\tau$  is an unknotting tunnel for  $k \cup c$ , by definition,  $W = S^3 - \text{int}V$  is also a genus two handlebody. Take a small tubular neighborhood  $N(c) \subset \text{int}V$  and perform  $-\frac{1}{n}$ -surgery on  $c$  using  $N(c)$ . Then we obtain a knot  $k_n$  as the image of  $k$  and obtain a genus two handlebody  $V(c; -\frac{1}{n})$ . Note that  $V(c; -\frac{1}{n})$  and  $W$  define a genus two Heegaard splitting of  $S^3$ , see Figure 6, where  $c_n^*$  denotes the core of the filled solid torus.

Then it is easy to see that an arc  $\tau_n$  given by Figure 6 is an unknotting tunnel for  $k_n$  as desired.  $\square$

Now we are ready to prove Theorem 1.1. Lemmas 2.1, 2.2 and 2.3 show that our knots  $K_n$  enjoy the required properties, except for  $n = 3, 4, 5$ . To prove

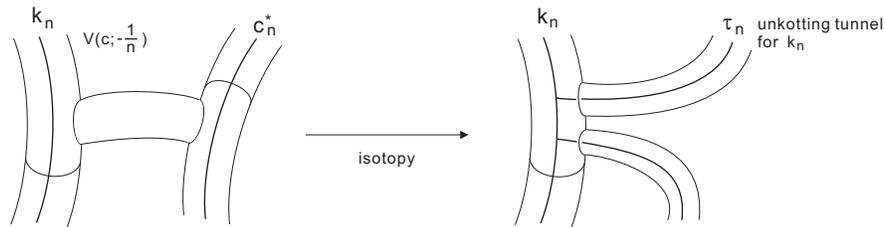


Figure 6

Theorem 1.1, we find hyperbolic knots  $K'_n$  so that  $(K'_n; n)$  is Seifert fibered for  $n = 3, 4, 5$  (instead of showing that  $K_3, K_5$  are hyperbolic). As the simplest way, let  $K'_3, K'_4$  and  $K'_5$  be the mirror image of  $K_{-3}, K_{-4}$  and  $K_{-5}$ , respectively. Since  $K_{-3}, K_{-4}$  and  $K_{-5}$  are tunnel number one, hyperbolic knots by Lemmas 2.2 and 2.3, their mirror images  $K'_3, K'_4$  and  $K'_5$  are also tunnel number one, hyperbolic knots. It is easy to observe that  $(K'_3; 3)$  (resp.  $(K'_4; 4), (K'_5; 5)$ ) is the mirror image of  $(K_{-3}; -3)$  (resp.  $(K_{-4}; -4), (K_{-5}; -5)$ ). By Lemma 2.1,  $(K_{-3}; -3), (K_{-4}; -4)$  and  $(K_{-5}; -5)$  are Seifert fibered, and hence  $(K'_3; 3), (K'_4; 4)$  and  $(K'_5; 5)$  are also Seifert fibered. Putting  $K_n$  as  $K'_n$  for  $n = 3, 4, 5$ , we finish a proof of Theorem 1.1.  $\square$

### 3 Identifying exceptional fibers

In [24], Miyazaki and Motegi conjectured that if  $K$  admits a Seifert fibered surgery, then there is a trivial knot  $c \subset S^3$  disjoint from  $K$  which becomes a Seifert fiber in the resulting Seifert fiber space, and verified the conjecture for several Seifert fibered surgeries [24, Section 6], see also [8]. Furthermore, computer experiments via “SnapPea” [31] suggest that such a knot  $c$  is realized by a short closed geodesic in the hyperbolic manifold  $S^3 - K$ , for details see [24, Section 9], [27].

In this section, we verify the conjecture for Seifert fibered surgeries given in Theorem 1.1.

Recall that  $K_n$  is obtained from  $k$  by  $\frac{1}{-n+4}$ -surgery on the trivial knot  $c$  (i.e.,  $(n - 4)$ -twist along  $c$ ), see Figure 1. Denote by  $c_n$  the core of the filled solid torus. Then  $K_n \cup c_n$  is a link in  $S^3$  such that  $c_n$  is a trivial knot.

**Lemma 3.1** *After  $n$ -surgery on  $K_n$ ,  $c_n$  becomes an exceptional fiber of index  $|4n - 15|$  in the resulting Seifert fiber space  $(K_n; n)$ .*

**Proof** Following the sequences given by Figures 3 and 4, we have a Montesinos link with three arcs  $\gamma$ ,  $\tau_1$  and  $\tau_2$  as in Figure 7, where  $n = 1$  in the final Montesinos link, and  $\gamma$ ,  $\tau_1$ ,  $\tau_2$  and  $\kappa$  are the images of  $c$ ,  $t_1$ ,  $t_2$  and  $k$ , respectively.

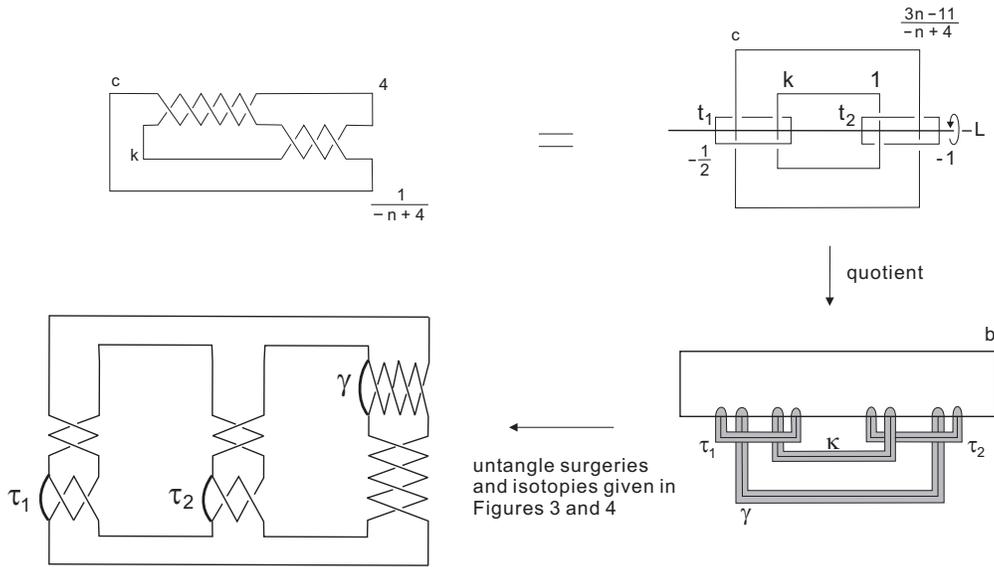


Figure 7: Positions of exceptional fibers

From Figure 7 we recognize that  $t_1, t_2$  and  $c$  become exceptional fibers of indices 5, 3 and  $|4n - 15|$ , respectively in  $(K_n; n)$ .  $\square$

For  $n \neq 3, 4, 5$ ,  $c_n$  becomes an exceptional fiber of index  $|4n - 15|$ , which is the unique maximal index, in  $(K_n; n)$ . Experiments via “SnapPea” [31] suggest that  $c_n$  is a shortest closed geodesic in  $S^3 - K_n$  ( $n \neq 3, 4, 5$ ). For sufficiently large  $|n|$ , hyperbolic Dehn surgery theorem [29], [30] shows that  $c_n$  is the unique shortest closed geodesic in  $S^3 - K_n$ .

Let us assume that  $n = 3, 4, 5$ . Then we have put  $K_n$  as the mirror image of  $K_{-n}$  in the proof of Theorem 1.1. Let  $k' \cup c'$  be the mirror image of the link  $k \cup c$ . Then  $K_n$  is obtained also from  $k'$  by  $\frac{1}{-n-4}$ -surgery on  $c'$  (i.e.,  $(n + 4)$ -twist along  $c'$ ); we denote the core of the filled solid torus by  $c'_n$ . Note that there is an orientation reversing diffeomorphism from  $(K_{-n}; -n)$  to  $(K_n; n)$  sending  $c_{-n}$  (regarded as a fiber in  $(K_{-n}; -n)$ ) to  $c'_n$  (regarded as a fiber in  $(K_n; n)$ ). Thus the above observation implies that  $c'_n$  becomes an exceptional fiber of index  $|4n + 15|$ , which is the unique maximal index, in  $(K_n; n)$  ( $n = 3, 4, 5$ ).

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