H-space structure on pointed mapping spaces

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Abstract We investigate the existence of an $H$-space structure on the function space, $\mathcal{F}_*(X,Y,\ast)$, of based maps in the component of the trivial map between two pointed connected CW-complexes $X$ and $Y$. For that, we introduce the notion of $H(n)$-space and prove that we have an $H$-space structure on $\mathcal{F}_*(X,Y,\ast)$ if $Y$ is an $H(n)$-space and $X$ is of Lusternik-Schnirelmann category less than or equal to $n$. When we consider the rational homotopy type of nilpotent finite type CW-complexes, the existence of an $H(n)$-space structure can be easily detected on the minimal model and coincides with the differential length considered by Y. Kotani.

When $X$ is finite, using the Haefliger’s model for function spaces, we can prove that the rational cohomology of $\mathcal{F}_*(X,Y,\ast)$ is free commutative if the rational Toomer invariant of $X$ is strictly less than the differential length of $Y$, extending a recent result of Y. Kotani.

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1 Introduction

Let $X$ and $Y$ be pointed connected CW-complexes. We study the occurrence of an $H$-space structure on the function space, $\mathcal{F}_*(X,Y,\ast)$, of based maps in the component of the trivial map. Of course when $X$ is a co-$H$-space or $Y$ is an $H$-space this mapping space is an $H$-space. Here, we are considering weaker conditions, both on $X$ and $Y$, which guarantee the existence of an $H$-space structure on the function space. In Definition 3, we introduce the notion of $H(n)$-space designed for this purpose and prove:

**Proposition 1** Let $Y$ be an $H(n)$-space and $X$ be a space of Lusternik-Schnirelmann category less than or equal to $n$. Then the space $\mathcal{F}_*(X,Y,\ast)$ is an $H$-space.
The existence of an $H(n)$-structure and the Lusternik-Schnirelmann category (LS-category in short) are hard to determine. We first study some properties of $H(n)$-spaces and give some examples. Concerning the second hypothesis, we are interested in replacing $\text{cat}(X) \leq n$ by an upper bound on an approximation of the LS-category (see [5, Chapter 2]). We succeed in Proposition 7 with an hypothesis on the dimension of $X$ but the most interesting replacement is obtained in the rational setting which constitutes the second part of this paper.

We use Sullivan minimal models for which we refer to [6]. We recall here that each finite type nilpotent CW-complex $X$ has a unique minimal model $(\wedge V, d)$ that characterises all the rational homotopy type of $X$. We first prove that the existence of an $H(n)$-structure on a rational space $X_0$ can be easily detected from its minimal model. It corresponds to a valuation of the differential of this model, introduced by Y. Kotani in [11]:

The differential $d$ of the minimal model $(\wedge V, d)$ can be written as $d = d_1 + d_2 + \cdots$ where $d_i$ increases the word length by $i$. The differential length of $(\wedge V, d)$, denoted $dl(X)$, is the least integer $n$ such that $d_{n-1}$ is non zero.

As a minimal model of $X$ is defined up to isomorphism, the differential length is a rational homotopy type invariant of $X$, see [11, Theorem 1.1]. Proposition 8 establishes a relation between $dl(X)$ and the existence of an $H(n)$-structure on the rationalisation of $X$.

Finally, recall that the rational cup-length $\text{cup}_0(X)$ of $X$ is the maximal length of a nonzero product in $H^{>0}(X; \mathbb{Q})$ and that the rational Toomer invariant $e_0(X)$ of $X$ can be defined as follows: if $(\wedge V, d)$ denotes the minimal model of $X$, then $e_0(X)$ is the least integer $r$ such that the projection $(\wedge V, d) \to (\wedge V/ \wedge^{>r} V, d)$ is injective in cohomology. In [11], by using the rational cup-length of $X$ and the differential length of $Y$, Y. Kotani gives a necessary and sufficient condition for the rational cohomology of $\mathcal{F}_s(X, Y, *)$ to be free commutative when $X$ is a rational formal space and when the dimension of $X$ is less than the connectivity of $Y$. We show here that a large part of the Kotani criterium remains valid, without hypothesis of formality and dimension. We prove:

Theorem 2 Let $X$ and $Y$ be nilpotent finite type CW-complexes, with $X$ finite.

1. If $e_0(X) < dl(Y)$, then the cohomology algebra $H^*(\mathcal{F}_s(X, Y, *); \mathbb{Q})$ is free commutative.
2. If $\dim(X) \leq \text{conn}(Y)$ and if the cohomology algebra $H^*(\mathcal{F}_s(X, Y, *); \mathbb{Q})$ is free commutative, then $\text{cup}_0(X) < dl(Y)$. 

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As an application, we describe in Theorem 12 the Postnikov tower of the rationalisation of \( F_*(X,Y,*) \) where \( X \) is a finite nilpotent space and \( Y \) a finite type CW-complex whose connectivity is greater than the dimension of \( X \). Our description implies the solvability of the rational Pontrjagin algebra of \( \Omega(F_*(X,Y,*)) \).

Section 2 contains the topological setting and the proof of Proposition 1. The link with rational models is done in Section 3. Our proof of Theorem 2 uses the Haefliger model for mapping spaces. In order to be self-contained, we recall briefly Haefliger’s construction in Section 4. The proof of Theorem 2 is contained in Section 5. Finally, Section 6 is devoted to the description of the Postnikov tower.

In this text, all spaces are supposed of the homotopy type of connected pointed CW-complexes and we will use cdga for *commutative differential graded algebra*. A quasi-isomorphism is a morphism of cdga’s which induces an isomorphism in cohomology.

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## 2 Structure of \( H(n) \)-space

First we recall the construction of Ganea fibrations, \( p^X_n: G_n(X) \to X \).

- Let \( F_0(X) \xrightarrow{i_0} G_0(X) \xrightarrow{p^X_0} X \) denote the path fibration on \( X \), \( \Omega X \to PX \to X \).
- Suppose a fibration \( F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p^X_n} X \) has been constructed. We extend \( p^X_n \) to a map \( q_n: G_n(X) \cup C(F_n(X)) \to X \), defined on the mapping cone of \( i_n \), by setting \( q_n(x) = p^X_n(x) \) for \( x \in G_n(X) \) and \( q_n([y,t]) = * \) for \( [y,t] \in C(F_n(X)) \).
- Now convert \( q_n \) into a fibration \( p^X_{n+1}: G_{n+1}(X) \to X \).

This construction is functorial and the space \( G_n(X) \) has the homotopy type of the \( n \)th-classifying space of Milnor [12]. We quote also from [8] that the direct limit \( G_\infty(X) \) of the maps \( G_n(X) \to G_{n+1}(X) \) has the homotopy type of \( X \). As spaces are pointed, one has two canonical applications \( i_n^*: G_n(X) \to G_n(X \times X) \) and \( i_n^*: G_n(X) \to G_n(X \times X) \) obtained from maps \( X \to X \times X \) defined respectively by \( x \mapsto (x,*) \) and \( x \mapsto (*,x) \).

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Definition 3 A space $X$ is an $H(n)$-space if there exists a map $\mu_n : G_n(X \times X) \to X$ such that $\mu_n \circ \iota_n = \mu_n \circ \iota'_n = p_n^X : G_n(X) \to X$.

Directly from the definition, we see that an $H(\infty)$-space is an $H$-space and that any space is a $H(1)$-space. Recall also that any co-$H$-space is of LS-category 1. Then, Proposition 1 contains the trivial cases of a co-$H$-space $X$ and of an $H$-space $Y$.

Proof of Proposition 1 From the hypothesis, we have a section $\sigma : X \to G_n(X)$ of the Ganea fibration $p_n^X$ and a map $\mu_n : G_n(Y \times Y) \to Y$ extending the Ganea fibration $p_n^Y$, as in Definition 3. If $f$ and $g$ are elements of $F(X, Y, \ast)$, we set $f \cdot g = \mu_n \circ G_n(f \times g) \circ G_n(\Delta_X) \circ \sigma$, where $\Delta_X$ denotes the diagonal map of $X$. One checks easily that $f \cdot \ast \simeq \ast \cdot f \simeq f$. □

In the rest of this section, we are interested in the existence of $H(n)$-structures on a given space. For the detection of an $H(n)$-space structure, one may replace the Ganea fibrations $p_n^X$ by any functorial construction of fibrations $\hat{p}_n : \hat{G}_n(X) \to X$ such that one has a functorial commutative diagram,

$$
\begin{array}{ccc}
\hat{G}_n(X) & \xrightarrow{\hat{p}_n} & \hat{G}_n(X) \\
\downarrow & & \downarrow \\
X & \xrightarrow{p_n^X} & X.
\end{array}
$$

Such maps $\hat{p}_n$ are called fibrations à la Ganea in [13] and substitutes to Ganea fibrations here. Moreover, as we are interested in product spaces, the following filtration of the space $G_\infty(X) \times G_\infty(Y)$ plays an important role:

$$(G(X) \times G(Y))_n = \bigcup_{i+j=n} G_i(X) \times G_j(Y).$$

In [10], N. Iwase proved the existence of a commutative diagram

$$
\begin{array}{ccc}
(G(X) \times G(Y))_n & \xrightarrow{\bigcup (p_i^X \times p_j^Y)} & G_n(X \times Y) \\
\downarrow & & \downarrow \\
X \times Y & \xrightarrow{p_n^{X \times Y}} & X \times Y
\end{array}
$$

and used it to settle a counter-example to the Ganea conjecture. Therefore, in Definition 3, we are allowed to replace the Ganea space $G_n(X \times X)$ by $(G(X) \times G(X))_n$. Moreover, if $\hat{p}_n : \hat{G}_n(X) \to X$ are substitutes to Ganea fibrations as above, we may also replace $G_n(X \times X)$ by

$$(\hat{G}(X) \times \hat{G}(Y))_n = \bigcup_{i+j=n} \hat{G}_i(X) \times \hat{G}_j(Y).$$

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We will use this possibility in the rational setting.

In the case \( n = 2 \), we have a cofibration sequence,

\[
\Sigma(G_1(X) \wedge G_1(X)) \xrightarrow{\text{Wh}} G_1(X) \vee G_1(X) \rightarrow G_1(X) \times G_1(X),
\]

coming from the Arkowitz generalisation of a Whitehead bracket, \([2]\). Therefore, the existence of an \( H(2) \)-structure on a space \( X \) is equivalent to the triviality of \((p_1^X \vee p_2^X) \circ \text{Wh}\). As the loop \( \Omega p_1^X \) of the Ganea fibration \( p_1^X : G_1(X) \rightarrow X \) admits a section, we get the following necessary condition:

– if there is an \( H(2) \)-structure on \( X \), then the homotopy Lie algebra of \( X \) is abelian, i.e. all Whitehead products vanish.

**Example 4** In the case \( X \) is a sphere \( S^n \), the existence of an \( H(2) \)-structure on \( S^n \) implies \( n = 1, 3 \) or \( 7 \), \([1]\). Therefore, only the spheres which are already \( H \)-spaces endow a structure of \( H(2) \) space. One can also observe that, in general, if a space \( X \) is both of category \( n \) and an \( H(2n) \)-space, then it is an \( H \)-space. The law is given by \( X \times X \xrightarrow{\sigma} G_{2n}(X \times X) \xrightarrow{\mu_{2n}} X \), where the existence of the section \( \sigma \) to \( p_{2n}^{X \times X} \) comes from \( \text{cat}(X \times X) \leq 2 \text{cat}(X) \).

**Example 5** If we restrict to spaces whose loop space is a product of spheres or of loop spaces on a sphere, the previous necessary condition becomes a criterion. For instance, it is proved in \([3]\) that all Whitehead products are zero in the complex projective 3-space. This implies that \( \mathbb{C}P^3 \) is an \( H(2) \)-space. (Observe that \( \mathbb{C}P^3 \) is not an \( H \)-space.) From \([3]\), we know also that the homotopy Lie algebra of \( \mathbb{C}P^2 \) is not abelian. Therefore \( \mathbb{C}P^2 \) is not an \( H(2) \)-space.

The following example shows that we can find \( H(n) \)-spaces, for any \( n > 1 \).

**Example 6** Denote by \( \varphi_r : K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 2r) \) the map corresponding to the class \( x^r \in H^{2r}(K(\mathbb{Z}, 2); \mathbb{Z}) \), where \( x \) is the generator of \( H^2(K(\mathbb{Z}, 2); \mathbb{Z}) \). Let \( E \) be the homotopy fibre of \( \varphi_r \). We prove below that \( E \) is an \( H(r - 1) \)-space.

First we derive, from the homotopy long exact sequence associated to the map \( \varphi_r \), that \( \Omega E \) has the homotopy type of \( S^1 \times K(\mathbb{Z}, 2r - 2) \). Therefore, the only obstruction to extend \( G_{r-1}(E) \vee G_{r-1}(E) \rightarrow E \) to \( (G(E) \times G(E))_{r-1} = \cup_{i+j=r-1} G_i(E) \times G_j(E) \) lies in \( \text{Hom}(H_{2r}((G(E) \times G(E))_{r-1}; \mathbb{Z}), \pi_{2r-2}(E)) \).

If \( A \) and \( B \) are CW-complexes, we denote by \( A \sim_n B \) the fact that \( A \) and \( B \) have the same \( n \)-skeleton. If we look at the Ganea total spaces and fibres, we get \( \Sigma \Omega E \sim_{2r} S^2 \vee S^{2r-1} \vee S^{2r} \), \( F_1(E) = \Omega E \ast \Omega E \sim_{2r} S^3 \vee S^{2r} \vee S^{2r} \), and

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more generally, \( F_s(E) \sim_{2r} S^{2s+1} \), for any \( s, 2 \leq s \leq r - 1 \). Observe also that \( H_{2r}(F_2(E); \mathbb{Z}) \to H_{2r}(G_1(E); \mathbb{Z}) \) is onto. (As we have only spherical classes in this degree, this comes from the homotopy long exact sequence.)

As a conclusion, we have no cell in degree \( 2r \) in \((G(E) \times G(E))_{r-1}\) and \( E \) is an \( H(r-1)\)-space.

We end this section with a reduction to a more computable invariant than the LS-category. Consider \( \rho^X_n : X \to G[n](X) \) the homotopy cofibre of the Ganea fibration \( p^X_n \). Recall that, by definition, \( \text{wcat}_G(X) \leq n \) if the map \( \rho^X_n \) is homotopically trivial. Observe that we always have \( \text{wcat}_G(X) \leq \text{cat}(X) \), see [5, Section 2.6] for more details on this invariant.

**Proposition 7** Let \( X \) be a CW-complex of dimension \( k \) and \( Y \) be a CW-complex \((c - 1)\)-connected with \( k \leq c - 1 \). If \( Y \) is an \( H(n)\)-space such that \( \text{wcat}_G(X) \leq n \), then \( \mathcal{F}_*(X,Y,*) \) is an \( H\)-space.

**Proof** Let \( f \) and \( g \) be elements of \( \mathcal{F}_*(X,Y,*) \). Denote by \( \tilde{\iota}^X_n : \tilde{F}_n(X) \to X \) the homotopy fibre of \( \rho^X_n : X \to G[n](X) \). This construction is functorial and the map \((f,g) : X \to Y \times Y\) induces a map \( \tilde{F}_n(f,g) : \tilde{F}_n(X) \to \tilde{F}_n(Y \times Y) \) such that \( \tilde{\iota}^Y_{gX} \circ \tilde{F}_n(f,g) = (f,g) \circ \tilde{\iota}^X_n \).

By hypothesis, we have a homotopy section \( \tilde{\sigma} : X \to \tilde{F}_n(X) \) of \( \tilde{\iota}^X_n \). Therefore, one gets a map \( X \to \tilde{F}_n(Y \times Y) \) as \( \tilde{F}_n(f,g) \circ \tilde{\sigma} \).

Recall now that, if \( A \to B \to C \) is a cofibration with \( A \) \((a - 1)\)-connected and \( C \) \((c - 1)\)-connected, then the canonical map \( A \to F \) in the homotopy fibre of \( B \to C \) is an \((a + c - 2)\)-equivalence. We apply it in the following situation:

\[
\begin{array}{ccc}
G_n(Y \times Y) & \xrightarrow{p_n^{Y \times Y}} & Y \times Y \\
\downarrow{\tilde{j}_n^{Y \times Y}} & & \downarrow{\tilde{i}_n^{Y \times Y}} \\
\tilde{F}_n(Y \times Y) & \xrightarrow{\iota_n^{Y \times Y}} & \tilde{F}_n(Y \times Y)
\end{array}
\]

The space \( G_n(Y \times Y) \) is \((c - 1)\)-connected and \( G[n](Y \times Y) \) is \( c \)-connected. Therefore the map \( j_n^{Y \times Y} \) is \((2c - 1)\)-connected. From the hypothesis, we get \( k \leq c - 1 < 2c - 1 \) and the map \( j_n^{Y \times Y} \) induces a bijection

\[
[X,G_n(Y \times Y)] \cong [X,\tilde{F}_n(Y \times Y)].
\]

Denote by \( g_n : X \to G_n(Y \times Y) \) the unique lifting of \( \tilde{F}_n(f,g) \circ \tilde{\sigma} \). The composition \( g \circ f \) is defined as \( \mu_n \circ g_n \) where \( \mu_n \) is the \( H(n)\)-structure on \( Y \).
If we set \( g = \ast \), then \( \tilde{F}_n(f, g) \) is obtained as the composite of \( \tilde{F}_n(f) \) with the map \( \tilde{F}_n(Y) \to \tilde{F}_n(Y \times Y) \) induced by \( y \mapsto (y, \ast) \). As before, one has an isomorphism \( [X, G_n(Y)] \cong [X, \tilde{F}_n(Y)] \). A chase in the following diagram shows that \( f \ast \ast = f \) as expected.

\[
\begin{array}{cccc}
G_n(Y) & \to & G_n(Y \times Y) & \downarrow \\
\tilde{F}_n(X) & \to & \tilde{F}_n(Y) & \to & \tilde{F}_n(Y \times Y) & \downarrow \\
\tilde{F}_n(f) & \downarrow & \tilde{F}_n(f) & \downarrow & \tilde{F}_n(f) & \downarrow \\
X & \to & \tilde{F}_n(Y) & \to & \tilde{F}_n(Y \times Y) & \downarrow \\
\& \end{array}
\]

\[\square\]

### 3 Rational characterisation of \( H(n) \)-spaces

Define \( m_H(X) \) as the greatest integer \( n \) such that \( X \) admits an \( H(n) \)-structure and denote by \( X_0 \) the rationalisation of a nilpotent finite type CW-complex \( X \). Recall that \( dl(X) \) is the valuation of the differential of the minimal model of \( X \), already defined in the introduction.

**Proposition 8** Let \( X \) be a nilpotent finite type CW-complex of rationalisation \( X_0 \). Then we have \( m_H(X_0) + 1 = dl(X) \).

**Proof** Let \( (\wedge V, d) \) be the minimal model of \( X \). Recall from [7] that a model of the Ganea fibration \( p^n \) is given by the following composition,

\[
(\wedge V, d) \to (\wedge V/ \wedge > n V, d) \leftarrow (\wedge V/ \wedge > n V, d) \oplus S,
\]

where the first map is the natural projection and the second one the canonical injection together with \( S \cdot S = S \cdot V = 0 \) and \( d(S) = 0 \). As the first map is functorial and the second one admits a left inverse over \( (\wedge V, d) \), we may use the realisation of \( (\wedge V, d) \to (\wedge V/ \wedge > n V, d) \) as substitute for the Ganea fibration.

Suppose \( dl(X) = r \). We consider the cdga \((\wedge V', d') \otimes (\wedge V'', d'')/I_r\) where \((\wedge V', d')\) and \((\wedge V'', d'')\) are copies of \((\wedge V, d)\) and where \( I_r \) is the ideal \( I_r = \oplus_{i+j \geq r} \wedge^i V' \otimes \wedge^j V'' \). Observe that this cdga has a zero differential and that the morphism

\[
\varphi : (\wedge V, d) \to (\wedge V', d') \otimes (\wedge V'', d'')/I_r
\]

defined by \( \varphi(v) = v' + v'' \) satisfies \( \varphi(dv) = 0 \). Therefore \( \varphi \) is a morphism of cdga’s and its realisation induces an \( H(n) \)-structure on the rationalisation \( X_0 \). That shows: \( m_H(X_0) + 1 \geq dl(X) \).

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Suppose now that $m_H(X_0) + 1 > dl(X) = r$. By hypothesis, we have a morphism of cdga’s

$$\varphi : (\wedge V, d) \rightarrow (\wedge V', d') \otimes (\wedge V'', d'')/I_{r+1}.$$ 

By construction, in this quotient, a cocycle of wedge degree $r$ cannot be a coboundary. Since the composition of $\varphi$ with the projection on the two factors is the natural projection, we have $\varphi(v) - v' - v'' \in \wedge^+ V' \otimes \wedge^+ V''$. Now let $v \in V$, of lowest degree with $d_r(v) \neq 0$. From $d_r(v) = \sum_{i_1, i_2, \ldots, i_r} c_{i_1 i_2 \ldots i_r} v_{i_1} v_{i_2} \cdots v_{i_r}$, we get

$$\varphi(dv) = \sum_{i_1, i_2, \ldots, i_r} c_{i_1 i_2 \ldots i_r} (v'_{i_1} + v''_{i_1}) \cdot (v'_{i_2} + v''_{i_2}) \cdots (v'_{i_r} + v''_{i_r}).$$ 

This expression cannot be a coboundary and the equation $d\varphi(x) = \varphi(dx)$ is impossible. We get a contradiction, therefore one has $m_H(X_0) + 1 = dl(X)$.

## 4 The Haefliger model

Let $X$ and $Y$ be finite type nilpotent CW-complexes with $X$ of finite dimension. Let $(\wedge V, d)$ be the minimal model of $Y$ and $(A, d_A)$ be a finite dimensional model for $X$, which means that $(A, d_A)$ is a finite dimensional cdga equipped with a quasi-isomorphism $\psi$ from the minimal model of $X$ into $(A, d_A)$. Denote by $A^\vee$ the dual vector space of $A$, graded by

$$(A^\vee)^{-n} = \text{Hom}(A^n, \mathbb{Q}).$$

We set $A^+ = \bigoplus_{i=1}^{\infty} A^i$, and we fix an homogeneous basis $(a_1, \ldots, a_p)$ of $A^+$. The dual basis $(a^s)_1 \leq s \leq p$ is a basis of $B = (A^+)^\vee$ defined by $\langle a^s; a_t \rangle = \delta_{st}$. We construct now a morphism of algebras $\varphi : \wedge V \rightarrow A \otimes \wedge (B \otimes V)$ by

$$\varphi(v) = \sum_{s=1}^{p} a_s \otimes (a^s \otimes v).$$

In [9] Haefliger proves that there is a unique differential $D$ on $\wedge (B \otimes V)$ such that $\varphi$ is a morphism of cdga’s, i.e. $(d_A \otimes D) \circ \varphi = \varphi \circ d$.

In general, the cdga $(\wedge (B \otimes V), D)$ is not positively graded. Denote by $D_0 : B \otimes V \rightarrow B \otimes V$ the linear part of the differential $D$. We define a cdga $(\wedge Z, D)$ by constructing $Z$ as the quotient of $B \otimes V$ by $\oplus_{j \leq 0} (B \otimes V)^j$ and their image by $D_0$. Haefliger proves:

**Theorem 9** [9] The commutative differential graded algebra $(\wedge Z, D)$ is a model of the mapping space $\mathcal{F}_*(X, Y, *)$. 

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5 Proof of Theorem 2

Proof We start with an explicit description of the Haefliger model, keeping the notation of Section 4. The cdga \((\wedge V, d)\) is a minimal model of \(Y\) and we choose for \(V\) a basis \((v_k)\), indexed by a well-ordered set and satisfying \(d(v_k) \in \wedge(V, v_r)\) for all \(k\). As homogeneous basis \((a_s)_{1 \leq s \leq p}\) of \(A\), we choose elements \(h_i\), \(e_j\) and \(b_j\) such that:

- the elements \(h_i\) are cocycles and their classes \([h_i]\) form a linear basis of the reduced cohomology of \(A\);
- the elements \(e_j\) form a linear basis of a supplement of the vector space of cocycles in \(A\), and \(b_j = d_A(e_j)\).

We denote by \(h^i\), \(e^j\) and \(b^j\) the corresponding elements of the basis of \(B = (A^+)^V\). By developing \(D_0(\sum_s a_s \otimes (a^s \otimes v)) = 0\), we get a direct description of the linear part \(D_0\) of the differentialid \(D\) of the Haefliger model:

\[
D_0(b^j \otimes v) = -(-1)^{|b^j|} e^j \otimes v \quad \text{and} \quad D_0(h^i \otimes v) = 0, \quad \text{for each} \quad v \in V.
\]

A linear basis of the graded vector space \(Z\) is therefore given by the elements:

\[
\begin{align*}
& b^j \otimes v_k, \quad \text{with} \quad |b^j \otimes v_k| \geq 1, \\
& e^j \otimes v_k, \quad \text{with} \quad |e^j \otimes v_k| \geq 2, \\
& h^i \otimes v_k, \quad \text{with} \quad |h^i \otimes v_k| \geq 1.
\end{align*}
\]

Now, from \(\varphi(dv) = (D - D_0)\varphi(v)\) and \(d(v) = \sum c_{i_1 i_2 \cdots i_r} v_{i_1} v_{i_2} \cdots v_{i_r}\), we deduce:

\[
(D - D_0)(a^s \otimes v) = \\
\pm \sum c_{i_1 i_2 \cdots i_r} \sum a_{i_1 a_{i_2} \cdots a_{i_r}} ^{a^s}(a_i \otimes v_i) \cdot (a_{i_2} \otimes v_{i_2}) \cdots (a_{i_r} \otimes v_{i_r})
\]

where, as usual, the sign \(\pm\) is entirely determined by a strict application of the Koszul rule for a permutation of graded objects.

Let \((A,d_A)\) be a finite dimensional model of \(X\), obtained as the quotient of the minimal model \((\wedge W, d)\) of \(X\) by the ideal \((\wedge W)^{>N} \oplus S\) where \(N\) is greater than the dimension of \(X\) and \(S\) is a supplement of the cocycles in degree \(N\). Denote by \(J_q\) the ideal of \(A\) generated by the products of \(q\) elements in \(A^+\). Then the Forman invariant \(e_0(X)\) is equal to the minimum \(q\) such that the quotient map \((A,d_A) \to (A/J_q, d_A)\) is injective in cohomology.

Suppose first that \(e_0(X) < c_d(Y)\). This inequality allows the choice of a basis \((h_j),(e_j),(b_j)\) such that \(\langle h^j;\alpha \rangle = 0\) for any \(\alpha \in J_q\) with \(q = e_0(X)\). The ideal \(I\) generated by the elements \(b^j \otimes v_s\) and \(D(b^j \otimes v_s)\) is a differential id \(D\) is a differential acyclic ideal. In the quotient \((\wedge Z,D)/I\), the elements \(b^j \otimes v_s\) disappear and the \(e^j \otimes v_s\) are
replaced by decomposable elements of the form $h^j \otimes v_s$. By the above remark and the Haefliger definition, the differential $D$ is zero on $(\wedge Z, D)/I$.

We consider now the case $\text{cup}_0(X) \geq d(Y)$ with $\text{dim}(X) \leq \text{conn}(Y)$. We choose linearly independent cocycles $z_1, \ldots, z_l$ of $A$, such that the cohomology class of the product $\omega = z_1^{i_1} \cdots z_l^{i_l}$ is not zero with $m = \sum_i i_i$. We choose the basis $(h_{i_j})$ such that it contains all the elements $z_1^{i_1} \cdots z_l^{i_l}$ with $i_i \leq q_i$. We choose also an element $v \in V$ that satisfies $dv = d_{r-1}v + \cdots$, with $d_{r-1}(v) \neq 0$ and $r \leq m$. As above we can kill all the elements $b^j \otimes v_s$ and $D(b^j \otimes v_s)$ and keep a quasi-isomorphism $\rho: (\wedge Z, D) \to (\wedge T, D) := (\wedge Z/I, D)$. If the differential $D$ is nonzero then the theorem is proved.

We give a weight at each variable $v_i \in V$ and denote by $\mu v_1 \cdots v_r$ the monomial of highest weight in $d_{r-1}(v)$. Let now $h_1, \ldots, h_r$ be $r$ elements in the family $(h_i)$ such that $\omega = h_1 \cdots h_r$. Let $\omega' \in A^V$ such that $\langle \omega'; \omega \rangle = 1$. Two permutations $\sigma$ and $\tau \in \Sigma_r$ are said equivalent if $h_{\sigma(i)} = h_{\tau(i)}$ for all $i$. We denote by $T \subset \Sigma_r$ a set of representatives of the equivalences classes and by $T' \subset T$, the set of $\sigma$ such that $v_{\sigma(i)} = v_i$ for each $i$. Then the component of $(h^1 \otimes v_1) \cdots (h^r \otimes v_r)$ in $D_{r-1}(\omega' \otimes v)$ is $|T'| \cdot \mu \neq 0$. This shows that the differential $D$ is nonzero. \[\Box\]

**Example 10** In assertion (1) of Theorem 2, we cannot replace $e_0(X)$ by cup$_0(X)$. Consider for instance the space $$X = S^2_0 \cup S^2_0 \cup e^5, \quad \text{with } \omega = [a, [a, b]].$$

A finite dimensional model for $X$ is given by the differential graded algebra $$(A, d) = (\wedge(a, b, c)/(a^2, b^2, bc), d)$$
with $|a| = |b| = 2, |c| = 3, d(a) = d(b) = 0, d(c) = ab$. A linear basis for $A$ is given by the elements $1, a, b, c, ab, ca$, and a linear basis for $A^V$ is given by $1^*, a^*, b^*, c^*, (ab)^*, (ca)^*$. Observe that cup$_0(X) = 1$, $dl(X) = e_0(X) = 2$. Let now $Y$ be the wedge $S^2 \cup S^7$ whose minimal model is $(\wedge V, d)$ with $V = \langle v, w, z, u, t, \ldots \rangle$, $|v| = |w| = 7, |z| = 13, |u| = |t| = 19$, the other generators having degrees $\geq 20$. The differential of the first generators satisfies $dv = dw = 0, dz = vw, du = zv, dt = zw$. In the Haefliger model for $\mathcal{F}_*(X, Y, *)$, if we take the quotient by the acyclic ideal $I$ generated by the elements $b^j \otimes v_s$ and $D(b^j \otimes v_s)$, we get a nonzero differential. In particular, $D((ca)^* \otimes u) = \pm (b^* \otimes v)(a^* \otimes w)(a^* \otimes v)$.

This implies that the cohomology of the mapping space is not free.

**Example 11** When the dimension of $X$ is greater than the connectivity of $Y$, the degrees of the elements have some importance. The cohomology can
be commutative free even if cup\(_0(X) \geq \text{dl}(Y)\). For instance, consider \(X = S^5 \times S^{11}\) and \(Y = S^8\). One has cup\(_0(X) = \text{dl}(Y) = 2\) and the function space \(F_*(X,Y,\ast)\) is a rational \(H\)-space with the rational homotopy type of \(K(\mathbb{Q}, 3) \times K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 10)\), as a direct computation with the Haefliger model shows.

6 Rationalisation of \(F_*(X,Y,\ast)\) for \(\dim(X) \leq \text{conn}(Y)\)

Let \(X\) be a finite nilpotent space with rational LS-category equal to \(m - 1\) and let \(Y\) be a finite type nilpotent CW-complex whose connectivity \(c\) is greater than the dimension of \(X\). We set \(r = \text{dl}(Y)\) and denote by \(s\) the maximal integer such that \(m/r^s \geq 1\), i.e. \(s\) is the integral part of \(\log_r m\).

**Theorem 12** There is a sequence of rational fibrations \(K_k \to F_k \to F_{k-1}\), for \(k = 1, \ldots, s\), with \(F_0 = \ast\), \(F_s\) is the rationalisation of \(F_*(X,Y,\ast)\) and each space \(K_k\) is a product of Eilenberg-MacLane spaces. In particular, the rational loop space homology of \(F_*(X,Y,\ast)\) is solvable with solvable index less than or equal to \(s\).

**Proof** By a result of Cornea [4], the space \(X\) admits a finite dimensional model \(A\) such that \(m\) is the maximal length of a nonzero product of elements of positive degree. We denote by \((\wedge V, d)\) the minimal model of \(Y\).

We consider the ideals \(I_k = A^{>m/r^k}\), and the short exact sequences of cdga’s

\[I_k/I_{k-1} \to A/I_{k-1} \to A/I_k.\]

These short exact sequences realise into cofibrations \(T_k \to T_{k-1} \to Z_k\) and the sequences

\[(\wedge((A^+/I_k)^\vee \otimes V), D) \to (\wedge((A^+/I_{k-1})^\vee \otimes V), D) \to (\wedge((I_k/I_{k-1})^\vee \otimes V), D)\]

are relative Sullivan models for the fibrations

\[F_*(Z_k, Y, \ast) \to F_*(T_{k-1}, Y, \ast) \to F_*(T_k, Y, \ast).\]

Now since the cup length of the space \(Z_k\) is strictly less than \(r\), the function spaces \(F_*(Z_k, Y, \ast)\) are rational \(H\)-spaces, and this proves Theorem 12. \(\square\)
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