

A Jones polynomial for braid-like isotopies of oriented links and its categorification

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Abstract A braid-like isotopy for links in 3-space is an isotopy which uses only those Reidemeister moves which occur in isotopies of braids. We define a refined Jones polynomial and its corresponding Khovanov homology which are, in general, only invariant under braid-like isotopies.

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1 Introduction

Links in 3-space are usually given by diagrams and isotopies of links by sequences of Reidemeister moves (see e.g. [BZ85]). For relatively oriented links (i.e. up to global orientation reversing), there are exactly two different local (i.e. without regarding the rest of the diagram) Reidemeister moves of type *II*, shown in Figure 1, and eight local Reidemeister moves of type *III*, shown in Figure 2.

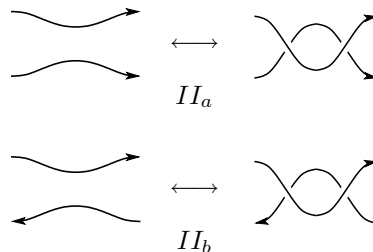
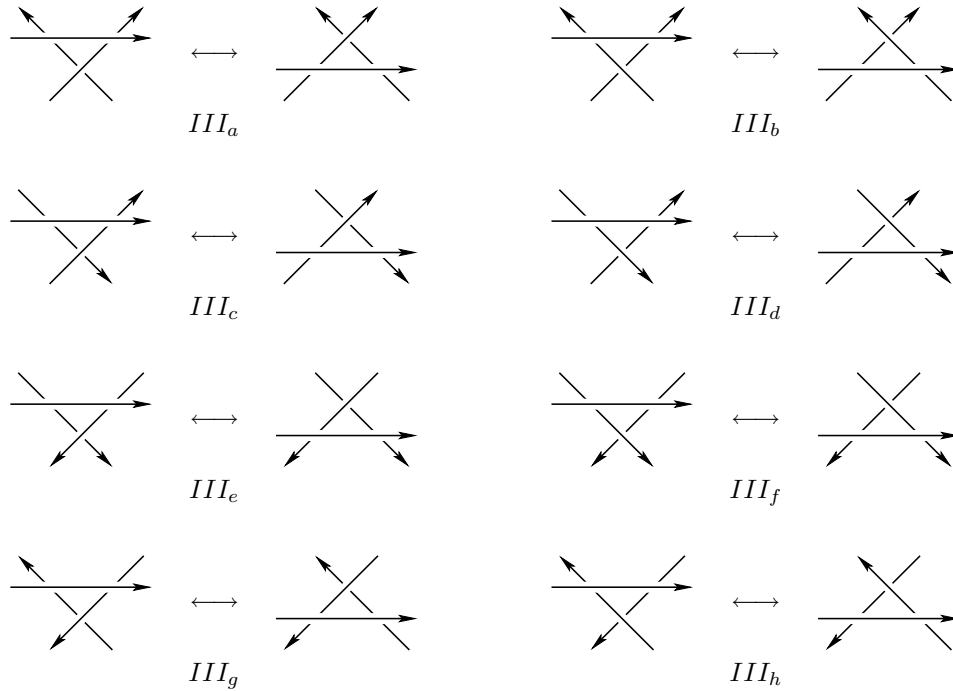


Figure 1: The 2 local Reidemeister moves of type *II*

Definition 1.1 The moves *II_a*, *III_a*, *III_b*, *III_c*, *III_d*, *III_e* and *III_f* are called braid-like Reidemeister moves. Two oriented links are called braid-like

Figure 2: The 8 local Reidemeister moves of type *III*

isotopic if their diagrams are related by a sequence of only braid-like Reidemeister moves.

Remark A well-known theorem of Artin (see e.g. [M78]) says that two closed braids in the solid torus are isotopic if and only if they are braid-like isotopic.

It is a natural and interesting question to know in what extend certain types of Reidemeister moves can be replaced by certain other types of Reidemeister moves in an isotopy of oriented links.

Using results from chapter 1.2 in [F01], one easily proves the following lemmata.

Let I denote any invariant of oriented diagrams.

Lemma 1.2 *If I is invariant under all Reidemeister moves of type *II* and under just one Reidemeister move of type *III*, then it is invariant under all other Reidemeister moves of type *III*.*

We have therefore to exclude some types of Reidemeister II moves in order to obtain a distinction between the different types of Reidemeister III moves.

Lemma 1.3 *If I is invariant (only) under the braid-like Reidemeister II move and under just one braid-like Reidemeister III move, then it is invariant under all other braid-like Reidemeister III moves.*

Question Let us consider an isotopy which uses only braid-like Reidemeister moves of type II and moves of type III_g and III_h (the non braid-like moves of type III). Can the isotopy be replaced by another one which uses only braid-like Reidemeister moves of type II and only exactly one of the types III_g or III_h ?

In other words, do the moves III_g and III_h become independent?

In this paper, we study oriented diagrams up to braid-like isotopies. Our main result says that the Jones polynomial can be refined in such a way that it becomes invariant in general only under braid-like isotopies. In the case of closed braids, our invariant coincides with Hoste's and Przytycki's refinement of the Jones polynomial [HP89].

For a lightened version of our invariant, we define a corresponding Khovanov link homology. Again, in the case of closed braids, it coincides with the Khovanov homology for links in the solid torus defined by Asaeda, Przytycki and Sikora [APS04].

In a forthcoming paper, we construct another generalization of the Jones polynomial for which the corresponding link homology turns out to be invariant under move III_h (as well as under the braid-like Reidemeister II move) but at the same time we have no proof (nor counterexample) for the invariance under move III_g . This difficulty sheds new light on braid-like isotopies, which form a rather natural class of restricted isotopies of oriented links in 3-space.

Question Is the natural analogue of Markov's theorem still true, i.e. do braid-like isotopies and Markov moves (i.e. the four types of Reidemeister I moves in the plane) generate isotopies of oriented links in 3-space?

It is a well-known open problem to know whether the Jones polynomial detects the unknot. One can ask whether the refined Jones polynomial or the corresponding link homology characterises the unknot. This might be an easier question than the original one, because there are infinitely many refined polynomials which are realised by the unknot but we have no characterisation of these polynomials.

2 Main results

Our starting point is the Kauffman bracket for a framed oriented link L in 3-space. We use Kauffman's notations and terminology (see [K87]).

Each circle $s: S^1 \hookrightarrow \mathbb{R}^2$ in a Kauffman state of a diagram D of L has a piecewise orientation induced by that of D . The points on s where the orientation changes are called *break points*. Obviously, the number of break points on s is always even.

Definition 2.1 A circle in a Kauffman state is of type d if one half of the number of break points on s is odd and it is of type h otherwise.

Let C denote the set of all configurations (up to ambient isotopies) of unoriented embedded circles in the plane, including the empty set.

Definition 2.2 For any state s of a diagram D , we define

$$\begin{aligned}\sigma(s) &= \#(A\text{-smoothings}) - \#(A^{-1}\text{-smoothings}), \\ d(s) &= \#(d\text{-circles}), \\ h(s) &= \#(h\text{-circles}),\end{aligned}$$

and $c(s) \in C$ which is the configuration of only the h -circles.

Let Γ denote the $\mathbb{Z}[A, A^{-1}]$ -module over C .

Definition 2.3 For any diagram D , the bracket $\langle D \rangle_{br} \in \Gamma$ is defined by

$$\langle D \rangle_{br} = \sum_{s \text{ state of } D} A^{\sigma(s)} (-A^2 - A^{-2})^{d(s)} c(s).$$

From this definition, it follows immediately:

Proposition 2.4 Let D be a diagram and v a crossing of D . Let D_0 and D_1 be the diagrams obtained from D by smoothing v respectively in the A -fashion and in the A^{-1} one and adding break points in the natural way. Then, we have

$$\langle D \rangle = A \langle D_0 \rangle + A^{-1} \langle D_1 \rangle.$$

Moreover, we have $\langle D \textcircled{d} \rangle = (-A^2 - A^{-2}) \langle D \rangle$.

Let $Sei(D)$ denote the Seifert state of a diagram D i.e. the state obtained by smoothing all the crossings with respect to the orientation of D , and let $w(D)$ denote the writhe of that diagram.

Theorem 2.5 *The bracket $\langle \cdot \rangle_{br}$ is invariant under global orientation reversing and braid-like isotopies. It has the following properties:*

- (1) $(-A)^{-3w(\cdot)} \langle \cdot \rangle_{br}$ satisfies the skein relation of the Jones polynomial (after replacing as usual A by $t^{\frac{1}{4}}$) but with infinitely many initial conditions;
- (2) in the case of closed braids, $(-A)^{-3w(\cdot)} \langle \cdot \rangle_{br}$ coincides with the invariant of Hoste and Przytycki [HP89];
- (3) for any diagram D , $\langle D \rangle_{br} = A^{w(D)} Sei(D) +$ terms with fewer circles in their configurations.

Remark Surprisingly, we have no proof that the properties 1., 2. and 3. are sufficient to determine $\langle \cdot \rangle_{br}$. The problem is that we have no classification of braid-like isotopy classes of the trivial knot (compare with Theorem 2 in [F93]).

Question Is it true that for each diagram D of a trivial knot, the oriented configuration of the Seifert state together with the writhe $w(D)$ determine $\langle D \rangle_{br}$?

Remark Replacing c by $\chi^{|c|}$ defines a map

$$\chi: \Gamma \longrightarrow \mathbb{Z}[A, A^{-1}, \chi].$$

Obviously, for any link L up to braid-like isotopy, $\langle L \rangle_{br}$ contains more information than $\chi(\langle L \rangle_{br})$ but it turns out that the latter can be easily categorified contrary to the former.

Theorem 2.6 *For any link L , there is a graded chain complex with integer coefficients $(\mathcal{C}_{i,j,k}, d)$ with differential d of tridegree $(-1, 0, 0)$ such that its homology groups are invariant under braid-like isotopy of the oriented link L . Moreover, the bigraded Euler characteristic of this homology coincides with $(-A)^{-3w(L)} \chi(\langle L \rangle_{br})$, that is*

$$(-A)^{-3w(L)} \chi(\langle L \rangle_{br}) = \sum_{i,j,k} (-1)^i (-A^2)^j \chi^k \dim_{\mathbb{Q}}(\mathcal{H}_{i,j,k}(L) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

In the case of closed braids, the groups $\mathcal{H}_{i,j,k}(L)$ coincide with those defined in [APS04].

Question Is it possible to categorify the full bracket $\langle \cdot \rangle_{br}$?

3 Proof of theorem 2.5

It is clear that reversing the orientation does not change break points nor A and A^{-1} -smoothings. Thus, $\langle \cdot \rangle_{br}$ is invariant under global orientation reversing.

From now on, we will denote break points by dots on the diagram and we will not always write brackets.

Lemma 3.1 *The bracket $\langle \cdot \rangle_{br}$ is invariant under braid-like Reidemeister move of type II.*

Proof Calculating the Kauffman bracket, we obtain

$$\text{Diagram 1} = A^2 \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + A^{-2} \text{Diagram 5}.$$

The circle is of type d and the usual identification $d = -A^2 - A^{-2}$ implies invariance. □

Remark We cannot achieve invariance under the other Reidemeister II move and distinguish the types d and h of the circles. Indeed, the calculation gives

$$\text{Diagram 1} = A^2 \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + A^{-2} \text{Diagram 5},$$

and the diagrams are completely different with respect to the types d or h of the circles!

Lemma 3.2 *The bracket $\langle \cdot \rangle_{br}$ is invariant under braid-like Reidemeister moves of type III.*

Proof According to lemma 1.3, it is sufficient to prove the invariance for the move shown in Figure 3.

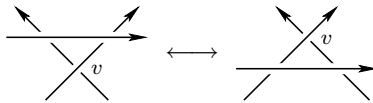


Figure 3: Braid like Reidemeister move of type III

The A -smoothing of v in the upper part of Figure 3 leads to isotopic diagrams. Hence, we have only to prove:

$$\text{Diagram 1} = \text{Diagram 2}$$

The calculation on the left-hand side gives

$$\begin{aligned}
 \text{Diagram} &= A^2 \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 + A^{-2} \text{Diagram}_4 \\
 &= \text{Diagram}_5,
 \end{aligned}$$

and on the right-hand side

$$\begin{aligned}
 \text{Diagram} &= A^2 \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 + A^{-2} \text{Diagram}_4 \\
 &= \text{Diagram}_5.
 \end{aligned}$$

Remark Note that we are forced to distinguish the d and the h types by $\frac{1}{2}\#(\text{break points}) \pmod 2$ and we would not be able to do better than that.

Property (1) follows immediately from proposition 2.4. There are infinitely many initial conditions because there are infinitely many diagrams of the trivial knot with the same writhe and the same Whitney index (i.e. that are regularly isotopic) but which are pairwise not braid-like isotopic (see Figure 4).

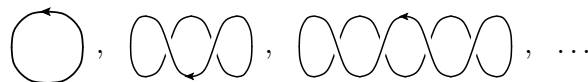


Figure 4: Regularly, but not braid-like, isotopic trivial knots

Property (2) follows from the following simple observation: a circle in a state for a closed braid in a solid torus is contractible in the solid torus if and only if it is of type d . Indeed, the critical points of the restriction of the radius function on a circle correspond exactly to the break points on that circle. Evidently, the number of such critical points is congruent to $2 \pmod 4$ if and only if the circle is contractible.

Property (3) follows from the fact that the Seifert state of a diagram D contributes $A^{w(D)}Sei(D)$ to $\langle D \rangle_{br}$. Let us prove that any other state s has strictly fewer circles of type h than the Seifert state.

Suppose that s maximizes the number of h -circles without being the Seifert state i.e. containing some break points. Then, remove as many break points as possible without changing the number of h -circles using the moves 1–4 in figure 5.

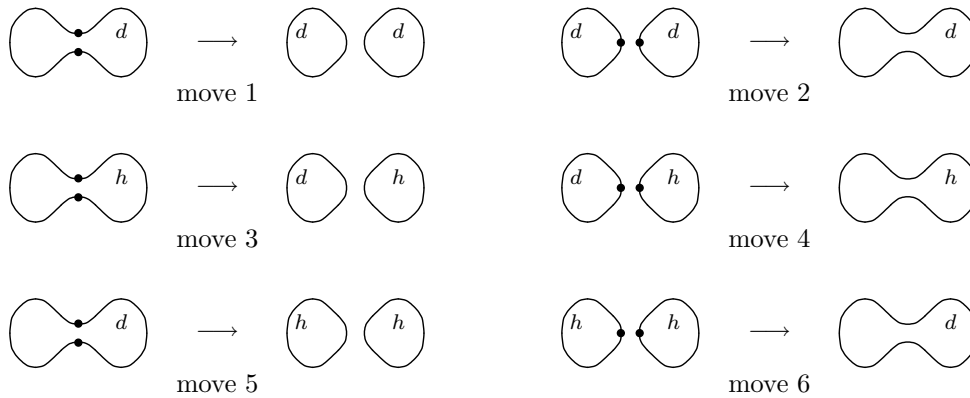


Figure 5: Moves removing break points

If a d -circle remains, then necessarily a move 5 can be performed on it. As the move 5 increases the number of h -circles, the resulting state must be the Seifert state by hypothesis. The result is then proven.

Otherwise, we get a configuration of only h -circles that we denote s' with the following properties:

- (i) no circle of s' contains any internal breaking-smoothed crossings i.e. any pair of break points linked by a crossing are on distinct circles;
- (ii) it contains some break points because, as the last move to get the Seifert state must be a move 5 in figure 5, s' cannot be the Seifert state by construction.

Then we construct a graph G by associating a vertex to each h -circle of s' and an edge to each pair of break points linked by a crossing. Property (i) assures that G is planar and property (ii) that it contains some connected component not reduced to a point. Let's choose such a component G' with $k \geq 2$ vertices and n edges corresponding to a sub-state s'' of s' with k h -circle and $2n$ break points.

Now change the smoothing of $(k - 1)$ breaking-smoothed crossing of s'' in such a way that the number of circle decreases each time by one. This can be done because G' is connected and the removal of two linked break points corresponds to the retraction of the corresponding edge in G' .

Then one circle remains with $2(n - k + 1)$ break points corresponding to a graph with a single vertex and $(n - k + 1)$ loops. As this graph is still planar,

we can find two adjacent break points linked by a crossing. Removing this two break points will create an h -circle with no break points. By repeating this operation, we will create one such h -circle for each pair of linked break points, except for the last one which will create two such h -circle.

At last, we get a state with $(n - k + 2)$ h -circles. Now, each initial h -circles in s'' was containing at least one break point, therefore at least four break points in order to be of type h . Thus, we have

$$2n \geq 4k,$$

and consequently $n - k + 2 \geq k + 2$.

We have therefore increased the number of h -circles, so, by our hypothesis on s , the last state must be the Seifert state. Property (3) is thus proven.

We have thus finished the proof of the theorem. □

4 Proof of theorem 2.6

4.1 Definition of the homology

Definition 4.1 An enhanced state S of a diagram D is a Kauffman state s of D enhanced by an assignment of a plus or a minus sign to each of the circles of s .

Let us fix a diagram D . Setting $\chi = -H^2 - H^{-2}$ and according to [V02], we rewrite our lightened bracket as

$$\chi((-A)^{-3w(D)}\langle D \rangle_{br}) = \sum_{\substack{S \text{ enhanced} \\ \text{state of } D}} (-1)^{\frac{\sigma(S)-w(D)}{2}} (-A^2)^{\frac{\sigma(S)-3w(D)+2\tau_d(S)}{2}} (-H^2)^{\tau_h(S)}, \tag{1}$$

where τ_d (resp. τ_h) is the difference between the number of pluses and minuses assigned to the d -circles (resp. h -circles).

Definition 4.2 For any enhanced state S of D , we define

$$\begin{aligned} i(S) &= \frac{\sigma(S)-w(D)}{2} \\ j(S) &= \frac{\sigma(S)-3w(D)+2\tau_d(S)}{2} \\ k(S) &= \tau_h(S). \end{aligned}$$

Then, we can define $\mathcal{C}_{i,j,k}$ to be the \mathbb{Z} -module spanned by

$$\{S \text{ enhanced state of } D \mid i(S) = i, j(S) = j, k(S) = k\}.$$

Strictly speaking, in accordance with the first definitions, we will not need the writhe as it is invariant under braid-like Reidemeister moves, but we will continue to use it, on the one hand for historical reasons and on the other for i, j and k to be integers.

Now, we assign an ordering of the crossings of D .

Definition 4.3 Let S and S' be two enhanced states of D and v a crossing of D . We define an incidence number $[S : S']_v$ by

- $[S : S']_v = 1$ if the following four conditions are satisfied:
 - (1) v is A -smoothed in S but A^{-1} -smoothed in S' ;
 - (2) all the other crossings are smoothed the same way in S and S' ;
 - (3) common circles of S and S' are labelled the same way;
 - (4) $j(S) = j(S')$ and $k(S) = k(S')$;
- otherwise, $[S : S']_v = 0$.

Definition 4.4 We define a differential of tridegree $(-1, 0, 0)$ on $(\mathcal{C}_{i,j,k})_{i,j,k \in \mathbb{Z}}$ by

$$d(S) = \sum_{\substack{v \text{ crossing} \\ A\text{-smoothed in } S}} d_v(S)$$

where the partial differential d_v is defined by

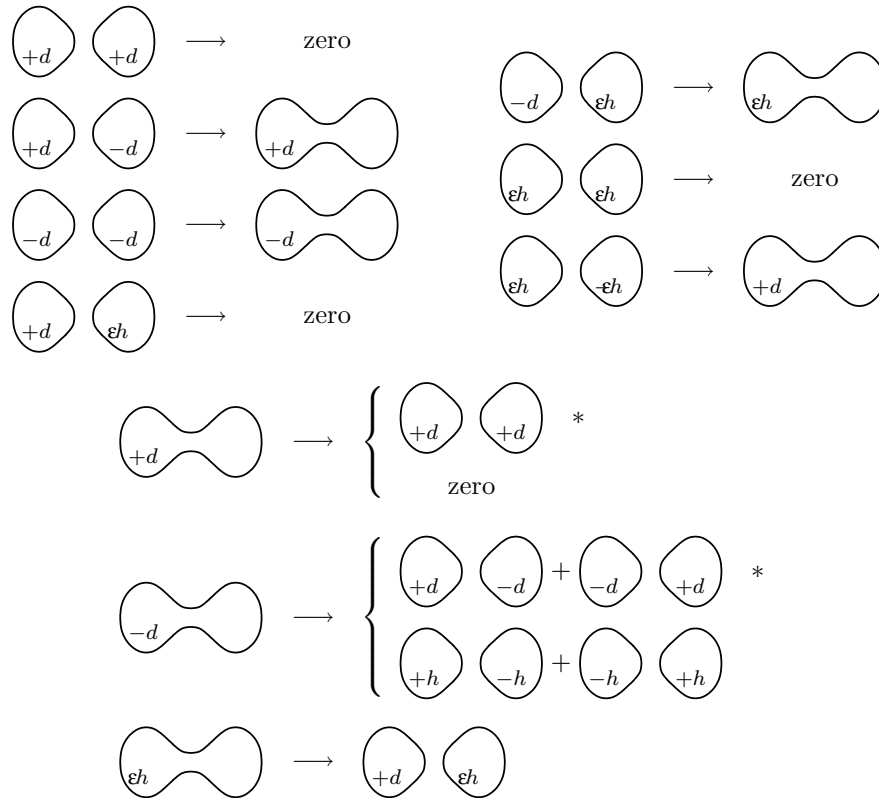
$$d_v(S) = (-1)^{t_{v,S}} \sum_{S' \text{ state of } D} [S, S']_v S',$$

with $t_{v,S}$ the number of A^{-1} -smoothed crossings in S labelled with numbers greater than the label of v .

Note that $d(S)$ is the alternating sum of enhanced states obtained by switching one A -smoothed crossing of S into a A^{-1} -smoothing and by locally relabeling in such a way that j and k are preserved and i is decreased by 1. Moreover the merge of two d -circles or two h -circles always gives a d -circle and the merge of a d -circle with an h -circle always an h -circle. Thus, one can explicit the action of d_v , as done in Figure 6.

Lemma 4.5 *Merging rules between d and h -circles are identical to the merging rules for the annulus in [APS04].*

Proof It follows from the identification between the two definitions of d and h -circles as given in the proof of Property (2) of theorem 2.5. □



* depending on the type of the resulting circles

Figure 6: Definition of the partial differential d_v

Lemma 4.6 Every two distinct partial differentials anti-commute. Thus $d^2 = 0$, that is d is a differential.

Proof According to lemma 4.5, one only has to check cases from 1A–1D to 5A–5D like in [APS04] as done in the proof of lemma 4.4 of the same article. \square

Lemma 4.7 The homology \mathcal{H}_{br} associated to d does not depend on the ordering of the crossings.

Proof This follows from the proof of theorem 6.1 in [APS04]. \square

Proposition 4.8 The extended Euler characteristic of \mathcal{H}_{br} associated to d coincides with $(-A)^{-3w(D)}\chi(\langle D \rangle_{br})$.

Proof The fact that the extended Euler characteristic of $(\mathcal{C}_{i,j,k})_{i,j,k \in \mathbb{Z}}$ coincides with $(-A)^{-3w(D)} \chi(\langle D \rangle_{br})$ is clear from equation 1 and the definition of the chain complex. Moreover, passing to homology does not change the extended Euler characteristic. \square

Although we still have a Viro-like exact sequence, the proof of invariance under Reidemeister moves of type *II* and *III* given by Asaeda, Przytycki and Sikora does not work in our context since we have no invariance under Reidemeister move of type I. Nevertheless, the original proof by Khovanov still works. We will check this using the point of view and the (intuitive) notations developed by the first author in [Au05].

For this, we introduce the operator $\cdot\{n\}$ (resp. $\cdot[n]$) which is the global uplifting of the grading (resp. homological) degree by n .

Definition 4.9 Let $f: \mathcal{C} \rightarrow \mathcal{C}'$ be a morphism of chain complexes. Then the cone of f is the chain complex \mathcal{C}'' defined by $\mathcal{C}'' = \mathcal{C} \oplus \mathcal{C}'[-1]$ and $d_{\mathcal{C}''} = (d_{\mathcal{C}} + f) \oplus (-d_{\mathcal{C}'})$.

Proposition 4.10 [Au05] Let D be a diagram of a link L and v a crossing of D . Let D_0 and D_1 be the diagrams obtained from D by smoothing v respectively in the A -fashion and in the A^{-1} one. Then d_v defines a grading preserving map of chain complexes from $\mathcal{C}(D_0)$ to $\mathcal{C}(D_1)\{-1\}$. Moreover, $\mathcal{C}(D)$ is the cone of this map.

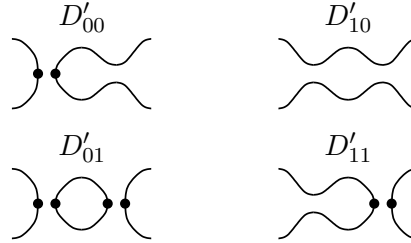
Proposition 4.11 The cone of an isomorphism between two chain complexes is acyclic.

4.2 Invariance under braid-like Reidemeister move of type *II*

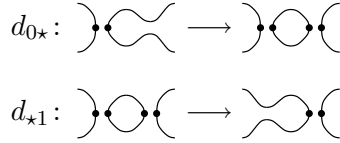
Consider two diagrams which differ only locally by a braid-like Reidemeister move of type *II*.



We order the crossings of D and D' in the same fashion, letting v and v' be the last two ones of D' . So we have



and two maps:



Lemma 4.12 *The morphisms d_{0*} and d_{*1} are respectively injective and surjective.*

Proof This can be checked by direct calculus on canonical basis. □

Lemma 4.13 *The cone of*

$$\begin{array}{ccc} \dots & \xleftarrow{d_{D'}^{i-1}} & \text{Im}(d_{*0}^i + d_{0*}^i)\{-1\} & \xleftarrow{d_{D'}^i} & \dots \\ & & \uparrow & & \\ \dots & \xleftarrow{d_{D'00}^{i-1}} & \mathbb{Z} \left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} & \xleftarrow{d_{D'00}^i} & \dots \end{array}$$

is an acyclic graded subcomplex of $\mathcal{C}(D')$ denoted \mathcal{C}_1 .

Proof To prove this lemma, we have to check the four following points:

- (1) stability of the upper and the lower chain complexes under their differentials;
- (2) grading-preserving;
- (3) inclusion in $\mathcal{C}(D')$;
- (4) acyclicity.

Stability is clear for the lower line. For the upper one, as $d_{0\star}$ is injective, the map $(d_{\star 0}^i + d_{0\star}^i)$ has a right inverse on its image, and by anti-commutativity of distinct partial differentials we have for any S in $Im(d_{\star 0}^i + d_{0\star}^i)\{-1\}$

$$d_{D'}^{i-1}(S) = -(d_{\star 0}^{i-1} + d_{0\star}^{i-1}) \circ d_{D'_{00}}^{i-1} \circ (d_{\star 0}^i + d_{0\star}^i)^{-1}(S) \in Im(d_{\star 0}^{i-1} + d_{0\star}^{i-1})\{-1\}.$$

The second point is clear by definition.

Inclusion of spaces is obvious and the identification of the differentials follows from proposition 4.10.

The last point is a consequence of proposition 4.11. □

Lemma 4.14 *The cone of*

$$\begin{array}{ccc} \dots \longleftarrow \mathbb{Z} \left\{ \text{diagram} \right\} \{-1\} \longleftarrow \dots & & \xleftarrow{d_{D'}^i} \dots \\ & \uparrow d_{\star 1}^i & \\ \dots \longleftarrow \mathbb{Z} \left\{ \text{diagram} \right\} \longleftarrow \dots & & \xleftarrow{d_{D'_{01}}^i} \dots \end{array}$$

is an acyclic graded subcomplex of $\mathcal{C}(D')$ denoted \mathcal{C}_2 .

Proof The stability of the upper line is clear. The differential $d_{D'_{01}}^i$ do not change the smoothing of the only two crossings contiguous to the middle circle labeled by $-d$, so this one is let unchanged and the lower line is stable too.

Arguments of lemma 4.13 are still valid to prove the last three points. □

Definition 4.15 We define the map:

$$\begin{array}{ccc} \cdot \otimes v_- : & \text{diagram} & \longrightarrow & \text{diagram} \\ & \text{diagram} & \longmapsto & \text{diagram} \end{array}$$

Remark The map $\cdot \otimes v_-$ is grading-preserving and is a right inverse for $d_{\star 1}$.

Lemma 4.16 *The complex*

$$\dots \longleftarrow \mathbb{Z} \left\{ \text{diagram} - (d_{1\star}(\text{diagram})) \otimes v_- \right\} \longleftarrow \dots$$

is a graded subcomplex of $\mathcal{C}(D')$ denoted \mathcal{C}_3 , isomorphic to $\mathcal{C}(D)$ via ψ_{II} defined by:

$$\psi_{II} \left(\text{diagram} \right) = \text{diagram} - (d_{1\star}(\text{diagram})) \otimes v_-$$

Proof To begin, we split the differential into the sum of the underlying partial differentials. First consider d_c where $c \neq v, v'$. As d_c let unchanged the middle circle in $\left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_-$ and as $\cdot \otimes v_-$ remove one A^{-1} -smoothed crossing labeled with a number greater than the label of c , we have:

$$\begin{aligned} d_c \left((d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \right) \otimes v_- &= - \left(d_c (d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \right) \otimes v_- \\ &= \left(d_{1\star} (d_c \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \right) \otimes v_- \end{aligned}$$

So we have:

$$d_c \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} - (d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \right) \otimes v_- = d_c \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) - \left(d_{1\star} (d_c \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \right) \otimes v_-$$

Then, $d_{1\star}$ and $d_{\star 1}$ act respectively on $\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$ and $(d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \otimes v_-$. But as $\cdot \otimes v_-$ is a right inverse for $d_{\star 1}$, we have:

$$\begin{aligned} d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) - d_{\star 1} \left((d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \right) \otimes v_- &= d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) - d_{1\star} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \right) \\ &= 0 \end{aligned}$$

Not only this two points show that the subcomplex is stable, but also, as, for $c \neq v, v'$, d_c acts the same on $\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$ and $\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$, they show that ψ_{II} is a chain complexes map. Then, it is clearly a grading-preserving isomorphism since $\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$ has only two more crossings than $\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$, one A -smoothed and the other A^{-1} -smoothed. \square

Lemma 4.17 $\mathcal{C}(D') \simeq \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3$.

Proof As $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 are subcomplexes of $\mathcal{C}(D')$, differentials can already been identified. Thus, one only has to prove it at the level of modules.

The subcomplex \mathcal{C}_2 contains all elements of the form $\left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_-$, thus we can clean all parasite terms in \mathcal{C}_3 and get

$$\mathcal{C}_2 \oplus \mathcal{C}_3 \simeq \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \oplus \left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_- \oplus \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$$

Moreover,

$$Im(d_{\star 0}^i + d_{0\star}^i) = \mathbb{Z} \left\{ \left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_- + \alpha_1, \left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_- + \alpha_2, \left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_- + \alpha_3 \right\},$$

with $\alpha_i \in \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$. So, with $\mathcal{C}_2 \oplus \mathcal{C}_3$ we can again clean and get

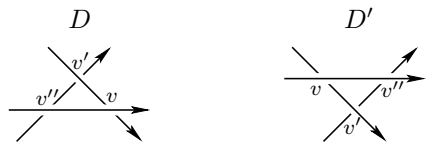
$$Im(d_{\star 0}^i + d_{0\star}^i) \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \simeq \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array} \oplus \left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_- \oplus \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowleft \end{array}$$

Now $\mathcal{C}_1 = \left. \begin{array}{c} \curvearrowright \\ \circlearrowleft \\ \curvearrowleft \end{array} \right\} \otimes v_- \oplus Im(d_{\star 0}^i + d_{0\star}^i)$, so the lemma is proven. \square

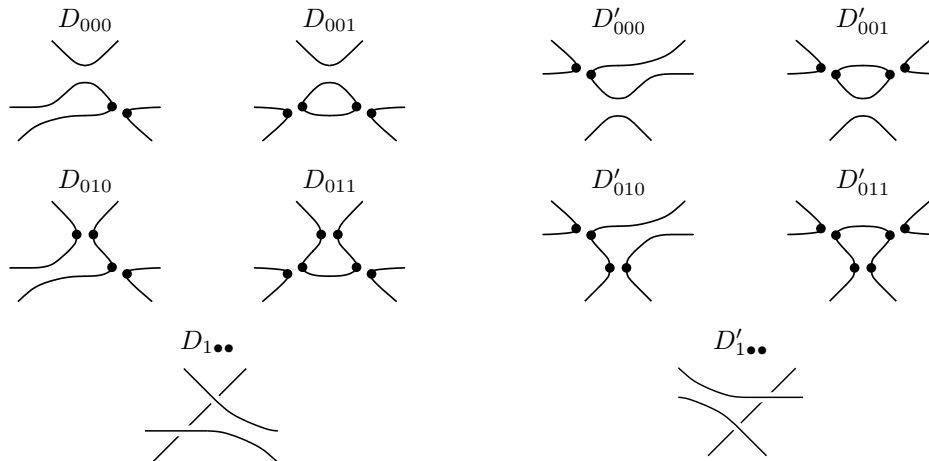
Finally, we achieve the invariance under star-like Reidemeister move of type *II* by passing to homology in lemma 4.17.

4.3 Invariance under braid-like Reidemeister moves of type III

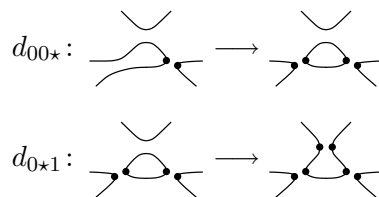
According to lemma 1.3, it is sufficient to prove the invariance for two diagrams which differ only locally by the following braid-like Reidemeister move of type III.



We order the crossings of D and D' in the same fashion, letting the three crossings involved in the Reidemeister move be the last three ones. Thus we have:



We can, immediatly, note that $D_{1..} \simeq D'_{1..}$ and $D_{010} \simeq D'_{010}$. With respect to D , we have two maps:



Lemma 4.18 *The maps d_{00*} and d_{0*1} are, respectively injective and surjective.*

Then, the rest follows as in the case of the Reidemeister move of type II.

Lemma 4.19 *The cone of*

$$\begin{array}{ccc} \cdots \longleftarrow \xrightarrow{d_D^{i-1}} & \text{Im}(d_{\star 00}^i + d_{0\star 0}^i + d_{00\star}^i)\{-1\} & \longleftarrow \xrightarrow{d_D^i} \cdots \\ & \uparrow d_{\star 00}^i + d_{0\star 0}^i + d_{00\star}^i & \\ \cdots \longleftarrow \xrightarrow{d_{D000}^{i-1}} & \mathbb{Z} \left\{ \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \right\} & \longleftarrow \xrightarrow{d_{D000}^i} \cdots \end{array}$$

is an acyclic graded subcomplex of $\mathcal{C}(D)$ denoted \mathcal{C}_1 .

Lemma 4.20 *The cone of*

$$\begin{array}{ccc} \cdots \longleftarrow \xrightarrow{d_D^{i-1}} & \mathbb{Z} \left\{ \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \right\} + \mathbb{Z} \left\{ \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \right\} \{-1\} & \longleftarrow \xrightarrow{d_D^i} \cdots \\ & \uparrow d_{\star 01}^i + d_{0\star 1}^i & \\ \cdots \longleftarrow \xrightarrow{d_{D001}^{i-1}} & \mathbb{Z} \left\{ \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \right\} & \longleftarrow \xrightarrow{d_{D001}^i} \cdots \end{array}$$

is an acyclic graded subcomplex of $\mathcal{C}(D')$ denoted \mathcal{C}_2 .

Definition 4.21 We define the map

$$\begin{array}{ccc} \cdot \otimes v_- : & \begin{array}{c} \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \\ \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \end{array} & \begin{array}{c} \longrightarrow \\ \mapsto \end{array} & \begin{array}{c} \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \\ - \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \end{array} \end{array}$$

Remark The map $\cdot \otimes v_-$ is grading-preserving and is a right inverse for $d_{0\star 1}$.

Lemma 4.22 *The complex*

$$\cdots \longleftarrow \xrightarrow{d_D^{i-1}} \mathbb{Z} \left\{ \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \right\} - (d_{01\star}(\begin{array}{c} \gamma \\ \alpha \quad \beta \end{array})) \otimes v_- \Big\} \oplus \begin{array}{c} \gamma \\ \alpha \quad \beta \end{array} \longleftarrow \xrightarrow{d_D^i} \cdots$$

is a graded subcomplex of $\mathcal{C}(D)$ denoted \mathcal{C}_3 .

Lemma 4.23 $\mathcal{C}(D) \simeq \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3$.

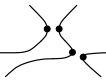
With the same reasoning on D' , we get:

Lemma 4.24 $\mathcal{C}(D') \simeq \mathcal{C}'_1 \oplus \mathcal{C}'_2 \oplus \mathcal{C}'_3$ with \mathcal{C}'_1 and \mathcal{C}'_2 acyclic and


$$\mathcal{C}'_3 = \mathbb{Z} \left\{ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right\} - (d_{01\star} \left(\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right)) \otimes v_- \left. \right\} \oplus \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}$$

Finally, we achieve the invariance under braid-like Reidemeister moves of type III thanks to the following lemma.

Lemma 4.25 The chain complexes \mathcal{C}_3 and \mathcal{C}'_3 are isomorphic via ψ_{III} defined

on  by

$$\psi \left(\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} - (d_{01\star} \left(\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right)) \otimes v_- \right) = \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} - (d_{01\star} \left(\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right)) \otimes v_-$$

and on  in the obvious way.

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