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The topological Hawaiian earring group does not embed in the inverse limit of free groups

Paul Fabel

Abstract Endowed with natural topologies, the fundamental group of the Hawaiian earring continuously injects into the inverse limit of free groups. This note shows the injection fails to have a continuous inverse. Such a phenomenon was unexpected and appears to contradict results of another author.

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1 Introduction

Quite generally the based fundamental group $\pi_1(X,p)$ of a space X becomes a topological group whose topology is invariant under the homotopy type of the underlying space X (Corollary 3.4 [1]). In the context of spaces complicated on the small scale the utility of this invariant is emerging. For example topological π_1 has the potential to distinguish spaces when the algebraic homotopy groups fail to do so [2]. Unfortunately even in the simplest cases the topological properties of $\pi_1(X,p)$ can be challenging to understand.

Consider the familiar Hawaiian earring $X=\cup_{n=1}^{\infty}S_n$, (the union of a null sequence of simple closed curves S_n joined at a common point) and the canonical homomorphism $\phi:\pi_1(X)\to \lim_{\leftarrow}\pi_1(\cup_{i=1}^{n}S_i)$.

The paper [1, page 370] seems to claim that ϕ is also a homeomorphism onto its image ("ψ^{-1} is surely continuous as well...”). The intent of this note is to show that such a claim is false. The monomorphism ϕ is not a homeomorphism onto its image, and thus ϕ fails to be a topological embedding (Theorem 2.1). To prove this we consider the sequence $[(y_1 * y_n * y_1^{-1} * y_n^{-1})^n]$ where y_i loops counterclockwise around the *ith* circle. The sequence diverges in $\pi_1(X,p)$ with the quotient topology but the sequence converges to the trivial element in the inverse limit space $\lim_{\leftarrow}\pi_1(\cup_{i=1}^{n}S_i)$.

2 Main Result

Suppose X is a topological space and $p \in X$. Endowed with the compact open topology, let $C_p(X) = \{f : [0,1] \to X$ such that f is continuous and $f(0) = f(1) = p\}$. Then the *topological fundamental group* $\pi_1(X,p)$ is the quotient space of $C_p(X)$ obtained by treating the path components of $C_p(X)$ as points. Thus, letting $q : C_p([0,1],X) \to \pi_1(X,p)$ denote the canonical surjection, a set $A \subset \pi_1(X,p)$ is closed in $\pi_1(X,p)$ if and only if $q^{-1}(A)$ is closed in $C_p([0,1],X)$.

The space Y is said to be T_1 if the one point subsets of Y are closed.

If A_1, A_2, are topological spaces and $f_n : A_{n+1} \to A_n$ is a continuous surjection then, (endowing $A_1 \times A_2 ...$ with the product topology) the *inverse limit space* $\lim_{\leftarrow} A_n = \{(a_1, a_2, ...) \in (A_1 \times A_2 ...) | f_n(a_{n+1}) = a_n\}$.

The map $f : [0,1] \to Y$ *is of the form* $\alpha_1 * \alpha_2 ... * \alpha_n$ if there exists a partition $t_0 \leq t_1 ... \leq t_n$ of $[0,1]$ such that for each $i \geq 1$ we have $f_{[t_{i-1},t_i]} = \alpha_i$.

For the remainder of the paper we use the following notation.

Let $X_n = \cup_{i=1}^n \{(x,y) \in R^2 | (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$. Note since X_n is locally contractible the path components of $C_p(X_n)$ are open in $C_p(X_n)$ and hence the topological group $\pi_1(X_n,p)$ has the discrete topology.

Let $r_n^* : \pi_1(X_n,p) \to \pi_1(X_{n-1},p)$ denote the epimorphism induced by the retraction $r_n : X_n \to X_{n-1}$ collapsing the n^{th} circle to the point $p = (0,0)$. Let $\lim_{\leftarrow} \pi_1(X_n,p)$ denote the inverse limit space under the maps r_n^*.

Let $X = \cup_{n=1}^\infty X_n$ denote the familiar Hawaiian and let $R_n : X \to X_n$ denote the retraction fixing X_n pointwise and collapsing $\cup_{i=n+1}^\infty X_i$ to the point p.

The formula $\phi([f]) = ([R_1(f)], [R_2(f)], ...)$ determines a canonical homomorphism $\phi : \pi_1(X,p) \to \lim_{\leftarrow} \pi_1(X_n,p)$.

Remark The homomorphism $\phi : \pi_1(X,p) \to \lim_{\leftarrow} \pi_1(X_n,p)$ is continuous (Proposition 3.3 [1]) and one to one (Theorem 4.1 [3]). Since $\pi_1(X_n,p)$ is discrete the space $\Pi_{n=1}^\infty \pi_1(X_n,p)$ is metrizable and in particular the subspace $\lim_{\leftarrow} \pi_1(X_n,p)$ is a T_1 space. Consequently $\pi_1(X,p)$ is a T_1 space since ϕ is continuous and one to one. Thus the path components of $C_p(X)$ are closed in $C_p(X)$.

Theorem 2.1 *The injection* $\phi : \pi_1(X,\{p\}) \hookrightarrow \lim_{\leftarrow} \pi_1(X_n,p)$ *is not a topological embedding.*

Proof Let $q = (2,0)$ in X_1. For a loop $f : [0,1] \to \cup_{i=1}^{\infty} X_i$ with base point $p = (0,0)$ define the oscillation number $O_q(f)$ as the maximal n such that there exist $0 = t_0 < t_1 \cdots t_{2n-1} < t_{2n} = 1$ with $f(t_{2i}) = p$ and $f(t_{2i-1}) = q$. Let $y_i \in C_p(X)$ loop once counterclockwise around the ith circle and let $y_i^{-1} \in C_p(X)$ loop once clockwise around the ith circle.

First note that if $f \in C_p(\cup_{i=1}^{\infty} X_i)$ is path homotopic to a map of the form $(y_1^{-1} * y_n^{-1} * y_1 * y_n)^n$ then $O_q(f) \geq 2n$ for $n \geq 2$. To see this first observe $O_q(f) = O_q(R_n f)$. Now recall $\pi_1(X_n, p)$ is canonically isomorphic to the free group on generators $\{y_1, ...y_n\}$. Thus if w is an (unreduced) word corresponding to $R_n f$ then each step of the algebraic reduction of w to $(y_1^{-1} y_n^{-1} y_1 y_n)^n$ never raises the oscillation number of the corresponding path in X_n. Hence $O_q(f) \geq O_q((y_1^{-1} * y_n^{-1} * y_1 * y_n)^n) = 2n$.

To prove ϕ is not an embedding consider the set $A \subset \pi_1(X,p)$ defined as $A = \{[f_2],[f_3]...]\}$ where f_n is of the form $(y_1^{-1} * y_n^{-1} * y_1 * y_n)^n$. To see that A is closed in $\pi_1(X,p)$ consider the union of (closed) path components $B = \cup_{n=2}^{\infty} [f_n] \subset C_p(X)$. Observe if $f \in C_p(X)$ there exists an open neighborhood $U \subset C_p(X)$ such that $O_q(f) \geq O_q(g)$ for each $g \in U$. Thus $U \cap [f_n] \neq \emptyset$ for at most finitely many of the closed sets $[f_n]$. Hence B is closed in $C_p(X)$ and consequently A is closed in $\pi_1(X,p)$. On the other hand $\phi(A)$ is not closed in the image of ϕ since the sequence $\{\phi([f_n])\}$ converges to the trivial element in $\lim_{\leftarrow} \pi_1(X_n,p)$. Hence ϕ is not a homeomorphism from $\pi_1(X,p)$ onto the image of ϕ. □

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Drawer MA, Department of Mathematics and Statistics
Mississippi State University, Mississippi State, MS 39762, USA

Email: `fabel@ra.msstate.edu`

URL: `http://www2.msstate.edu/~fabel/`

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