

# Legendrian knots and monopoles

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We prove a generalization of Bennequin’s inequality for Legendrian knots in a 3–dimensional contact manifold  $(Y, \xi)$ , under the assumption that  $Y$  is the boundary of a 4–dimensional manifold  $M$  and the version of Seiberg–Witten invariants introduced by Kronheimer and Mrowka in [10] is nonvanishing. The proof requires an excision result for Seiberg–Witten moduli spaces; then the Bennequin inequality becomes a special case of the adjunction inequality for surfaces lying inside  $M$ .

[57R17](#), [57M25](#), [57M27](#), [57R57](#)

## 1 Introduction

This paper is a sequel to the article [10] by Kronheimer and the first author, where the Seiberg–Witten invariants were generalized to invariants of connected oriented smooth 4–manifolds carrying a contact structure on their boundary. An oriented contact structure  $\xi$  (or more generally an oriented 2–plane field) induces a *canonical*  $\text{Spin}^c$ –structure  $\mathfrak{s}_\xi$  on  $Y$ . In [10], the Seiberg–Witten invariants were defined for 4–manifolds with boundary endowed with a contact structure. The domain of these invariants is the set  $\text{Spin}^c(\bar{M}, \xi)$  of isomorphism classes of pairs  $(\mathfrak{s}, h)$  where  $\mathfrak{s}$  is a  $\text{Spin}^c$ –structure on  $\bar{M}$  and  $h$  is an isomorphism between  $\mathfrak{s}|_Y$  and  $\mathfrak{s}_\xi$ . The Seiberg–Witten invariant is a map

$$\text{sw}: \text{Spin}^c(\bar{M}, \xi) \rightarrow \mathbb{Z}$$

well defined up to an overall sign. The main result of this paper is an excision property for these invariants. As a corollary we derive that for overtwisted contact structures these invariants are trivial and combining this with extension of the Taubes nonvanishing theorem [14] to this context we derive a pseudoholomorphic curve free proof of Eliashberg’s theorem [2] that weakly symplectic fillable contact structures are tight.

### 1.1 Some recollections and conventions

All 4–manifolds will be connected, oriented and smooth unless otherwise noted. All 3–manifolds will be oriented and smooth but not necessarily connected.

A contact structure on a compact 3–dimensional manifold  $Y^3$  is a totally nonintegrable 2–plane field  $\xi \subset TY$  which we assume to be orientable. Thus there is a nonvanishing differential 1–form  $\eta$ , such that  $\xi = \ker \eta$  and  $\eta \wedge d\eta \neq 0$  and so  $Y$  gets an orientation provided by the volume form  $\eta \wedge d\eta > 0$ .

A knot  $K$  in  $Y$  is called *Legendrian* if its tangent space is contained in the contact plane field  $\xi$ . Suppose that  $Y$  is the oriented boundary of an oriented connected smooth 4–manifold  $M$  and that  $K$  is the boundary of a smooth connected orientable surface  $\Sigma$ . Then  $\Sigma$  determines, up to sign, a homology class

$$\sigma \in H_2(M, K; \mathbb{Z})$$

which maps to a generator of  $H_1(K; \mathbb{Z})$  under the coboundary map.

The Thurston–Bennequin invariant and the rotation number of a knot in  $S^3$  generalize to invariants of the pair  $K$  and  $\sigma$ . The rotation number is only defined up to sign until an orientation of  $K$  is chosen. Both invariants arise because a Legendrian knot has a *canonical framing* obtained by choosing a vector field  $V$  transverse to the contact distribution along  $K$ . To generalize the Thurston–Bennequin invariant choose an arbitrary orientation for  $K$  give  $\Sigma$  the compatible orientation. Push  $K$  off slightly in the  $V$ –direction, thus obtaining a disjoint oriented knot  $K'$ . Then push  $\Sigma$  off itself to get a surface  $\Sigma'$  so that its boundary coincides with  $K'$ . Since the  $\Sigma$  and  $\Sigma'$  are disjoint along their boundary they have a well defined intersection number. The Thurston–Bennequin invariant relative to  $\Sigma$ ,  $\text{tb}(K, \sigma)$  is defined to be this self–intersection number. If  $\Sigma$  is contained in  $Y$  (so that  $K$  is nullhomologous in  $Y$ ) then

$$\text{tb}(K, \sigma) = \text{lk}(K, K') := \text{tb}(K).$$

Notice that  $\text{tb}(K)$  does not depend on the choice of the initial orientation for  $K$  or  $\Sigma$ .

The generalization of the rotation number is obtained as follows. After choosing an orientation, the contact distribution can be endowed with an almost complex structure  $J_\xi$ , which is unique up to homotopy. Therefore  $\xi \rightarrow Y$  has the structure of a complex line bundle, hence it has a well defined Chern class. The isomorphism  $h$  induces an isomorphism of the determinant line  $L_\mathfrak{s} = \det(W_\mathfrak{s}^+)$  of the bundle of positive spinors for the  $\text{Spin}^c$ –structure  $\mathfrak{s}$  with  $\xi$  on the boundary. If we also fix an orientation for  $K$  then we get a preferred nonvanishing tangent vector field  $v$  and so, by the Legendrian property, a nonvanishing section of  $L_\mathfrak{s}$ . Then the rotation number of  $K$  relative to  $\Sigma$  is by definition

$$r(K, \sigma, \mathfrak{s}, h) := \langle c_1(L_\mathfrak{s}, v), \sigma \rangle$$

where  $c_1(L_\mathfrak{s}, v)$  is the relative Chern class with respect to the trivialization of  $\xi$  along  $K$  induced by  $v$ . Notice that the rotation number depends *a priori* on the homology

class of  $\Sigma$  and on the orientation of  $K$ . This definition coincides with the usual rotation number defined on  $(\mathbb{R}^3, \xi_{\text{std}})$  as the winding number of  $v$  in  $\xi_{\text{std}}$ .

## 1.2 Main results

With these definitions and notation in place we can state our generalization of Bennequin's Inequality.

**Theorem A** *Let  $(Y, \xi)$  be a 3-dimensional closed manifold endowed with a contact structure  $\xi$  and let  $\bar{M}$  be a compact 4-dimensional manifold with boundary  $Y$ . Suppose we have a Legendrian knot  $K \subset Y$ , and a connected, orientable compact surface  $\Sigma \subset \bar{M}$  with boundary  $\partial\Sigma = K$ . Then for every relative  $\text{Spin}^c$ -structure  $(\mathfrak{s}, h)$  with  $\text{sw}_{\bar{M}, \xi}(\mathfrak{s}, h) \neq 0$  we have*

$$\chi(\Sigma) + \text{tb}(K, \sigma) + |r(K, \sigma, \mathfrak{s}, h)| \leq 0,$$

where  $\chi$  denotes the Euler characteristic.

Notice that this result was known before in the case of compact Stein complex surfaces with pseudoconvex boundary (see Akbulut–Matveyev and Lisca–Matić [1; 11]).

Here are two corollaries of Theorem A. A contact manifold  $(Y, \xi)$  is called *weakly symplectically fillable* if it is the boundary of a symplectic manifold  $(\bar{M}, \omega)$  such that  $\omega$  is positive on  $\xi$ . We say that  $(Y, \xi)$  is *weakly symplectically semi-fillable* if it is a component of a weakly symplectically fillable contact manifold. A contact structure  $\xi$  on  $Y$  is called *overtwisted* if  $Y$  contains an embedded disk  $D^2$  whose boundary  $K = \partial D^2$  is tangent to  $\xi$  while  $D^2$  is transverse to  $\xi$  at the boundary — such a disk is called an *overtwisted disk*. Otherwise we say that the contact structure is *tight*. If  $\xi$  is overtwisted, we see that the Bennequin inequality does not hold since  $\text{tb}(\partial D^2) = 0$  and  $\chi(D^2) = 1$ . So we have the following result.

**Corollary B** *The Seiberg–Witten invariant  $\text{sw}_{\bar{M}, \xi}$  is identically zero for a manifold with an overtwisted contact boundary.*

Using the fact that  $\text{sw}_{\bar{M}, \xi} \neq 0$  for weakly symplectically semi-fillable contact structures [10, Theorem 1.1], we have a pseudoholomorphic curve free proof of the the following theorem of Eliashberg's theorem result without using curves.

**Corollary C** *If  $(Y^3, \xi)$  is a weakly symplectically semi-fillable contact manifold, it is necessarily tight.*

This result could also be deduced from Taubes nonvanishing theorem and Eliashberg's recent result on concave fillings [4].

These results highlight an interesting question. Are there contact 3-manifolds  $(Y, \xi)$  which are not weakly symplectically fillable and yet there is a 4-manifold  $M$  which bounds  $Y$  so that  $\text{sw}_{\bar{M}, \xi}$  is not identically zero? On the other hand there are contact structures which are tight but not weakly symplectically fillable. The first examples are due to Etnyre and Honda [5] on certain Seifert fibered space and later infinite families were discovered by Lisca and Stipsicz [12; 13].

All these results rely on an excision property for Seiberg–Witten invariants. Recall that a *symplectic cobordism* between contact manifolds  $(Y, \xi)$  and  $(Y', \xi')$  is a compact symplectic manifold  $(\bar{Z}, \omega)$  so that, with the symplectic orientation, its boundary is  $\partial\bar{Z} = -Y \sqcup Y'$ , where  $Y$  and  $Y'$  have their orientations induced by the contact structures.  $Y$  is called the *concave* end of the cobordism and  $Y'$  is called the *convex* end. In addition, it is required that  $\omega$  is strictly positive on  $\xi$  and  $\xi'$  with their induced orientations. By convention the boundary components will always be given in the order concave, convex.

A symplectic cobordism is said to be *special* if

- the symplectic form is given in a collar neighborhood of the concave boundary by a symplectization of  $(Y, \xi)$ ;
- the map induced by the inclusion

$$(1.1) \quad i^*: H^1(\bar{Z}, Y') \rightarrow H^1(Y)$$

is the zero map.

A symplectic cobordism carries a canonical  $\text{Spin}^c$ -structures  $\mathfrak{s}_\omega$ . Moreover, there are isomorphisms, unique up to homotopy, which identify the restrictions of  $\mathfrak{s}_\omega$  to  $Y$  and  $Y'$  with the canonical  $\text{Spin}^c$ -structures  $\mathfrak{s}_\xi$  and  $\mathfrak{s}_{\xi'}$ .

Let  $(\mathfrak{s}, h)$  be an element in  $\text{Spin}^c(\bar{M}, \xi)$ . There is a canonical way to extend  $(\mathfrak{s}, h)$  to a  $\text{Spin}^c$  structure  $\mathfrak{t}$  on  $\bar{M}' = \bar{M} \cup \bar{Z}$  together with an isomorphism  $h'$  between  $\mathfrak{t}|_{Y'}$  and  $\mathfrak{s}_{\xi'}$ . We declare  $\mathfrak{t} := \mathfrak{s}_\omega$  on  $\bar{Z}$ . The data of  $h$  identifies the restriction  $\mathfrak{s}|_Y$  with  $\mathfrak{s}_\xi$  while  $\mathfrak{s}_\omega|_Y$  is identified canonically with  $\mathfrak{s}_\xi$ . Together these provides a gluing map and defines  $\mathfrak{t}$ . Thus we have defined a canonical map

$$j: \text{Spin}^c(\bar{M}, \xi) \rightarrow \text{Spin}^c(\bar{M}', \xi').$$

The main technical result of this paper is the following.

**Theorem D** Let  $\bar{M}$  be a manifold with contact boundary  $(Y, \xi)$ , and let  $\bar{Z}$  be a special symplectic cobordism between  $(Y, \xi)$  and a second contact manifold  $(Y', \xi')$ . Let  $\bar{M}' = \bar{M} \cup_Y \bar{Z}$  be the manifold obtained by gluing  $\bar{Z}$  on  $\bar{M}$  along  $Y$ . Then

$$(1.2) \quad \text{sw}_{\bar{M}, \xi} \circ j = \pm \text{sw}_{\bar{M}', \xi'} .$$

**Remark 1.2.1** Recall the invariants themselves are only defined up to an overall sign. We can refine the statement above to the following. For every pair of  $\text{Spin}^c$ -structures  $(\mathfrak{s}_1, h_1)$  and  $(\mathfrak{s}_2, h_2)$  in  $\text{Spin}^c(\bar{M}, \xi)$  we have:

$$\text{sw}_{\bar{M}', \xi'} \circ j(\mathfrak{s}_1, h_1) \text{sw}_{\bar{M}, \xi}(\mathfrak{s}_2, h_2) = \text{sw}_{\bar{M}, \xi}(\mathfrak{s}_1, h_1) \text{sw}_{\bar{M}', \xi'} \circ j(\mathfrak{s}_2, h_2) .$$

**Remark 1.2.2** Assumption (1.1) can be reformulated as follows. If  $u \in \text{Map}(\bar{Z}, S^1)$  is homotopic to the identity along  $Y'$ , then  $u$  must be homotopic to the identity along  $Y$ . Without the assumption (1.1), the gauge transformation  $u|_Y$  may not extend to  $\bar{M}$ . In this case the map  $j$  is not generally injective anymore.

All the cobordisms of interest in this paper will be shown to verify assumption (1.1), that is to say 1 and 2-handle surgeries. Assumption (1.1) may be removed, but conclusion (1.2) of Theorem D has to be replaced by

$$(1.3) \quad \text{sw}_{\bar{M}', \xi'}(\mathfrak{s}', h') = \pm \sum_{(\mathfrak{s}, h) \in j^{-1}(\mathfrak{s}', h')} \text{sw}_{\bar{M}, \xi}(\mathfrak{s}, h) .$$

This generalization is proved by refining the gluing Theorem E as explained in Remark 2.4.3. Indeed if two pairs  $(s_1, h_1)$  and  $(s_2, h_2)$  have the same image under  $j$  then we can assume, up to isomorphism, that  $s_1 = s_2$  and  $h_1 = uh_2$  where  $u$  is an automorphism of  $\mathfrak{s}_\xi$  which extends to an automorphism of  $\mathfrak{s}_\omega$  which is the identity at infinity.

A result of Weinstein [15] shows that a 1-handle surgery, or a 2-handle surgery along a Legendrian knot  $K$  with framing coefficient  $-1$  relative to the canonical framing, on the boundary of  $\bar{M}$  leads to a manifold  $\bar{M}'$  given by

$$\bar{M}' = \bar{M} \cup_Y \bar{Z}$$

where  $(\bar{Z}, \omega)$  is a special symplectic cobordism between  $(Y, \xi)$  and a contact boundary  $(Y', \xi')$  obtained by the surgery.

The strategy to prove Theorem A is to cap  $\Sigma$  doing a Weinstein surgery along  $K$ . Bennequin's inequality is then obtained by applying the adjunction inequality [9] to the resulting closed surface  $\Sigma' \subset \bar{M}'$ ; however the adjunction inequality holds provided  $\text{sw}_{\bar{M}', \xi'} \circ j(\mathfrak{s}, h)$  does not vanish. This is true under the assumption that  $\text{sw}_{\bar{M}, \xi}(\mathfrak{s}, h)$  does not vanish thanks to the excision Theorem D.

### 1.3 Monopoles and contact structures

We briefly describe the Seiberg–Witten theory on a 4–manifold  $\bar{M}$  with contact boundary  $(Y, \xi)$  developed by Kronheimer and the first author in [10].

**1.3.1 An almost Kähler cone** We pick a contact 1–form  $\eta$  with  $\ker \eta = \xi$ . The contact form determines the Reeb vector field  $R$  by the properties that  $\iota_R d\eta = 0$  and  $\eta(R) = 1$ . Then  $(0, +\infty) \times Y$  is endowed with a symplectic structure called the symplectization of  $(Y, \eta)$ , and the symplectic form is defined by the formula

$$\omega = \frac{1}{2}d(t^2\eta).$$

Choose an almost complex structure  $J$  on  $(0, \infty) \times Y$  where  $\xi$  and  $d\eta$  are invariant under  $J$ , and such that

$$\iota J \partial_t = R.$$

The almost complex structure  $J$  is clearly compatible with the symplectic form  $\omega$  in the sense that  $g = \omega(J\cdot, \cdot)$  is a Riemannian metric and that  $\omega$  is  $J$  invariant; thus we have defined an almost Kähler structure  $(g, \omega, J)$  on  $(0, \infty) \times Y$  called the *almost Kähler cone* on  $(Y, \eta, J|_\xi)$ .

The metric  $g$  has an expression of the form

$$g = dt^2 + t^2\eta^2 + t^2\gamma,$$

where  $\gamma$  is the positive symmetric bilinear form on  $Y$  defined by  $\gamma = \frac{1}{2}d\eta(J\cdot, \cdot)$ .

Let  $M$  be the manifold

$$M = \bar{M} \cup C_M,$$

where  $C_M = (T, \infty) \times Y$  is endowed with the structure of almost Kähler cone on  $(Y, \eta, J|_\xi)$  described above. Then we arbitrarily extend the Riemannian metric  $g$  defined on the end  $C_M$  to the compact set  $\bar{M}$ .

The  $\text{Spin}^c$ –structure  $\mathfrak{s}_\xi$  induced by the contact structure on  $Y$  is canonically identified with the restriction to  $\{T\} \times Y$  of the  $\text{Spin}^c$ –structure  $\mathfrak{s}_\omega$  induced by the symplectic form  $\omega$  on  $C_M$ . Therefore, an element  $(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi)$  admits a natural extension over  $M$ . By a slight abuse of language, we will still denote it  $(\mathfrak{s}, h)$ , where  $h$  is now an isomorphism between  $\mathfrak{s}$  and  $\mathfrak{s}_\omega$  over  $C_M$ .

The spinor bundle  $W = W^+ \oplus W^-$  of  $\mathfrak{s}$  is identified over  $C_M$  with the canonical Spin–bundle  $W_J = W_J^+ \oplus W_J^-$  of  $\mathfrak{s}_\omega$ ; we recall that

$$W_J^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad W_J^- = \Lambda^{0,1}.$$

**1.3.2 Monopole equations** Let  $A$  be a spin connection on  $W$  and  $\nabla^A$  be its covariant derivative. The Dirac operator is then defined by  $D_A^g = \sum_i e^i \cdot \nabla_{e_j}^A$ , where  $e_j$  is an oriented orthonormal local frame on  $M$ .  $e^i$  is the dual coframe and acts by Clifford multiplication. It is a first order elliptic operator of order 1 between the space of spinor fields  $D_A: \Gamma(W^\pm) \rightarrow \Gamma(W^\mp)$ .

Put  $\Psi = (1, 0) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$ . Let  $B$  be the spin-connection in the spinor bundle  $W_J^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$  so that  $\nabla^B \Psi$  is a section of  $T^*X \otimes \Lambda^{0,2}$ .

The Dirac operator  $D^{can}$  on  $W_J$  is obtained by replacing  $\nabla^A$  with  $\widehat{\nabla}$  in the definition. For  $\Phi \in \Gamma(W_J^+)$ , we have explicitly  $D^{can} \Phi = \sqrt{2}(\partial\beta + \bar{\partial}^* \gamma)$  for  $\Phi = (\beta, \gamma) \in \Gamma(\Lambda^{0,0} \oplus \Lambda^{0,2})$ . Then  $D^{can} \Psi = 0$ .

The Chern connection  $\widehat{\nabla}$  in  $\Lambda^{0,0} \oplus \Lambda^{0,2}$  induces a connection in  $\Lambda^{0,2} = \det(W_J^+)$  and this connection agrees with the one induced from  $B$ . There is also an identity

$$D_B = D^{can}.$$

Since the spinor bundle  $W$  is identified with  $W_J$  via the isomorphism  $h$  over  $C_M$ ,  $B$  may be considered as a spin connection over  $W|_{C_M}$  and  $\Psi|_{C_M}$  as a section of  $W^+|_{C_M}$ . We extend them arbitrarily over  $\bar{M}$ .

The domain of the Seiberg–Witten equations on  $(M, g, \mathfrak{s})$  is the space of configurations

$$\mathcal{C} = \{(A, \Phi) \in \text{Conn}(W) \times \Gamma(W^+)\},$$

where  $\text{Conn}(W)$  is the space of spin connections on  $W$ . Recall that  $\text{Conn}(W)$  is an affine space modeled on  $\Gamma(i\Lambda^1)$ : for two spin connections  $A$  and  $\tilde{A}$ , we have

$$\tilde{A} = A + a \otimes \text{id}|_W,$$

where  $a$  is a purely imaginary 1-form. We will simply write the above identity  $\tilde{A} = A + a$  in the sequel. We introduce the curvature form

$$(1.4) \quad F_A(X, Y) = \frac{1}{4} \text{trace}_{\mathbb{C}} R_A(X, Y),$$

where the full curvature tensor  $R_A$  of  $A$  is viewed as a section of  $\Lambda^2 \otimes \text{End}_{\mathbb{C}}(W)$ . With this convention, we have

$$F_A = \frac{1}{2} F_{\widehat{A}}$$

where  $\widehat{A}$  is the unitary connection induced by  $A$  on the determinant line bundle<sup>1</sup>  $L$ , and  $F_{\widehat{A}}$  is its usual curvature form. The Seiberg–Witten equations are

$$(1.5) \quad F_A^+ - \{\Phi \otimes \Phi^*\}_0 = F_B^+ - \{\Psi \otimes \Psi^*\}_0 + \varpi$$

<sup>1</sup>Alternatively  $F_A$  can be viewed as the curvature of the unitary connection induced by  $A$  on the virtual line bundle  $L^{1/2}$

$$(1.6) \quad D_A \Phi = 0,$$

where  $\varpi$  is a self-dual purely imaginary 2-form on  $M$  and  $\{\cdot\}_0$  represents the trace free part of an endomorphism; the first equation makes sense since purely imaginary self-dual 2-forms are identified with the traceless endomorphisms of  $W^+$  via Clifford multiplication. The configuration  $(B, \Psi)$  is clearly a solution of the equations over  $C_M$ .

The space of solutions  $\mathcal{Z}_\varpi$  is acted on by a gauge group  $\mathcal{G} = \text{Map}(M, S^1)$  and the action is defined on  $\mathcal{C}$  by

$$(1.7) \quad u \cdot (A, \Phi) = (A - u^{-1} du, u\Phi) \quad \text{for all } u \in \mathcal{G}.$$

The moduli space  $\mathcal{M}_\varpi(M, g, \mathfrak{s}) = \mathcal{Z}_\varpi / \mathcal{G}$ , for suitable generic  $\varpi$ , is a compact smooth manifold of dimension

$$d = \langle e(W^+, \Psi), [M, C_M] \rangle,$$

where  $e(W^+, \Psi)$  is the relative Euler class of  $W^+$ . If  $d \neq 0$ , then the Seiberg–Witten invariant is always 0; if  $d = 0$ , the Seiberg–Witten invariant is the number of points of  $\mathcal{M}_\varpi(M, \mathfrak{s})$  counted with signs. Following [10] there is a trivial bundle over the configuration space (the determinant line bundle of the appropriate deformation operator) which is identified with the orientation bundle of moduli space. Furthermore a trivialization of this determinant line bundle for one relative  $\text{Spin}^c$ -structure determines a trivialization for all others in a canonical manner and determines a *consistent orientation*. Thus the set of consistent orientations is a two-element set. In particular unlike the closed case the sign of the invariant cannot be pinned down by a homology orientation rather the ratio of the signs of the values of the invariant for different relative  $\text{Spin}^c$ -structures is well defined. It turns out that this number depends only on  $\bar{M}$ , on the contact structure at the boundary, and on the choice of  $(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi)$ ; this explains the notation  $\text{SW}_{\bar{M}, \xi}$ .

**Remark** The details concerning the regularity of  $\mathcal{C}$  and  $\mathcal{G}$  have been omitted at the moment; morally,  $(A, \Phi)$  is supposed to behave like  $(B, \Psi)$  near infinity and gauge transformations should be close to identity as well. The relevant Sobolev spaces will be introduced later on.

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## 2 Excision

The goal of this section is to prove [Theorem D](#). The strategy is to show, in a more general setting, that the Seiberg–Witten moduli spaces associated to  $\bar{M}$  and  $\bar{M}'$  are diffeomorphic for a suitable choice of metrics and perturbations. This is achieved thanks to a gluing technique in [Theorem E](#).

### 2.1 Families of AFAK manifolds

We set up the analytical framework for our gluing problem. The first step is to construct suitable families of asymptotically flat almost Kähler metrics.

Let  $\bar{M}$  be a manifold with a contact boundary  $(Y, \xi)$ . We choose a particular contact form  $\eta$  and glue an almost Kähler cone on the boundary as in [Section 1.3.1](#); furthermore the Riemannian metric is extended arbitrarily to  $\bar{M}$ . Hence we have obtained  $(M, g, \omega, J)$  where  $g$  is Riemannian metric and  $M$  splits as

$$(2.1) \quad M = \bar{M} \cup_Y C_M$$

with an almost Kähler structure defined outside a compact set. More generally, we recall the definition [\[10, Condition 3.1\]](#) of an *asymptotically flat almost Kähler* manifold (AFAK manifold in short).

**Definition 2.1.1** (AFAK manifolds) A manifold  $M$  with an almost Kähler structure  $(M, \omega, J)$  defined outside a compact set  $K \subset M$ , a Riemannian metric  $g$  extending the metric on the end, and a proper function  $\sigma: M \rightarrow (0, \infty)$  is called an asymptotically flat almost Kähler manifold if it satisfies the following conditions:

- (i) There is a constant  $\kappa > 0$ , such that the injectivity radius satisfies  $\kappa \operatorname{inj}(x) > \sigma(x)$  for all  $x \in M$ .
- (ii) For each  $x \in M$ , let  $e_x$  be the map  $e_x: v \mapsto \exp_x(\sigma(x)v/\kappa)$  and  $\gamma_x$  be the metric on the unit ball in  $T_x M$  defined as  $e_x^*g/\sigma(x)^2$ . Then these metrics have bounded geometry in the sense that all covariant derivatives of the curvature are bounded by some constants independent of  $x$ .
- (iii) For each  $x \in M \setminus K$ , let  $o_x$  similarly be the symplectic form  $e_z^*\omega/\sigma(z)^2$  on the unit ball. Then  $o_x$  similarly approximates the translation-invariant form, along with all its derivatives.

- (iv) For all  $\varepsilon > 0$ , the function  $e^{-\varepsilon\sigma}$  is integrable on  $M$ .
- (v) The symplectic form  $\omega$  extends as a closed form on  $K$ .

**Convention** When the end of  $M$  has a structure of symplectic cone as in (2.1) we choose  $\sigma(t, y) = t$  on the end  $C_M$  and extend it arbitrarily with the condition  $0 < \sigma(x) \leq T$  on  $\bar{M}$ .

**Definition 2.1.2** (AFAK ends) An asymptotically flat almost Kähler end is a manifold  $Z$  which admits a decomposition  $C_Z \cup_Y N$ , where  $N$  is a not necessarily compact 4-dimensional manifold, with a contact boundary  $Y$ , endowed with a fixed contact form  $\eta$ , and  $C_Z = (0, T] \times Y$  for some  $T > 0$ .

In addition,  $Z$  is endowed with an almost Kähler structure  $(\omega_Z, J_Z)$  and a proper function  $\sigma_Z: Z \rightarrow (0, \infty)$  satisfying:

- on  $(0, T] \times Y \subset Z$ , we have  $\sigma_Z(t, y) = t$ ;
- the almost Kähler structure on  $C_Z$  is the the one of an almost Kähler cone on  $(Y, \eta)$  as defined in Section 1.3.1;
- if we substitute  $M$  by  $Z$ ,  $K$  by  $\emptyset$  and  $\sigma$  by  $\sigma_Z$ , in Definition 2.1.1, the the properties (i) – (iv) are verified;
- the map induced by the inclusion  $Y = \partial N \subset N$

$$(2.2) \quad H_c^1(N) \longrightarrow H^1(Y)$$

where  $H_c^*(N)$  is the compactly supported DeRham cohomology, is identically 0.

### Remarks 2.1.3

- The last condition is a technical condition, reminiscent of the assumption (1.1) (see Remark 1.2.2). More precisely, we will construct some particular AFAK ends out of special symplectic cobordisms in this section, and the property (2.2) will be a consequence of the assumption (1.1). This property will be used for the compactness results of Section 2.2.4. However this assumption is not essential as explained in Remark 2.2.3.
- If we scale the metric by  $\lambda^2$ , and accordingly scale  $T, \sigma$  into  $\lambda T, \lambda\sigma$ , then the constant  $\kappa$  and the constants controlling  $\gamma_x$  and  $o_x$  remain unchanged. In particular, we may always assume  $T = 1$  in Definitions 2.1.1 and 2.1.2.
- In Definition 2.1.2, the contact form  $\eta$  can be replaced by any other 1-form representing the same contact structure. This is shown in the next lemma which allows us to modify the symplectic form  $\omega_Z$  near the sharp end of the cone  $C_Z$ .

**Lemma 2.1.4** *Let  $Y$  be a compact 3-manifold endowed with a contact structure  $\xi$ . Let  $\eta_1$  and  $\eta_2$  be two 1-forms such that  $\ker \eta_j = \xi$ . Then, for every  $\varepsilon > 0$ , there exist  $\alpha \in (0, \varepsilon)$  and a symplectic form  $\omega$  on  $(0, +\infty) \times Y$  such that*

- $\omega = \frac{1}{2}d(t^2\eta_1)$  on  $(0, \alpha) \times Y$ ,
- $\omega = \frac{1}{2}d(t^2\eta_2)$  on  $(\varepsilon, +\infty) \times Y$ .

**Proof** We write  $\eta_2 = e^\mu\eta_1$  where  $\mu$  is a real function on  $Y$ . We consider the exact 2-form

$$\omega = \frac{1}{2}d(t^2e^{\mu f(t)}\eta_1),$$

where  $f(t)$  is a smooth increasing function equals to 1 for  $t \geq 2\varepsilon$  and to 0 for  $t \leq \varepsilon$ . If we show that we can choose  $f$  in such a way that  $\omega$  is definite, then,  $\omega$  is a symplectic form and is a solution for the lemma.

A direct computation shows that

$$\omega^2 = \frac{1}{2}t^3(2 + t\mu f')dt \wedge \eta \wedge d\eta.$$

Let  $c$  be the minimum of the function  $\mu$  on  $Y$ . If  $c \geq 0$ , we just require that  $f$  is an increasing function of  $t$ . If  $c < 0$ , a sufficient additional condition for having  $\omega^2 > 0$  is

$$(2.3) \quad f'(t) < -\frac{2}{tc}.$$

The function  $f_0 = -\frac{1}{c} \ln(\frac{2et}{\varepsilon})$  verifies the above identity. Then we define

- $f_1(t) = 0$  for  $t \in (0, \frac{\varepsilon}{2e}]$ ,
- $f_1(t) = f_0(t)$  for  $t \in [\frac{\varepsilon}{2e}, \frac{\varepsilon}{2}]$ ,
- $f_1(t) = 1$  for  $t \in [\frac{\varepsilon}{2}, +\infty)$ .

By definition  $f_1$  is a piecewise  $C^1$  function which verifies the condition (2.3) on each interval. We can regularize  $f_1$  into a positive smooth increasing function  $f$  by making a perturbation on the interval  $[\frac{\varepsilon}{4e}, \varepsilon]$  in such a way that we have  $f' \leq f'_1$  on each interval where  $f'_1$  is defined. Therefore, the condition  $f' < -\frac{2}{c}t$  is preserved hence  $\omega^2 > 0$ .  $\square$

**2.1.5 Gluing an AFAK end on  $\bar{M}$**  Thanks to [Remarks 2.1.3](#), we may assume from now on that  $T = 1$  and that the contact form  $\eta$  is the same in the definition of the almost Kähler cone  $C_M = (T, \infty) \times Y$  and in the definition of the AFAK end ([Definition 2.1.2](#)). We identify an annulus in  $C_M \subset M$  with an annulus in  $C_Z \subset Z$  using the dilation map

$$\begin{aligned} M \supset C_M \simeq (1, +\infty) \times Y \supset (1, \tau) \times Y &\xrightarrow{\nu^\tau} (1/\tau, 1) \times Y \subset (0, 1) \times Y \simeq C_Z \subset Z \\ (t, y) &\longmapsto (t/\tau, y) \end{aligned}$$

and define the manifold  $M_\tau$  as the union of  $M \cap \{\sigma_M < \tau\}$  and  $Z \cap \{\sigma_Z > 1/\tau\}$  and with the identify along the annuli given by the dilation  $\nu^\tau$ . The operation of connected sum along  $Y$  we just defined is represented in the figure below. The gray regions represent the annuli, the arrows suggest that they are identified by a dilation, and the dashed regions are the parts of  $M$  and  $Z$  that are taken off for the construction of  $M_\tau$ .

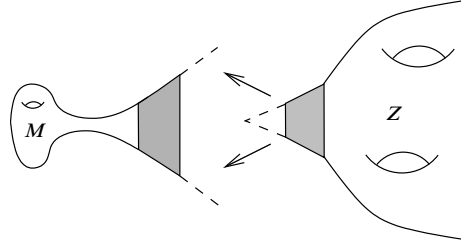


Figure 1: Construction of AFAK  $M_\tau$

Now  $\nu_*^\tau \omega_{C_M} = \tau^2 \omega_Z$  and  $\sigma_Z \circ \nu^\tau = \sigma_M / \tau$ , hence, if we scale  $\omega_Z$  by  $\tau^2$  and  $\sigma_Z$  by  $\tau$ , all the structure will match on the annuli. In conclusion  $M_\tau$  carries an almost Kähler structure  $(\omega_\tau, J_\tau)$  defined outside the compact set  $\bar{M} \subset M_\tau$  and functions  $\sigma_\tau$ .

**Remark** Every compact set of  $K \subset M$  is also, by definition, a compact set of  $M_\tau$  provided  $\tau$  is large enough. Similarly, the structures  $g_\tau, \sigma_\tau, J_\tau, \omega_\tau$  are equal on every compact set when  $\tau$  is large enough.

The following lemma is satisfied by construction.

**Lemma 2.1.6** *The manifolds with almost Kähler structure defined outside a compact set and a proper function  $(M_\tau, g_\tau, J_\tau, \sigma_\tau)$  satisfy [Definition 2.1.1](#) uniformly, in the sense that the constant  $\kappa, \varepsilon$ , the bounds on  $\gamma_x, o_x$  and on the integral of  $e^{-\varepsilon \sigma_\tau}$  can be chosen independently of  $\tau$ .*

### Remarks

- A simple consequence of the lemma is the following: for all  $\varepsilon > 0$ , there exists  $T_k$  large enough such that, for every  $\tau$ , the pull-back of the almost Kähler structure on a unit balls in  $M_\tau \cap \{\sigma_\tau \geq T_k\}$ , via the exponential map  $v \rightarrow \exp_x v$ , on the tangent space  $T_x M_\tau$ , is  $\varepsilon$ -close in  $C^k$ -norm to the euclidean structure  $(g_\tau, \omega_\tau, J_\tau)|_{T_x M_\tau}$ .
- For all the remainder of Section 2,  $M$  and  $M_\tau$  will denote the the manifolds that we just constructed at Section 2.1.5 together with their additional structures (identification of  $C_M$  with an almost Kähler cone, proper function  $\sigma$ , almost Kähler structure and Riemannian metric).

**2.1.7 Spin<sup>c</sup>-structures on AFAK manifolds** Similarly to the case of a manifold with a contact boundary, we define the space  $\text{Spin}^c(X, \omega)$  for an AFAK manifold  $X$  as the set of equivalence classes of pairs  $(\mathfrak{s}, h)$ , where  $\mathfrak{s}$  is a Spin<sup>c</sup>-structure on  $X$  and  $h$  is an isomorphism, defined outside a compact set  $K_1 \subset X$ , between  $\mathfrak{s}_\omega|_{X \setminus K_1}$  and  $\mathfrak{s}|_{X \setminus K_1}$ . As we saw in Section 1.3.1, there is a well defined identification of  $\text{Spin}^c(\overline{M}, \xi)$  with  $\text{Spin}^c(M, \omega)$ ; more generally, there is a natural identification  $j: \text{Spin}^c(\overline{M}, \xi) \rightarrow \text{Spin}^c(M_\tau, \omega_\tau)$  when  $M_\tau$  is obtained by adding an AFAK end  $Z$  to the end of  $\overline{M}$ . Notice that the set of equivalence classes  $\text{Spin}^c(M_\tau, \omega_\tau)$  do not depend on  $\tau$ . However the realization of the Spin<sup>c</sup>-structure by a spinor bundle is sensitive to the choice of  $\tau$ : let  $(\mathfrak{s}, h) \in \text{Spin}^c(M, \omega)$ , and suppose that  $h$  is defined on  $M \cap \{\sigma > 1\}$  for simplicity. Alternatively,  $h$  can be thought of as an isomorphism between the spinor bundle  $W$  of  $\mathfrak{s}$  and the spinor bundle  $W_J$  of  $\mathfrak{s}_\omega$ . We define a family of spinor bundles  $W_\tau$  on  $M_\tau$  by:

$$\begin{cases} W_\tau := W & \text{over } M_\tau \cap \{\sigma_\tau < \tau\} \subset M \\ W_\tau := W_{J_\tau} & \text{over } M_\tau \cap \{\sigma_\tau > 1\} \subset Z, \end{cases}$$

and the transition map from  $W$  to  $W_{J_\tau}$  is given by  $h$  over the annulus  $\{1 < \sigma_\tau < \tau\} \cap M_\tau \subset C_M$  (where  $W_{J_\tau} = W_J$ ). At the level of equivalence classes of Spin<sup>c</sup>-structures, this procedure defines an identification of  $\text{Spin}^c(M, \omega)$  with  $\text{Spin}^c(M_\tau, \omega_\tau)$ .

We stress the fact that  $W_\tau$  is identified with the canonical spinor bundle  $W_{J_\tau}$  for it is, by construction, equal to it on  $M_\tau \cap \{\sigma_\tau > 1\}$ . Moreover, the spinor bundles  $W_\tau$  restricted to any compact set  $K \subset M$  are all identified provided  $\tau$  is large enough.

In Section 1.3.2 we defined a configuration  $(B, \Psi)$  for the spinor bundle  $W \rightarrow M$ . Similarly, we can define a configuration  $(B_\tau, \Psi_\tau)$  for the spinor bundle  $W_\tau \rightarrow M_\tau$ : outside  $\overline{M}$ , the bundle  $W_\tau$  is identified with the canonical spinor bundle  $W_{J_\tau}$ . The almost Kähler structure induces a Chern connection  $\widehat{\nabla}_\tau$  on  $L_{J_\tau} = \det W_{J_\tau}^+ = \Lambda^{0,2}$ .

Hence we deduce a spin connection  $B_\tau$  from the Levi–Civita connection of  $g_\tau$  and  $\widehat{\nabla}_\tau$ . Let  $\Psi_\tau = (1, 0)$  be the spinor of  $\Lambda^{0,0} \oplus \Lambda^{0,2} = W_{J_\tau}^+$ . Then we choose a common extension (independent of  $\tau$ ) of the configuration  $(B_\tau, \Psi_\tau)$  to the compact  $\overline{M} \subset M_\tau$ .

In conclusion we have constructed a family of spinor bundles  $W_\tau \rightarrow M_\tau$  which are identified with  $W_{J_\tau}$  outside  $\overline{M}$  and which are realizations of the same element  $j(\mathfrak{s}, h) \in \text{Spin}^c(M_\tau, \omega_\tau)$ .

On a compact set  $K \subset M$ ,  $W$  is canonically identified with  $W_\tau$  for every  $\tau$  large enough and the configurations  $(B_\tau, \Psi_\tau)$  agree; hence we will write  $W$  and  $(B, \Psi)$  instead of  $W_\tau, (B_\tau, \Psi_\tau)$  for simplicity of notation.

## 2.2 A family of Seiberg–Witten moduli spaces

We introduce now Seiberg–Witten equations for the AFAK manifolds  $M$  and  $M_\tau$ . We show that in some sense the moduli space of Seiberg–Witten equations on  $M$  is a limit of the moduli spaces on  $M_\tau$  as  $\tau \rightarrow +\infty$ .

**2.2.1 A family of Seiberg–Witten equations** Starting from an element  $(\mathfrak{s}, h) \in \text{Spin}^c(\overline{M}, \xi)$ , we consider the  $\text{Spin}^c$ –structure induced on  $M$  and  $j(\mathfrak{s}, h)$  on  $M_\tau$ . The Seiberg–Witten equations were introduced in [Section 1.3.2](#) on  $M$ , which has an end modeled on an almost Kähler cone  $C_M$ . The equations are given on  $M_\tau$  in the same way by

$$(2.4) \quad F_A^+ - \{\Phi \otimes \Phi^*\}_0 = F_B^+ - \{\Psi \otimes \Psi^*\}_0 + \varpi_\tau$$

$$(2.5) \quad D_A \Phi = 0,$$

where  $\Phi$  is a section of  $W_\tau^+$ ,  $A$  is a spin connection on  $W_\tau$  and  $\varpi_\tau$  is a perturbation in  $\Gamma(i\Lambda^+ M_\tau)$ .

As in the case of an almost Kähler conical end, the almost Kähler structure defined outside  $\overline{M}$  induces a Chern connection  $\widehat{\nabla}$  on  $W_\tau$ , with a corresponding canonical Dirac operator  $D^{can} = \sqrt{2}(\bar{\partial} \oplus \bar{\partial}^*)$  and we have  $D_B^{g_\tau} = D^{can}$ , for  $B$  the spin connection deduced from the Levi–Civita connection on  $M_\tau$  and the Chern connection on  $L_\tau \simeq L_{J_\tau}$ . Therefore  $(B, \Psi)$  solves the Seiberg–Witten equations restricted to  $M_\tau \setminus \overline{M}$  with  $\varpi_\tau = 0$ .  $(B, \Psi)$  is called the *canonical solution*. Notice that the Dirac operator, the projection  $+$  and  $(B, \Psi) = (B_\tau, \Psi_\tau)$  depend on  $\tau$ .

A suitable family of cut-off functions is now needed. Let  $\chi(t)$  be a smooth decreasing function such that

$$\begin{cases} \chi(t) = 0 & \text{if } t \geq 1 \\ \chi(t) = 1 & \text{if } t \leq 0 \end{cases}$$

Put

$$(2.6) \quad \boxed{\chi_\tau = \chi \left( \frac{t-\tau}{N_0} + 1 \right)}$$

where  $N_0$  is any number with  $N_0 \geq 1$ ;  $N_0$  will be fixed later on to make the derivatives of  $\chi_\tau$  as small as required in our constructions. We define a cut-off function on  $M_\tau$  by the formula  $\chi_\tau(\sigma_\tau)$ . By a slight abuse of notation, the latter function will be denoted  $\chi_\tau$  as well.

For a given perturbation of Seiberg–Witten equation  $\varpi$  on  $M$ , the perturbation of the equations on  $M_\tau$  is defined by

$$(2.7) \quad \varpi_\tau = \chi_\tau \varpi.$$

**2.2.2 Linear theory** The Study of Seiberg–Witten equations requires introducing suitable Sobolev spaces rather than using naive smooth objects defined in [Section 1.3.2](#). We recall very quickly the results of [\[10\]](#) in this section.

The configuration  $(B, \Psi)$  is a solution of Seiberg–Witten equations on the almost Kähler end of  $M_\tau$ ; hence we study solutions  $(A, \Phi)$  with the same asymptotic behavior. We introduce the configuration space

$$\mathcal{C}_l(M_\tau) = \{(A, \Phi) \in \text{Conn}(W_\tau) \times \Gamma(W_\tau^+) / A - B \in L_l^2(g_\tau), \\ \text{and } \Phi - \Psi \in L_l^2(g_\tau, B)\},$$

the gauge group

$$\mathcal{G}_l(M_\tau) = \{u : M_\tau \rightarrow \mathbb{C} / |u| = 1, \text{ and } 1 - u \in L_{l+1}^2(g_\tau)\},$$

acting on  $\mathcal{C}_l$  by  $u \cdot (A, \Phi) = (A - u^{-1} du, u\Phi)$ , and, for some fixed  $\varepsilon_0 > 0$ , the perturbation space

$$\mathcal{N}(M_\tau) = e^{-\varepsilon_0 \sigma_\tau} C^r(i\mathfrak{su}(W_\tau^+)),$$

equipped with the norm

$$\|\varpi\|_{\mathcal{N}_\tau} = \|e^{\varepsilon_0 \sigma_\tau} \varpi\|_{C^r(g_\tau)}.$$

The  $L_k^2(g_\tau)$ -norm is the usual  $L^2$  norm with  $k$  derivatives on  $M_\tau$  defined using the metric  $g_\tau$ . In order to define a similar norm on the spinor fields, a unitary connection  $A$  is needed. We put

$$\|\phi\|_{L_l^2(g_\tau, A)}^2 = \int_{M_\tau} \left( |\phi|^2 + |\nabla_A \phi|^2 + \cdots + |\nabla_A^l \phi|^2 \right) \text{vol}^{g_\tau},$$

and define  $L_l^2(g_\tau, A)$  as the completion of the space of smooth sections for this norm. For two different connections,  $A$  and  $A'$  with  $A - A' \in L_l^2(g_\tau)$ , Sobolev multiplication theorems show that the norms  $L_l^2(A, g_\tau)$  and  $L_l^2(A', g_\tau)$  are commensurate.

**Remark** For any pair  $\tau, \tau'$ , it is easy to construct a diffeomorphism  $f: M_\tau \rightarrow M_{\tau'}$  covered by an isomorphism  $F$  between  $W_\tau$  and  $W_{\tau'}$  which are dilations near infinity. Therefore  $F^*C_l(M_{\tau'}) = C_l(M_\tau)$  and  $f^*\mathcal{G}_l(M_{\tau'}) = \mathcal{G}_l(M_\tau)$ . In this sense, the spaces  $C_l$  and  $\mathcal{G}_l$  are in fact independent of  $\tau$ . However, the fact that the norms depend on  $\tau$  will become crucial for analyzing the compactness properties of the family of moduli spaces on  $M_\tau$ .

Of course the choice of  $l \geq 2$  is actually perfectly arbitrary thanks to elliptic regularity. However it will be chosen with  $l \geq 4$  so that we have the inclusion  $L_l^2 \subset C^1$ . The Sobolev multiplication theorem shows that  $\mathcal{G}_l$  is a Hilbert Lie group acting smoothly on the Hilbert affine space  $C_l$ . Furthermore action of  $\mathcal{G}_l$  is *free*: if we have  $u \cdot (A, \Phi) = (A, \Phi)$ , then  $du = 0$  hence  $u$  must be constant. Now  $u - 1 \in L_{l+1}^2$  therefore  $u = 1$ .

Let  $\mathcal{Z}_l^\tau$  be the space of configurations  $(A, \Phi) \in C_l$  which verify the Seiberg–Witten equations (2.4) on  $M_\tau$  with perturbation  $\varpi_\tau$ . Then  $\mathcal{Z}_l^\tau$  is invariant under the gauge group action and we define

$$\mathcal{M}_l(M_\tau) = \mathcal{Z}_l^\tau / \mathcal{G}_l.$$

We drop the reference to the index  $\tau$  at the moment, for simplicity of notation. All of what we say in the rest of Section 2.2.2 holds for  $M$  and  $M_\tau$  or indeed any AFAM manifold. The linearized action of the gauge group at an arbitrary configuration  $(A, \Phi) \in C_l$  is given by a differential operator

$$\begin{aligned} \delta_{1,(A,\Phi)}: L_{l+1}^2(i\mathbb{R}) &\longrightarrow L_l^2(i\Lambda^1) \oplus L_{l,B}^2(W^+) \\ v &\longmapsto (-dv, v\Phi) \end{aligned}$$

and its formal adjoint is given by

$$\delta_{1,(A,\Phi)}^*(a, \phi) = -d^*a + i\mathfrak{S}\langle \Phi, \phi \rangle;$$

notice that with our convention, the Hermitian product  $\langle \cdot, \cdot \rangle$  is anti-complex linear in the first variable.

A tangent vector  $(a, \phi)$  is  $L^2$ -orthogonal to the orbit through  $(A, \Phi)$  if and only if  $\delta_{1,(A,\Phi)}^*(a, \phi) = 0$ ; the orbit space  $C_l/\mathcal{G}_l$  is a smooth Hilbert manifold, and its tangent space at  $(A, \Phi)$  is identified with  $\ker \delta_{1,(A,\Phi)}^*$ .



The linearized Seiberg–Witten equations at  $(A, \Phi)$  are given as well by a differential operator:

$$\begin{aligned} \delta_{2,(A,\Phi)}: L_{l+1}^2(i\Lambda^1) \oplus L_{l+1,B}^2(W^+) &\rightarrow L_l^2(i\mathfrak{su}W^+) \oplus L_{l,B}^2(W^-) \\ (a, \phi) &\mapsto (d^+a - \{\Phi \otimes \phi^* + \phi \otimes \Phi^*\}_0, D_A\phi + a \cdot \Phi) \end{aligned}$$

**Remark** There is a slight inconsistency in the conventions of [10]. The linear theory studied there is exactly the one presented in the current paper. However, this is not the one of the equations written in [10] where  $F_A$  is replaced by the curvature  $F_{\hat{A}}$  of the unitary connection  $\hat{A}$  induced by  $A$  on the determinant line bundle  $L$ . Since  $F_A = 2F_{\hat{A}}$ , the corresponding linearized equations should be  $\delta_{2,(A,\Phi)}(a, \phi) = (2d^+a - \{\Phi \otimes \phi^* + \phi \otimes \Phi^*\}_0, D_A\phi + a \cdot \Phi)$ .

At a solution of Seiberg–Witten equations  $(A, \Phi)$ , the operators verify  $\delta_{2,(A,\Phi)} \circ \delta_{1,(A,\Phi)} = 0$ , so we have an elliptic complex:

$$\begin{aligned} 0 \rightarrow L_{l+2}^2(i\mathbb{R}) &\xrightarrow{\delta_{1,(A,\Phi)}} L_{l+1}^2(i\Lambda^1) \oplus L_{l+1,B}^2(W^+) \\ &\xrightarrow{\delta_{2,(A,\Phi)}} L_l^2(i\mathfrak{su}W^+) \oplus L_{l,B}^2(W^-) \rightarrow 0 \end{aligned}$$

$H_0 = 0$  since the action of the gauge group is free,  $H^1 = \ker \delta_2 / \mathfrak{S}\delta_1$  is the virtual tangent space to the Seiberg–Witten moduli space at  $(A, \Phi)$ , and  $H^2 = \text{coker } \delta_2$  is the obstruction space. Equivalently  $H_1$  can be viewed as the kernel of the elliptic operator

$$(2.8) \quad \mathcal{D}_{(A,\Phi)} = \delta_{1,(A,\Phi)}^* \oplus \delta_{2,(A,\Phi)}.$$

### Facts

- The moduli space  $\mathcal{M}_l$  is compact.
- By elliptic regularity,  $\mathcal{M}_l = \mathcal{M}_{l+1} = \mathcal{M}_{l+2} = \dots = \mathcal{M}_{r+1}$ ; this is precisely why the choice of  $l$  does not matter. So the moduli space is simply referred to by  $\mathcal{M}$ .
- By Sard–Smale theory, we may always assume that  $H^2 = 0$  after choosing a suitable *generic* perturbation  $\varpi_\tau$ . Then  $\mathcal{M}$  is unobstructed; it is a smooth manifold of dimension equal to the virtual dimension

$$d = \langle e(W^+, \Psi), [\bar{M}, Y] \rangle,$$

which is nothing else but the index of  $\mathcal{D}_{(A,\Phi)}$ .

From now on,  $\varpi$  will be a generic perturbation of Seiberg–Witten equations on  $M$ ; then we define  $\varpi_\tau$  by (2.7) on  $M_\tau$ . We will show in Corollary 3.1.5 that for  $\tau$  large

enough, then the moduli space on  $M_\tau$  becomes unobstructed. We may now state the general version of our excision theorem.

**Theorem E** *Let  $\bar{M}$  be a manifold with a contact boundary  $(Y, \xi)$  and an element  $(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi)$ . Let  $Z$  and  $Z'$  be two AFAK ends compatible with the contact structure of  $Y$ . Let  $M_\tau$  and  $M'_\tau$  be the AFAK manifolds obtained as the connected sum of  $\bar{M}$  with  $Z$  or  $Z'$  along  $Y$ , together with the Riemannian metrics  $g_\tau$  and  $g'_\tau$  and the particular Seiberg–Witten equations with perturbation  $\varpi_\tau$  constructed in [Section 2.2.1](#).*

*Then, for  $\tau$  large enough, the moduli spaces  $\mathcal{M}(M_\tau)$  and  $\mathcal{M}(M'_\tau)$  are generic, and there is a diffeomorphism*

$$\mathfrak{G}: \mathcal{M}_{\varpi_\tau}(M_\tau) \rightarrow \mathcal{M}_{\varpi_\tau}(M'_\tau).$$

*Furthermore there is a canonical identification of the set of consistent orientations for  $M_\tau$  and  $M'_\tau$ . Using this canonical identification the above diffeomorphism becomes orientation preserving.*

**Remark 2.2.3** *If we remove the assumption (2.2) for the cobordisms  $Z$  and  $Z'$ , then the extension maps  $j: \text{Spin}^c(\bar{M}, \xi) \rightarrow \text{Spin}^c(M_\tau, \omega_\tau)$  and  $j': \text{Spin}^c(\bar{M}, \xi) \rightarrow \text{Spin}^c(M'_\tau, \omega'_\tau)$  are not injective in general.*

Then we may still prove a generalization of [Theorem E](#): assume that  $Z$  is just the symplectization of  $(Y, \xi)$  and that  $Z'$  is an AFAK end as before, but without assuming the property (2.2). To discuss the generalization we need to make the notation more precise. We denote by  $\mathcal{M}_{\varpi_\tau}(M_\tau, j(\mathfrak{s}, h))$  the moduli space for some choice of  $(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi)$ . Similarly we have the moduli space  $\mathcal{M}_{\varpi_\tau}(M'_\tau, j'(\mathfrak{s}, h))$ . Then, we may form the moduli space  $\tilde{\mathcal{M}}_{\varpi_\tau}(M_\tau, (\mathfrak{s}', h'))$  for some choice of  $(\mathfrak{s}', h') \in \text{Spin}^c(M'_\tau, \omega'_\tau)$  defined by

$$(2.9) \quad \tilde{\mathcal{M}}_{\varpi_\tau}(M_\tau, (\mathfrak{s}', h')) := \bigsqcup_{\substack{(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi) \\ j'(\mathfrak{s}, h) = (\mathfrak{s}', h')}} \mathcal{M}_{\varpi_\tau}(M_\tau, j(\mathfrak{s}, h)).$$

Then the conclusion of [Theorem E](#) is the same if we replace  $\mathcal{M}_{\varpi_\tau}(M'_\tau)$  by  $\mathcal{M}_{\varpi_\tau}(M'_\tau, (\mathfrak{s}', h'))$  and  $\mathcal{M}_{\varpi_\tau}(M_\tau)$  by  $\tilde{\mathcal{M}}_{\varpi_\tau}(M_\tau, (\mathfrak{s}', h'))$ . In particular, there is an orientation preserving diffeomorphism

$$\mathfrak{G}: \tilde{\mathcal{M}}_{\varpi_\tau}(M_\tau, (\mathfrak{s}', h')) \rightarrow \mathcal{M}_{\varpi_\tau}(M'_\tau, (\mathfrak{s}', h')).$$

The rest of [Section 2](#) is devoted to constructing the map  $\mathfrak{G}$  by a gluing technique and to showing that it is a diffeomorphism.

**2.2.4 Compactness** In this section, we refine the result of compactness for one fixed moduli space  $\mathcal{M}(M_\tau)$ , by showing that a sequence of solutions  $(A_\tau, \Phi_\tau)$  of Seiberg–Witten equations on  $M_\tau$  converge in some sense as  $\tau \rightarrow \infty$ , up to extraction of a subsequence, and modulo gauge transformations, to a solution of Seiberg–Witten equations on  $M$ .

We review the arguments proving the compactness of one particular moduli space in [10] and explain how to apply them to the family  $M_\tau$ .

**Lemma 2.2.5** *There exist constants  $\kappa_1, \kappa_2$  such that for every  $\tau$  and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$  we have the estimate*

$$\|\Phi\|_{C^0} \leq \kappa_1 + \kappa_2 \|\varpi_\tau\|_{C^0}.$$

**Proof** By construction of the moduli space,  $\Phi - \Psi \in L^2_3$ ; now the Sobolev inclusion  $C^0 \subset L^2_3$  together with Lemma 2.1.6 tells us that the pointwise norm  $|\Phi - \Psi|_{C^0} \rightarrow 0$  near infinity on  $M_\tau$ .

Hence, either  $|\Phi| \leq 1$ , and we are done, either this is not true, and  $|\Phi|$  must have a local maximum at a point  $x \in M_\tau$ . We apply the maximum principle at  $x$ :

$$\begin{aligned} 0 &\leq \frac{1}{2} \Delta |\Phi|^2 = \langle \nabla_A^* \nabla_A \Phi, \Phi \rangle - |\nabla_A \Phi|^2 \\ &\leq \langle \nabla_A^* \nabla_A \Phi, \Phi \rangle = \langle D_A^2 \Phi, \Phi \rangle - \langle F_A^+ \cdot \Phi, \Phi \rangle - \frac{s}{4} |\Phi|^2 \end{aligned}$$

where the last identity follows from the Lichnerowicz formula. Using Seiberg–Witten equations, we have

$$0 \leq -\frac{1}{2} |\Phi|^4 - \left\langle (F_B^+ - \{\Psi \otimes \Psi^*\}_0) \cdot \Phi, \Phi \right\rangle - \langle \varpi_\tau \cdot \Phi, \Phi \rangle - \frac{s}{4} |\Phi|^2.$$

The lemma follows from the fact that the pointwise norm of  $s$ ,  $F_B$ ,  $\Psi$  is bounded independently of  $\tau$  by Lemma 2.1.6.  $\square$

As we saw, the AFAK structure induces a Chern connection  $\widehat{\nabla}$  and a spin connection  $B$  on the bundle  $W_\tau$  restricted to  $M_\tau \setminus \overline{M}$ . In the rest of this paper, we will constantly use the following notation: for every spin connection  $A = B + a$  on  $L_\tau$ , we define the *twisted Chern connection* on  $W_\tau|_{M_\tau \setminus \overline{M}}$  by:

$$\widehat{\nabla}_A \Phi := \widehat{\nabla} \Phi + a \otimes \Phi$$

In particular  $\widehat{\nabla}_B := \widehat{\nabla}$  with this notation. Notice that  $\nabla_B \neq \widehat{\nabla}_B$  unless the almost complex structure is integrable.

**Proposition 2.2.6** *There exists a compact  $K \subset M$  large enough and  $\delta > 0$  such that for every integer  $k$  there is a constant  $c_k > 0$  so that, for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have the pointwise estimate on  $M_\tau \setminus K$*

$$(2.10) \quad |1 - |\beta|^2 - |\gamma|^2|, \quad |\gamma|, \quad |\widehat{\nabla}_A \Phi|, \quad |\widehat{\nabla}_A^2 \Phi|, \dots, \quad |\widehat{\nabla}_A^k \Phi| \leq c_k e^{-\delta \sigma_\tau},$$

where  $\Phi = (\beta, \gamma) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$ .

**Remark** The quantities (2.10) controlled by the lemma are gauge invariant.

**Proof** The lemma was proved in [10, Proposition 3.15] for  $\tau$  fixed. It is readily checked that it extends as stated for the family  $M_\tau$ . We recall what the ingredients of the proof are.

A configuration  $(A, \Phi)$  on  $M_\tau$  has an *energy* which is a gauge invariant quantity defined by

$$E_\tau(A, \Phi) = \int_{M_\tau \setminus K_0} \left( (|\beta|^2 + |\gamma|^2 - 1)^2 + |\gamma|^2 + |\widehat{\nabla}_A \Phi|^2 \right) \text{vol}^{g_\tau},$$

where all the norms, connections are taken with respect to the structures defined on  $M_\tau$  and  $K_0$  is a compact in  $M$  containing  $\bar{M}$ .

**Lemma 2.2.7** *There exist a compact  $K_0 \subset M$  large enough, and some constants  $\kappa_3$  and  $\kappa_4$ , such that for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have*

$$E_\tau(A, \Phi) \leq \kappa_3 + \kappa_4 \|\varpi_\tau\|_{\mathcal{N}_\tau}^2.$$

**Proof** The proof of this lemma is the same proof than for [10, Lemma 3.17]. The fact that  $\kappa_3$  and  $\kappa_4$  do not depend on  $\tau$  is insured by Lemma 2.1.6.  $\square$

If we look carefully at the proof, using the notation of [10, page 232], we read the claim that

$$\int_{K_3} da \wedge \omega$$

can be controlled, for  $K_3$  a compact domain large enough,  $\omega$  a closed form extending the symplectic form on the whole manifold as in Definition 2.1.1, and  $a = A - B$  decaying exponentially fast. No explanation of this is given and we provide one now.

Pick an arbitrarily small  $\varepsilon > 0$ . We have

$$\left| \int_{K_3} da \wedge \omega \right| \leq \frac{\varepsilon}{2} \int_{K_3} |da|^2 + \frac{1}{2\varepsilon} \int_{K_3} |\omega|^2 \leq \frac{\varepsilon}{2} \int_M |da|^2 + \frac{1}{2\varepsilon} \int_{K_3} |\omega|^2.$$

Using the exponential decay of  $a$ , we have as in the case of a compact manifold

$$\int_M |da|^2 = 2 \int_M |d^+a|^2.$$

Therefore

$$\left| \int_{K_3} da \wedge \omega \right| \leq \varepsilon \int_{M \setminus K_3} |d^+a|^2 + \varepsilon \int_{K_3} |d^+a|^2 + \frac{1}{2\varepsilon} \int_{K_3} |\omega|^2.$$

Using the  $C^0$  bound on  $d^+a$ , we deduce a constant  $C > 0$ , such that

$$\left| \int_{K_3} da \wedge \omega \right| \leq \varepsilon \int_{M \setminus K_3} |d^+a|^2 + C,$$

and the proof of [10, Lemma 3.17] is now complete.

**Lemma 2.2.8** *For every  $\varepsilon > 0$  there exists a compact set  $K_1 \subset M$  large enough, such that for every  $\tau$  large enough, every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$  verifies*

$$|\beta| \geq 1 - \varepsilon, \text{ on } M_\tau \setminus K_1, \text{ where } \Phi = (\beta, \gamma) \in \Lambda^{0,0} \oplus \Lambda^{0,2}.$$

**Proof** Assume the lemma is not true. Then, there is a sequence of  $\tau_j \rightarrow \infty$ , a sequence of Seiberg–Witten solutions  $(A_j, \Phi_j)$  on  $M_{\tau_j}$  such that we have  $|\beta_j(x_j)| < 1 - \varepsilon$  (using the decomposition  $\Phi_j = (\beta_j, \gamma_j)$ ) at some points  $x_j \in M_{\tau_j}$  verifying  $\sigma_{\tau_j}(x_j) \rightarrow \infty$ .

We restrict  $(A_j, \Phi_j)$  to the ball of center  $x_j$  and radius  $\sigma_{\tau_j}(x_j)/\kappa$ . Thus we have a sequence of Seiberg–Witten solutions on a sequence of balls of increasing radius. By Lemma 2.1.6, the Riemannian metric and symplectic form on the balls converge to the standard structures on  $\mathbb{R}^4$ .

After taking suitable gauge transformations, we can extract a subsequence converging smoothly on every compact set to a solution  $(A, \Phi)$  of Seiberg–Witten equations on  $\mathbb{R}^4$ , with its standard metric, symplectic form, and perturbation  $\varpi = 0$ . In particular  $\Phi = (\beta, \gamma)$ , with  $|\beta| \leq 1 - \varepsilon$  at the origin since  $|\beta_j(x_j)| < 1 - \varepsilon$  by assumption. Using Lemma 2.2.7, we deduce that  $(A, \Phi)$  has a bounded energy. By Lemma [10, 3.20],  $(A, \Phi)$  must be gauge equivalent to the standard solution  $(B, \Psi)$  of Seiberg–Witten equations on the Euclidean space; this is a contradiction since  $\Psi = (1, 0) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$ , hence we should have  $|\beta| = 1$ .  $\square$

Proposition 2.2.6 now follows from Lemma 2.1.6, Lemma 2.2.8 and the local behavior of Seiberg–Witten solutions [10, Proposition 3.22].  $\square$

**2.2.9 Gauge with uniform exponential decay** The next corollary is essential to study the compactness property of  $\mathcal{M}_{\varpi_\tau}(M_\tau)$ .

**Corollary 2.2.10** *There exist constants  $c_k, \delta > 0$  and a compact set  $K \subset M$  such that for every  $\tau$  large enough, every solution of Seiberg–Witten equations on  $M_\tau$  admits a particular gauge representative  $(A, \Phi)$  such that we have the pointwise estimates*

$$(2.11) \quad |A - B|, \quad |\nabla(A - B)|, \dots, |\nabla^k(A - B)|, \\ |\Phi - \Psi|, \quad |\widehat{\nabla}_A \Phi|, \quad |\widehat{\nabla}_A^2 \Phi|, \dots, |\widehat{\nabla}_A^{k-1} \Phi| \quad \text{and} \quad |\widehat{\nabla}_A^k \Phi| \leq c_k e^{-\delta \sigma_\tau}$$

on  $M_\tau \setminus K$ .

**Remark** We could state a similar Corollary by replacing  $\widehat{\nabla}_A^j \Phi$  in (2.11) with  $\nabla_A^j(\Phi - \Psi)$ . Indeed, the spin connection  $\nabla_B$  tends uniformly to the Chern connection  $\widehat{\nabla}$  on unit balls with center going to infinity thanks to [Lemma 2.1.6](#).

**Proof** Let  $(A, \Phi) \in \mathcal{Z}_I^\tau$  be a solution of Seiberg–Witten equations on  $M_\tau$ . We follow closely the proof of [\[10, Corollary 3.16\]](#): we begin to increase the regularity of  $(A, \Phi) \in \mathcal{C}_I$ . For that purpose, we can apply a gauge transformation  $u_1 \in \mathcal{G}_I$  after which  $(A, \Phi)$  satisfies the Coulomb condition

$$\delta_{1, (A, \Phi)}^*(A - B, \Phi - \Psi) = 0$$

near infinity. The linear theory for Seiberg–Witten equations shows that in this gauge we have an additional regularity  $(A - B, \Phi - \Psi) \in L_{I+1}^2$ . From this point, we can define the map  $u: M_\tau \setminus K \rightarrow S^1$  by

$$u = \frac{|\beta|}{\beta}.$$

Kato inequality and Sobolev multiplication theorem show that  $1 - u \in L_{I+1}^2$ . In particular, it is homotopic to 1 outside a very large compact set. By the assumption (2.2), we deduce that  $u$  is also homotopic to 1 on  $M_\tau \cap \{1 \leq \sigma_\tau \leq \tau\}$ . In particular,  $u$  can be extended to  $\overline{M}$  thus defining a gauge transformation in  $\mathcal{G}_I(M_\tau)$ .

The gauge transformed solution  $(a - u^{-1} du, u\beta, u\gamma)$  has exponential decay thanks to [Proposition 2.2.6](#), using the fact that  $u\beta = |\beta|$  and the identity

$$a - u^{-1} du = (u\widehat{\nabla}_A \beta - d|\beta|)/|\beta|. \quad \square$$

**Theorem 2.2.11** *Let  $(A_\tau, \Phi_\tau)_{\tau \in I}$  be a sequence of solutions of Seiberg–Witten equations on  $M_\tau$  where  $I$  is an unbounded subset of  $\mathbb{R}^+$ . Then, we can extract*

a subsequence  $\tau_j \rightarrow \infty$  and apply gauge transformations  $u_j$ , such that the gauge transformed solutions  $u_j \cdot (A_{\tau_j}, \Phi_{\tau_j})$  have uniform exponential decay, in the sense that they verify (2.11), and converge smoothly on every compact set of  $M$  toward a solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M$ .

**Proof** As suggested in the Theorem, we apply first Corollary 2.2.10, i.e. we apply a gauge transformations  $u_\tau$  to  $(A_\tau, \Phi_\tau)$  such that Corollary 2.2.10 is verified by the gauge transformed solutions  $(A'_\tau, \Phi'_\tau) = u_\tau \cdot (A_\tau, \Phi_\tau)$ . Now,  $(A'_\tau, \Phi'_\tau)$  have uniform exponential decay outside a certain compact set  $K \subset M$ ; in particular,  $\|(A, \Phi) - (B, \Psi)\|_{L^2_l(g_\tau, B)}$  is uniformly bounded; a diagonal argument shows that we can extract a subsequence  $\tau_j \rightarrow \infty$  such that  $(A'_{\tau_j}, \Phi'_{\tau_j})$  converge strongly on every compact set of  $M \setminus K$  in the  $L^2_{l-1}$  sense to a weak limit  $(A', \Phi') \in L^2_l(g, B)$  solution of Seiberg–Witten equation on  $M \setminus K$ . Elliptic regularity shows that, in fact,  $(A'_{\tau_j}, \Phi'_{\tau_j})$  converge smoothly to  $(A', \Phi')$  on every compact of  $M \setminus K$ .

Consider the compact manifold with boundary  $K_2 = M \cap \{\sigma \leq T_2\}$  and choose  $T_2$  large enough, so that  $K$  is properly contained in  $K_2$ . There exists a sequence of gauge transformations  $v_j \in L^2_{l+1}(K_2)$  such that after passing to a subsequence, the transformed solutions  $v_j \cdot (A_{\tau_j}, \Phi_{\tau_j})$  converge smoothly in  $K_2$  (see [9] for a proof of this statement).

Now, the ratio  $w_j = u_j v_j^{-1}$  must converge to a gauge transformation  $w$  on  $K_2 \setminus K$ . Choose  $j_0$  sufficiently large so that  $|w_j - w_{j_0}| \leq 1/2$  for all  $j \geq j_0$ ; then  $j$  large, we may write  $w_j = \exp(2i\pi\theta_j)w_{j_0}$  where  $\theta_j$  is a real valued function with  $|\theta_j| < 1$ . The gauge transformation  $u_{j_0}$  on  $M_{\tau_{j_0}}$  is homotopic to 1 near infinity. Therefore, by assumption (2.2), it must be homotopic to 1 on  $K_2 \setminus K$  and we write  $u_{j_0} = \exp(2i\pi\mu)$  for some real function  $\mu$  on  $K_2 \setminus K$ . Then, we define gauge transformations on  $M_{\tau_j}$  by

$$f_j = \begin{cases} v_j v_{j_0}^{-1} \exp(2i\pi(1 - \chi_{T_2})(\theta_j + \mu)) & \text{on } K_2 \setminus K \\ u_j & \text{on } M_\tau \setminus K_2. \end{cases}$$

now the  $f_j \cdot (A_{\tau_j}, \Phi_{\tau_j})$  converge smoothly on every compact set and have the uniform exponential decay property as required in the theorem.  $\square$

**Remark 2.2.12** Notice that the assumption (2.2) was used for the first time in the proof of this compactness theorem. Without this assumption, the gauge transformation  $u_{j_0}$  could be non homotopic to 1. Then, a sequence of solutions of Seiberg–Witten equations on  $M_\tau$  for some choice of  $\text{Spin}^c$ –structure  $(\tilde{\mathfrak{s}}, \tilde{h}) \in \text{Spin}^c(M_\tau, \omega_\tau)$  still converge up to gauge transformation on every compact to a solution of Seiberg–Witten equations on  $M$ . However, the relevant  $\text{Spin}^c$ –structure on  $M$  is not uniquely defined

and the limit is now an element of the enlarged moduli space

$$\tilde{\mathcal{M}}_{\varpi}(M, (\tilde{\mathfrak{s}}, \tilde{h})) = \bigsqcup_{\substack{(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi) \\ j(\mathfrak{s}, h) = (\tilde{\mathfrak{s}}, \tilde{h})}} \mathcal{M}_{\varpi}(M, j(\mathfrak{s}, h)).$$

**Corollary 2.2.13** *For every integer  $k$ , there exists a constant  $c_k > 0$ , such that for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have the pointwise estimates*

$$|\nabla_A \Phi|, |\nabla_A^2 \Phi|, \dots, |\nabla_A^k \Phi| \leq c_k.$$

Moreover, we may assume after applying a suitable gauge transformation to any other solution that we have

$$|A - B|, |\nabla(A - B)|, \dots, |\nabla^k(A - B)| \leq c_k.$$

**Proof** Suppose first assertion is false. Then, we have a sequence  $\tau_j \rightarrow \infty$ , solutions  $(A_j, \Phi_j)$  of Seiberg–Witten equations and a sequence of points  $x_j \in M_{\tau_j}$  with  $|\nabla_{A_j}^r \Phi|(x_j) \rightarrow \infty$ . After applying gauge transformations and extracting a subsequence, we can assume that  $(A_j, \Phi_j)$  converge smoothly on every compact set. Therefore, the point  $x_j$  must go to infinity on the end of  $M_{\tau_j}$  in the sense that  $\sigma_{\tau_j}(x_j) \rightarrow \infty$ . But this is a contradiction according to [Corollary 2.2.10](#) and the first part of the corollary is proved.

Suppose the second part is false for  $k = 0$ . Then, we have a sequence  $\tau_j \rightarrow \infty$ , and solutions  $(A_j, \Phi_j)$  of Seiberg–Witten on  $M_{\tau_j}$  such that, for every sequence of gauge transformations  $u_j$ , we have  $\|u_j \cdot A_j - B\|_{C^k(g_{\tau_j})} \rightarrow \infty$ . This is in contradiction with [Theorem 2.2.11](#) and [Corollary 2.2.10](#).  $\square$

As an immediate, although important, consequence of [Lemma 2.2.5](#), [Corollary 2.2.13](#) and the Sobolev embedding  $L_1^2 \subset L^4$ , we have the following corollary.

**Corollary 2.2.14** *For every  $k$ , there exists a constant  $a_k \in (0, 1)$ , such that for every  $\tau$  large enough, every solution of Seiberg–Witten equations on  $M_\tau$  admits a gauge representative  $(A, \Phi)$  such that*

$$a_k \|\cdot\|_{L_k^2(g_\tau, A)} \leq \|\cdot\|_{L_k^2(g_\tau, B)} \leq \frac{1}{a_k} \|\cdot\|_{L_k^2(g_\tau, A)}.$$

Another important consequence of uniform exponential decay is that the constant involved estimates for local elliptic regularity can be chosen uniformly, as in the next proposition.



**Proposition 2.2.15** *There exist and a compact  $K \subset M$ , and for every  $k \geq 0$  a constant  $c_k > 0$ , such that for every  $\tau$  large enough and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , we have*

$$\|\mathcal{D}_{(A,\Phi)}(a, \phi)\|_{L_k^2(g_\tau, A)} + \|(a, \phi)\|_{L^2(g_\tau, K)} \geq c_k \|(a, \phi)\|_{L_{k+1}^2(g_\tau, A)},$$

for all  $(a, \phi) \in \Lambda^+(M_\tau) \times W^+(M_\tau)$ , where  $L^2(g_\tau, K)$  is the  $L^2$ -norm restricted to  $K$ .

**Proof** We use the Weitzenbock formula derived in [10, Proposition 3.8]: for every section  $(a, \phi)$  on  $M_\tau$ , we have outside a compact set  $K \subset M$ ,

$$\begin{aligned} \|\mathcal{D}_{(A,\Phi)}(a, \phi)\|_{L^2(g_\tau, K')}^2 &= \int_{M_\tau \setminus K'} \left( |\nabla a|^2 + |\nabla_A \phi|^2 + |\Phi|^2(|\phi|^2 + |a|^2) \right. \\ &\quad \left. + \text{Ric}(a, a) + \frac{s}{4}|\phi|^2 + \langle F_A^+ \cdot \phi, \phi \rangle - 2\langle a \otimes \phi, \nabla_A \Phi \rangle \right) \text{vol}^{g_\tau}. \end{aligned}$$

Using the uniform exponential decay and the fact that the metrics  $g_\tau$  are uniformly asymptotically flat, we deduce that, provided  $K$  is large enough, we have control

$$\|\mathcal{D}_{(A,\Phi)}(a, \phi)\|_{L^2(g_\tau, M_\tau \setminus K)} \geq c_1 \|(a, \phi)\|_{L^2(g_\tau, M_\tau \setminus K)}$$

where  $L^2(g_\tau, M_\tau \setminus K)$  means the  $L^2$ -norm restricted to the set  $M_\tau \setminus K$ , and where  $c_1 > 0$  is a constant independent of  $\tau$  and  $(A, \Phi)$ . Clearly

$$(2.12) \quad \|\mathcal{D}_{(A,\Phi)}(a, \phi)\|_{L^2(g_\tau)} + \|(a, \phi)\|_{L^2(g_\tau, K)} \geq c_2 \|(a, \phi)\|_{L^2(g_\tau)},$$

for  $c_2 = \min(1, c_1)$ .

Now, consider two balls  $B_1 \subset B_2$  in  $M_\tau$  with small radii  $\alpha/2$  and  $\alpha$ , centered at a point  $x$ . Then, by local elliptic regularity, there exists a constant  $c_3 > 0$  such that

$$\|\mathcal{D}_{(A,\Phi)}(a, \phi)\|_{L_k^2(g_\tau, A, B_2)} + \|(a, \phi)\|_{L^2(g_\tau, B_2)} \geq c_3 \|(a, \phi)\|_{L_{k+1}^2(g_\tau, A, B_1)},$$

where  $L_k^2(g_\tau, A, B_j)$  is the  $L_k^2(g_\tau, A)$ -norm restricted to the ball  $B_j$ .

Now, using the fact that the geometry of the  $M_\tau$  is uniformly bounded in the sense of Lemma 2.1.6, Corollary 2.2.13 and Corollary 2.2.14, we conclude that  $c_3$  can be chosen in such a way that it does not depend neither on  $\tau$ ,  $(A, \Phi)$  nor on the center of the balls  $x$ .

Therefore, we can find a constant  $c_4 > 0$ , independent of  $\tau$  and  $(A, \Phi)$  such that globally

$$\|\mathcal{D}_{(A,\Phi)}(a, \phi)\|_{L_k^2(g_\tau, A)} + \|(a, \phi)\|_{L^2(g_\tau)} \geq c_4 \|(a, \phi)\|_{L_{k+1}^2(g_\tau, A)};$$

the latter inequality together with the control (2.12) proves the proposition.  $\square$

### 2.3 Refined slice theorem

An important step in developing Seiberg–Witten theory, and in particular, in order to give a differentiable structure to the moduli space, consists in proving a *slice theorem*.

Let  $[A, \Phi]$  be the orbit under the gauge group action of a configuration  $(A, \Phi) \in \mathcal{C}(M_\tau)$ . Then we consider the equivariant smooth map

$$(2.13) \quad \begin{aligned} \nu: U \times \mathcal{G} &\longrightarrow \mathcal{C} \\ ((a, \phi), u) &\longmapsto u \cdot (A + a, \Phi + \phi), \end{aligned}$$

where  $U$  is a neighborhood of 0 in  $\ker \delta_{1, (A, \Phi)}^*$ . The usual slice theorem says that, if  $U$  is an open ball small enough centered at 0, then  $\nu$  is a diffeomorphism onto a  $\mathcal{G}$ -invariant open neighborhood of the orbit  $[A, \Phi]$ . Therefore  $U \times \mathcal{G}$  defines an equivariant local chart on  $\mathcal{C}$ . Such coordinates  $\nu$  endow  $\mathcal{C}/\mathcal{G}$  with a structure of smooth Hilbert manifold.

In this section, we show that  $U$  can be taken to be a ball of fixed size (independent of both  $\tau$  and the solution to the Seiberg–Witten equations). Furthermore  $\nu(U \times \mathcal{G})$  contains a ball about the solution whose size is also independent of both  $\tau$  and the solution. This refinement of the slice theorem will actually be crucial in the proof that the gluing map is an embedding.

Before going further, we need to define a metric on the orbit space  $\mathcal{C}/\mathcal{G}$ : let  $(A, \Phi), (\tilde{A}, \tilde{\Phi}) \in \mathcal{C}_l$ . Put

$$d_{l, [A, \Phi]}([A, \Phi], [\tilde{A}, \tilde{\Phi}]) = \inf_{u \in \mathcal{G}_l} \|(A, \Phi) - u \cdot (\tilde{A}, \tilde{\Phi})\|_{L_l^2(g_\tau, A)}.$$

Then we have the following theorem:

**Theorem 2.3.1** *For every  $\alpha_1 > 0$  small enough, there exists  $\alpha_2 \in (0, \alpha_1]$  such that for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , the equivariant map  $\nu: U_{\alpha_1} \times \mathcal{G}_2 \rightarrow \mathcal{C}_2$  defined by (2.13), where*

$$U_{\alpha_1} = \{(a, \phi) \in \ker \delta_{1, (A, \Phi)}^* \subset L_2^2 \mid \|(a, \phi)\|_{L_2^2(g_\tau, A)} < \alpha_1\}$$

*is a diffeomorphism onto a  $\mathcal{G}$ -invariant open neighborhood of  $(A, \Phi)$ . Furthermore  $\nu(U \times \mathcal{G}_2)$  contains a ball of radius  $\alpha_2$ ; more precisely, we have  $B_{\alpha_2} \subset \nu(U_{\alpha_1} \times \mathcal{G}_2)$  where*

$$B_{\alpha_2} = \{[\tilde{A}, \tilde{\Phi}] \mid (\tilde{A}, \tilde{\Phi}) \in \mathcal{C}_2 \text{ with } d_{2, [A, \Phi]}([A, \Phi], [\tilde{A}, \tilde{\Phi}]) < \alpha_2\}.$$

**Proof** Let  $\alpha_3 > 0$  and

$$G_{\alpha_3} = \{v \in L_3^2(i\mathbb{R}) \mid \|v\|_{L_3^2(g_\tau)} < \alpha_3\}.$$

We consider the map  $\bar{v}$  rather than  $v$  defined by

$$(2.14) \quad \begin{aligned} \bar{v}: U_{\alpha_3} \times G_{\alpha_3} &\longrightarrow \mathcal{C}_2 \\ ((a, \phi), v) &\longmapsto e^v \cdot (A + a, \Phi + \phi). \end{aligned}$$

The map  $\bar{v}$  has the decomposition

$$\bar{v}((a, \phi), v) = (A, \Phi) + (a, \phi) + \delta_{1,(A,\Phi)}(v) + Q_{\bar{v}}(v, \phi)$$

where

$$Q_{\bar{v}}(\phi, v) = \left(0, (e^v - v - 1)\Phi + (e^v - 1)\phi\right)$$

is the nonlinear part of  $\bar{v}$ . Clearly

$$(2.15) \quad d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v}) = (\dot{a}, \dot{\phi}) + \delta_{1,(A,\Phi)}(\dot{v}).$$

We study first the linearized problem for finding an inverse to  $\bar{v}$ . We define the Laplacian

$$\Delta_{1,(A,\Phi)} v = \delta_{1,(A,\Phi)}^* \delta_{1,(A,\Phi)}(v) = d^* dv + |\Phi|^2 v.$$

Then, the spectrum of  $\Delta_{1,(A,\Phi)}$  is bounded from below according to the next lemma.

**Lemma 2.3.2** *For every integer  $k \geq 0$ , there exists a constant  $b_k > 0$  such that for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , the operator  $\Delta_{1,(A,\Phi)}$  satisfies*

$$(2.16) \quad b_k \|v\|_{L_{k+2}^2(g_\tau)} \leq \|\Delta_{1,(A,\Phi)}(v)\|_{L_k^2(g_\tau)} \leq \frac{1}{b_k} \|v\|_{L_{k+2}^2(g_\tau)}.$$

We finish the proof of [Theorem 2.3.1](#) before proving [Lemma 2.3.2](#). In particular, we have

$$(2.17) \quad b_1 \|v\|_{L_3^2(g_\tau)} \leq \|\Delta_{1,(A,\Phi)}(v)\|_{L_1^2(g_\tau)}$$

so  $\Delta_{1,(A,\Phi)}: L_3^2 \rightarrow L_1^2$  is Fredholm and injective. On the other hand  $\Delta_{1,(A,\Phi)}$  is self-adjoint, hence its index is 0, therefore it must be an isomorphism. It is now readily seen that the differential  $d_0 \bar{v}$  (see (2.15)) has an inverse of the form

$$d_0 \bar{v}^{-1}(\tilde{a}, \tilde{\phi}) = \left( (\tilde{a}, \tilde{\phi}) - \delta_{1,(A,\Phi)} \Delta_{1,(A,\Phi)}^{-1} \delta_{1,(A,\Phi)}^* (\tilde{a}, \tilde{\phi}), \Delta_{1,(A,\Phi)}^{-1} \delta_{1,(A,\Phi)}^* (\tilde{a}, \tilde{\phi}) \right).$$

We define a map

$$F: U_{\alpha_3} \times G_{\alpha_3} \rightarrow \{(\tilde{a}, \tilde{\phi}) \in L_2^2, \delta_{1,(A,\Phi)}^*(\tilde{a}, \tilde{\phi}) = 0\} \times L_3^2(i\mathbb{R})$$

by

$$(2.18) \quad F((a, \phi), v) = d_0 \bar{v}^{-1} \left( \bar{v}((a, \phi), v) - (A, \Phi) \right).$$

Then, a straightforward computation shows that

$$F((a, \phi), v) = ((a, \phi), v) + Q(\phi, v)$$

where

$$Q(\phi, v) = \left( -\delta_{1, (A, \Phi)} \Delta_{1, (A, \Phi)}^{-1} R(\phi, v), \Delta_{1, (A, \Phi)}^{-1} R(\phi, v) \right)$$

with

$$R(\phi, v) = \delta_{1, (A, \Phi)}^* Q_{\bar{v}}(\phi, v) = i \Im \left( \langle \Phi, (e^v - 1)\phi \rangle + |\Phi|^2 (e^v - v - 1) \right).$$

We are going to show that  $F$  is a local diffeomorphism about 0 using the fixed point theorem. In order to apply it, we need to show that  $Q$  is locally contracting in the sense of [Lemma 2.3.4](#). Before hand, it is required to start with the following technical lemma.

**Lemma 2.3.3** *For every  $\kappa > 0$  there exists  $\alpha > 0$  such that for every  $\tau$  large enough and every pair of complex valued functions  $w$  and  $\tilde{w}$  on  $M_\tau$  and every open set  $D \subset M_\tau$ , we have*

$$(2.19) \quad \|w\|_{L_3^2(g_\tau, D)}, \|\tilde{w}\|_{L_3^2(g_\tau, D)} \leq \alpha \Rightarrow \\ \|e^w - e^{\tilde{w}} - (w - \tilde{w})\|_{L_k^2(g_\tau, D)} \leq \kappa \|w - \tilde{w}\|_{L_k^2(g_\tau, D)}, \text{ for } k = 0, 1, 2, 3,$$

where  $L_k^2(g_\tau, D)$  is the usual  $L_k^2(g_\tau)$ -norm taken on the open set  $D$ .

**Proof** For  $\tau$  fixed, the property (2.19) is readily deduced from the Sobolev multiplication theorems, the inclusion  $L_3^2 \hookrightarrow C^0$  and the fact that the function  $f(w) = e^w - w - 1$  is analytic, with a zero of multiplicity 2 at  $w = 0$ .

The geometry of  $M_\tau$  is uniformly controlled at infinity by [Lemma 2.1.6](#), hence the Sobolev constants involved in the above inclusion or multiplication theorems can be chosen independently of  $\tau$ . This shows that  $\alpha$  can be chosen independently of  $\tau$ , and the lemma is proved.  $\square$

Before going further, we need to introduce a suitable complete Euclidean norm on  $L_2^2(i\Lambda^1 \oplus W^+) \oplus L_3^2(i\mathbb{R})$  defined by

$$\|((a, \phi), v)\|_{L_{2,3}^2(g_\tau, A)} := \max \left( \|a, \phi\|_{L_2^2(g_\tau, A)}, \|v\|_{L_3^2(g_\tau)} \right).$$

We will use the shorthand

$$\|(\phi, v)\|_{L^2_{2,3}(g_\tau, A)} := \|((0, \phi), v)\|_{L^2_{2,3}(g_\tau, A)}.$$

**Lemma 2.3.4** *For all  $\kappa > 0$  there exists  $\alpha > 0$ , such that for every  $\tau$  large enough, every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have for every pair  $(\phi, v)$  and  $(\tilde{\phi}, \tilde{v})$*

$$(2.20) \quad \|(\phi, v)\|_{L^2_{2,3}(g_\tau, A)}, \|(\tilde{\phi}, \tilde{v})\|_{L^2_{2,3}(g_\tau, A)} \leq \alpha \Rightarrow \\ \|Q(\tilde{\phi}, \tilde{v}) - Q(\phi, v)\|_{L^2_{2,3}(g_\tau, A)} \leq \kappa \|(\tilde{\phi}, \tilde{v}) - (\phi, v)\|_{L^2_{2,3}(g_\tau, A)},$$

**Proof** Let  $\kappa > 0$ , and two pairs  $(\phi, v)$  and  $(\tilde{\phi}, \tilde{v})$ . Then

$$R(\tilde{\phi}, \tilde{v}) - R(\phi, v) = i \Im \left( \Phi, \left( (e^{\tilde{v}} - e^v - (\tilde{v} - v)) \Phi + (e^{\tilde{v}} - e^v) \tilde{\phi} + (e^v - 1)(\tilde{\phi} - \phi) \right) \right).$$

Using [Corollary 2.2.13](#), we deduce that for some constant  $c_1 > 0$  independent of  $\tau$  and  $(A, \Phi)$  we have

$$c_1 \|R(\tilde{\phi}, \tilde{v}) - R(\phi, v)\|_{L^2_1(g_\tau)} \leq \|e^{\tilde{v}} - e^v - (\tilde{v} - v)\|_{L^2_1(g_\tau)} \\ + \|(e^{\tilde{v}} - e^v) \tilde{\phi}\|_{L^2_1(g_\tau, A)} + \|(e^{\tilde{v}} - 1)(\tilde{\phi} - \phi)\|_{L^2_1(g_\tau, A)}.$$

The Sobolev multiplication theorems show that for some constant  $c_2 > 0$ :

$$\|(e^{\tilde{v}} - e^v) \tilde{\phi}\|_{L^2_1(g_\tau, A)} \leq c_2 \|e^{\tilde{v}} - e^v\|_{L^2_2(g_\tau)} \|\tilde{\phi}\|_{L^2_2(g_\tau, A)};$$

$$\text{and} \quad \|(e^{\tilde{v}} - 1)(\tilde{\phi} - \phi)\|_{L^2_1(g_\tau, A)} \leq c_2 \|e^{\tilde{v}} - 1\|_{L^2_2(g_\tau)} \|\tilde{\phi} - \phi\|_{L^2_2(g_\tau, A)};$$

thanks to [Lemma 2.1.6](#) and [Corollary 2.2.13](#), the constant  $c_2$  can be chosen independently of  $\tau$ .

Eventually, using [Lemma 2.3.3](#), we see that for  $\alpha > 0$  small enough, we have

$$(2.21) \quad \|(\phi, v)\|_{L^2_{2,3}(g_\tau, A)}, \|(\tilde{\phi}, \tilde{v})\|_{L^2_{2,3}(g_\tau, A)} \leq \alpha \Rightarrow \\ \|R(\tilde{\phi}, \tilde{v}) - R(\phi, v)\|_{L^2_1(g_\tau)} \leq \kappa_0 \|(\tilde{\phi}, \tilde{v}) - (\phi, v)\|_{L^2_{2,3}(g_\tau, A)}.$$

The estimate [\(2.17\)](#) implies that

$$\|\Delta_{1, (A, \Phi)}^{-1} \left( R(\tilde{\phi}, \tilde{v}) - R(\phi, v) \right)\|_{L^2_3(g_\tau)} \leq \frac{\kappa_0}{b_1} \|(\tilde{\phi}, \tilde{v}) - (\phi, v)\|_{L^2_{2,3}(g_\tau, A)}.$$

By [Corollary 2.2.13](#), there is a constant  $c_3 > 0$  independent of  $\tau$  and of the solution  $(A, \Phi)$  of Seiberg–Witten equations such that for every  $(a, \phi)$

$$\|\delta_{1,(A,\Phi)}(a, \phi)\|_{L^2_2(g_\tau, A)} \leq c_3 \|(a, \phi)\|_{L^2_3(g_\tau, A)}.$$

In particular

$$\|\delta_{1,(A,\Phi)} \Delta_{1,(A,\Phi)}^{-1} \left( R(\tilde{\phi}, \tilde{v}) - R(\phi, v) \right)\|_{L^2_2(g_\tau, A)} \leq \frac{c_3 \kappa_0}{b_1} \|(\tilde{\phi}, \tilde{v}) - (\phi, v)\|_{L^2_2(g_\tau, A)}.$$

Now, if we take  $\kappa_0 = \min(\frac{b_1 \kappa}{2c_3}, \frac{b_1 \kappa}{2})$  from the beginning, we have

$$\begin{aligned} \|Q(\tilde{\phi}, \tilde{v}) - Q(\phi, v)\|_{L^2_{2,3}(g_\tau, A)} &= \|\delta_{1,(A,\Phi)} \Delta_{1,(A,\Phi)}^{-1} \left( R(\tilde{\phi}, \tilde{v}) - R(\phi, v) \right)\|_{L^2_2(g_\tau, A)} \\ &\quad + \|\Delta_{1,(A,\Phi)}^{-1} \left( R(\tilde{\phi}, \tilde{v}) - R(\phi, v) \right)\|_{L^2_3(g_\tau)} \leq \kappa \|(\tilde{\phi}, \tilde{v}) - (\phi, v)\|_{L^2_2(g_\tau, A)}, \end{aligned}$$

and the lemma holds.  $\square$

Recall that there is an effective version of the contraction mapping theorem.

**Proposition 2.3.5** *Let  $S: \mathbb{E} \rightarrow \mathbb{E}$  be a smooth function on a Banach space  $(\mathbb{E}, \|\cdot\|)$  such that  $S(0) = 0$ ; assume that there exist constants  $\alpha > 0$  and  $\kappa \in (0, 1/2)$  such that*

$$\text{for all } x, y \in \mathbb{E}, \quad \|x\|, \|y\| \leq \alpha \Rightarrow \|S(y) - S(x)\| \leq \kappa \|y - x\|.$$

*Then, for every  $y$  in the open ball  $B_{\alpha/2}(0)$ , the equation  $y = x + S(x)$  has a unique solution  $x(y) \in B_\alpha(0)$ ; moreover, we have  $\|x(y) - y\| \leq \frac{\kappa}{1-\kappa} \|y\|$ , and the smooth function*

$$\begin{aligned} F: B_\alpha(0) &\longrightarrow B_{3\alpha/2}(0) \\ x &\longmapsto x + S(x), \end{aligned}$$

*restricted to  $F: F^{-1}(B_{\alpha/2}(0)) \rightarrow B_{\alpha/2}(0)$  is a diffeomorphism. In addition  $B_{\alpha/3}(0) \subset F^{-1}(B_{\alpha/2}(0))$  and  $F(B_{\alpha/3}(0))$  is an open neighborhood of 0 containing the ball  $B_{\alpha/6}(0)$ .*

**Proof** Under these assumptions  $x \mapsto y - S(x)$  maps the ball of radius  $\alpha$  in  $\mathbb{E}$  to itself and is a contraction mapping there. These estimates for the fixed points follow immediately. The last claim is deduced easily by replacing  $\alpha$  with  $\alpha/3$  in the first part of the proposition.  $\square$

An important ingredient in the refined slice theorem is to study the size of the open sets on  $\mathcal{G}$  provided by the exponential map. More specifically, we have the following corollary.

**Corollary 2.3.6** For every  $\alpha > 0$  small enough and every  $\tau$  large enough and any open set  $D \subset M_\tau$ , the exponential defines a smooth map

$$\exp: B_\alpha(0) \rightarrow B_{3\alpha/2}(1)$$

where  $B_r(u_0)$  is the space of functions  $u \in L^2_3(\mathbb{C})$  on  $D$  such that,  $\|u - u_0\|_{L^2_3(g_\tau, D)} < r$ . Moreover,  $\exp$  is a diffeomorphism onto an open neighborhood of 1 which contains  $B_{\alpha/2}(1)$ .

**Proof** This is a direct consequence of [Lemma 2.3.3](#) and [Proposition 2.3.5](#).  $\square$

As a second application, we have the following result for  $F$ .

**Corollary 2.3.7** For every  $\alpha_3 > 0$  small enough, every  $\tau$  large enough and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , the map

$$F: U_{\alpha_3} \times G_{\alpha_3} \rightarrow \{(\tilde{a}, \tilde{\phi}) \in L^2_2, \delta^*_{1, (A, \Phi)}(\tilde{a}, \tilde{\phi}) = 0\} \times L^2_3(i\mathbb{R})$$

defined at [\(2.18\)](#) is diffeomorphism onto an open neighborhood of 0. Moreover, we have

$$U_{\alpha_3/2} \times G_{\alpha_3/2} \subset F(U_{\alpha_3} \times G_{\alpha_3}) \subset U_{3\alpha_3/2} \times G_{3\alpha_3/2}.$$

**Proof** Put  $x = ((a, \phi), v)$ ,  $y = ((\tilde{a}, \tilde{\phi}), \tilde{v})$  and  $S(x) = Q(\phi, v)$ . Then  $F(x) = x + S(x)$  and we use the  $L^2_{2,3}(g_\tau, A)$ -norm. Thanks to [Lemma 2.3.4](#), for every  $\alpha > 0$  small enough, every  $\tau$  large enough and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , the assumption of [Proposition 2.3.5](#) holds. Then  $\alpha_3 = \alpha/3$  satisfies the conclusions of the corollary.  $\square$

We state a serie of preparation lemmas for proving [Theorem 2.3.1](#).

**Lemma 2.3.8** There exists a constant  $c_4 > 0$  such that for every  $\tau$  large enough and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , we have for all  $((\dot{a}, \dot{\phi}), \dot{v})$ , with  $(\dot{a}, \dot{\phi}) \in \ker \delta^*_{1, (A, \Phi)}$  the estimate

$$\|((\dot{a}, \dot{\phi}), \dot{v})\|_{L^2_{2,3}(g_\tau, A)} \leq c_4 \|d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v})\|_{L^2_2(g_\tau, A)}.$$

**Proof** Thanks to [Lemmas 2.2.13](#) and [2.1.6](#), there is a constants  $c_5, c_6 > 0$  independent of  $\tau$  and of the solution  $(A, \Phi)$  of Seiberg–Witten equations such that

$$\|\delta^*_{1, (A, \Phi)} d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v})\|_{L^2_1(g_\tau)} \leq c_5 \|d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v})\|_{L^2_2(g_\tau, A)}$$

and

$$c_6 \|\delta_{1, (A, \Phi)}(\dot{v})\|_{L^2_2(g_\tau, A)} \leq \|\dot{v}\|_{L^2_3(g_\tau)}.$$

On the other hand, using the fact that  $d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v}) = (\dot{a}, \dot{\phi}) + \delta_{1, (A, \Phi)}(\dot{v})$ , we have

$$\delta_{1, (A, \Phi)}^* d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v}) = \Delta_{1, (A, \Phi)} \dot{v}.$$

Using the estimate (2.17), we obtain

$$\|\dot{v}\|_{L^2_3(g_\tau)} \leq \frac{c_5}{b_1} \|d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v})\|_{L^2_2(g_\tau, A)},$$

hence

$$\|\delta_{1, (A, \Phi)}(\dot{v})\|_{L^2_2(g_\tau, A)} \leq \frac{c_5}{b_1 c_6} \|d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v})\|_{L^2_2(g_\tau, A)}.$$

Therefore,

$$\begin{aligned} (1 + \frac{c_5}{b_1 c_6} + \frac{c_5}{b_1}) \|d_0 \bar{v}((\dot{a}, \dot{\phi}), \dot{v})\|_{L^2_2(g_\tau, A)} &\geq \\ \left( \|(\dot{a}, \dot{\phi})\|_{L^2_2(g_\tau, A)} - \|\delta_{1, (A, \Phi)}(\dot{v})\|_{L^2_2(g_\tau, A)} \right) + \|\delta_{1, (A, \Phi)}(\dot{v})\|_{L^2_2(g_\tau, A)} + \|\dot{v}\|_{L^2_3(g_\tau)} &= \\ \|(\dot{a}, \dot{\phi})\|_{L^2_2(g_\tau, A)} + \|\dot{v}\|_{L^2_3(g_\tau)}, & \end{aligned}$$

and the lemma is proved.  $\square$

**Lemma 2.3.9** *For every  $\alpha_3 > 0$ , there exists a constant  $\alpha'_1 > 0$  such that for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have for all  $(a, \phi), (\tilde{a}, \tilde{\phi}) \in \ker \delta_{1, (A, \Phi)}^*$  and  $u \in \mathcal{G}_2$*

$$\left. \begin{aligned} \|(a, \phi)\|_{L^2_2(g_\tau)}, \|(a, \phi)\|_{L^2_2(g_\tau)} &\leq \alpha'_1 \text{ and } \\ u \cdot (A + a, \Phi + \phi) &= (A + \tilde{a}, \Phi + \tilde{\phi}) \end{aligned} \right\} \Rightarrow \|u\|_{L^2_3} < \alpha_3/2$$

**Proof** By assumption  $\|\tilde{a} - a\|_{L^2_2(g_\tau)} \leq 2\alpha'_1$  and  $u^{-1} du = \tilde{a} - a$ , hence

$$(2.22) \quad \left\| \frac{du}{u} \right\|_{L^2_2(g_\tau)} = \|du\|_{L^2_2(g_\tau)} \leq 2\alpha'_1.$$

First, we derive an  $L^2_2$  bound on  $du$ .

$$(2.23) \quad \nabla(\tilde{a} - a) = \nabla \left( \frac{du}{u} \right) = -\frac{du \otimes du}{u^2} + \frac{\nabla^2 u}{u}$$

$$(2.24) \quad = -(\tilde{a} - a) \otimes (\tilde{a} - a) + \frac{\nabla^2 u}{u}$$

The Sobolev multiplication theorem  $L^2_1 \otimes L^2_1 \subset L^2$  says that for some constant  $c > 0$  we have

$$\|(\tilde{a} - a) \otimes (\tilde{a} - a)\|_{L^2(g_\tau)} \leq c_7 \|\tilde{a} - a\|_{L^2_1(g_\tau)}^2 \leq 4c_7 \alpha_1^2.$$



Thanks to [Lemma 2.1.6](#), the constant  $c_7$  can be chosen independently of  $\tau$ . We deduce a bound from [\(2.24\)](#)

$$\|\nabla^2 u\|_{L^2(g_\tau)} \leq (4c_7\alpha'_1 + 2)\alpha'_1.$$

Then, we take care of derivatives of order 2. We have

$$(2.25) \quad \nabla^2(\tilde{a} - a) = \nabla^2\left(\frac{du}{u}\right) = -\nabla\left((\tilde{a} - a) \otimes (\tilde{a} - a)\right) - 2\frac{du \otimes \nabla^2 u}{u^2} + \frac{\nabla^3 u}{u}$$

Since

$$\nabla\left((\tilde{a} - a) \otimes (\tilde{a} - a)\right) = \nabla(\tilde{a} - a) \otimes (\tilde{a} - a) + (\tilde{a} - a) \otimes \nabla(\tilde{a} - a),$$

we can use again the Sobolev embedding theorem to deduce the estimate

$$(2.26) \quad \left\| \nabla\left((\tilde{a} - a) \otimes (\tilde{a} - a)\right) \right\|_{L^2(g_\tau)} \leq 8c_7(\alpha'_1)^2.$$

The next term of [\(2.25\)](#) can be written

$$-2\frac{du \otimes \nabla^2 u}{u^2} = -2\frac{du}{u} \otimes \nabla\left(\frac{du}{u}\right) - \frac{2}{u}\frac{du \otimes du}{u^2}.$$

Again the Sobolev multiplication theorem gives control

$$\begin{aligned} \left\| -2\frac{du \otimes \nabla^2 u}{u^2} \right\|_{L^2(g_\tau)} &\leq 2c_7 \left\| \frac{du}{u} \right\|_{L^2_1(g_\tau)} \left\| \nabla\left(\frac{du}{u}\right) \right\|_{L^2_1(g_\tau)} + 2c_7 \left\| \frac{du}{u} \right\|_{L^2_1(g_\tau)}^2 \\ &\leq 16c_7(\alpha'_1)^2. \end{aligned}$$

Returning to the identity [\(2.25\)](#), we derive the estimate

$$\left\| \nabla^3 u \right\|_{L^2(g_\tau)} \leq 2\alpha'_1 + 8c_7(\alpha'_1)^2 + 16c_7(\alpha'_1)^2 = 2\alpha'_1(1 + 12c_7\alpha'_1).$$

In conclusion we have proved the estimate

$$(2.27) \quad \|du\|_{L^2_2(g_\tau)} \leq 2\alpha'_1(3 + 14c_7\alpha'_1).$$

We show now that we have an  $L^2$ -estimate on  $u - 1$  outside a compact set. By [Proposition 2.2.6](#), there is a compact set  $K \subset M$  such that for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have  $|\Phi| \geq \frac{1}{2}$ . We write  $(1 - u)\Phi = (u - 1)\phi + (\phi - \tilde{\phi})$  and take the  $L^2(g_\tau)$ -norm on the noncompact set  $M_\tau \setminus K$ . We have the estimate

$$(2.28) \quad \frac{1}{2}\|u - 1\|_{L^2(g_\tau, M_\tau \setminus K)} \leq \|(u - 1)\phi\|_{L^2(g_\tau, M_\tau \setminus K)} + \|\phi - \tilde{\phi}\|_{L^2(g_\tau, M_\tau \setminus K)}.$$

By the Sobolev multiplication theorem  $L_1^2 \otimes L_1^2 \hookrightarrow L^2$ , we deduce that for some constant  $c_8 > 0$  (independent of  $\tau$  and the choice of a solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$  by [Lemma 2.1.6](#) and [Corollary 2.2.14](#))

$$\begin{aligned} \|(1-u)\tilde{\phi}\|_{L^2(g_\tau, M_\tau \setminus K)} &\leq c_8 \|1-u\|_{L_1^2(g_\tau, M_\tau \setminus K)} \|\tilde{\phi}\|_{L_1^2(g_\tau, A, M_\tau \setminus K)} \\ &\leq c_8 \alpha'_1 \|1-u\|_{L_1^2(g_\tau, M_\tau \setminus K)}. \end{aligned}$$

So if we chose  $\alpha'_1 \leq 1/4c_8$ , we deduce from [\(2.28\)](#)

$$\frac{1}{4} \|u-1\|_{L^2(g_\tau, M_\tau \setminus K)} \leq \frac{1}{4} \|du\|_{L^2(g_\tau, M_\tau \setminus K)} + \|\tilde{\phi} - \phi\|_{L^2(g_\tau, A, M_\tau \setminus K)}.$$

Using the estimate [\(2.22\)](#), we have

$$(2.29) \quad \|u-1\|_{L^2(g_\tau, M_\tau \setminus K)} \leq 10\alpha'_1.$$

In order to exploit the estimates [\(2.27\)](#) and [\(2.29\)](#), the following lemma is needed.

**Lemma 2.3.10** *Let  $K \subset M$  be a compact set. There exists a constant  $c > 0$  such that for every  $\tau$  large enough and every complex valued function  $w$  on  $M_\tau$  we have*

$$c \|w\|_{L_1^2(g_\tau)} \leq \|dw\|_{L^2(g_\tau)} + \|w\|_{L^2(g_\tau, M_\tau \setminus K)}.$$

**Proof** Suppose this is not true. Then we have a compact set  $K$  and a sequence  $\tau_j \rightarrow \infty$ , together with a sequence of functions  $w_j$  on  $M_{\tau_j}$  such that

$$\|w_j\|_{L_1^2(g_{\tau_j})} = 1, \quad \|dw_j\|_{L^2(g_{\tau_j})} \rightarrow 0, \quad \text{and} \quad \|w_j\|_{L^2(g_{\tau_j}, M_{\tau_j} \setminus K)} \rightarrow 0.$$

Hence, up to extraction of a subsequence,  $w_j$  converge in the strong  $L^2$  sense on every compact toward a weak limit  $w \in L_2^2(g, M)$ . Moreover  $w$  satisfies  $dw = 0$  on  $M$  and  $w = 0$  on  $M \setminus K$ . Therefore  $w \equiv 0$ .

Thus, we have

$$\|dw_j\|_{L^2(g_{\tau_j})} + \|w_j\|_{L^2(g_{\tau_j}, M_{\tau_j} \setminus K)} + \|w_j\|_{L^2(g_{\tau_j}, K)} = \|w_j\|_{L_1^2(g_{\tau_j})}.$$

The first two terms converge to 0 by assumption. The third one converge to 0 since  $w_j \rightarrow 0$  in the  $L^2$  sense on every compact. This is a contradiction since RHS equals 1 and the lemma is proved.  $\square$

We return now to the proof of [Lemma 2.3.9](#). Thanks to [Lemma 2.3.10](#), there is a constant  $c_9 > 0$  such that for every  $\tau$  large enough

$$c_9 \|u\|_{L_1^2(g_\tau)} \leq \|du\|_{L^2(g_\tau)} + \|u\|_{L^2(g_\tau, M_\tau \setminus K)}.$$

Using the estimates (2.22) and (2.29), we deduce that

$$\|u - 1\|_{L^2_1(g_\tau)} \leq 12\alpha'_1/c_9.$$

Together with (2.27), it gives

$$(2.30) \quad \|u - 1\|_{L^2_3(g_\tau)} \leq 12\alpha'_1/c_9 + 2\alpha'_1(3 + 14c_7\alpha'_1).$$

For  $\alpha'_1$  small enough, we have  $12\alpha'_1/c_9 + 2\alpha'_1(3 + 14c_7\alpha'_1) < \alpha_3/2$  and the lemma is proved.  $\square$

We can complete now the proof of [Theorem 2.3.1](#). Let  $\alpha_3 > 0$  be a constant as in [Corollary 2.3.7](#). Then any constant  $\alpha_1 \in (0, \alpha_3]$  also satisfies the Corollary. We will in addition assume that  $\alpha_1 \leq \alpha'_1$ , where  $\alpha'_1$  satisfies [Lemma 2.3.9](#).

In particular, we have  $U_{\alpha_1/2} \times G_{\alpha_1/2} \subset F(U_{\alpha_1} \times G_{\alpha_1})$ . Let  $c_4$  be the constant from [Lemma 2.3.8](#). Then for every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have

$$\left\{ (a, \phi) \in L^2_2, \quad \|(a, \phi)\|_{L^2_2(g_\tau)} < c_4\alpha_1/2 \right\} \subset d_0\bar{v}(U_{\alpha_1/2} \times G_{\alpha_1/2}).$$

Since  $\bar{v} = d_0\bar{v} \circ F + (A, \Phi)$ , we have

$$\left\{ (\tilde{A}, \tilde{\Phi}) \in \mathcal{C}_2, \quad \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} < c_4\alpha_1/2 \right\} \subset \bar{v}(U_{\alpha_1} \times G_{\alpha_1}).$$

In conclusion,  $\bar{v}: U_{\alpha_3} \times G_{\alpha_3} \rightarrow \mathcal{C}_2$  is a diffeomorphism onto an open neighborhood of  $(A, \Phi)$  and  $\bar{v}(U_{\alpha_1} \times G_{\alpha_1})$  contains a ball of radius  $\alpha_2 := c_4\alpha_1/2$  as above.

For a choice of  $\alpha_3 > 0$  small enough, [Corollary 2.3.6](#) says that the exponential map

$$\exp: G_{\alpha_3} \rightarrow \mathcal{G}_2,$$

is a diffeomorphism onto an open neighborhood  $G'_{\alpha_3} = \exp(G_{\alpha_3})$  of 1 containing a ball centered at 1 with  $L^2_3(g_\tau)$ -radius  $\alpha_3/2$ .

By equivariance, it follows that  $v: U_{\alpha_3} \times \mathcal{G}_2 \rightarrow \mathcal{C}_2$  is a local diffeomorphism onto an open neighborhood of the orbit of  $(A, \Phi)$ . Moreover, the ball  $B_{\alpha_2}$  defined in [Theorem 2.3.1](#), is contained in  $v(U_{\alpha_1} \times \mathcal{G}_2)$ .

We just need to show that  $v: U_{\alpha_1} \times \mathcal{G}_2 \rightarrow \mathcal{C}_2$  is injective in order to prove that  $\alpha_1$  and  $\alpha_2$  satisfy the theorem. Suppose that  $v((a, \phi), u) = v((\tilde{a}, \tilde{\phi}), \tilde{u})$ , for some  $(a, \phi), (\tilde{a}, \tilde{\phi}) \in U_{\alpha_1}$  and  $u, \tilde{u} \in \mathcal{G}_2$ . By equivariance, we have  $v((a, \phi), u\tilde{u}^{-1}) = v((\tilde{a}, \tilde{\phi}), 1)$ . Since  $\alpha_1 \leq \alpha'_1$ , the assumption of [Lemma 2.3.9](#) are verified, therefore we must have  $\|u\tilde{u}^{-1}\|_{L^2_3(g_\tau)} < \alpha_3/2$ , hence  $u\tilde{u}^{-1} \in G'_{\alpha_3}$ . By injectivity of  $\bar{v}: U_{\alpha_3} \times G'_{\alpha_3} \rightarrow \mathcal{C}_2$  we conclude that  $u\tilde{u}^{-1} = 1$ , therefore  $u = \tilde{u}$  and  $(a, \phi) = (\tilde{a}, \tilde{\phi})$ .  $\square$

We now return to the proof of [Lemma 2.3.2](#).

**Proof** According to [Proposition 2.2.6](#), there exists a compact  $K \subset M$  such that  $|\Phi|^2 \geq 1/2$  outside  $K$  for every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$  and for every  $\tau$  large enough; let  $\chi_K$  be the characteristic function of  $K$ . Then, for every smooth function  $v$ , we have

$$(2.31) \quad \int \langle \Delta_{1,(A,\Phi)}(v), v \rangle \operatorname{vol}^{g_\tau} = \int (|dv|^2 + |\Phi|^2|v|^2) \operatorname{vol}^{g_\tau} \\ \geq \|dv\|_{L^2(g_\tau)}^2 + \frac{1}{2} \|(1 - \chi_K)v\|_{L^2(g_\tau)}^2.$$

Therefore, according to [Lemma 2.3.10](#), there exists a constant  $c > 0$  (depending only on  $K$ ) such that for every  $\tau$  large enough, we have

$$(2.32) \quad \|\Delta_{1,(A,\Phi)}(v)\|_{L^2(g_\tau)} \geq c \|v\|_{L^2_1(g_\tau)}.$$

The next step is to obtain a control on higher derivatives. For this purpose, we use the fact that the operator  $\Delta_1$  is elliptic; we consider two balls  $B_1 \subset B_2$  in  $M_\tau$  with small radii  $\alpha/2$  and  $\alpha$ , centered at a point  $x$ . Then there exists a constant  $c'_k > 0$  such that

$$\|\Delta_{1,(A,\Phi)}(v)\|_{L^2_k(g_\tau, B_2)} + \|v\|_{L^2_1(g_\tau, B_2)} \geq c'_k \|v\|_{L^2_{k+2}(g_\tau, B_1)},$$

where  $L^2_k(g_\tau, B_j)$  is the  $L^2_k(g_\tau, A)$ -norm restricted to the ball  $B_j$ .

Now, using the fact that the geometry of the  $M_\tau$  is uniformly bounded in the sense of [Lemma 2.1.6](#) and the bounds on  $(A, \Phi)$  obtained from [Lemma 2.2.5](#) and [Corollary 2.2.13](#), we deduce that the coefficients of  $\Delta_{1,(A,\Phi)}$  are bounded independently of  $(A, \Phi)$  and  $\tau$ . We conclude that  $c'_k$  can be chosen in such a way that it does not depend neither on the center of the balls  $x$ ,  $\tau$  nor on  $(A, \Phi)$ . Therefore, we can find constants  $c''_k > 0$ , independent of  $\tau$  and  $(A, \Phi)$  such that globally

$$\|\Delta_{1,(A,\Phi)}(v)\|_{L^2_k(g_\tau)} + \|v\|_{L^2_1(g_\tau)} \geq c''_k \|v\|_{L^2_{k+2}(g_\tau)}.$$

Hence

$$(1 + 1/c) \|\Delta_{1,(A,\Phi)}(v)\|_{L^2_1(g_\tau)} \geq c''_k \|v\|_{L^2_{k+2}(g_\tau)},$$

where  $c$  is the constant of the control [\(2.32\)](#). Then  $b_k = \frac{cc''_k}{c+1}$  satisfies

$$b_k \|v\|_{L^2_{k+2}(g_\tau)} \leq \|\Delta_{1,(A,\Phi)}(v)\|_{L^2_k(g_\tau)}.$$

Finally, we can always take a smaller value for  $b_k > 0$  such that the second inequality of the lemma is verified. This is a trivial consequence of [Corollary 2.2.13](#).  $\square$

## 2.4 Approximate solutions

This section deals with the first step for constructing the gluing map  $\mathfrak{G}$  of [Theorem E](#). Recall that the family of AFAK manifolds  $M_\tau$  was constructed in [Section 2.1](#) by adding an AFAK end  $Z$  to a manifold with  $\bar{M}$  with a contact boundary  $(Y, \xi)$ . Let  $Z'$  be another AFAK end compatible with the contact boundary  $(Y, \eta)$  and, similarly we have,  $(M'_\tau, g'_\tau, \omega'_\tau, J'_\tau, \sigma'_\tau)$  the family of AFAK manifolds constructed out of  $\bar{M}$  and  $Z'$ .

Moreover, an element  $(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi)$  gives rise to a  $\text{Spin}^c$ -structure with an identification with  $\mathfrak{s}_{\omega'_\tau}$  outside  $\bar{M}$ , that is to say an element  $j'(\mathfrak{s}, h) \in \text{Spin}^c(M'_\tau, \omega'_\tau)$ . Let  $W'_\tau$  be the spinor bundle of  $j'(\mathfrak{s}, h)$  on  $M'_\tau$  constructed similarly to  $W_\tau \rightarrow M_\tau$  (see [Section 2.1.7](#)).

The compact domains  $\{\sigma \leq \tau\} \subset M$ ,  $\{\sigma_\tau \leq \tau\} \subset M_\tau$  and  $\{\sigma'_\tau \leq \tau\} \subset M'_\tau$  are equal by construction and on these sets, all the structures match (Riemannian metrics, functions  $\sigma$ , almost Kähler structures,  $\text{Spin}^c$ -structures, spinor bundles, canonical solutions  $(B, \Psi)$  and  $(B', \Psi')$ ).

We will now explain how to construct an approximate solution of Seiberg–Witten equations on  $M'_\tau$  from a solution of Seiberg–Witten equations on  $M_\tau$ .

**2.4.1 Construction of the spinor bundle** On the end  $\{1 \leq \sigma_\tau\} \subset M_\tau$ , the spinor bundle  $W_\tau$  is by definition identified with the spinor bundle  $W_{J_\tau}$  of  $\mathfrak{s}_{\omega_\tau}$ . Let  $(A, \Phi)$  be a solution of Seiberg–Witten equations on  $M_\tau$ . The spinor  $\Phi$  can be regarded as  $(\beta, \gamma) \in W_{J_\tau}^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ .

By [Proposition 2.2.6](#), there is a  $T$  large enough, independent of  $\tau$  and  $(A, \Phi)$ , such that, say,  $|\beta| \geq 1/2$  on  $M_\tau \setminus K$  where  $K = \{T \leq \sigma_\tau\}$ . Hence we may define the map

$$h_{(A, \Phi)}: M_\tau \setminus K \rightarrow S^1$$

by

$$h_{(A, \Phi)} = \frac{|\beta|}{\beta}.$$

By assumption,  $(A, \Phi) \in \mathcal{C}_l$  with  $l \geq 4$ . Therefore  $1 - h_{(A, \Phi)} \in L_{l-1}^2$ ; using the Sobolev inclusion  $L_3^2 \subset C^0$ , we see that  $h_{(A, \Phi)}$  tends to 1 in  $C^0$ -norm near infinity. So  $h_{(A, \Phi)}$  is homotopic to 1 near infinity, and by assumption [\(2.2\)](#), it implies that the restriction of  $h_{(A, \Phi)}$  to the annulus  $\{T \leq \sigma_\tau \leq \tau\}$  is homotopic to 1.

We define now a spinor bundle  $W_{(A, \Phi)}$  on  $M'_\tau$  as follows:

$$\begin{cases} W_{(A, \Phi)} := W & \text{over } M'_\tau \cap \{\sigma'_\tau < \tau\} \subset M \\ W_{(A, \Phi)} := W_{J'_\tau} & \text{over } M'_\tau \cap \{\sigma'_\tau > T\} \subset Z' \end{cases}$$

and the transition map from  $W$  to  $W_{J'_\tau}$  is given by  $h'_{(A,\Phi)} = h_{(A,\Phi)} \circ h$  over the annulus  $\{T < \sigma_\tau < \tau\}$ .

The spinor bundle  $W_{(A,\Phi)}$  together with its preferred identification with  $W_{J'_\tau}$  on the end of  $M'_\tau$  define an element of  $\text{Spin}^c(M'_\tau, \omega'_\tau)$ . In fact, this is nothing else but  $j'(\mathfrak{s}, h)$ , since  $h_{(A,\Phi)}$  is homotopic to 1 over the patching region.

The construction is compatible with the the gauge group action, in the sense that for every  $u \in \mathcal{G}(M_\tau)$ , then

$$h_{u \cdot (A,\Phi)} = u^{-1} h_{(A,\Phi)};$$

therefore  $u$  induces an isomorphism  $u^\sharp: W_{(A,\Phi)} \rightarrow W_{u \cdot (A,\Phi)}$  equal to 1 on the end once the spinor bundles are identified with  $W_{J'_\tau}$ .

**2.4.2 Definition of the approximate solutions** We suppose from now on that  $\tau > T$  as in the previous section, so that for any solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we can construct the spinor bundle  $W_{(A,\Phi)}$  on  $M'_\tau$ .

The bundles  $W_{(A,\Phi)}$  and  $W_\tau$  restricted to the region  $K_T = \{\sigma \leq \tau\} \subset M$  are equal by construction. Hence  $(A, \Phi)$  defines a configuration for the spinor bundle  $W_{(A,\Phi)}|_{K_T}$ . We explain now how to extend it to the entire manifold  $M'_\tau$ .

As before, we express  $\Phi$  as a pair  $(\beta, \gamma) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$  using the identification between  $W_\tau$  and  $W_{J_\tau}$  on the end on  $M_\tau$ . By construction, the bundle  $W_{(A,\Phi)}$  is identified via  $h_{(A,\Phi)}$  to  $W_{J'_\tau}$  on the end  $\{T < \sigma'_\tau\}$  furthermore and we can write modulo this identification

$$h'_{(A,\Phi)} \cdot (A, \Phi) = h_{(A,\Phi)} \cdot (B + a, (\beta, \gamma)) := (B + \hat{a}, (\hat{\beta}, \hat{\gamma})).$$

The main effect of the isomorphism  $h_{(A,\Phi)}$ , is that  $\hat{\beta}$  is now a real function and  $\hat{\beta} \geq \frac{1}{2}$ ; hence the following definition makes sense:

$$(A, \Phi)^\sharp = (B' + \chi_\tau \hat{a}, (\hat{\beta}^{\chi_\tau}, \chi_\tau \hat{\gamma})).$$

Here  $\chi_\tau$  is the function defined in Equation (2.6). This extends naturally as a configuration relative to the spinor bundle  $W_{(A,\Phi)}$  on  $M'_\tau$  by setting  $(A, \Phi)^\sharp := (B, \Psi)$  on the end  $\{\sigma'_\tau \geq \tau\} \subset M'_\tau$ . We will also use the notation  $(A^\sharp, \Phi^\sharp) := (A, \Phi)^\sharp$ . We stress on the fact that if  $(\tilde{A}, \tilde{\Phi})$  is another solution of Seiberg–Witten equations on  $M_\tau$ , then  $(A, \Phi)^\sharp$  and  $(\tilde{A}, \tilde{\Phi})^\sharp$  are not *a priori* defined on the same bundles. However this construction is compatible with the gauge group action in the sense that

$$u^\sharp \cdot (A, \Phi)^\sharp = (u \cdot (A, \Phi))^\sharp;$$

thus we have defined a map, called the *pregluing map*,

$$\sharp: \mathcal{M}_{\varpi_\tau}(M_\tau) \rightarrow \mathcal{C}/\mathcal{G}(M'_\tau).$$

We will see shortly that in fact this is a smooth map.

**Remark 2.4.3** In case we want to remove assumption (2.2) for the AFAK end  $Z'$  and  $Z$  is just a symplectic cone, we can construct similarly a pregluing map

$$\sharp: \tilde{\mathcal{M}}_{\varpi_\tau}(M, (s', h')) \rightarrow \mathcal{C}/\mathcal{G}(M'_\tau, (s', h')),$$

where  $(s', h') \in \text{Spin}^c(M'_\tau, \omega'_\tau)$  and  $\tilde{\mathcal{M}}_{\varpi_\tau}(M, (s', h'))$  is the enlarged moduli space defined at (2.9). The reader can check that the gluing theory applied starting from this pregluing map leads to the result mentioned in Remark 1.2.2.

## 2.5 Rough gauge fixing on the target

Given two solutions of Seiberg–Witten equations  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  on  $M_\tau$ , the approximate solutions  $(A, \Phi)^\sharp$  and  $(\tilde{A}, \tilde{\Phi})^\sharp$  are defined on different bundles  $W_{(A, \Phi)}$  and  $W_{(\tilde{A}, \tilde{\Phi})}$  on  $M'_\tau$ . In this section, we explain how to construct a preferred isomorphism provided  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  are close enough.

**2.5.1 Definition** Using the identification between  $W_\tau$  and  $W_{J_\tau}$  on the end  $\{\sigma_\tau \geq T\} \subset M_\tau$ , we may write  $\Phi = (\beta, \gamma)$ ,  $\tilde{\Phi} = (\tilde{\beta}, \tilde{\gamma}) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$ ,  $A = B + a$  and  $\tilde{A} = B + \tilde{a}$ .

In order to simplify notation, we may assume after applying a gauge transformation that  $(A, \Phi)$  is already in a gauge with exponential decay as in Corollary 2.2.10. In particular, we have  $\beta = |\beta|$  hence  $h_{(A, \Phi)} = 1$ , therefore  $W_{(A, \Phi)}$  is equal to the spinor bundle  $W'_\tau$  constructed in Section 2.1.7.

If we assume that  $\tilde{\Phi}$  is sufficiently close to  $\Phi$  in  $C^0$ -norm, then  $h_{(\tilde{A}, \tilde{\Phi})} = |\tilde{\beta}|/|\tilde{\beta}|$  is also close to 1; in particular, we can write  $h_{(\tilde{A}, \tilde{\Phi})} = \exp(-\tilde{v})$ , where  $\tilde{v}$  is a purely imaginary function completely determined by the requirement  $|\tilde{v}| < \pi$ . With this notation, we have  $\tilde{\beta} = \exp(\tilde{u} + \tilde{v})$ , where  $\tilde{u}$  is a real function such that  $|\tilde{\beta}| = \exp \tilde{u}$ .

Put

$$k_{(\tilde{A}, \tilde{\Phi})} := \exp(-\chi_\tau \tilde{v});$$

this is an isomorphism of  $W_{J'_\tau}$ , restricted to the annulus  $M_\tau \cap \{T < \sigma'_\tau < \tau\}$ , equals to  $h_{(\tilde{A}, \tilde{\Phi})}$  in a neighborhood of  $\{\sigma'_\tau = T\}$ , and to 1 along  $\{\sigma'_\tau = \tau\}$ . Therefore, we can extend  $k_{(\tilde{A}, \tilde{\Phi})}$  to an isomorphism

$$k_{(\tilde{A}, \tilde{\Phi})}: W_{(A, \Phi)} \rightarrow W_{(\tilde{A}, \tilde{\Phi})}$$

by setting

- $k_{(\tilde{A}, \tilde{\Phi})} := \text{id}|_{W_{J'_\tau}}$  for  $\sigma'_\tau \geq \tau$  and
- $k_{(\tilde{A}, \tilde{\Phi})} := \text{id}|_W$  for  $\sigma'_\tau \leq T$ .

**2.5.2 Estimates for the pregluing in rough gauge**  $(A, \Phi)^\sharp$  and  $k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\sharp$  are two configurations defined with respect to the same spinor bundle  $W_{(A, \Phi)}$ . We are going to show that, if  $(\tilde{A}, \tilde{\Phi})$  is in Coulomb gauge with respect to  $(A, \Phi)$ , then,  $k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\sharp$  is in some sense very close to be in Coulomb gauge with respect to  $(A, \Phi)^\sharp$ . We will also prove similar estimates for the linearized Seiberg–Witten equations.

Since the isomorphism  $k_{(\tilde{A}, \tilde{\Phi})}$  apparently differs from the identity only on the annulus  $\{T \leq \sigma'_\tau \leq \tau\}$ , we may focus our study of the pregluing map on this region. With our particular choice of gauge for  $(A, \Phi)$ , we have  $|\beta| = \beta$  hence  $W_{(A, \Phi)} = W'_\tau$ ; using the the identification  $W'_\tau \simeq W_{J'_\tau}$  for  $\sigma'_\tau \geq T$ , we have by definition of  $\sharp$ ,

$$(2.33) \quad (A, \Phi)^\sharp = (B' + \chi_\tau a, \exp(\chi_\tau u), \chi_\tau \gamma),$$

where  $u$  is a real function such that  $\beta = \exp u$ .

**Remark 2.5.3** The fact that  $(A, \Phi)$  is in a gauge with exponential decay together with the identity (2.33) show that for some constant  $c > 0$ , we have for every  $N_0 \geq 1$  (cf definition (2.6) of  $\chi_\tau$ ), every  $\tau$  large enough and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$

$$\chi_\tau \left| (A, \Phi)^\sharp - (A, \Phi) \right| \leq c \chi_\tau (1 - \chi_{(\tau - N_0)}) e^{-\delta \sigma_\tau}.$$

Furthermore, similar estimates hold for all the covariant derivatives of  $\chi_\tau((A, \Phi)^\sharp - (A, \Phi))$ . Notice that the constants involved do not depend on  $N_0 \geq 1$  either, for the simple reason that the derivatives of  $\chi_\tau$  are uniformly bounded as  $N_0$  varies.

Indeed, the map  $\sharp$  provides some *very good* approximate solutions of Seiberg–Witten equations on  $M'_\tau$ . This claim is made precise in the next lemma. Beforehand, it will be convenient to package the equations on  $M'_\tau$  into a single equation:

$$\text{SW}(A', \Phi') = (F_{A'}^+ - \{\Phi' \otimes (\Phi')^*\}_0 - F_{B'_\tau}^+ + \{\Psi'_\tau \otimes (\Psi'_\tau)^*\}_0 - \varpi_\tau, D_{A'} \Phi').$$



**Lemma 2.5.4** *There exist  $\delta > 0$  and  $T$  large enough such that for every  $N_0 \geq 1$ ,  $k \in \mathbb{N}$ ,  $\tau \geq T + N_0$  and every solutions  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , we have*

$$\text{SW}(A, \Phi)^\# = 0 \text{ on } \{\sigma'_\tau \leq T\} \subset M'_\tau$$

and

$$|\text{SW}(A, \Phi)^\#|_{C^k(g'_\tau, A^\#)} \leq c_k e^{-\delta\sigma_\tau} \text{ on } \{\sigma'_\tau \geq T\} \subset M'_\tau$$

**Proof** It is trivial that  $\text{SW}(A, \Phi)^\# = \text{SW}(A, \Phi) = 0$  on  $\{\sigma'_\tau \leq \tau - N_0\}$  and  $\text{SW}(A, \Phi)^\# = \text{SW}(B', \Psi') = 0$  on  $\{\sigma'_\tau \geq \tau\}$ .

If we put  $(A, \Phi)$  in a gauge with exponential decay, we obtain the estimate of the lemma thanks to [Remark 2.5.3](#). It follows that the lemma is true for any gauge representative.  $\square$

A direct computation shows that

$$k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\# = \left( B' + \chi_\tau \tilde{a} - \tilde{v} d\chi_\tau, \exp(\chi_\tau(\tilde{u} + \tilde{v})), \chi_\tau \tilde{\gamma} \exp((\chi_\tau - 1)\tilde{v}) \right),$$

hence

$$(2.34) \quad k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\# - (A, \Phi)^\# = (\chi_\tau(\tilde{a} - a) + \zeta_1, \chi_\tau(\tilde{\beta} - \beta) + \zeta_2, \chi_\tau(\tilde{\gamma} - \gamma) + \zeta_3),$$

where

$$(2.35) \quad \begin{aligned} \zeta_1 &= -\tilde{v} d\chi_\tau, \\ \zeta_2 &= \exp(\chi_\tau(\tilde{u} + \tilde{v})) - \exp(\chi_\tau u) - \chi_\tau(\exp(\tilde{u} + \tilde{v}) - \exp u), \\ \zeta_3 &= \chi_\tau \tilde{\gamma}(\exp((\chi_\tau - 1)\tilde{v}) - 1), \end{aligned}$$

are some ‘smaller’ terms. More precisely,  $\zeta_j$  is controlled by  $\tilde{\beta} - \beta$  thanks to [Lemma 2.3.3](#). A direct consequence is that  $\#$  is uniformly locally Lipschitz in the following sense:

**Lemma 2.5.5** *There exist  $c, \alpha > 0$ , such that, for every  $N_0 \geq 1$ , every  $\tau$  large enough and every solution of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have for every configuration  $(\tilde{A}, \tilde{\Phi})$  on  $M_\tau$*

$$\begin{aligned} \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_{\frac{2}{3}}(g_\tau, A)} \leq \alpha &\Rightarrow \\ \|k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\# - (A, \Phi)^\# \|_{L^2_k(g'_\tau, A^\#)} &\leq c \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_k(g_\tau, A)} \\ &\text{for all } k = 0, 1, 2, 3. \end{aligned}$$

A key estimate in order to study the local injectivity of the (pre) gluing is given in the next Lemma.

**Lemma 2.5.6** *For every  $N_0$  large enough and  $\varepsilon > 0$ , there exists  $\alpha > 0$ , such that, for every  $\tau$  large enough, and, every pair of Seiberg–Witten solutions  $(A, \Phi)$ ,  $(\tilde{A}, \tilde{\Phi})$  on  $M_\tau$  with the conditions*

$$\delta_{1,(A,\Phi)}^*(\tilde{A} - A, \tilde{\Phi} - \Phi) = 0 \text{ and } \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_3(g_\tau, A)} \leq \alpha,$$

then

(2.36)

$$\left\| \delta_{1,(A,\Phi)}^* \left( k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\# - (A, \Phi)^\# \right) \right\|_{L^2_1(g'_\tau)} \leq \varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)},$$

(2.37)

$$\left\| \delta_{2,(A,\Phi)^\#} \left( k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\# - (A, \Phi)^\# \right) \right\|_{L^2_1(g'_\tau, A^\#)} \leq \varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}.$$

**Proof** We will only prove the estimate (2.37). The operator  $\delta_1^*$  is dealt with in completely similar way.

In order to prove the estimate (2.37), we inspect the contributions of each term of identity (2.34). Let  $\varepsilon > 0$ . We estimate the contribution of the terms  $\zeta_j$  defined at (2.35) first. According to Corollary 2.2.13, there exists a constant  $c_1 > 0$  independent of  $\tau$  and the solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , such that

$$\|\delta_{2,(A,\Phi)^\#} \zeta_1\|_{L^2_1(g'_\tau, A^\#)} \leq c_1 \|\zeta_1\|_{L^2_2(g'_\tau, A^\#)}.$$

The derivatives of  $\chi_\tau$  can be made arbitrarily small for  $N_0$  large enough (see (2.6)) so that

$$\|\zeta_1\|_{L^2_2(g'_\tau, A^\#)} \leq \frac{\varepsilon}{16c_1} \|\tilde{v}\|_{L^2_2(g_\tau, \{T < \sigma_\tau < \tau\})},$$

where  $L^2_2(g_\tau, \{T < \sigma_\tau < \tau\})$  is the  $L^2_2$  norm taken on the open set  $\{T < \sigma_\tau < \tau\}$ . Using the uniform exponential decay of  $(A, \Phi)$ , we deduce that for every  $\tau$  large enough, we have, say

$$\|\beta - 1\|_{L^2_3(g_\tau, \{T < \sigma_\tau < \tau\})} \leq \alpha,$$

hence

$$\|\tilde{\beta} - 1\|_{L^2_3(g_\tau, \{T < \sigma_\tau < \tau\})} \leq 2\alpha.$$

For  $\alpha$  small enough and  $\tau$  large enough, Corollary 2.3.6 apply hence

$$\|u\|_{L^2_3(g_\tau, \{T < \sigma_\tau < \tau\})}, \|\tilde{u} + \tilde{v}\|_{L^2_3(g_\tau, \{T < \sigma_\tau < \tau\})} \leq 4\alpha.$$

Thank to [Lemma 2.3.3](#) we have the estimate

$$\|\tilde{v} + \tilde{u} - u\|_{L^2_2(g_\tau, \{T < \sigma_\tau < \tau\})} \leq 2 \|\exp(\tilde{v} + \tilde{u}) - \exp u\|_{L^2_2(g_\tau, \{T < \sigma_\tau < \tau\})}.$$

Using the fact, the fact that  $A$  decays exponentially fast toward  $B$  again, the RHS is a lower bound for  $2\|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}$  for every  $\tau$  large enough.

Since  $\tilde{v} = \Im(\tilde{u} + \tilde{v} - u)$ , we have

$$\|\tilde{v}\|_{L^2_2(g_\tau, \{T < \sigma_\tau < \tau\})} \leq 4\|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}.$$

In conclusion we have we have shown that

$$(2.38) \quad \|\delta_{2, (A, \Phi)^\#} \zeta_1\|_{L^2_1(g'_\tau, A^\#)} \leq \frac{\varepsilon}{4} \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}.$$

Similar estimates show that for  $\alpha$  small enough, we also have

$$(2.39) \quad \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_3(g_\tau, A)} \leq \alpha \Rightarrow \|\delta_{2, (A, \Phi)^\#} \zeta_j\|_{L^2_1(g'_\tau, A^\#)} \leq \frac{\varepsilon}{4} \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)},$$

for  $j = 2, 3$ .

We need to estimate the contribution of the remaining linear terms displayed in [\(2.34\)](#):

$$\delta_{2, (A, \Phi)^\#} \left( \chi_\tau \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right) \right) = \chi_\tau \delta_{2, (A, \Phi)^\#} \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right) + \left( \left( d\chi_\tau \wedge (\tilde{A} - A) \right)^+, d\chi_\tau \cdot (\tilde{\Phi} - \Phi) \right).$$

Notice that  $\delta_{2, (A, \Phi)^\#} \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right)$  makes sense only on the domain  $\{\sigma'_\tau \leq \tau\} = \{\sigma_\tau \leq \tau\}$  where all bundles, metrics, etc... are equal.

Similarly to  $\zeta_1 = \tilde{v} d\chi_\tau$ , we can by increasing  $N_0$  make  $d\chi_\tau$  as small as required in order to have the estimate

$$(2.40) \quad \left\| \left( \left( d\chi_\tau \wedge (\tilde{A} - A) \right)^+, d\chi_\tau \cdot (\tilde{\Phi} - \Phi) \right) \right\|_{L^2_1(g'_\tau, A^\#)} \leq \frac{\varepsilon}{12} \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_1(g_\tau, A)}.$$

The operator  $\chi_\tau \left( \delta_{2, (A, \Phi)^\#} - \delta_{2, (A, \Phi)} \right)$  is linear operator of degree 0 and we have

$$\chi_\tau \left( \delta_{2, (A, \Phi)^\#} - \delta_{2, (A, \Phi)} \right) \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right) = \chi_\tau \left\{ (A, \Phi)^\# - (A, \Phi), (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right\},$$

where  $\{\cdot, \cdot\}$  is some bilinear pairing whose explicit expression is irrelevant to us. The uniform exponential decay property of  $\chi_\tau \left( (A, \Phi)^\# - (A, \Phi) \right)$  mentioned in [Remark](#)

2.5.3 show that for  $\tau$  large enough, we have for every pair of solutions  $(A, \Phi)$ ,  $(\tilde{A}, \tilde{\Phi})$  of Seiberg–Witten equations

$$\begin{aligned} \left\| \chi_\tau \left( \delta_{2, (A, \Phi)^\#} - \delta_{2, (A, \Phi)} \right) \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right) \right\|_{L_1^2(g'_\tau, A^\#)} \\ \leq \frac{\varepsilon}{12} \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L_1^2(g_\tau, A)}. \end{aligned}$$

Eventually, using the fact that  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  are both solution of Seiberg–Witten equations we have the identity

$$\delta_{2, (A, \Phi)} \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right) + \mathcal{Q} \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right) = 0,$$

where  $\mathcal{Q}$  is the quadratic term

$$(2.41) \quad \mathcal{Q}(a, \phi) = (\{\phi \otimes \phi^*\}_0, a \cdot \phi).$$

The Sobolev multiplication theorem  $L_2^2 \otimes L_2^2 \hookrightarrow L_1^2$  on every compact set together Lemmas 2.1.6 and 2.2.13 tell us that there exists a constant  $c_2 > 0$  independent of  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$ , such that for every  $\tau$  large enough we have

$$\|\mathcal{Q} \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right)\|_{L_1^2(g_\tau, A)} \leq c_2 \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L_2^2(g_\tau, A)}^2.$$

So, if we choose  $\alpha$  small enough, we have

$$(2.42) \quad \|\chi_\tau \delta_{2, (A, \Phi)} \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right)\|_{L_1^2(g'_\tau, A^\#)} \leq \frac{\varepsilon}{12} \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L_2^2(g_\tau, A)}^2.$$

Remark that the above estimate is true for every  $N_0 \geq 1$ , since  $\|d\chi_\tau\|_{C^0}$  decays as  $N_0$  increases. Adding up the estimate (2.38), (2.39), (2.40) and (2.42), we conclude

$$\left\| \delta_{2, (A, \Phi)^\#} \left( k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\# - (A, \Phi)^\# \right) \right\|_{L_1^2(g'_\tau, A^\#)} \leq \varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L_2^2(g_\tau, A)},$$

which proves the lemma.  $\square$

### 3 Gluing

Our goal is to perturb  $\sharp: \mathcal{M}(M_\tau) \rightarrow \mathcal{C}/\mathcal{G}(M'_\tau)$ , in order to get a *gluing map*

$$\mathfrak{G}: \mathcal{M}_{\varpi_\tau}(M_\tau) \rightarrow \mathcal{M}_{\varpi_\tau}(M'_\tau).$$

Moreover we will show that  $\mathfrak{G}$  is a diffeomorphism.

### 3.1 Construction of the gluing map

We seek a solution of Seiberg–Witten equations of the form

$$(3.1) \quad (A', \Phi') = (A, \Phi)^\# + \delta_{2, (A, \Phi)^\#}^*(b', \psi'),$$

where  $b'$  is a purely imaginary self-dual form and  $\psi'$  a section of the negative spinor bundle  $W_{(A, \Phi)}^-$  on  $M'_\tau$ . The Seiberg–Witten equations for  $(A', \Phi')$  viewed as equations bearing on  $(b', \psi')$  have the form

$$(3.2) \quad \Delta_{2, (A, \Phi)^\#}(b', \psi') + \mathcal{Q}(\delta_{2, (A, \Phi)^\#}^*(b', \psi')) = -\text{SW}(A, \Phi)^\#,$$

where

$$\Delta_{2, (A, \Phi)^\#} := \delta_{2, (A, \Phi)^\#} \delta_{2, (A, \Phi)^\#}^*$$

and  $\mathcal{Q}(a', \phi')$  is nonlinear part defined at (2.41).

**3.1.1 The linear problem** As usual we start by solving the linear problem.

**Proposition 3.1.2** *Assume that the moduli space,  $\mathcal{M}_{\overline{\omega}}(M)$  is unobstructed. Then, for each  $k \geq 0$ , there exists a constant  $c_k > 0$ , such that for every  $\tau$  large enough, every  $N_0 \geq 1$  (see definition of  $\chi_\tau$ ) and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , we have for every pairs  $(b, \psi)$  on  $M_\tau$  and  $(b', \psi')$  on  $M'_\tau$*

$$(3.3) \quad \|\Delta_{2, (A, \Phi)}(\psi, b)\|_{L_k^2(g_\tau, A)} \geq c_k \|(\psi, b)\|_{L_{k+2}^2(g_\tau, A)},$$

where  $\Delta_{2, (A, \Phi)} := \delta_{2, (A, \Phi)} \delta_{2, (A, \Phi)}^*$ , and

$$(3.4) \quad \|\Delta_{2, (A, \Phi)^\#}(b', \psi')\|_{L_k^2(g'_\tau, A)} \geq c_k \|(b', \psi')\|_{L_{k+2}^2(g'_\tau, A^\#)}.$$

**Proof** We will just prove the second statement (3.4), since the proof of first part of the proposition is the same with less complications.

Let  $\square_{(A, \Phi)^\#}$  be the operator  $\mathcal{D}_{(A, \Phi)^\#} \mathcal{D}_{(A, \Phi)^\#}^*$ , where  $\mathcal{D}_{(A, \Phi)^\#} = \delta_{1, (A, \Phi)^\#}^* \oplus \delta_{2, (A, \Phi)^\#}$  as defined in Section 2.2.2. Suppose that for some constant  $c > 0$ , we have for every  $\tau$  large enough, every  $N_0 \geq 1$ , we have the inequality

$$(3.5) \quad \|\square_{(A, \Phi)^\#}(u', b', \psi')\|_{L^2(g'_\tau)} \geq c \|(u', b', \psi')\|_{L_2^2(g'_\tau, A^\#)} \quad \text{for all } (u', b', \psi').$$

Then, the estimate on the operator  $\Delta_{2, (A, \Phi)^\#}$  follows. To see it we write

$$(3.6) \quad \square_{(A, \Phi)^\#}(u', b', \psi') = \left( \Delta_{1, (A, \Phi)^\#} u' + \mathcal{L}_{(A, \Phi)^\#}^*(b', \psi') \right) \oplus \left( \Delta_{2, (A, \Phi)^\#}(b', \psi') + \mathcal{L}_{(A, \Phi)^\#} u' \right)$$

where  $\mathcal{L}_{(A, \Phi)^\#} = \delta_{2, (A, \Phi)^\#} \delta_{1, (A, \Phi)^\#}$ .

If  $(A, \Phi)^\#$  were an exact solution of Seiberg–Witten equations, we would simply have  $\mathcal{L}_{(A, \Phi)^\#} = 0$  for

$$(3.7) \quad \mathcal{L}_{(A, \Phi)^\#} u' = (u' D_{A^\#} \Phi^\#, 0);$$

nevertheless, the operator  $\mathcal{L}_{(A, \Phi)^\#}$  has operator norm very close to 0 since  $(A, \Phi)^\#$  is an approximate solution in the sense of [Lemma 2.5.4](#). More precisely, we have for every  $N_0 \geq 1$  and every  $\tau$  large enough

$$\|\mathcal{L}_{(A, \Phi)^\#} u'\|_{L^2(g'_\tau)} + \|\mathcal{L}_{(A, \Phi)^\#}^*(b', \psi')\|_{L^2(g'_\tau)} \leq \frac{c}{2} \|(u', b', \psi')\|_{L^2(g'_\tau)}.$$

It follows from [\(3.6\)](#) and [\(3.5\)](#) that

$$(3.8) \quad \|\Delta_{2, (A, \Phi)^\#}(b', \psi')\|_{L^2(g'_\tau)} + \|\Delta_{1, (A, \Phi)^\#} u'\|_{L^2(g'_\tau)} \geq \frac{c}{2} \|(u', b', \psi')\|_{L^2_2(g'_\tau, A^\#)}.$$

This gives the control that we wanted by setting  $u' = 0$ .

It remains to show that [\(3.5\)](#) holds to finish the proof of the proposition. First we observe that the operator  $\mathcal{D}^*$  gives uniform control on sections supported near infinity.

**Lemma 3.1.3** *There exist  $T > 0$  large enough and a constant  $c > 0$  such that for every  $\tau > T$ , every  $N_0 \geq 1$  and every approximate solution  $(A, \Phi)^\#$  of Seiberg–Witten equations on  $M'_\tau$ , we have for all sections  $(u', b', \psi')$  with compact support in  $\{\sigma'_\tau > T\} \subset M'_\tau$*

$$\|\mathcal{D}_{(A, \Phi)^\#}^*(u', b', \psi')\|_{L^2(g'_\tau)} \geq c \|(u', b', \psi')\|_{L^2_1(g'_\tau, A^\#)}.$$

**Proof** We use the Weitzenböck formula derived in [\[10, Proposition 3.8\]](#) which says that

$$\begin{aligned} \|\mathcal{D}_{(A, \Phi)^\#}^*(u', b', \psi')\|_{L^2(g'_\tau)} = & \int_{\sigma'_\tau \geq T} \left( |du'|^2 + 4|\nabla b'|^2 + |\nabla_{A^\#} \psi'|^2 + |\Phi^\#|^2(|u'|^2 + |b'|^2 + |\psi'|^2) \right. \\ & + 4 \left\langle \left( \frac{s}{6} \text{id} - \mathcal{W}^+ \right) \cdot b', b' \right\rangle + \frac{s}{4} |\psi'|^2 + \frac{1}{2} \langle F_{A^\#}^- \cdot \psi', \psi' \rangle \\ & \left. + 4 \langle b' \otimes \psi', \nabla_{A^\#} \Phi^\# \rangle + 2 \langle \psi', u' D_{A^\#} \Phi^\# \rangle + 2 \langle b', \psi' \cdot D_{A^\#} \Phi^\# \rangle \right) \text{vol}^{g'_\tau}, \end{aligned}$$

where  $\mathcal{W}^+$  is the positive part of the Weyl curvature. Thanks to [Lemma 2.5.4](#), and the fact that the geometries for all  $M'_\tau$  are uniformly controlled by [Lemma 2.1.6](#), we know that  $s$ ,  $\mathcal{W}$ ,  $|\Phi^\#| - 1$ ,  $F_{A^\#}$ ,  $\nabla_{A^\#} \Phi^\#$ ,  $D_{A^\#} \Phi^\#$  are uniformly close to zero. Therefore, the terms of first line control all the others and the lemma is proved.  $\square$

Next we prove that similar estimates hold for large  $\tau$  and for global sections.

**Lemma 3.1.4** *There exists  $\kappa > 0$  such that for every  $\tau$  large enough, for every  $N_0 \geq 1$ , for every solution of Seiberg–Witten equations  $(A, \Phi)$  and for every (global) section  $(u', b', \psi')$  on  $M'_\tau$ , we have*

$$\|\mathcal{D}_{(A, \Phi)^\#}^*(u', b', \psi')\|_{L^2(g'_\tau)} \geq \kappa \| (u', b', \psi') \|_{L^2_1(g'_\tau, A^\#)}.$$

**Proof** Suppose this is not true. Then, we have sequences  $N_{0,j} \geq 1$  and  $\tau_j \rightarrow +\infty$  with  $\tau_j \geq T + N_{0,j}$ , some solutions of Seiberg–Witten equations  $(A_j, \Phi_j)$  on  $M_{\tau_j}$ , and  $(u'_j, b'_j, \psi'_j)$  such that

$$\|\mathcal{D}_{(A_j, \Phi_j)^\#}^*(u'_j, b'_j, \psi'_j)\|_{L^2(g'_{\tau_j})} \rightarrow 0, \quad \|(u'_j, b'_j, \psi'_j)\|_{L^2_1(g'_{\tau_j}, A_j^\#)} = 1.$$

After applying gauge transformations and extracting a subsequence, we may assume that  $(A_j, \Phi_j)$  converge on every compact set to a solution of Seiberg–Witten equations  $(A_\infty, \Phi_\infty)$  on  $M$ . Then, using the  $L^2_1$  bound on  $(u'_j, b'_j, \psi'_j)$ , the sequence converges on every compact subset to a weak limit  $(u, b, \psi) \in L^2_1(g, A_\infty)$  on  $M$  verifying  $\mathcal{D}_{(A_\infty, \Phi_\infty)^\#}^*(u, b, \psi) = 0$ . Using the assumption that the moduli space on  $M$  is unobstructed, this implies that  $(u, b, \psi) = 0$ . Using the compactness of the inclusion  $L^2_1 \subset L^2$  on compact sets, we see that the sequence  $(u'_j, b'_j, \psi'_j)$  converge strongly toward 0 in the  $L^2$ -sense on every compact set after further extraction.

Let  $\chi$  be a cut-off function equal to 1 on  $\{\sigma'_\tau \leq T\}$  and to 0 outside  $\{\sigma'_\tau \leq T + 1\}$ . Then

$$(3.9) \quad \|(u'_j, b'_j, \psi'_j)\|_{L^2(g'_\tau)} \leq \|\chi(u'_j, b'_j, \psi'_j)\|_{L^2(g'_\tau)} + \|(1 - \chi)(u'_j, b'_j, \psi'_j)\|_{L^2(g'_\tau)};$$

the first term in the RHS tends to 0 since  $(u'_j, b'_j, \psi'_j)$  converges to 0 on the compact set  $\sigma'_\tau \leq T + 1$ . The second term is supported in  $\sigma'_\tau \geq T$ , hence we can apply [Lemma 3.1.3](#) which says that it is controlled by

$$c \|\mathcal{D}_{(A_j, \Phi_j)^\#}^*(1 - \chi)(u'_j, b'_j, \psi'_j)\|_{L^2(g'_{\tau_j})}.$$

The derivatives of  $\chi$  are compactly supported; so the Leibniz rule for  $\mathcal{D}^*$  and the fact that  $\|\mathcal{D}_{(A_j, \Phi_j)^\#}^*(u'_j, b'_j, \psi'_j)\|_{L^2(g'_{\tau_j})} \rightarrow 0$ , shows that

$$\|\mathcal{D}_{(A_j, \Phi_j)^\#}^*(1 - \chi)(u'_j, b'_j, \psi'_j)\|_{L^2(g'_{\tau_j})} \rightarrow 0.$$

Therefore  $\|(1 - \chi)(u'_j, b'_j, \psi'_j)\|_{L^2(g'_{\tau_j})} \rightarrow 0$ , hence  $\|(u'_j, b'_j, \psi'_j)\|_{L^2(g'_{\tau_j})} \rightarrow 0$  by (3.9). This is a contradiction and the lemma is proved.  $\square$

We are ready to prove that the estimate (3.5) holds. Under the assumption of Lemma 3.1.4, we have

$$\int_{M'_\tau} \langle \square_{(A, \Phi)^\#}(u', b', \psi'), (u', b', \psi') \rangle \operatorname{vol}^{g'_\tau} = \|\mathcal{D}_{(A, \Phi)^\#}^*(u', b', \psi')\|_{L^2(g'_\tau)}^2 \geq \kappa^2 \|(u', b', \psi')\|_{L^2_1(g'_\tau, A^\#)}^2.$$

By Cauchy–Schwarz inequality, the LHS is a lower bound for

$$\left\| \square_{(A, \Phi)^\#}(u', b', \psi') \right\|_{L^2(g'_\tau)} \left\| (u', b', \psi') \right\|_{L^2_1(g'_\tau, A^\#)},$$

and we conclude that

$$(3.10) \quad \left\| \square_{(A, \Phi)^\#}(u', b', \psi') \right\|_{L^2(g'_\tau)} \geq \kappa^2 \|(u', b', \psi')\|_{L^2_1(g'_\tau, A^\#)}.$$

The operator  $\square_{(A, \Phi)^\#}$  is elliptic of order 2. Hence, for some constant  $c_2 > 0$ , we have on a small ball  $B_\alpha \subset M'_\tau$  of radius  $\alpha$ , the estimate

$$\left\| \square_{(A, \Phi)^\#}(u', b', \psi') \right\|_{L^2(g'_\tau, B_\alpha)} + \|(u', b', \psi')\|_{L^2(g'_\tau, B_\alpha)} \geq c_2 \|(u', b', \psi')\|_{L^2_2(g'_\tau, A^\#, B_{\alpha/2})}.$$

The fact that the approximate solution of Seiberg–Witten equations  $(A, \Phi)^\#$  on  $M'_\tau$  is controlled thanks to Corollary 2.2.13 and Remark 2.5.3, that the geometry of the  $M'_\tau$  is uniformly AFAK in the sense of Lemma 2.1.6 implies that the constant  $c'_2$  can be chosen independently of  $\tau$  (provided it is large enough), of the approximate solution  $(A, \Phi)^\#$ , of the constant  $N_0 \geq 1$  involved in our construction, and of the center of the ball  $B_\alpha$ . It follows that there exists a constant  $c'_2 > 0$ , independent of all data as  $c_2$ , such that

$$\left\| \square_{(A, \Phi)^\#}(u', b', \psi') \right\|_{L^2(g'_\tau)} + \|(u', b', \psi')\|_{L^2(g'_\tau)} \geq c'_2 \|(u', b', \psi')\|_{L^2_2(g'_\tau, A^\#)}.$$

Together with (3.10), this implies that we have an estimate of the form (3.3), thus Proposition 3.1.2 is proved for  $k = 0$ . For  $k \geq 1$ , similar estimates hold by elliptic regularity as in the proof of Lemma 2.3.2.  $\square$

As an immediate consequence, we have the following Corollary:

**Corollary 3.1.5** *If the Seiberg–Witten equations on  $M$  are unobstructed, the Seiberg–Witten equations on  $M_\tau$  (or  $M'_\tau$ ) are unobstructed for every  $\tau$  large enough.*

**Proof** Proposition 3.1.2 together with (3.3) shows in particular that for every  $\tau$  large enough,  $\delta_{2, (A, \Phi)}$  is surjective for every  $(A, \Phi)$ , which means that the moduli spaces  $\mathcal{M}_{\varpi_\tau}(M_\tau)$  and  $\mathcal{M}_{\varpi_\tau}(M'_\tau)$  are unobstructed.  $\square$



**Remark** In fact, we already assumed in [Section 2.2.2](#) that the Seiberg–Witten equation on  $M$  are unobstructed, by choosing a generic perturbation  $\varpi$ . We will always suppose that it is the case from now on, unless stated.

Here is another immediate consequence of [Proposition 3.1.2](#), directly related to our gluing problem.

**Corollary 3.1.6** *There exists a constant  $c > 0$  such that for every  $\tau$  large enough, every  $N_0 \geq 1$  and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , the operator  $\Delta_{2,(\mathcal{A},\Phi)\#}: L_2^2 \rightarrow L^2$  is an isomorphism, and, moreover, its inverse  $\Delta_{2,(\mathcal{A},\Phi)\#}^{-1}$  verifies*

$$(3.11) \quad c \|(b', \psi')\|_{L_1^2(g'_\tau, \mathcal{A}\#)} \geq \|\Delta_{2,(\mathcal{A},\Phi)\#}^{-1}(b', \psi')\|_{L_3^2(g'_\tau, \mathcal{A}\#)} \quad \text{for all } (b', \psi').$$

**3.1.7 The nonlinear problem** Using the substitution  $V = \Delta_{2,(\mathcal{A},\Phi)\#}(b', \psi')$ , we can rewrite the equation [\(3.2\)](#) in the form

$$(3.12) \quad V = S_{(\mathcal{A},\Phi)\#}(V) - \text{SW}(A, \Phi)\#,$$

where

$$S_{(\mathcal{A},\Phi)\#}(V) = -\mathcal{Q}(\delta_{2,(\mathcal{A},\Phi)\#}^* \Delta_{2,(\mathcal{A},\Phi)\#}^{-1} V).$$

The key argument for solving Equation [\(3.12\)](#) is that the operator  $S$  is a uniform contraction in the sense of the next lemma.

**Lemma 3.1.8** *There exist constants  $\alpha > 0$ ,  $\kappa \in (0, 1/2)$ , such that for every  $\tau$  large enough, every  $N_0 \geq 1$  and every approximate solution of Seiberg–Witten equations on  $M'_\tau$  we have*

$$\begin{aligned} \text{for all } V_1, V_2 \quad \|V_1\|_{L^2(g'_\tau)}, \|V_2\|_{L_1^2(g'_\tau, \mathcal{A}\#)} \leq \alpha \Rightarrow \\ \|S_{(\mathcal{A},\Phi)\#}(V_2) - S_{(\mathcal{A},\Phi)\#}(V_1)\|_{L_1^2(g'_\tau, \mathcal{A}\#)} \leq \kappa \|V_2 - V_1\|_{L_1^2(g'_\tau, \mathcal{A}\#)}. \end{aligned}$$

**Proof** We choose  $\tau$  large enough and a constant  $c > 0$  according to [Corollary 3.1.6](#).

$$S_{(\mathcal{A},\Phi)\#}(V_2) - S_{(\mathcal{A},\Phi)\#}(V_1) = \mathcal{Q}\left(\delta_{2,(\mathcal{A},\Phi)\#}^* \Delta_{2,(\mathcal{A},\Phi)\#}^{-1} V_2\right) - \mathcal{Q}\left(\delta_{2,(\mathcal{A},\Phi)\#}^* \Delta_{2,(\mathcal{A},\Phi)\#}^{-1} V_1\right).$$

Since  $\mathcal{Q}(a, \phi)$  is a quadratic polynomial in  $(a, \phi)$ , we deduce using the Sobolev multiplication Theorem  $L_2^2 \otimes L_2^2 \rightarrow L_1^2$ , a control

$$(3.13) \quad \begin{aligned} & \left\| \mathcal{Q}\left(\delta_{2,(\mathcal{A},\Phi)\#}^* \Delta_{2,(\mathcal{A},\Phi)\#}^{-1} V_2\right) - \mathcal{Q}\left(\delta_{2,(\mathcal{A},\Phi)\#}^* \Delta_{2,(\mathcal{A},\Phi)\#}^{-1} V_1\right) \right\|_{L_1^2(g'_\tau, \mathcal{A}\#)} \\ & \leq C \|\delta_{2,(\mathcal{A},\Phi)\#}^* \Delta_{2,(\mathcal{A},\Phi)\#}^{-1} (V_1 + V_2)\|_{L_2^2(g'_\tau, \mathcal{A}\#)} \|\delta_{2,(\mathcal{A},\Phi)\#}^* \Delta_{2,(\mathcal{A},\Phi)\#}^{-1} (V_1 - V_2)\|_{L_2^2(g'_\tau, \mathcal{A}\#)}, \end{aligned}$$

where  $C$  is a constant which depend neither on  $\tau$  (large enough) nor on the approximate solution of Seiberg–Witten equations  $(A, \Phi)^\sharp$  on  $M'_\tau$ , nor on the constant  $N_0 \geq 1$  involved in the construction of  $\sharp$ .

The bounds on the approximate solution  $(A, \Phi)^\sharp$  deduced from those of  $(A, \Phi)$ , show that there is a constant  $C_2 > 0$  independent of  $\tau$  (large enough),  $(A, \Phi)^\sharp$  and  $N_0 \geq 1$ , such that

$$(3.14) \quad \text{for all } (b', \psi'), \quad \|\delta_{2,(A,\Phi)^\sharp}(b', \psi')\|_{L^2_2(g_\tau, A^\sharp)} \leq C_2 \|(b', \psi')\|_{L^2_3(g_\tau, A^\sharp)}.$$

In particular this holds for  $(b', \psi') = \Delta_{2,(A,\Phi)^\sharp}^{-1}(V_1 \pm V_2)$ . Together with (3.11) we deduce from (3.13) the estimate

$$\begin{aligned} & \left\| \mathcal{Q}(\delta_{2,(A,\Phi)^\sharp}^* \Delta_{2,(A,\Phi)^\sharp}^{-1} V_2) - \mathcal{Q}(\delta_{2,(A,\Phi)^\sharp}^* \Delta_{2,(A,\Phi)^\sharp}^{-1} V_1) \right\|_{L^2_1(g'_\tau, A^\sharp)} \\ & \leq c C C_2 \|V_2 + V_1\|_{L^2_1(g'_\tau, A^\sharp)} \|V_2 - V_1\|_{L^2_1(g'_\tau, A^\sharp)}, \end{aligned}$$

and the lemma holds for  $\alpha = \frac{\kappa}{2c C C_2}$ .  $\square$

In Since  $S$  is contractant in a suitable sense and  $\text{SW}(A, \Phi)^\sharp$  converges uniformly to 0 in the sense of Lemma 2.5.4, we can solve equation (3.12) thanks to Proposition 2.3.5 for  $V \in L^2_1$ . Then we obtain a solution of (3.1) given by  $(b', \psi') = \Delta_{2,(A,\Phi)^\sharp}^{-1} V$ , hence  $(b', \psi') \in L^2_3$ . More precisely, we have the following theorem:

**Theorem 3.1.9** (Definition of the gluing map) *There exist constants  $\alpha, c > 0$  such that for every  $\tau$  large enough, every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$  and every constant  $N_0 \geq 1$  (see definition of  $\sharp$ ), there is a unique section  $(b', \psi')$  on  $M'_\tau$  such that*

$$\mathfrak{G}(A, \Phi) = (A, \Phi)^\sharp + \delta_{2,(A,\Phi)^\sharp}^*(b', \psi')$$

is a solution of Seiberg–Witten equations on  $M'_\tau$ , with  $\|(b', \psi')\|_{L^2_3(g, A^\sharp)} \leq \alpha$ .

The map  $\mathfrak{G}$  is smooth and gauge equivariant and induces map

$$\mathfrak{G}: \mathcal{M}_{\varpi_\tau}(M_\tau) \rightarrow \mathcal{M}_{\varpi_\tau}(M'_\tau),$$

furthermore

(3.15)

$$\|(b', \psi')\|_{L^2_3(g_\tau, A^\sharp)}, \|\mathfrak{G}(A, \Phi) - (A, \Phi)^\sharp\|_{L^2_2(g'_\tau, A^\sharp)} \leq c \|\text{SW}(A, \Phi)^\sharp\|_{L^2_1(g'_\tau, A^\sharp)}.$$

**Proof** We have already proved the existence of a solution  $(b', \psi') \in L^2_3$ . Therefore  $\mathfrak{G}(A, \Phi) \in \mathcal{C}_2(M'_\tau)$ . If  $(A, \Phi) \in \mathcal{C}_l(M_\tau)$ , we have  $\text{SW}(A, \Phi)^\sharp \in L^2_{l-1}$  and it follows

by elliptic regularity of Seiberg–Witten equations that  $\mathfrak{G}(A, \Phi) \in \mathcal{C}_l(M'_\tau)$ . If we choose  $l \geq 4$  (which is required to define the moduli space), we have then a well defined induced map  $\mathfrak{G}: \mathcal{M}_{l, \varpi_\tau}(M_\tau) \rightarrow \mathcal{M}_{l, \varpi_\tau}(M'_\tau)$ .

The smoothness of  $\mathfrak{G}$  follows from the smoothness of  $\sharp$  and from the fact that  $S_{(A, \Phi)^\sharp}$  in (3.12) depends smoothly on the parameter  $(A, \Phi)^\sharp$ .

The only part of the theorem left to be proved is the estimate (3.15). Recall that  $V = \Delta_{2, (A, \Phi)^\sharp}(b', \psi')$  where  $V$  is a solution of (3.12) provided by Proposition 2.3.5. Therefore,  $V$  verifies

$$\|V\|_{L^2_1(g'_\tau, A^\sharp)} \leq 2\|\text{SW}(A, \Phi)^\sharp\|_{L^2_1(g_\tau, A^\sharp)}.$$

Let  $c_1 > 0$  be a constant obtained by Proposition 3.1.2 such that

$$c_1\|(b', \psi')\|_{L^2_3(g'_\tau, A^\sharp)} \leq \|\Delta_{2, (A, \Phi)^\sharp}(b', \psi')\|_{L^2_1(g'_\tau)} = \|V\|_{L^2_1(g'_\tau, A^\sharp)}$$

Similarly to estimate (3.14), there is a constant  $c_2 > 0$  independent of  $N_0 \geq 1$ ,  $\tau$  (large enough) and  $(A, \Phi)^\sharp$ , such that

$$(3.16) \quad \text{for all } (b', \psi'), \quad \|\delta_{2, (A, \Phi)^\sharp}^*(b', \psi')\|_{L^2_2(g_\tau, A^\sharp)} \leq c_2\|(b', \psi')\|_{L^2_3(g_\tau, A^\sharp)}.$$

Eventually (3.15) is verified for  $c = \max(2c_2/c_1, 2/c_1)$ .  $\square$

### 3.2 From local to global study of the gluing map

We will prove that the gluing is locally injective in the following sense:

**Proposition 3.2.1** *There exist  $N_0 \geq 1$  (for the construction the pregluing map) large enough, and a constants  $\alpha_2 > 0$ , such that for every  $\tau$  large enough and every solution  $(A, \Phi)$  of Seiberg–Witten equations on  $M_\tau$ , the gluing map  $\mathfrak{G}: \mathcal{M}_{\varpi_\tau}(M_\tau) \rightarrow \mathcal{M}_{\varpi_\tau}(M'_\tau)$  restricted to the open set*

$$B([A, \Phi], \alpha_2) = \{[\tilde{A}, \tilde{\Phi}] \in \mathcal{M}_{\varpi_\tau}(M_\tau) \mid \exists u \in \mathcal{G}(M_\tau), \\ \|u \cdot (\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} < \alpha_2\}$$

*is an embedding.*

**Remark** The set  $B([A, \Phi], \alpha)$  is just a ball of center  $[A, \Phi]$  and ‘ $L^2_2$ -radius’  $\alpha_2$  in  $\mathcal{M}_l(M_\tau)$ .

Before giving the proof of this proposition we show how it implies Theorem E.

**Corollary 3.2.2** *For every  $\tau$  and  $N_0$  large enough, the gluing map*

$$\mathfrak{G}: \mathcal{M}_{\varpi_\tau}(M_\tau) \rightarrow \mathcal{M}_{\varpi_\tau}(M'_\tau)$$

*is a diffeomorphism.*

**Proof** We already know that  $\mathfrak{G}$  is a local diffeomorphism, so we just need to prove that it is 1 : 1. We show first that  $\mathfrak{G}$  is globally injective for  $\tau$  large enough. Suppose it is not true: then we have a sequence  $\tau_j \rightarrow \infty$  and solutions of Seiberg–Witten equations  $(A_j, \Phi_j)$  and  $(\tilde{A}_j, \tilde{\Phi}_j)$  on  $M_{\tau_j}$  such that  $[A_j, \Phi_j] \neq [\tilde{A}_j, \tilde{\Phi}_j]$  and  $\mathfrak{G}(A_j, \Phi_j) = u'_j \cdot \mathfrak{G}(\tilde{A}_j, \tilde{\Phi}_j)$  for some gauge transformations  $u'_j \in \mathcal{G}_l(M'_{\tau_j})$ .

After applying gauge transformations and extracting a subsequence, we may assume that  $(A_j, \Phi_j)$  and  $(\tilde{A}_j, \tilde{\Phi}_j)$  have exponential decay and converge on every compact to some solutions  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  of Seiberg–Witten equations on  $M$ . Notice that if  $(A, \Phi) = (\tilde{A}, \tilde{\Phi})$ , it implies

$$\|(A, \Phi) - (\tilde{A}, \tilde{\Phi})\|_{L^2_2(g_\tau, A_j)} \rightarrow 0.$$

We are going to see that it is indeed the case:  $\|\mathfrak{G}(A_j, \Phi_j) - (A_j, \Phi_j)^\# \|_{L^2_2(g'_\tau, A_j^\#)} \rightarrow 0$  according to [Theorem 3.1.9](#) and [Lemma 2.5.4](#). Hence  $\mathfrak{G}(A_j, \Phi_j)$  converges on every compact toward  $(A, \Phi)$  since  $(A_j, \Phi_j)^\#$  does. Similarly  $\mathfrak{G}(\tilde{A}_j, \tilde{\Phi}_j)$  converges to  $(\tilde{A}, \tilde{\Phi})$ . The fact that  $\mathfrak{G}(A_j, \Phi_j)$  and  $\mathfrak{G}(\tilde{A}_j, \tilde{\Phi}_j)$  are gauge equivalent for each  $j$  implies that the limits are also gauge equivalent. After making further gauge transformations, we can assume that  $(A_j, \Phi_j)$  and  $(\tilde{A}_j, \tilde{\Phi}_j)$  converge toward the same limit  $(A, \Phi)$  on  $M$ .

Therefore we have for  $j$  large enough

$$\|(\tilde{A}_j, \tilde{\Phi}_j) - (A_j, \Phi_j)\|_{L^2_2(g_\tau, A_j)} < \alpha,$$

where  $\alpha$  is chosen according to [Proposition 3.2.1](#). The fact that  $[\mathfrak{G}(\tilde{A}_j, \tilde{\Phi}_j)] = [\mathfrak{G}(A_j, \Phi_j)]$  should then imply  $[\tilde{A}_j, \tilde{\Phi}_j] = [A_j, \Phi_j]$ . This is a contradiction, hence  $\mathfrak{G}$  must be injective.

We show now that  $\mathfrak{G}$  is surjective: for  $\tau$  large enough, there is a second gluing map

$$\mathfrak{G}': \mathcal{M}_{\varpi_\tau}(M'_\tau) \rightarrow \mathcal{M}_{\varpi_\tau}(M_\tau)$$

since  $M'_\tau$  and  $M_\tau$  play symmetric roles. The map  $\mathfrak{G}'$  enjoys all the properties of  $\mathfrak{G}$ . In particular  $\mathfrak{G}' \circ \mathfrak{G}$  is an embedding of  $\mathcal{M}_{\varpi_\tau}(M_\tau)$  into itself. Using the fact that  $\mathcal{M}_{\varpi_\tau}(M_\tau)$  is a finite dimensional compact manifold, we conclude that  $\mathfrak{G}' \circ \mathfrak{G}$  is therefore a diffeomorphism. Therefore,  $\mathfrak{G}$  must be surjective, otherwise, we would

have an  $[A', \Phi'] \notin \mathfrak{S}\mathfrak{G}$ . On the other hand, using the fact that  $\mathfrak{G}' \circ \mathfrak{G}$  is surjective, there exists  $[A, \Phi] \in \mathcal{M}_{\varpi_\tau}(M_\tau)$  such that  $\mathfrak{G}' \circ \mathfrak{G}[A, \Phi] = \mathfrak{G}'[A', \Phi']$  which contradicts the injectivity of  $\mathfrak{G}'$ .  $\square$

**Remarks 3.2.3**

- An immediate consequence of [Corollary 3.2.2](#) is that  $\sharp$  is an embedding for every  $\tau$  and  $N_0$  large enough.
- More generally, under the assumption of [Theorem 3.1.9](#) and [Corollary 3.2.2](#), we can introduce the map

$$\mathfrak{G}_s(A, \Phi) = (A, \Phi)^\sharp + s\delta_{2,(A,\Phi)^\sharp}^*(b', \psi'),$$

for a parameter  $s \in [0, 1]$ . Then,  $\mathfrak{G}_s(\mathcal{M}_{\varpi_\tau}(M_\tau))$  realizes an isotopy between  $\sharp(\mathcal{M}_{\varpi_\tau}(M_\tau))$  and  $\mathcal{M}_{\varpi_\tau}(M'_\tau)$  in  $\mathcal{C}/\mathcal{G}(M'_\tau)$ .

We return now to the proof of [Proposition 3.2.1](#).

**Proof** It is convenient to study locally  $\mathfrak{G}$  in the charts provided by the slice theorem. We begin by observing that we have charts of radius uniformly bounded from below in [Theorem 2.3.1](#).

Let  $\alpha_1, \alpha_2 \in (0, \alpha_1]$  and  $U_{\alpha_1}$  be as in [Theorem 2.3.1](#), for a solution  $(A, \Phi)$  of the Seiberg–Witten equations on  $M_\tau$ . Put  $(A', \Phi') := \mathfrak{G}(A, \Phi)$  and denote  $\alpha'_1, \alpha'_2$  and  $U'_{\alpha'_1}$  the analogous data on  $M'_\tau$ .

Let  $(\tilde{A}, \tilde{\Phi})$  be a solution of Seiberg–Witten equations such that

$$[\tilde{A}, \tilde{\Phi}] \in B([A, \Phi], \alpha_2).$$

Then, we may assume, thanks to the slice theorem, that up to a gauge transformation, we have  $(\tilde{A}, \tilde{\Phi}) \in U_{\alpha_1}$ .

We are going to show that if  $\alpha_1$  is chosen small enough, independently of  $(A, \Phi)$ ,  $N_0 \geq 1$  and  $\tau$  large enough, then we automatically have  $[\mathfrak{G}(\tilde{A}, \tilde{\Phi})] \in B([\mathfrak{G}(A, \Phi)], \alpha'_2)$ .

The next lemma is a classical application of elliptic regularity for Seiberg–Witten equations and [Proposition 2.2.15](#). Again, the constant involved can be chosen uniformly thanks to the fact that it is the case for Sobolev constants due to [Lemma 2.1.6](#), and that the moduli spaces are uniformly bounded in the sense of [Corollary 2.2.13](#).

**Lemma 3.2.4** *There exist constants  $C, \alpha_1 > 0$  and a compact set  $K \subset M$  large enough such that for every  $\tau$  large enough and every solutions  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  of Seiberg–Witten equations on  $M_\tau$  with*

$$\delta_{1,(A,\Phi)}^* \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right) = 0 \quad \text{and} \quad \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \leq \alpha_1,$$

we have

$$\|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_3(g_\tau, A)} \leq C \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2(g_\tau, K)}.$$

The solutions provided by the gluing map have the form

$$\begin{aligned} (A', \Phi') &:= \mathfrak{G}(A, \Phi) = (A, \Phi)^\# + \delta_{2,(A,\Phi)^\#}^*(b', \psi') \\ \mathfrak{G}(\tilde{A}, \tilde{\Phi}) &= (\tilde{A}, \tilde{\Phi})^\# + \delta_{2,(\tilde{A},\tilde{\Phi})^\#}^*(\tilde{b}', \tilde{\psi}') \end{aligned}$$

for some  $(\psi, b)$  and  $(\tilde{\psi}', \tilde{b}')$  given by [Theorem 3.1.9](#). Moreover we have

$$(3.17) \quad \|\delta_{2,(A,\Phi)^\#}^*(b', \psi')\|_{L^2_2(g_\tau, A^\#)} \leq c \|\text{SW}(A, \Phi)^\#\|_{L^2_1(g'_\tau, A^\#)},$$

(3.18)

$$\|(\tilde{b}', \tilde{\psi}')\|_{L^2_3(g_\tau, \tilde{A}^\#)}, \quad \|\delta_{2,(\tilde{A},\tilde{\Phi})^\#}^*(\tilde{b}', \tilde{\psi}')\|_{L^2_2(g_\tau, \tilde{A}^\#)} \leq c \|\text{SW}(\tilde{A}, \tilde{\Phi})^\#\|_{L^2_1(g'_\tau, \tilde{A}^\#)},$$

and we may assume that the above quantities are uniformly small for  $\tau$  large enough according to [Lemma 2.5.4](#).

Let  $k_{(\tilde{A}, \tilde{\Phi})}: W_{(A, \Phi)} \rightarrow W_{(\tilde{A}, \tilde{\Phi})}$  be the isomorphism defined in [Section 2.5](#) and put

$$(\hat{A}, \hat{\Phi}) = k_{(\tilde{A}, \tilde{\Phi})}^{-1} \cdot (\tilde{A}, \tilde{\Phi})^\#$$

so that  $(A, \Phi)^\#$  and  $(\hat{A}, \hat{\Phi})$  are now defined for the same spin bundle  $W_{(A, \Phi)}$ . Then

$$k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) = (\hat{A}, \hat{\Phi}) + \delta_{2,(\hat{A}, \hat{\Phi})}^*(\hat{b}', \hat{\psi}'),$$

where  $(\hat{b}', \hat{\psi}') := k_{(\tilde{A}, \tilde{\Phi})}^{-1}(\tilde{b}', \tilde{\psi}')$ . Notice that with this notation,  $k_{(\tilde{A}, \tilde{\Phi})}$  is the identity on the bundle of self-dual forms. By gauge invariance

$$(3.19) \quad \|\delta_{2,(\tilde{A}, \tilde{\Phi})^\#}^*(\tilde{b}', \tilde{\psi}')\|_{L^2_2(g_\tau, \tilde{A}^\#)} = \|\delta_{2,(\hat{A}, \hat{\Phi})}^*(\hat{b}', \hat{\psi}')\|_{L^2_2(g_\tau, \hat{A})},$$

$$(3.20) \quad \|(\tilde{b}', \tilde{\psi}')\|_{L^2_3(g_\tau, \tilde{A}^\#)} = \|(\hat{b}', \hat{\psi}')\|_{L^2_3(g_\tau, \hat{A})}.$$

Put  $(a', \phi') = (\hat{A}, \hat{\Phi}) - (A, \Phi)^\#$ ; this is just the variation of  $\sharp$  with our rough gauge choice. If  $\alpha_1$  is small enough, we get an  $L^2_3$ -estimate on  $(\tilde{A}, \tilde{\Phi}) - (A, \Phi)$  by [Lemma](#)

3.2.4, hence we can assume that [Lemma 2.5.5](#) applies and we have eventually an estimate

$$(3.21) \quad \|(a', \phi')\|_{L^2_3(g'_\tau, A^\#)} \leq cC \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \leq cC\alpha_1.$$

Therefore, the Sobolev multiplication theorems show that for a choice of  $\alpha_1$  small enough the  $L^2_k(g'_\tau, A^\#)$  and  $L^2_k(g'_\tau, \hat{A})$  are commensurate up to  $k = 3$ , so that, say

$$(3.22) \quad \|\delta^*_{2,(\hat{A}, \hat{\Phi})}(\hat{b}', \hat{\psi}')\|_{L^2_2(g_\tau, A^\#)} \leq 2\|\delta^*_{2,(\hat{A}, \hat{\Phi})}(\hat{b}', \hat{\psi}')\|_{L^2_2(g_\tau, \hat{A})}$$

$$(3.23) \quad \|(\hat{b}', \hat{\psi}')\|_{L^2_3(g_\tau, A^\#)} \leq 2\|(\hat{b}', \hat{\psi}')\|_{L^2_3(g_\tau, \hat{A})}.$$

Eventually,  $\alpha_1$  controls the  $L^2_2(g_\tau, A^\#)$ -norm of  $(a', \phi')$  by (3.21). The  $L^2_2(g_\tau, A^\#)$  norm of  $\delta^*_{2,(\hat{A}, \hat{\Phi})}(\hat{b}', \hat{\psi}')$  is controlled by  $\|\text{SW}(\tilde{A}, \tilde{\Phi})^\# \|_{L^2_1(g_\tau, \tilde{A}^\#)}$  thanks to (3.18) (3.19) and (3.22), while the one of  $\delta^*_{2,(A, \Phi)^\#}(b', \psi')$  is controlled via (3.17) by  $\|\text{SW}(A, \Phi)^\# \|_{L^2_1(g_\tau, A^\#)}$ .

Similarly, the  $L^2_2(g_\tau, A^\#)$  and  $L^2_2(g_\tau, A')$  are also commensurate, and if  $\tau$  is large enough, we may replace  $A^\#$  by  $A'$  in the estimates (3.17) and (3.22). That means that for a suitable choice of  $\alpha_1$ , we will have automatically  $[\mathfrak{G}(\tilde{A}, \tilde{\Phi})] \in B([\mathfrak{G}(A, \Phi)], \alpha'_2)$ .

Therefore, using the slice theorem about  $[A', \Phi'] = [\mathfrak{G}(A, \Phi)]$ , we can recast  $\mathfrak{G}$  into a map

$$\mathfrak{G}_1: \mathcal{U} \rightarrow U'_{\alpha'_1},$$

where  $\mathcal{U}$  is a finite dimensional submanifold of  $U_{\alpha_1}$  corresponding to Seiberg–Witten moduli space  $\mathcal{M}_{\varpi_\tau}(M_\tau)$  in this local neighborhood of  $[A, \Phi]$ .

Everything is in order to study the variations of  $\mathfrak{G}$  about  $[A, \Phi]$ , that is to say the variations of  $\mathfrak{G}_1$  about  $0 \in \mathcal{U}$ .

**Lemma 3.2.5** *For every  $N_0$  large enough and  $\varepsilon > 0$ , there exists  $\alpha_1 > 0$  such that for every  $\tau$  large enough and every pair  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  of solutions of Seiberg–Witten equations with  $(\tilde{A}, \tilde{\Phi})$  in the slice about  $(A, \Phi)$  and with  $\|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \leq \alpha_1$ ,*

$$\begin{aligned} \|\Delta_{2,(A, \Phi)^\#} \left( (\hat{b}', \hat{\psi}') - (b', \psi') \right) \|_{L^2_1(g'_\tau, A^\#)} \\ \leq \varepsilon \left( \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} + \|(\hat{b}', \hat{\psi}') - (b', \psi')\|_{L^2_3(g'_\tau, A^\#)} \right) \end{aligned}$$

where  $(b', \hat{\psi}')$  and  $(b', \psi')$  are defined as above.

**Proof** Notice first that we have

$$\delta_{2,(\widehat{A},\widehat{\Phi})\#}^*(\widehat{b}',\widehat{\psi}') = \delta_{2,(A,\Phi)\#}^*(\widehat{b}',\widehat{\psi}') + \{(a',\phi'),(\widehat{b}',\widehat{\psi}')\},$$

where  $\{\cdot, \cdot\}$  is a bilinear pairing with fixed coefficients, whose particular expression is irrelevant to us. More generally, we will denote any bilinear pairing in this way in the proof of the lemma.

The Seiberg–Witten equations for  $k_{(\widetilde{A},\widetilde{\Phi})}^{-1} \mathfrak{G}(\widetilde{A},\widetilde{\Phi})$  give the identity

$$\begin{aligned} & \delta_{2,(A,\Phi)\#}(a',\phi') + \Delta_{2,(A,\Phi)\#}(\widehat{b}',\widehat{\psi}') + \delta_{2,(A,\Phi)\#} \{(a',\phi'),(\widehat{b}',\widehat{\psi}')\} \\ & + \mathcal{Q}((a',\phi') + \delta_{2,(A,\Phi)\#}^*(\widehat{b}',\widehat{\psi}') + \{(a',\phi'),(\widehat{b}',\widehat{\psi}')\}) = -\text{SW}(A,\Phi)\#, \end{aligned}$$

where  $\mathcal{Q}$  is the quadratic term of the Seiberg–Witten defined at (2.41). Taking the difference with (3.2), we have

$$\begin{aligned} \Delta_{2,(A,\Phi)\#}((\psi',b') - (\widehat{b}',\widehat{\psi}')) &= \delta_{2,(A,\Phi)\#}(\dot{a},\dot{\phi}) + \delta_{2,(A,\Phi)\#} \{(a',\phi'),(\widehat{b}',\widehat{\psi}')\} \\ & - \mathcal{Q}(\delta_{2,(A,\Phi)\#}^*(b',\psi')) + \mathcal{Q}((a',\phi') + \delta_{2,(A,\Phi)\#}^*(\widehat{b}',\widehat{\psi}') + \{(a',\phi'),(\widehat{b}',\widehat{\psi}')\}) \end{aligned}$$

Developing the last line, we find an expression which is formally

(3.24)

$$\Delta_{2,(A,\Phi)\#}((\psi',b') - (\widehat{b}',\widehat{\psi}')) = \delta_{2,(A,\Phi)\#}((a',\phi')$$

(3.25)

$$+ \mathcal{Q}(\delta_{2,(A,\Phi)\#}^*(\widehat{b}',\widehat{\psi}')) - \mathcal{Q}(\delta_{2,(A,\Phi)\#}^*(b',\psi'))$$

(3.26)

$$+ \mathcal{Q}((a',\phi'))$$

(3.27)

$$+ \mathcal{Q}(\{(a',\phi'),(\widehat{b}',\widehat{\psi}')\})$$

(3.28)

$$+ \delta_{2,(A,\Phi)\#} \{(a',\phi'),(\widehat{b}',\widehat{\psi}')\}$$

(3.29)

$$+ \{(a',\phi'),\{(a',\phi'),(\widehat{b}',\widehat{\psi}')\}\}$$

(3.30)

$$+ \{\delta_{2,(A,\Phi)\#}^*(\widehat{b}',\widehat{\psi}'),\{(a',\phi'),(\widehat{b}',\widehat{\psi}')\}\}$$

(3.31)

$$+ \{\delta_{2,(A,\Phi)\#}^*(\widehat{b}',\widehat{\psi}'),(a',\phi')\}$$

We explain how to control each term of the RHS: we know that the  $L_3^2(g_\tau, A^\#)$ -norm of  $(a',\phi')$  is controlled by its  $\alpha_1$  (see (3.21)). Hence we may assume that Lemma 2.5.6 applies. Therefore, for  $N_0 \geq 1$  large enough, we have

$$\|\delta_{2,(A,\Phi)\#}^*(a',\phi')\|_{L_1^2(g_\tau, A^\#)} \leq \frac{\varepsilon}{7} \|(\widetilde{A},\widetilde{\Phi}) - (A,\Phi)\|_{L_2^2(g_\tau, A)}.$$



The term of line (3.25) is controlled using the fact that  $\mathcal{Q}$  is a quadratic polynomial. Similarly to (3.13), we have an estimate

$$\begin{aligned} & \left\| \mathcal{Q} \left( \delta_{2, (A, \Phi)^\#}^* (\widehat{b}', \widehat{\psi}') \right) - \mathcal{Q} \left( \delta_{2, (A, \Phi)^\#}^* (b', \psi') \right) \right\|_{L_1^2(g'_\tau, A^\#)} \\ & \leq c \| (b', \psi') - (\widehat{b}', \widehat{\psi}') \|_{L_3^2(g'_\tau, A^\#)} \| (b', \psi') + (\widehat{b}', \widehat{\psi}') \|_{L_3^2(g'_\tau, A^\#)}. \end{aligned}$$

Since the  $L_3^2(g'_\tau, A^\#)$ -norm of  $(b', \psi')$  and  $(\widehat{b}', \widehat{\psi}')$  is arbitrarily small for  $\alpha_1$  small enough and  $\tau$  large enough, we deduce that for a suitable choice of  $\alpha_1$ , we will have

$$\begin{aligned} & \left\| \mathcal{Q} \left( \delta_{2, (A, \Phi)^\#}^* (\widehat{b}', \widehat{\psi}') \right) - \mathcal{Q} \left( \delta_{2, (A, \Phi)^\#}^* (b', \psi') \right) \right\|_{L_1^2(g'_\tau, A^\#)} \\ & \leq \varepsilon \| (b', \psi') - (\widehat{b}', \widehat{\psi}') \|_{L_3^2(g'_\tau, A^\#)}, \end{aligned}$$

for every  $\tau$  large enough.

A similar technique together with Lemma 2.5.5 and (3.21) shows that the term at line (3.26) is controlled under the same circumstances by

$$\| \mathcal{Q}((a', \phi')) \|_{L_1^2(g'_\tau, A^\#)} \leq \frac{\varepsilon}{7} \| (\widetilde{A}, \widetilde{\Phi}) - (A, \Phi) \|_{L_2^2(g_\tau, A)}.$$

For the term at line (3.27), we use the fact that it is this time a homogeneous polynomial expression of degree 4 in  $(a', \psi')$  and  $(\widehat{b}', \widehat{\psi}')$ . Thanks to the Sobolev embedding theorem  $L_2^2 \hookrightarrow L^8$ , we deduce that

$$\| \mathcal{Q} \left( \left\{ (a', \phi'), (\widehat{b}', \widehat{\psi}') \right\} \right) \|_{L_1^2(g'_\tau, A^\#)} \leq c \| (a', \phi') \|_{L_3^2(g'_\tau, A^\#)}^2 \| (b', \psi') \|_{L_3^2(g'_\tau, A^\#)}^2.$$

The estimate (3.17) and the fact that the  $L_3^2(g'_\tau, A^\#)$ -norm of  $(\widehat{b}', \widehat{\psi}')$  is arbitrarily small for  $\alpha_1$  small enough and  $\tau$  large lead to an estimate

$$\| \mathcal{Q} \left( \left\{ (a', \phi'), (\widehat{b}', \widehat{\psi}') \right\} \right) \|_{L_1^2(g'_\tau, A^\#)} \leq \frac{\varepsilon}{7} \| (\widetilde{A}, \widetilde{\Phi}) - (A, \Phi) \|_{L_2^2(g_\tau, A)}.$$

Arguing in the same manner, it is easy to show that the  $L_1^2(g'_\tau, A^\#)$ -norms of (3.28), (3.29), (3.30) and (3.31) are lower bounds for

$$\frac{\varepsilon}{7} \| (\widetilde{A}, \widetilde{\Phi}) - (A, \Phi) \|_{L_2^2(g_\tau, A)}$$

if  $\alpha_1$  is chosen small enough and  $\tau$  large enough. Summing up all the estimates, we obtain the lemma.  $\square$

We return to the proof of [Proposition 3.2.1](#) Applying [Proposition 3.1.2](#), we deduce that for some universal constant  $c_1 > 0$ , we have

$$c_1 \|(b', \hat{\psi}') - (b', \psi')\|_{L^2_3(g'_\tau, A^\#)} \leq \|\Delta_{2, (A, \Phi)^\#} \left( (b', \hat{\psi}') - (b', \psi') \right)\|_{L^2_1(g'_\tau, A^\#)}.$$

Together with the [Lemma 3.2.5](#) this implies

$$(3.32) \quad (c_1 - \varepsilon) \|(\hat{\psi}', \hat{b}') - (b', \psi')\|_{L^2_3(g'_\tau, A^\#)} \leq \varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}.$$

Using the estimate [\(3.16\)](#), that we have

$$\|\delta_{2, (A, \Phi)^\#}^* \left( (\hat{b}', \hat{\phi}') - (b', \psi') \right)\|_{L^2_2(g'_\tau, A^\#)} \leq c_2 \|(\hat{b}', \hat{\phi}') - (b', \psi')\|_{L^2_3(g'_\tau, A^\#)},$$

and it follows, once we made sure that we started with  $\varepsilon < c_1$ , that

$$\|\delta_{2, (A, \Phi)^\#}^* \left( (\hat{b}', \hat{\phi}') - (b', \psi') \right)\|_{L^2_2(g'_\tau, A^\#)} \leq \frac{c_2 \varepsilon}{c_1 - \varepsilon} \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}.$$

Now

$$(3.33) \quad k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - \mathfrak{G}(A, \Phi) \\ = (a', \phi') + \delta_{2, (A, \Phi)^\#}^* \left( (\hat{b}', \hat{\psi}') - (b', \psi') \right) + \left\{ (a', \phi'), (\hat{b}', \hat{\phi}') \right\},$$

and the last term is controlled as in the proof of [Lemma 3.2.5](#), i.e. for  $\tau$  large enough, we have say

$$\left\| \left\{ (a', \phi'), (\hat{b}', \hat{\phi}') \right\} \right\|_{L^2_2(g'_\tau, A^\#)} \leq \varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}.$$

Hence we have the estimate

$$(3.34) \quad \|(a', \phi')\|_{L^2_2(g'_\tau, A^\#)} - \left( \varepsilon + \frac{c_2 \varepsilon}{c_1 - \varepsilon} \right) \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \\ \leq \|k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - \mathfrak{G}(A, \Phi)\|_{L^2_2(g'_\tau, A^\#)}.$$

Then, we can apply [Lemma 3.2.4](#), and since the identity  $(a', \phi') = (\tilde{A}, \tilde{\Phi}) - (A, \Phi)$  holds on every compact set for  $\tau$  large enough, we have

$$(3.35) \quad \left( \frac{1}{C} - \varepsilon - \frac{c_2 \varepsilon}{c_1 - \varepsilon} \right) \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \\ \leq \|k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - \mathfrak{G}(A, \Phi)\|_{L^2_2(g'_\tau, A^\#)}.$$

If we take  $\varepsilon$  small enough in the first place, so that  $0 < \varepsilon + \frac{c_2 \varepsilon}{c_1 - \varepsilon} < \frac{1}{2C}$ , we see immediately that  $\mathfrak{G}_1$  is injective.

However, a little more work is needed to see that  $\mathfrak{G}_1$  is an immersion at the origin. We estimate first how far is  $k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi})$  from being in a Coulomb gauge w.r.t.  $\mathfrak{G}(A, \Phi)$  in the next lemma.

**Lemma 3.2.6** *For every  $N_0$  large enough and  $\varepsilon > 0$ , there exists  $\alpha_1 > 0$  such that for every  $\tau$  large enough and every pair  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  of solutions of Seiberg–Witten equations on  $M_\tau$  with  $(\tilde{A}, \tilde{\Phi})$  in the slice about  $(A, \Phi)$  and with  $\|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \leq \alpha_1$ ,*

$$(3.36) \quad \|\delta_{1, (A', \Phi')}^* \left( k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - (A', \Phi') \right)\|_{L^2_1(g'_\tau)} \leq \varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}$$

where  $(A', \Phi') := \mathfrak{G}(A, \Phi)$ .

**Proof** We have

$$(3.37)$$

$$\delta_{1, (A, \Phi)^\#}^* \left( k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - (A', \Phi') \right) = \delta_{1, (A, \Phi)^\#}^* (a', \phi')$$

$$(3.38) \quad + \delta_{1, (A, \Phi)^\#}^* \delta_{2, (A, \Phi)^\#}^* \left( (\hat{b}', \hat{\psi}') - (b', \psi') \right)$$

$$(3.39) \quad + \delta_{1, (A, \Phi)^\#}^* \left\{ (a', \phi'), (\hat{b}', \hat{\phi}') \right\}.$$

The  $L^2_1(g'_\tau)$  norm of the RHS of (3.37) is controlled via Lemma 3.2.5. The operator  $\mathcal{L}_{(A, \Phi)^\#} = \delta_{2, (A, \Phi)^\#} \delta_{1, (A, \Phi)^\#}$  has norm very close to 0 by Lemma 2.5.4 and (3.7), therefore, the  $L^2_1(g'_\tau)$ -norm of the term in (3.38) is controlled using (3.32). The term (3.39) is controlled using Sobolev multiplication as in Lemma 3.2.5.  $\square$

An immediate consequence of Lemma 3.2.6 concerns the problem of fixing a Coulomb gauge:

**Corollary 3.2.7** *For every  $N_0$  large enough and  $\varepsilon > 0$ , there exists  $\alpha_1 > 0$  such that for every  $\tau$  large enough and every pair  $(A, \Phi)$  and  $(\tilde{A}, \tilde{\Phi})$  of solutions of Seiberg–Witten equations on  $M_\tau$  with  $(\tilde{A}, \tilde{\Phi})$  in the slice about  $(A, \Phi)$  and with  $\|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \leq \alpha_1$ , there exists a gauge transformation  $u' \in \mathcal{G}_2(M'_\tau)$  such that*

$$\begin{aligned} \delta_{1, (A', \Phi')}^* \left( u' \cdot k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - (A', \Phi') \right) &= 0 \\ \|1 - u'\|_{L^2_3(g'_\tau)} &\leq \varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}. \end{aligned}$$

**Proof** Using [Lemma 3.2.6](#), we can show that the  $L^2_{2,3}(g_\tau, A')$ -norm of

$$d_0 v^{-1} \left( k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - (A', \Phi') \right)$$

is controlled by  $\varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}$  (see [Section 2.3](#)).

Then [Corollary 2.3.7](#) implies that  $u' = e^{v'}$ , with the  $L^2_3(g'_\tau)$ -norm of  $v'$  controlled by  $\varepsilon \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)}$ . The corollary follows using [Corollary 2.3.6](#).  $\square$

We can finish the proof of [Proposition 3.2.1](#). Applying [Corollary 3.2.7](#) with a choice of  $\varepsilon$  small enough, we deduce from [\(3.35\)](#) that for some constant  $c > 0$ , we have

$$c \|(\tilde{A}, \tilde{\Phi}) - (A, \Phi)\|_{L^2_2(g_\tau, A)} \leq \|u' \cdot k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - (A', \Phi')\|_{L^2_2(g_\tau, A)}.$$

Since  $u' \cdot k_{(\tilde{A}, \tilde{\Phi})}^{-1} \mathfrak{G}(\tilde{A}, \tilde{\Phi}) - (A', \Phi') = \mathfrak{G}_1 \left( (\tilde{A}, \tilde{\Phi}) - (A, \Phi) \right)$ , the above inequality shows that  $\mathfrak{G}_1$  is an immersion at the origin.  $\square$

### 3.3 Orientations

In this section, we show that with suitable conventions, the gluing map  $\mathfrak{G}: \mathcal{M}_{\varpi_\tau}(M_\tau) \rightarrow \mathcal{M}_{\varpi_\tau}(M'_\tau)$ , is orientation preserving.

Suppose that we are given an orientation of  $\mathcal{M}_{\varpi_\tau}(M_\tau)$  that is to say an orientation of the index bundle  $\Omega(\mathcal{D})$  where  $\mathcal{D}_{(A, \Phi)}$  is the operator defined at [\(2.8\)](#). The aim of the next section is to explain how to deduce an orientation of  $\mathcal{M}_{\varpi_\tau}(M'_\tau)$  from the orientation of  $\mathcal{M}_{\varpi_\tau}(M_\tau)$ .

**3.3.1 Excision principle for the index** Let  $\bar{N}$  be an almost complex manifold with boundary  $\partial \bar{N} = \partial \bar{M} = Y$ . Such a manifold always exists (see for instance [\[7, Lemma 4.4\]](#)). From this data, we can construct  $N_\tau$  and  $N'_\tau$  by gluing the AFAK ends  $Z$  and  $Z'$  similarly to  $M_\tau$  and  $M'_\tau$ . Moreover  $N_\tau$  and  $N'_\tau$  are endowed with their canonical  $\text{Spin}^c$ -structures of almost complex type and suitable Hermitian metrics  $\tilde{g}_\tau$  and  $\tilde{g}'_\tau$ . Remark that the canonical solutions  $(B, \Psi)$  and  $(B', \Psi')$  have now a natural extension to the compact parts of  $N_\tau$  and  $N'_\tau$ . The identity  $\Psi = (1, 0) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$  and the spin connection  $B$  deduced from the Hermitian metric and the Chern connection on the determinant line bundle of the  $\text{Spin}^c$ -structure now make sense globally on  $N_\tau$ . Similarly,  $(B', \Psi')$  is defined globally on  $N'_\tau$ .

The almost complex structure and the Hermitian metric  $\tilde{g}_\tau$  on  $N_\tau$  induce a nondegenerate 2-form  $\omega_\tau$ . However, this 2-form is not closed in general. It has torsion (or a Lee form)  $\theta$  defined by

$$d\omega_\tau = \theta \wedge \omega_\tau.$$

Notice that by construction of  $\tilde{g}_\tau$ ,  $d\omega_\tau$  is identified with a compactly supported form on  $N$ , independent of  $\tau$ , and so can  $\theta$ .

According to a computation of Taubes [14] and later Gauduchon [6], we have in this case

$$D_B \Phi = D^{can} \Phi + \frac{1}{4} \theta \cdot \Phi,$$

where  $\theta$  acts by Clifford product on spinors and  $D^{can} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ . Therefore,  $(B, \Psi)$  is solution of the perturbed Seiberg–Witten equations

$$(3.40) \quad F_A^+ - \{\Phi \otimes \Phi^*\}_0 = F_B^+ - \{\Psi \otimes \Psi^*\}_0$$

$$(3.41) \quad D_A \Phi = \frac{1}{4} \theta \cdot \Phi.$$

The linearization  $\tilde{\mathcal{D}}_{(B, \Psi)}$  of the above equations differs from the linearization  $\mathcal{D}_{(B, \Psi)}$  of the standard one by a 0–order term. Precisely

$$(3.42) \quad \tilde{\mathcal{D}}_{(B, \Psi)}(a, \phi) = \mathcal{D}_{(B, \Psi)}(a, \phi) - (0, 0, \frac{1}{4} \theta \cdot \phi).$$

According to the computation [10, Proposition 3.8], we see that for every  $\lambda > 0$  large enough, the operator  $\tilde{\mathcal{D}}_{(B, \lambda \Psi)}$  is injective (equivalently we could just scale the metric). Replacing  $\Psi = (1, 0)$  by  $\lambda \Psi = (\lambda, 0)$  for such a large  $\lambda$  does not affect in anything to the theory developed so far since the normalization for  $\Psi$  was purely arbitrary. However, for simplicity of notation, we will assume that  $\lambda = 1$  is a suitable value in the sequel.

The index of  $\tilde{\mathcal{D}}_{(B, \Psi)}$  equals the index of  $\mathcal{D}_{(B, \Psi)}$  for the two operator are homotopic according to (3.42). But  $d := \tilde{\mathcal{D}}_{(B, \Psi)}$  is by definition the virtual dimension of the moduli space for a  $\text{Spin}^c$ –structure of almost complex type. This is zero, therefore  $\tilde{\mathcal{D}}_{(B, \Psi)}$  is an isomorphism. Hence the index bundle  $\Omega(\tilde{\mathcal{D}})$  is canonically trivial at  $(B, \Psi)$  and we deduce a compatible orientation of the index bundle.

The same remarks apply to  $N'_\tau$  where  $(B', \Psi')$  is solution of some modified Seiberg–Witten equation similar to (3.40) whose linearization  $\tilde{\mathcal{D}}_{(B', \Psi')}$  may be assumed to be an isomorphism.

We are now going to construct a generalization of the pregluing map  $\sharp$  of Section 2.4.2. Let  $U \subset \mathcal{C}_I(M_\tau) \times \mathcal{C}_I(N'_\tau)$  be the gauge invariant open set that consists into pairs of configurations  $((A, \Phi), (\tilde{A}', \tilde{\Phi}'))$ , such that  $|\beta|, |\tilde{\beta}'| > 0$  (with  $\Phi = (\beta, \gamma)$  and  $\tilde{\Phi} = (\tilde{\beta}, \tilde{\gamma})$ ) on the end  $\{\sigma_\tau \geq T\} \cap \{\tilde{\sigma}'_\tau \geq T\}$ . Because of the uniform exponential decay of Seiberg–Witten solutions, we may choose  $T$  large enough so that for every solutions of Seiberg–Witten equations  $(A, \Phi)$  on  $M_\tau$ , we have  $((A, \Phi), (B', \Psi')) \in U$ .

Remark that the cut-off function  $\chi_\tau$  defined at (2.6) makes sense over  $M_\tau$ ,  $M'_\tau$ ,  $N_\tau$  and  $N'_\tau$ . Then we define a positive function  $\mu_\tau$  on these manifolds by the condition that

$$\chi_\tau^2 + \mu_\tau^2 = 1.$$

For any pair  $((A, \Phi), (\tilde{A}', \tilde{\Phi}')) \in U$  we may assume that, after making a suitable gauge transformation,  $(A, \Phi)$  and  $(\tilde{A}', \tilde{\Phi}')$  are in real gauge on the end. Then we can define a configuration on  $M'_\tau$  by

$$(A', \Phi') = (B' + \chi_\tau a + \mu_\tau \tilde{a}', (\beta^{\chi_\tau} (\tilde{\beta}')^{\mu_\tau}, \chi_\tau \gamma + \mu_\tau \tilde{\gamma}'))$$

and on  $N_\tau$  by

$$(\tilde{A}, \tilde{\Phi}) = (B - \mu_\tau a + \chi_\tau \tilde{a}', (\beta^{-\mu_\tau} (\tilde{\beta}')^{\chi_\tau}, -\mu_\tau \gamma + \chi_\tau \tilde{\gamma}'))$$

where  $A = B + a$  and  $\tilde{A}' = B' + \tilde{a}'$  on the end. The configurations  $(\tilde{A}, \tilde{\Phi})$  and  $(A', \Phi')$  extend in a natural way to  $M'_\tau$  and  $N_\tau$ .

Thus we have defined a map

$$f: \mathcal{C}/\mathcal{G}(M_\tau) \times \mathcal{C}/\mathcal{G}(N'_\tau) \supset U/(\mathcal{G}(M_\tau) \times \mathcal{G}(N'_\tau)) \rightarrow \mathcal{C}/\mathcal{G}(M'_\tau) \times \mathcal{C}/\mathcal{G}(N_\tau)$$

by  $f([A, \Phi], [\tilde{A}', \tilde{\Phi}']) = ([A', \Phi'], [\tilde{A}, \tilde{\Phi}])$ . In order to study the variations of  $f$  we need to make a gauge choice. The definition of a rough gauge  $k$  as in Section 2.5 can be extended trivially to this setting and we can study the variations of  $k^{-1} \circ f$  around an element of  $U$  as we did for the map  $\sharp$ . A computation similar to (2.34) shows that the variation of  $f$  about  $((A, \Phi), (\tilde{A}', \tilde{\Phi}'))$  with a rough gauge fixing is given by the linear map

$$F: T_{(A, \Phi)}\mathcal{C}(M_\tau) \oplus T_{(\tilde{A}', \tilde{\Phi}')} \mathcal{C}(N'_\tau) \longrightarrow T_{(A', \Phi')} \mathcal{C}(M'_\tau) \oplus T_{(\tilde{A}, \tilde{\Phi})} \mathcal{C}(N_\tau)$$

of the form

$$F = F_0 + d\chi_\tau F_1 + d\mu_\tau F_2,$$

where  $F_1$  and  $F_2$  are two matrices with constant coefficients whose expression is here irrelevant to us and

$$F_0 = \begin{pmatrix} \chi_\tau & \mu_\tau \\ -\mu_\tau & \chi_\tau \end{pmatrix}.$$

This latter matrix is invertible. Hence for a choice of  $N_0$  large enough, we have  $d\chi_\tau$  and  $d\mu_\tau$  get arbitrarily small. Hence we can assume that  $F$  is invertible and is homotopic to  $F_0$  through isomorphisms.

Let  $P$  and  $R$  be the operators

$$P = \mathcal{D}_{(A, \Phi)} \oplus \tilde{\mathcal{D}}_{(\tilde{A}', \tilde{\Phi}')} , \quad R = \mathcal{D}_{(A', \Phi')} \oplus \tilde{\mathcal{D}}_{(\tilde{A}, \tilde{\Phi})}.$$

Since we assume that we are given an orientation of the index bundle of  $\mathcal{D}_{(A,\Phi)}$  and that we have a canonical way for orienting the index bundle of  $\mathcal{D}_{(\tilde{A}',\tilde{\Phi}')}$ , we deduce that the index bundle of  $P$  comes with an orientation.

Since  $F$  is invertible, it makes sense to talk about the differential operator  $f_*P := F \circ f \circ F^{-1}$  on  $M'_\tau \times N_\tau$ . We will deduce an orientation for the index bundle of  $R$  thanks to the next lemma.

**Lemma 3.3.2** *The operator  $F_0PF_0^{-1} - R$  is linear of order 0 and has compactly supported coefficients.*

**Proof** The fact that  $\chi_\tau^2 + \mu_\tau^2 = 1$  implies

$$F_0^{-1} = \begin{pmatrix} \chi_\tau & -\mu_\tau \\ \mu_\tau & \chi_\tau \end{pmatrix},$$

hence

$$\begin{aligned} F_0PF_0^{-1} &= \begin{pmatrix} \chi_\tau & \mu_\tau \\ -\mu_\tau & \chi_\tau \end{pmatrix} \begin{pmatrix} \mathcal{D}_{(A,\Phi)} & 0 \\ 0 & \tilde{\mathcal{D}}_{(\tilde{A}',\tilde{\Phi}')} \end{pmatrix} \begin{pmatrix} \chi_\tau & -\mu_\tau \\ \mu_\tau & \chi_\tau \end{pmatrix} \\ &= \begin{pmatrix} \chi_\tau \mathcal{D}_{(A,\Phi)} \chi_\tau + \mu_\tau \tilde{\mathcal{D}}_{(\tilde{A}',\tilde{\Phi}')} \mu_\tau & -\chi_\tau \mathcal{D}_{(A,\Phi)} \mu_\tau + \mu_\tau \tilde{\mathcal{D}}_{(\tilde{A}',\tilde{\Phi}')} \chi_\tau \\ -\mu_\tau \mathcal{D}_{(A,\Phi)} \chi_\tau + \chi_\tau \tilde{\mathcal{D}}_{(\tilde{A}',\tilde{\Phi}')} \mu_\tau & \mu_\tau \mathcal{D}_{(A,\Phi)} \mu_\tau + \chi_\tau \tilde{\mathcal{D}}_{(\tilde{A}',\tilde{\Phi}')} \chi_\tau \end{pmatrix} \end{aligned}$$

We apply the Leibniz rule together with the fact that  $d\chi_\tau$  and  $d\mu_\tau$  are supported in the annulus  $\chi_\tau\mu_\tau \neq 0$ . Moreover, the operators  $\mathcal{D}_{(A,\Phi)}$ ,  $\mathcal{D}_{(A',\Phi')}$ ,  $\tilde{\mathcal{D}}_{(\tilde{A},\tilde{\Phi})}$  and  $\tilde{\mathcal{D}}_{(\tilde{A}',\tilde{\Phi}')}$  differ only by some 0-order terms on the annulus where  $\chi_\tau\mu_\tau \neq 0$ . We deduce that the above matrix is equal to

$$\begin{pmatrix} \mathcal{D}_{(A',\Phi')} & 0 \\ 0 & \tilde{\mathcal{D}}_{(\tilde{A},\tilde{\Phi})} \end{pmatrix} + K$$

where  $K$  is a matrix with coefficients of order 0, compactly supported in the annulus  $\chi_\tau\mu_\tau \neq 0$ , and the lemma is proved.  $\square$

It follows that the operators  $f_*P$  and  $R$  are homotopic by [Lemma 3.3.2](#). So an orientation of  $\Omega(P)$  induces an orientation of  $\Omega(R)$ . Remark that this definition is consistent since  $U$  is path-connected. The index bundle of  $\mathcal{D}_{(\tilde{A},\tilde{\Phi})}$  is canonically oriented on  $N_\tau$ , so the orientation of  $\Omega(P)$  induces finally an orientation of  $\Omega(\mathcal{D}_{(A',\Phi')})$ .

**3.3.3 Gluing** We restrict now the map  $f$  to  $\mathcal{M}_{\varpi_\tau}(M_\tau) \times [B, \Psi]$ . We have clearly  $f = \sharp \times e$ , where  $e$  is a map from  $\mathcal{M}_{\varpi_\tau}$  into  $\mathcal{C}/\mathcal{G}(N_\tau)$ . Because the solutions of Seiberg–Witten equations have uniform exponential decay, we can assume that the image of  $e$  is contained in an arbitrarily small neighborhood of  $[B, \Psi]$ . Hence  $e$  is homotopic to the constant map  $e([A, \Phi]) = [B, \Psi]$ .

On the other hand, the pregluing  $\sharp$  is isotopic to the gluing map  $\mathfrak{G}$  (see [Remarks 3.2.3](#)). It follows that  $f(\mathcal{M}_{\varpi_\tau}(M_\tau) \times [B', \Psi'])$  is isotopic to  $\mathcal{M}_{\varpi_\tau}(M'_\tau) \times [B, \Psi]$ .

The first manifold is oriented by the choice of an orientation for  $\Omega(f_*P) = f_*\Omega(P)$  while the latter is oriented by orienting  $\Omega(R)$ . Since those orientations are compatible and the two manifolds are isotopic, it follows that the gluing map is orientation preserving.

**Lemma 3.3.4** *The gluing map  $\mathfrak{G}$  is orientation preserving.*

## 4 Proof of the main theorems

### 4.1 The proof of [Theorem D](#)

Let  $\bar{Z}$  be a special symplectic cobordism from  $(Y, \xi)$  to  $(Y', \xi')$ . Since it is *special*, there is a collar neighborhood  $[T_0, T_1) \times Y$  of  $Y \subset \partial\bar{Z}$  and a contact form  $\eta$  on  $Y$  such that the symplectic form  $\omega$  is given by  $\frac{1}{2}d(t^2\eta)$ . We can glue a sharp cone on the boundary  $Y$  by extending the collar neighborhood into  $(0, T_1) \times Y$  together with its symplectic form.

The next argument now follows closely [\[10, Lemma 4.1\]](#). Let  $(0, 1] \times Y'$  be a collar neighborhood of  $Y' \subset \bar{Z}$  and  $f(t)$  be a smooth increasing function equals to 0 on  $(0, 1/2]$  and which tends to infinity as  $t$  goes to 1. We perturb the symplectic form to

$$\omega_Z = \omega + \frac{1}{2}d(f^2\eta'),$$

where  $\eta'$  is a contact form on  $Y'$ . Now, we have a noncompact manifold  $Z = ((0, T_0) \times Y) \cup (\bar{Z} \setminus Y')$  together with a symplectic form  $\omega_Z$ . We define a function  $\sigma_Z$  on  $Z$  by

- $\sigma(t, y) = t$  on  $(0, T_0) \times Y$ ,
- $\sigma(t, y') = f(t)$  on  $f^{-1}[T_0 + 1, \infty) \times Y'$ ,

and we extend  $\sigma$  to the remaining part of  $Z$  with the condition that  $T_0 \leq \sigma \leq T_0 + 1$ .



With a suitable choice of compatible almost complex structure  $J_Z$ , the data

$$(Z, \omega_Z, J_Z, \sigma_Z)$$

is an AFAK end (see [Definition 2.1.2](#)). Notice that the condition [\(2.2\)](#) is verified as a direct consequence of the assumption [\(1.1\)](#) for special cobordims.

The piece  $(0, T_0) \times Y \subset Z$  has now a structure of almost Kähler cone. We extend it to the infinite almost Kähler cone  $Z' = (0, \infty) \times Y$  as defined in [Section 1.3.1](#).

Let  $\bar{M}$  be a compact manifold whose boundary  $Y$  is endowed with an element  $\mathfrak{s} \in \text{Spin}^c(\bar{M}, \xi)$ . Put  $\bar{M}' = \bar{M} \cup \bar{Z}$ . We apply the construction of [Section 2.1.5](#)

$$\bar{M}, Z \rightsquigarrow M_\tau \quad \text{and} \quad \bar{M}, Z' \rightsquigarrow M'_\tau.$$

The moduli space of Seiberg–Witten equations on  $M'_\tau$  leads to the invariant  $\text{sw}_{\bar{M}, \xi}(\mathfrak{s})$  while the moduli space on  $M_\tau$  leads to  $\text{sw}_{\bar{M}', \xi'}(j(\mathfrak{s}))$  (beware of the notation). On the other hand, the gluing [Theorem E](#) says that for  $\tau$  large enough, a suitable choice of perturbation and given compatible consistent orientations, the Seiberg–Witten moduli spaces for  $M_\tau$  and  $M'_\tau$  are generic and orientation preserving diffeomorphic via the gluing map  $\mathfrak{G}$ . Therefore  $\text{sw}_{\bar{M}, \xi}(\mathfrak{s}) = \pm \text{sw}_{\bar{M}', \xi'}(j(\mathfrak{s}))$  as does the more precise formulation hence [Theorem D](#) holds.

## 4.2 Surgery and monopoles

Let  $\bar{M}^4$ ,  $(Y, \xi)$  and  $K$  a Legendrian knot be as in [Theorem A](#).

**Result [15]** Let  $\bar{Z}$  be the cobordism between  $Y$  and the manifold  $Y'$  obtained by a 1–handle surgery on  $Y$ , or, from a 2–handle surgery along  $K$  with framing coefficient relative to the canonical framing is  $-1$ . Then  $Y'$  is endowed with a contact structure  $\xi'$  and  $\bar{Z}$  has a structure of symplectic cobordism between the contact manifolds  $Y$  and  $Y'$ . Moreover the symplectic form of  $\bar{Z}$  is equal to a symplectization of the contact structures in a collar neighborhood of  $Y$  and  $Y'$ .

This result of Weinstein generalizes to the symplectic category some techniques developed by Eliashberg in the case of Stein domains [\[3\]](#). The Weinstein surgeries are particular special symplectic cobordisms. The only thing that has to be checked is the property [\(1.1\)](#), namely that  $i^*: H^1(\bar{Z}, Y') \rightarrow H^1(Y)$  is the zero map: take a 1–cycle  $\mathcal{S}$  in  $Y$ . We can always perturb  $\mathcal{S}$  so that it avoids a neighborhood of the locus along which the surgery is performed. Then  $\mathcal{S}$  is homologous to a 1–cycle  $\mathcal{S}'$  in  $Y'$ . Let  $\Omega$  be a cohomology class in  $H^1(\bar{Z}, Y')$ . Then  $\Omega \cdot \mathcal{S} = \Omega \cdot \mathcal{S}' = 0$  hence  $i^*\Omega = 0$ .

Before we prove [Theorem A](#), notice that together with [Theorem D](#), they imply an improved version of [Corollary B](#).

**Corollary 4.2.1** *Let  $Y$  be a manifold with a contact structure  $\xi$ , filled by a manifold  $\bar{M}$  with  $\text{sw}_{\bar{M},\xi} \neq 0$ . Then, any contact manifold  $(Y', \xi')$  constructed by applying a sequence of Weinstein surgeries to  $(Y, \xi)$  is also tight.*

**Remark** It is not true in general that the category of tight contact structures is stable under Weinstein surgery (see for instance [8]). However the theorem of Gromov–Eliashberg asserts that the category of tight contact structures that are weakly symplectically semi-fillable is stable under Weinstein surgery. The above corollary gives a new proof of this fact, since the Seiberg–Witten invariant of a weakly symplectically semi-fillable manifold is nonvanishing [10], together with a more general statement.

There are examples of tight nonsymplectically semi-fillable contact structure [13]. However, to the best of our knowledge, these examples always have a vanishing gauge theoretic (Seiberg–Witten or Ozsváth–Szabó) invariant. Hence, it would be most interesting to find some new tight contact structures that are not symplectically semi-fillable although fillable by a manifold with a nonvanishing Seiberg–Witten invariant.

**4.2.2 Invariants of connected sums** Let  $\bar{M}_1$  and  $\bar{M}_2$  be two compact manifolds with contact boundaries  $(Y_1, \xi_1)$  and  $(Y_2, \xi_2)$ . The connected sum  $Y_1 \# Y_2$  is a particular case of 1–handle surgery and therefore carries a contact structure  $\xi_1 \# \xi_2$  by Weinstein’s result. Using the identification  $\text{Spin}^c(\bar{M}_1, \xi_1) \times \text{Spin}^c(\bar{M}_2, \xi_2) \simeq \text{Spin}^c(\bar{M}_1 \sqcup \bar{M}_2, \xi_1 \sqcup \xi_2)$  and **Theorem D**, we deduce that

$$\text{sw}_{\bar{M}_1, \xi_1} \text{sw}_{\bar{M}_2, \xi_2} = \text{sw}_{\bar{M}_1 \# \bar{M}_2, \xi_1 \# \xi_2} \circ j.$$

Similarly to the compact case, one can easily prove that the Seiberg–Witten invariants must vanish if the connected sum is performed in the interior of the manifolds  $\bar{M}_j$ . The above identity shows that Seiberg–Witten invariants behave in utterly different way for connected sum at the boundary.

**4.2.3 The proof of Theorem A** The proof will be based on the adjunction inequality.

**Proposition 4.2.4** (Adjunction inequality) *Let  $\bar{M}^4$  be a manifold with a contact boundary  $(Y, \xi)$  and an element  $(\mathfrak{s}, h) \in \text{Spin}^c(\bar{M}, \xi)$  such that*

$$\text{sw}_{\bar{M}, \xi}(\mathfrak{s}, h) \neq 0.$$

*Then every closed surfaces  $\Sigma \subset \bar{M}$  with  $[\Sigma]^2 = 0$  and genus at least 1 verifies*

$$|c_1(\mathfrak{s}) \cdot \Sigma| \leq -\chi(\Sigma).$$

**Proof** We refer the reader to [9] and check that the argument proving the adjunction inequality on a compact manifold applies in the same way in our case.  $\square$

We return to the assumptions and notation of Theorem A. Let  $\bar{Z}$  be a special symplectic cobordism obtained by performing Weinstein surgery along  $K$ . We have a new manifold with contact boundary  $(Y', \xi')$

$$\bar{M}' = \bar{M} \cup_Y \bar{Z}.$$

Let  $(s, h)$  be an element of  $\text{Spin}^c(\bar{M}, \xi)$  such that  $\text{sw}_{\bar{M}, \xi}(s, h) \neq 0$  and let  $(t, k) = j(s, h) \in \text{Spin}^c(\bar{M}', \xi')$ . Let  $\Sigma'$  be the closed connected surface obtained as the union of  $\Sigma$  and the core the 2–handle added along  $K$ . Then  $\Sigma'$  verifies

$$\chi(\Sigma') = \chi(\Sigma) + 1, \quad [\Sigma'] \cdot [\Sigma'] = \text{tb}(K, \sigma) - 1, \quad \langle c_1(t), [\Sigma'] \rangle = r(K, \sigma, s, h).$$

If  $[\Sigma']^2 = 0$  and  $\chi(\Sigma') \geq 0$  we are done by the adjunction inequality. We can reduce to this case by some standard tricks. We first make the assumption that  $\text{tb}(K, \sigma) \geq 1$  so that  $[\Sigma']^2 \geq 0$ . The case of equality is equivalent to the fact that the normal bundle of  $\Sigma'$  is trivial. To make it trivial, we blow up  $\text{tb}(K, \sigma) - 1$  points on the core of the 2–handle in the interior of  $\bar{Z}$  and denote by  $E_j$  the exceptional divisors with selfintersection  $-1$ . The blow-up  $\pi: \hat{Z} \rightarrow \bar{Z}$  has still a structure of special symplectic cobordism between  $(Y, \xi)$  and  $(Y', \xi')$ . We denote  $\hat{M}' = \bar{M} \cup \hat{Z}$ ; similarly to  $\bar{M}'$ , it is endowed with an element  $(\hat{t}, k)$  of  $\text{Spin}^c(\hat{M}', \xi')$  deduced from  $(s, h)$ . The adjunction formula shows that the relation between  $t, \hat{t}$  and the proper transform  $\hat{\Sigma}$  of  $\Sigma'$  are

$$[\hat{\Sigma}] = \pi^*[\Sigma'] - \sum_{j=1}^{\text{tb}(K)-1} E_j, \quad c_1(\hat{t}) = c_1(\pi^*t) + \sum_{j=1}^{\text{tb}(K)-1} E_j,$$

where  $E_j$  are Poincaré dual to the exceptional divisors (of self-intersection  $-1$ ). Then

$$[\hat{\Sigma}] \cdot [\hat{\Sigma}] = 0, \quad c_1(\hat{s}) \cdot [\hat{\Sigma}] = r(K, \sigma) + \text{tb}(K) - 1, \quad \chi(\hat{\Sigma}) = \chi(\Sigma') = \chi(\Sigma) + 1,$$

and the normal bundle of  $\hat{\Sigma}$  is trivial.

We have proved the theorem in the case where  $\text{tb}(K, \sigma) \geq 1$  and  $\chi(\Sigma) \leq -1$ . We now show that the theorem holds in general. For this purpose, we modify  $K$  and  $\Sigma$  as follow: let  $P$  be a point on  $K$ . Then  $P$  has a neighborhood in  $Y$  contactomorphic to  $\mathbb{R}^3$  with its standard contact structure. We make a connected sum between  $K$  and a right handed Legendrian trefoil knot at  $P$  as shown in Figure 2; accordingly, add two 1–handles to  $\Sigma$  by forming the connected sum at cusps as suggested by the gray region of Figure 2. This operation decreases  $\chi(\Sigma)$  by two, increases  $\text{tb}(K, \sigma)$  by 2 but does not change the rotation number  $r(K, \sigma, s, h)$ .

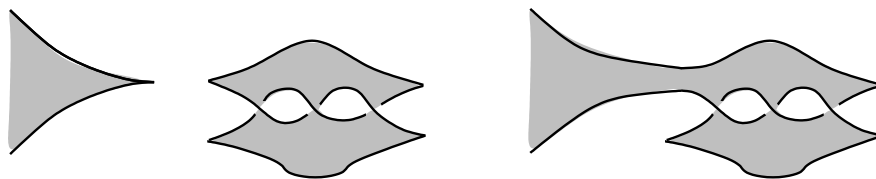


Figure 2: Connected sum with a right handed Legendrian trefoil knot

Overall the quantity  $\chi(\Sigma) + \text{tb}(K, \sigma) + |r(K, \sigma, \mathfrak{s}, h)|$  remains unchanged. So we can always assume that  $\text{tb}(K, \sigma) \geq 1$  and  $\chi(\Sigma) < 0$  and Theorem A is proved.

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