A rational splitting of a based mapping space

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Let $\mathcal{F}_*(X,Y)$ be the space of base-point-preserving maps from a connected finite CW complex X to a connected space Y. Consider a CW complex of the form $X\cup_\alpha e^{k+1}$ and a space Y whose connectivity exceeds the dimension of the adjunction space. Using a Quillen–Sullivan mixed type model for a based mapping space, we prove that, if the *bracket length* of the attaching map $\alpha : S^k \to X$ is greater than the Whitehead length $\mathrm{WL}(Y)$ of Y, then $\mathcal{F}_*(X\cup_\alpha e^{k+1}, Y)$ has the rational homotopy type of the product space $\mathcal{F}_*(X,Y)\times\Omega^{k+1}Y$. This result yields that if the bracket lengths of all the attaching maps constructing a finite CW complex X are greater than $\mathrm{WL}(Y)$ and the connectivity of Y is greater than or equal to $\dim X$, then the mapping space $\mathcal{F}_*(X,Y)$ can be decomposed rationally as the product of iterated loop spaces.

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1 Introduction

Let X be a connected finite CW complex with basepoint and $X\cup_\alpha e^{k+1}$ the adjunction space obtained by attaching the cell e^{k+1} to X along a cellular map $\alpha\colon S^k\to X$. Let $\mathcal{F}_*(X,Y)$ denote the space of base-point-preserving maps from X to a connected space Y with basepoint. The cofibre sequence $X\xrightarrow{i} X\cup_\alpha e^{k+1}\xrightarrow{j} S^{k+1}$ gives rise to the fibration

$$\Omega^{k+1}Y=\mathcal{F}_*(S^{k+1},Y)\xrightarrow{j^\sharp}\mathcal{F}_*(X\cup_\alpha e^{k+1},Y)\xrightarrow{i^\sharp}\mathcal{F}_*(X,Y).$$

The aim of this article is to consider when the above fibration splits after localization at zero. Roughly speaking, our main theorem described below asserts that such a splitting is possible if a number which expresses complexity of the attaching map $\alpha\colon S^k\to X$ is greater than the nilpotency of the rational homotopy Lie algebra of Y. In order to state the theorem more precisely, we first introduce the number associated with a map $\alpha\colon S^k\to X$. Let L be a graded Lie algebra. We define a subspace $[L,L]^{(l)}$ of L by $[L,L]^{(l)}=[L,[L,[...,[L,L]...]]$ (l–times) and $[L,L]^{(0)}=L$, where [,] denotes the Lie bracket of L. Observe that $[L,L]^{(l+1)}$ is a subspace of $[L,L]^{(l)}$.

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Definition 1.1 Let X be a simply-connected space. The *bracket length* of a map $\alpha\colon S^k \to X$, written $\mathrm{bl}(\alpha)$, is the greatest integer n such that the class of the adjoint map $\mathrm{ad}(\alpha)\colon S^{k-1} \to \Omega X$ to α is in $[L_X, L_X]^{(n)}$, where L_X denotes the homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$. If the map $\mathrm{ad}(\alpha)$ is in $[L_X, L_X]^{(n)}$ for any n, then $\mathrm{bl}(\alpha) = \infty$.

Recall the *Whitehead length* $\mathrm{WL}(Y)$ of Y which is the greatest integer n such that $[L_Y, L_Y]^{(n)} \neq 0$ (see for example Berstein and Ganea [1]).

In what follows, we assume that a space is based and its rational cohomology is locally finite. The connectivity of a space Y may be denoted by $\mathrm{Conn}(Y)$. For a nilpotent space X, we denote by $X_{\mathbb{Q}}$ the $\mathbb{Q}$–localization of X. Our main theorem can be stated as follows:

Theorem 1.2 *Let $\alpha\colon S^k \to X$ be a cellular map from the k–dimensional sphere to a simply-connected finite CW complex X, where $k > 0$. Let Y be a space such that* $\mathrm{Conn}(Y) \geq \max\{k+1, \dim X\}$. *If $\mathrm{bl}(\alpha) > \mathrm{WL}(Y)$, then the fibration*

$$\Omega^{k+1} Y = \mathcal{F}_*(S^{k+1}, Y) \xrightarrow{j^\sharp} \mathcal{F}_*(X \cup_\alpha e^{k+1}, Y) \xrightarrow{i^\sharp} \mathcal{F}_*(X, Y) \tag{1–1}$$

is rationally trivial; that is, there is a homotopy equivalence

$$\mathcal{F}_*(X \cup_\alpha e^{k+1}, Y)_{\mathbb{Q}} \xrightarrow{\simeq} (\mathcal{F}_*(X, Y) \times \Omega^{k+1} Y)_{\mathbb{Q}}$$

which covers the identity map on $\mathcal{F}_*(X, Y)_{\mathbb{Q}}$.

Suppose that Y is a connected nilpotent space and X is a finite CW complex. Then $\mathcal{F}_*(X, Y)$ is a connected nilpotent space (Hilton, Mislin and Roitberg [6, Theorem 2.5, Chapter II]). Moreover, $\mathcal{F}_*(X, Y)_{\mathbb{Q}}$ is homotopy equivalent to $\mathcal{F}_*(X, Y_{\mathbb{Q}})$ [6, Theorem 3.11, Chapter II].

Suppose that $\alpha\colon S^k \to X$ is homotopic to the constant map. Then it is evident that $\mathcal{F}_*(X \cup_\alpha e^{k+1}, Y)_{\mathbb{Q}} \simeq (\mathcal{F}_*(X, Y) \times \Omega^{k+1} Y)_{\mathbb{Q}}$. In this case, the bracket length of α is infinity. Thus we can regard that Theorem 1.2 explains such decomposition phenomena of mapping spaces more precisely from the rational homotopy theory point of view.

As an immediate corollary, we have the following result on rational decomposition of a mapping space.

Theorem 1.3 *Let X be a simply-connected finite CW complex and Y a space such that* $\mathrm{Conn}(Y) \geq \dim X$. *Suppose that the bracket length of each attaching map which constructs X is greater than* $\mathrm{WL}(Y)$. *Then $\mathcal{F}_*(X, Y)$ is rationally homotopy equivalent to the product space $\times_k (\Omega^k Y)^{n_k}$, where n_k denotes the number of the k–cells of X. In particular, $\mathcal{F}_*(X, Y)_{\mathbb{Q}}$ is a Hopf space.*

In fact, by looking at the attaching maps with higher dimension in order and by applying Theorem 1.2 repeatedly, we have the result.

As an example, we give a mapping space $\mathcal{F}_*(X,Y)$ which admits the decomposition described in Theorem 1.3. Construct a CW complex X_n $(n\geq 0)$ inductively as follows: Let X_0 be the m_0–sphere S^{m_0}, where $m_0\geq 2$. Suppose that X_i is defined. We fix k integers $m(i)_j$ $(1\leq j\leq k)$ greater than 1. Moreover we choose an element $\alpha_i\in\pi_{\deg\alpha_i}(X_i)$ and the generators $\iota_{m(i)_j}\in\pi_{m(i)_j}(S^{m(i)_j})$ $(1\leq j\leq k)$. Define a CW complex X_{i+1} by

$$X_{i+1}=(X_i\vee S^{m(i)_1}\vee\cdots\vee S^{m(i)_k})\cup_{[\alpha_i,[\iota_{m(i)_1}\cdots[\iota_{m(i)_{k-1}},\iota_{m(i)_k}]\cdots]]}e^{l_i},$$

where $l_i=\deg\alpha_i+m(i)_1+\cdots+m(i)_k-k+1$. It follows that the bracket length of each attaching map is greater than or equal to k. Let Y be a space which satisfies the condition that $k>\mathrm{WL}(Y)$ and $\dim X_n\leq\mathrm{Conn}(Y)$. Then Theorem 1.3 enables us to conclude that

$$\mathcal{F}_*(X_n,Y)\simeq_{\mathbb{Q}}\times_{i=0}^{n-1}(\Omega^{l_i}Y\times\Omega^{m(i)_1}Y\times\cdots\times\Omega^{m(i)_k}Y)\times\Omega^{m_0}Y.$$

We here describe an application of Theorem 1.3.

Corollary 1.4 *Let X and Y be the spaces which satisfy the conditions in Theorem 1.3. Then, for any space Z, there exist bijections of sets*

$$[Z\wedge X,Y_{\mathbb{Q}}]_*\cong[Z,\mathcal{F}_*(X,Y_{\mathbb{Q}})]_*\cong[Z,\times_k(\Omega^kY)^{n_k}_{\mathbb{Q}}]_*$$
$$\cong\bigoplus_{m,k\geq 0,\pi_m(Y)\otimes\mathbb{Q}\neq 0}\widetilde{H}^{m-k}(Z;\mathbb{Q})^{\oplus n_k},$$

where n_k denotes the number of the k–cells of X.

We emphasize that *a Quillen–Sullivan mixed type model* for a based mapping space, which is constructed out of a model for a free mapping space due to Brown and Szczarba [2] (see Section 2), plays a crucial role in proving Theorem 1.2.

The paper is organized as follows: In Section 2, we recall a Sullivan model for a mapping space constructed by Brown and Szczarba. The mixed type model mentioned above is described in this section. Moreover, we introduce a numerical invariant d_1–depth(Y), which is called the d_1–depth for a simply-connected space Y, using a filtration defined by the quadratic part of the differential of the minimal model for Y. This invariant is equal to the Whitehead length of Y. Section 3 is devoted to proving Theorem 1.2. In the appendix (Section 4), we prove that d_1–depth$(Y)=\mathrm{WL}(Y)$. It seems that the important equality is well known. However, we could not find until recently any reference in which the equality has been proved explicitly. Kaji [7] has also proved it by looking at the nilpotency of the loop space ΩY. We wish to stress that our proof of the equality in the appendix contains also a careful consideration on the filtration which defines the d_1–depth.

We end this section by fixing some notations and terminology for this article. A graded algebra A is defined over the rational field $\mathbb{Q}$ and is locally finite in the sense that each vector space A^i is finite dimensional. Moreover it is assumed that an graded algebra A is connected; that is, $A^0 = \mathbb{Q}$ and $A^i = 0$ for $i < 0$. We denote by $\mathbb{Q}\{x_i\}$ the vector space with a basis $\{x_i\}$. The free algebra generated by a graded vector space V is denoted by $\wedge V$ or $\mathbb{Q}[V]$. For an algebra A and its dual coalgebra C, we define A^+ and C^+ by $A^+ = \oplus_{i>0} A^i$ and $C^+ = \oplus_{i<0} C_i$, respectively. Let (B, d_B) be a differential graded algebra (DGA). We call a DGA $(B \otimes \wedge V, d)$ is a relative Sullivan algebra over (B, d_B) if $d|_B = d_B$ and there exists an increasing filtration $\{V(k)\}_{k\geq 0}$ such that $V = \cup_k V(k)$ and $d(V(k)) \subset B \otimes \wedge V(k-1)$.

2 A Quillen–Sullivan mixed type model for a mapping space

Let (B, d_B) be a DGA and $(\wedge V, d)$ a minimal DGA; that is, dv is decomposable for any $v \in V$. Let B_* denote the differential graded coalgebra defined by $B_q = \mathrm{Hom}(B^{-q}, \mathbb{Q})$ for $q \leq 0$ together with the coproduct D and the differential d_{B*}, which are dual to the multiplication of B and to the differential d_B, respectively. Let I be the ideal of the free algebra $\mathbb{Q}[\wedge V \otimes B_*]$ generated by $1 \otimes 1 - 1$ and all elements of the form

$$a_1 a_2 \otimes \beta_* - \sum_i (-1)^{|a_2||\beta_i'|}(a_1 \otimes \beta_{i*}')(a_2 \otimes \beta_{i*}''),$$

where $a_1, a_2 \in \wedge V$, $\beta_* \in B_*$ and $D(\beta_*) = \sum_i \beta_{i*}' \otimes \beta_{i*}''$. Observe that $\mathbb{Q}[\wedge V \otimes B_*]$ is a DGA with the differential $d := d_A \otimes 1 \pm 1 \otimes d_{B*}$. The result of Brown and Szczarba [2, Theorem 3.3] yields that $(d_A \otimes 1 \pm 1 \otimes d_{B*})(I) \subset I$. Moreover it follows from [2, Theorem 3.5] that the composition map

$$\rho\colon \mathbb{Q}[V \otimes B_*] \hookrightarrow \mathbb{Q}[\wedge V \otimes B_*] \to \mathbb{Q}[\wedge V \otimes B_*]/I$$

is an isomorphism of graded algebras. Thus we define a differential δ on $\mathbb{Q}[V \otimes B_*]$ by $\rho^{-1}\widetilde{d}\rho$, where $\widetilde{d}$ is the differential on $\mathbb{Q}[\wedge V \otimes B_*]/I$ induced by d. The differential δ is described explicitly as follows: For an element $v \in V$ and a cycle $\beta_* \in B_*$, if $d(v) = v_1 \cdots v_m$ with $v_i \in V$, then

$$\text{(2–1)} \qquad \begin{aligned} \delta(v \otimes (\beta_*)) &= \textstyle\sum_j v_1 \cdots v_m \cdot \beta_{j_1 *} \otimes \cdots \otimes \beta_{j_m *} \\ &= \textstyle\sum_j (-1)^{\varepsilon(v_1, \dots, v_m, \beta_{j_1 *}, \dots, \beta_{j_m *})} v_1 \otimes \beta_{j_1 *} \cdots v_m \otimes \beta_{j_m *} \end{aligned}$$

where $D^{(m-1)}(\beta_*) = \sum_j \beta_{j_1 *} \otimes \cdots \otimes \beta_{j_m *}$ with the iterated coproduct $D^{(m-1)}$ and the integer $(-1)^{\varepsilon(v_1,..,v_m,\beta_{j_1 *},..,\beta_{j_m *})}$ is defined by the formula

$$(-1)^{\varepsilon(v_1,..,v_m,\beta_{j_1 *},..,\beta_{j_m *})} v_1\beta_{j_1} \cdots v_m\beta_{j_m} = v_1 \cdots v_m \beta_{j_1} \cdots \beta_{j_m}$$

in the graded algebra $(\wedge V) \otimes B$ using elements β_{j_s} $(a \le s \le m)$ with $\deg \beta_{j_s} = -\deg \beta_{j_s *}$.

We denote by $A_{PL}(X)$ the DGA of the polynomial differential forms on a space X. Let X be a connected finite CW complex and Y a connected space with $\dim X \le \mathrm{Conn}(Y)$. We take a quasi-isomorphism $(B, d_B) \to A_{PL}(X)$ and a minimal model $(\wedge V, d)$ for Y. By applying the construction mentioned above, we obtain a DGA of the form $(\mathbb{Q}[V \otimes B_*], \delta)$, which gives an algebraic model (not minimal in general) for $\mathcal{F}(X, Y)$ the space of *free* maps from X to Y [2]. In fact, there exists a quasi-isomorphism which connects $A_{PL}(\mathcal{F}(X, Y))$ with the DGA $(\mathbb{Q}[V \otimes B_*], \delta)$. Moreover, the realization of $(\mathbb{Q}[V \otimes B_*], \delta)$ is homotopy equivalent to $\mathcal{F}(X, Y_{\mathbb{Q}})$ [2, Theorem 1.3] and hence to $\mathcal{F}(X, Y)_{\mathbb{Q}}$. The result of the first author [9, Proposition 5.3] asserts that $(\mathbb{Q}[V \otimes B_*], \delta)$ is a relative Sullivan algebra with the base $\mathbb{Q}[V]$. Observe that $(\mathbb{Q}[V \otimes B_*], \delta)$ itself is a Sullivan algebra [9, Reamrk 5.4]. Moreover the model for $\mathcal{F}(X, Y)$ leads to that for the based mapping space $\mathcal{F}_*(X, Y)$.

Theorem 2.1 [9, Theorem 4.3] *There exist a quasi-isomorphism from a Sullivan algebra of the form* $(\mathbb{Q}[V \otimes B_*]/(\mathbb{Q}[V]^+), \overline{\delta}) = (\mathbb{Q} \otimes_{\mathbb{Q}[V]} \mathbb{Q}[V \otimes B_*], 1 \otimes \delta)$ *to* $A_{PL}(\mathcal{F}_*(X, Y))$. *Here* $(\mathbb{Q}[V]^+)$ *is the ideal of* $\mathbb{Q}[V \otimes B_*]$ *generated by* $\mathbb{Q}[V]^+$.

From the explicit form (2–1) of the differential δ, we can deduce the following lemma. The proof is left to the reader.

Lemma 2.2 *Suppose that, for an element* $v \otimes \beta_* \in V \otimes B_*^+$, dv *is in* $\wedge^{\ge m} V$ *and* $\overline{D}^{m-1}(\beta_*) = 0$, *where* $\overline{D}^{m-1}\colon B_*^+ \to (B_*^+)^{\otimes m}$ *denotes the* $(m-1)$ *fold reduced coproduct. Then* $\overline{\delta}(v \otimes \beta_*) = 0$. *In particular,* $\overline{\delta}(v \otimes \beta_*) = 0$ *if* $\beta_* \in B_*$ *is a primitive element.*

We here recall, from Félix–Halperin–Thomas [4, Section 22], Quillen's functor $C_*(\ )$ from the category of connected differential graded Lie algebras (DGL's) to the category of simply-connected cocommutative differential graded coalgebras (DGC's). Let (L, d_L) be a DGL and $\wedge(sL)$ be the primitively generated coalgebra over the vector space sL. We define the differentials d_v and d_h on $\wedge(sL)$ by

$$d_v(sx_1 \wedge \cdots \wedge sx_k) = -\sum_{i=1}^{k} (-1)^{n_i} sx_1 \wedge \cdots \wedge sd_L x_i \wedge \cdots \wedge sx_k$$

and

$$d_h(sx_1 \wedge \cdots \wedge sx_k) = \sum_{1 \le i < j \le k} (-1)^{|sx_i| + n_{ij}} s[x_i, x_j] \wedge sx_1 \cdots \widehat{sx_i} \cdots \widehat{sx_j} \cdots \wedge sx_k,$$

respectively. Here $n_i = \sum_{j<i} |sx_j|$ and $sx_1 \wedge \cdots \wedge sx_k = (-1)^{n_{ij}} sx_i \wedge sx_j \wedge sx_1 \wedge \cdots \widehat{sx_i} \cdots \widehat{sx_j} \cdots \wedge sx_k$. We see that $C_*(L, d_L) = (\wedge(sL), d_v + d_h)$ is a DGC. To simplify, we may write $C_*(L)$ for $C_*(L, d_L)$. By using the above DGC, we can construct a more explicit model for a mapping space. Let (L, d_L) be a Lie model for a space X; that is, there exists a quasi-isomorphism $C^*(L, d_L) = \text{dual } C_*(L, d_L) \xrightarrow{\simeq} A_{PL}(X)$. We choose a minimal model $(\wedge V, d)$ for Y. Then Theorem 2.1 implies that the Sullivan algebra of the form $(\mathbb{Q}[V \otimes C_*(L, d_L)]/(\mathbb{Q}[V]^+), \overline{\delta}) = (\mathbb{Q}[V \otimes C_*(L, d_L)^+], \overline{\delta})$ is a model for the mapping space $\mathcal{F}_*(X, Y)$. This model, which is called *a Quillen–Sullivan mixed type model* for the based mapping space, is an important ingredient for the proof of Theorem 1.2.

Remark 2.3 The Sullivan algebra of the form $(\mathbb{Q}[V \otimes C_*(L, d_L)], \delta)$ is regarded as a mixed type model for the free mapping space $\mathcal{F}(X, Y)$.

We close this section by introducing a numerical invariant which is called the d_1*–depth* of a given space. We use the invariant to prove Theorem 1.2.

Let $(\wedge V, d)$ be a minimal model for a simply-connected space Y. Then the differential d is decomposed uniquely as $d = d_1 + d_2 + \cdots$ in which d_i is a derivation raising the wordlength by i. We call d_1 the quadratic part of d. We define a subspace V_0 of V by $V_0 = \{v \in V \mid d_1(v) = 0\}$ and put $V_{-1} = 0$. Moreover, define a subspace V_i inductively by $V_i = \{v \in V \mid d_1(v) \in \wedge V_{i-1}\}$. It is readily seen that $V_{k-1} \subset V_k$ and that if $V_l = V_{l-1}$, then $V_k = V_{k+1}$ for $k \ge l$.

Definition 2.4 The d_1*–depth* of Y, denoted d_1–depth(Y), is the greatest integer k such that V_{k-1} is a proper subspace of V_k.

It suffices to prove Theorem 1.2 by assuming that $\mathrm{bl}(\alpha) > d_1$–depth$(Y)$ instead of the sufficient condition $\mathrm{bl}(\alpha) > \mathrm{WL}(Y)$. The following theorem guarantees that the replacement is valid.

Theorem 2.5 *Let Y be a simply-connected space. Then d_1–depth$(Y) = \mathrm{WL}(Y)$.*

Proof See the appendix. □

Since the Whitehead length is a numerical topological invariant in the category of the rational spaces, it follows that the d_1–depth of Y does not depend on the choice of minimal models for Y and is also a topological invariant.

3 A minimal model and Proof of Theorem 1.2

Before proving Theorem 1.2, we recall from [2] a result concerning construction of a minimal model for a mapping space. Though the construction is for a free mapping space, it is applicable to the model $(\mathbb{Q}[V\otimes B_*]/(\mathbb{Q}[V]^+),\overline{\delta})$ for a based mapping space $\mathcal{F}_*(X,Y)$ which is described in Theorem 2.1. With the notation in Section 2, we write $\mathbb{Q}[V\otimes B_*]/(\mathbb{Q}[V]^+)=\mathbb{Q}[V\otimes B_*^+]$. Let $\{a_k,b_k,c_j\}$ be a basis for B_*^+ such that $d_{B_*^+}(a_k)=b_k$ and $d_{B_*^+}(c_j)=0$. Choose a basis $\{v_i\}$ for V so that $|v_i|\le|v_{i+1}|$ and $dv_{i+1}\in\mathbb{Q}[V_i]$, where V_i is the subspace spanned by the elements $v_1,\dots,v_i$. The result [2, Lemma 5.1] states that there exist free algebra generators w_{ij}, u_{ik} and v_{ik} such that

$$w_{ij}=v_i\otimes c_j+x_{ij},\ \text{where } x_{ij}\in\mathbb{Q}[V_{i-1}\otimes B_*^+], \tag{3–1}$$

$$\overline{\delta}w_{ij}\ \text{is decomposable and in } \mathbb{Q}[\{w_{sl};s<i\}], \tag{3–2}$$

$$u_{ik}=v_i\otimes a_k\ \text{and}\ \overline{\delta}u_{ik}=v_{ik}. \tag{3–3}$$

Thus we have a decomposition $\mathbb{Q}[V\otimes B_*^+]=\mathbb{Q}[w_{ij}]\otimes\mathbb{Q}[u_{ik},v_{ik}]$ of a differential graded algebra. Since $\mathbb{Q}[u_{ik},v_{ik}]$ is contractible, it follows that the inclusion $(\mathbb{Q}[w_{ij}],\overline{\delta})\to(\mathbb{Q}[V\otimes B_*^+],\overline{\delta})$ is a quasi-isomorphism. In consequence, we get a minimal model of the form $(\mathbb{Q}[w_{ij}],\overline{\delta})$ for the mapping space $\mathcal{F}_*(X,Y)$. Observe that the vector space generated by the elements w_{ij} is isomorphic to the reduced homology $H_*(B_*)^+$ as a vector space.

We rely on the following result to construct a minimal model for the mapping space $\mathcal{F}_*(X,Y)$ from the Sullivan algebra $(\mathbb{Q}[V\otimes C_*(L,d_L)^+],\overline{\delta})$ in Section 2.

Lemma 3.1 [4, Proposition 22.8] *For a DGL of the form $(\mathbb{L}_W,d_L)$, let $\rho_1\colon C_*(\mathbb{L}_W)=\wedge s\mathbb{L}_W\to s\mathbb{L}_W\oplus\mathbb{Q}$ and $\rho_2: s\mathbb{L}_W\oplus\mathbb{Q}\to sW\oplus\mathbb{Q}$ be the maps obtained by annihilating the factors $\wedge^{\ge2}s\mathbb{L}_W$ and $s(\mathbb{L}_W^{\ge2})$, respectively. Then the composition map $\rho_2\circ\rho_1\colon C_*(\mathbb{L}_W,d_L)\to(sW\oplus\mathbb{Q},d_0)$ is a quasi-isomorphism of complexes, where d_0 denotes the linear part of d_L.*

Recall a Lie model for an adjunction space. Let $(\mathbb{L}_W,d_L)$ be a minimal Lie model for X. By definition, there exists a quasi-isomorphism $C^*(\mathbb{L}_W,d_L)\xrightarrow{\simeq}A_{PL}(X)$. Moreover, we have an isomorphism $\sigma_L\colon H(\mathbb{L}_W,d_L)\xrightarrow{\cong}\pi_*(\Omega X)\otimes\mathbb{Q}$ of graded Lie algebras. Define an isomorphism $\tau_L\colon sH(\mathbb{L}_W,d_L)\to\pi_*(X)\otimes\mathbb{Q}$ by composing the map σ_L with the inverse of the connecting isomorphism $\partial\colon\pi_{*+1}(X)\otimes\mathbb{Q}\to\pi_*(\Omega X)\otimes\mathbb{Q}$. Let z_α be a cycle of $\mathbb{L}_W$ such that τ_L sends the class $s[z_\alpha]\in sH(\mathbb{L}_W,d_L)$ to $[\alpha]\in\pi_*(X)\otimes\mathbb{Q}$. Then, as a Lie model for the adjunction space $X\cup_\alpha e^{k+1}$, we can choose the graded

Lie algebra $(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}}, d)$ with $d_{|W} = d_L$ and $d(w_\alpha) = z_\alpha$ [4, Theorem 24.7]. By applying the construction described in Section 2, we obtain a Sullivan model for $\mathcal{F}(X \cup_\alpha e^{k+1}, Y)$ of the form $(\wedge(V \otimes C_*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}}, d)), \delta)$.

We need the following lemma to prove Theorem 1.2.

Lemma 3.2 *Let*

$$m_1\colon \mathbb{Q}[V] \to \mathbb{Q}[V \otimes C_*(\mathbb{L}_W)]$$

and

$$m_2\colon \mathbb{Q}[V] \to \mathbb{Q}[V \otimes C_*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}})]$$

be the inclusions of relative Sullivan algebras. Let

$$\eta\colon \mathbb{Q}[V \otimes C_*(\mathbb{L}_W)] \to \mathbb{Q}[V \otimes C_*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}})]$$

be the map induced by the inclusion $(\mathbb{L}_W, d) \to (\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}}, d)$ of DGL's. Then there exists a commutative diagram

$$\begin{tikzcd}
 & \mathbb{Q}[V] \arrow[rr, "\simeq"] \arrow[dl, "m_1"'] \arrow[dd, "m_2"] & & A_{PL}(Y) \arrow[dl, "A_{PL}(ev_*)"'] \arrow[dd, "A_{PL}(ev_*)"] \\
\mathbb{Q}[V \otimes C_*(\mathbb{L}_W)] \arrow[rr, "\simeq"] \arrow[dr, "\eta"'] & & A_{PL}(\mathcal{F}(X,Y)) \arrow[dr, "A_{PL}(i^\sharp)"] & \\
 & \mathbb{Q}[V \otimes C_*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}})] \arrow[rr, "\simeq"] & & A_{PL}(\mathcal{F}(X \cup_\alpha e^{k+1}, Y))
\end{tikzcd}$$

in the category of DGA's in which three horizontal arrows are quasi-isomorphisms. Hence the map $\overline{\eta}\colon \mathbb{Q}[V \otimes C_*(\mathbb{L}_W)^+] \to \mathbb{Q}[V \otimes C_*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}})^+]$ induced by η is a Sullivan model for the map $i^\sharp\colon \mathcal{F}_*(X \cup_\alpha e^{k+1}, Y) \to \mathcal{F}_*(X,Y)$ [4, Definition, page 182].

Proof See the appendix. □

Proof of Theorem 1.2 Under the hypotheses in Theorem 1.2, we see that the space $\mathcal{F}_*(X,Y)$ is simply-connected and $\mathcal{F}_*(X \cup_\alpha e^{k+1}, Y)$ is connected. We shall prove the fibration (1–1) is rationally trivial if the inequality $\mathrm{bl}(\alpha) > d_1$–depth$(Y)$ holds.

Under the notation mentioned above, we assume that

$$z_\alpha = \sum_i [x_{i_n}[x_{i_{n-1}}[x_{i_{n-2}}, \dots, [x_{i_1}, x_{i_0}], \dots]]]$$

with appropriate cycles x_{i_j} in $\mathbb{L}_W$, where $n = \mathrm{bl}(\alpha)$. By virtue of Lemma 3.2, we see that the inclusion $\overline{\eta}\colon \wedge(V \otimes C_*(\mathbb{L}_W, d)^+) \to \wedge(V \otimes C_*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}}, d)^+)$ is a model for the projection $i^\sharp$ of the fibration (1–1). Let $\varphi\colon (\wedge(Z), d) \to (\wedge(V \otimes$

$C_*(\mathbb{L}_W, d)^+), \delta)$ be the minimal model described before Lemma 3.1. Observe that φ is an inclusion and $Z \cong V \otimes H_*(C_*(\mathbb{L}_W, d)^+) \cong V \otimes sW$. If $\wedge(\widetilde{Z}')$ is a minimal model for the Sullivan algebra $(\wedge(V \otimes C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}, d)^+), \delta)$, then $\widetilde{Z}'$ is isomorphic to $V \otimes H_*(C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}, d)^+)$ and hence to $V \otimes s(W \oplus \mathbb{Q}\{w_\alpha\})$. With this in mind, we define a Sullivan algebra $(\wedge \widetilde{Z}, \widetilde{d})$ by $\widetilde{Z} = V \otimes s(W \oplus \mathbb{Q}\{w_\alpha\}) \cong Z \oplus (V \otimes sw_\alpha)$, $\widetilde{d}_{|Z} = d$ and $\widetilde{d}_{|V \otimes sw_\alpha} \equiv 0$. In order to prove Theorem 1.2, it suffices to show that there exists a quasi-isomorphism $\psi\colon (\wedge \widetilde{Z}, \widetilde{d}) \to (\wedge(V \otimes C_*(\mathbb{L}_W, d)^+), \delta)$ such that the diagram

$$\begin{tikzcd}
(\wedge Z, d) \arrow[rr, "I"] \arrow[d, hook, "\varphi", "\simeq"'] & & (\wedge \widetilde{Z}, \widetilde{d}) \arrow[d, "\psi"', "\simeq"] \\
(\wedge(V \otimes C_*(\mathbb{L}_W, d)^+), \overline{\delta}) \arrow[rr, "\eta"'] & & (\wedge(V \otimes C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}, d)^+), \overline{\delta})
\end{tikzcd}$$

is commutative, where I is the inclusion. In fact, we then see that the map I is regarded as a Sullivan model for $i^\sharp$. Moreover the Sullivan algebra $(\wedge \widetilde{Z}, \widetilde{d})$ is isomorphic to $(\wedge Z, d) \otimes (\wedge(V \otimes sw_\alpha), 0)$ as a DGA. Observe that $(\wedge(V \otimes sw_\alpha), 0)$ is the minimal model for $\Omega^{k+1} Y$.

We shall construct the required map ψ. Put $\wedge U = \wedge(V \otimes C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}, d)^+)$. Let $\wedge^s U$ be the vector subspace of $\wedge U$ consisting of elements with wordlength s and $\wedge^{\geq s} U$ the ideal of $\wedge U$ generated by $\wedge^s U$. Assume that $v \in V_m$, where $m = d_1$–depth(Y). We first choose a cycle

$$c_\alpha = sw_\alpha - \sum_i sx_{i_n} \wedge s[x_{i_{n-1}}[x_{i_{n-2}}, \ldots, [x_{i_1}, x_{i_0}]\ldots]]$$

in $C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}, d)$ and define an element γ_1 of $\wedge U$ by $\gamma_1 = v \otimes c_\alpha$. Observe that $n > m$ by assumption. We set $x_{i_{n-1},..,i_0} = [x_{i_{n-1}}[x_{i_{n-2}}, \ldots, [x_{i_1}, x_{i_0}]\ldots]]$. It follows from (2–1) that, in $\wedge^{\geq 2} U$,

$$\begin{aligned}
\overline{\delta}(\gamma_1) = -\Big(&\sum_{i, j_1} (-1)^{|sx_{i_n}||v'_{j_1}|} (v_{j_1} \otimes sx_{i_n}) \cdot (v'_{j_1} \otimes sx_{i_{n-1},..,i_0}) \\
&+ \sum_{i, j_1} (-1)^{|sx_{i_{n-1},\ldots,i_0}||sx_{i_n}| + |sx_{i_{n-1},\ldots,i_0}||v'_{j_1}|} (v_{j_1} \otimes sx_{i_{n-1},..,i_0}) \cdot (v'_{j_1} \otimes sx_{i_n}) \Big)
\end{aligned}$$

if $d_1(v) = \sum_{j_1} v_{j_1} v'_{j_1}$. We see that $\overline{\delta}(\gamma_1)$ belongs to $\wedge^2 U$ and is determined without depending on the term of $(d - d_1)(v)$ because sx_{i_n} and $sx_{i_{n-1},..,i_0}$ are primitive. Observe that v_{j_1} and v'_{j_1} are in V_{m-1} (see Lemma 4.4 for more polished result on the image of d_1).

We next define an element $\gamma_2 \in \wedge^2 U$ by

$$\gamma_2 = \sum_{i,j_1} (-1)^{\varepsilon_{i_n,\dots,i_0}} (v_{j_1} \otimes sx_{i_n}) \cdot (v'_{j_1} \otimes sx_{i_{n-1}} \wedge sx_{i_{n-2},..,i_0})$$
$$+ \sum_{i,j_1} (-1)^{\varepsilon'_{i_n,\dots,i_0}} (v_{j_1} \otimes sx_{i_{n-1}} \wedge sx_{i_{n-2},..,i_0}) \cdot (v'_{j_1} \otimes sx_{i_n}),$$

where $\varepsilon_{i_n,..,i_0}$ and $\varepsilon'_{i_n,..,i_0}$ denote the integers $|sx_{i_n}||v'_{j_1}|+|v_{j_1} \otimes sx_{i_n}|+|v'_{j_1}|+|sx_{i_{n-1}}|$ and $|sx_{i_{n-1},..,i_0}||sx_{i_n}|+|sx_{i_{n-1},..,i_0}||v'_{j_1}|+|v_{j_1}|+|sx_{i_{n-1}}|$, respectively. Since sx_{i_n} is primitive, it follows from Lemma 2.2 that $\overline{\delta}(\gamma_1) = -\overline{\delta}(\gamma_2)$ in $\wedge^2 U$.

In a similar fashion, we can define elements $\gamma_l \in \wedge^l U$ so that $\overline{\delta}(\gamma_{l-1}) = -\overline{\delta}(\gamma_l)$ in $\wedge^l U$ and each term of γ_l has the form

$$y \cdot \big(v_{j_l} \otimes (sx_{i_{n-l+1}} \wedge sx_{i_{n-l},..,i_0})\big),$$

where $v_{j_l} \in V_{m-l+1}$ and y is an element in the ideal of $\wedge U$ generated by elements of the form $u \otimes sx_{i_s}$ for some $u \in V$. Since $\overline{\delta}(\gamma_l) \in \wedge^l U \oplus \wedge^{l+1} U$ and $\overline{\delta}(\gamma_{m+1}) = 0$ in $\wedge^{m+2} U$, it follows that $\gamma_v := \gamma_1 + \cdots + \gamma_{m+1}$ is a $\overline{\delta}$–cycle in $\wedge U$ (see (3–4) below in which $\overline{\delta}_1$ denotes the linear part of the differential $\overline{\delta}$ and $\overline{\delta}_2 = \overline{\delta} - \overline{\delta}_1$).

(3–4)
$$\begin{tikzcd}
0 & & & \\
\gamma_1 \arrow[u, "\overline{\delta}_1"] \arrow[r, "\overline{\delta}_2"] & 0 & & \\
& \gamma_2 \arrow[u, "\overline{\delta}_1"] \arrow[r, dotted, "\overline{\delta}_2"] & 0 & \\
& & \gamma_{m+1} \arrow[u, dotted, "\overline{\delta}_1"] \arrow[r, "\overline{\delta}_2"] & 0
\end{tikzcd}$$

Observe that the element $\gamma_2 + \cdots + \gamma_{m+1}$ can be regarded as the element x_{ij} in condition (3–1).

The same argument above works well to show that $v \otimes sw_\alpha$ is a cycle when $v \in V_l$ for $l < m$ since $\mathrm{bl}(\alpha) = n > m = d_1$–depth$(Y)$.

We here define a map $\psi\colon (\wedge \widetilde{Z}, \widetilde{d}) \to (\wedge(V \otimes C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}, d)^+), \delta)$ by $\psi_{|Z} = \overline{\eta}\varphi$ and $\psi(v \otimes sw_\alpha) = \gamma_v$ for $v \otimes sw_\alpha \in V \otimes sw_\alpha$. The construction of $\mathbb{Q}[w_{ij}]$ described before Lemma 3.1 tells us that ψ is a minimal model. Moreover we see that all the required conditions for ψ hold. This completes the proof of Theorem 1.2. □

Example 3.3 Let us consider the projective space $\mathbb{C}P^2 = S^2 \cup_\gamma e^4$, where γ denotes the Hopf map. Let Y be a 4–connected space with a minimal model $(\wedge V, d)$ for which V is a vector space with a basis $\{x_1, x_2, x_3, y\}$, $d(x_i) = 0$ and $d(y) = x_1 x_2 x_3$.

Since γ is decomposable in $\pi_*(S^2)\otimes\mathbb{Q}$, it is evident that $\mathrm{bl}(\gamma)=\mathrm{bl}([\iota,\iota])=1>0=d_1$–depth$(Y)$, where ι is the generator of $\pi_2(S^2)$. Thus Theorem 1.2 allows us to conclude that the fibration $\Omega^4 Y\to\mathcal{F}_*(\mathbb{C}P^2,Y)\to\Omega^2 Y$ is rationally trivial.

Example 3.4 Let $\mathcal{L}P^2$ be the Cayley plane and $\mathbb{C}P^2_i$ a copy of the complex projective plane for $i=1,2$. Let ι_i denote the generator of $\pi_2(\mathbb{C}P^2_i)$. The space $\mathbb{C}P^2_1\vee\mathbb{C}P^2_2\cup_{[\iota_1,\iota_2]}e^4$ has a CW–decomposition for which the bracket length of each attaching map is greater than or equal to 1. Since $H^*(\mathcal{L}P^2;\mathbb{Q})\cong\mathbb{Q}[x_8]/(x_8^3)$, where $\deg x_8=8$, it follows that $\mathrm{WL}(\mathcal{L}P^2)=d_1$–depth$(\mathcal{L}P^2)=0$. Corollary 1.4 yields that, for any based space Z,

$$[Z\wedge(\mathbb{C}P^2_1\vee\mathbb{C}P^2_2\cup_{[\iota_1,\iota_2]}e^4),\mathcal{L}P^2_{\mathbb{Q}}]_*\cong(H^{8-4}(Z;\mathbb{Q})\oplus H^{23-4}(Z;\mathbb{Q}))^{\oplus 3}\\ \oplus(H^{8-2}(Z;\mathbb{Q})\oplus H^{23-2}(Z;\mathbb{Q}))^{\oplus 2}.$$

Example 3.5 Let G and H be a compact connected Lie group and a closed subgroup of G, respectively. By considering the K.S–extension of the fibration $G\to G/H\to BH$, we see that the minimal model $(\wedge V,d)$ for G/H satisfies the condtions: $dV^{even}=0$ and $dV^{odd}\subset\wedge V^{even}$. This implies that d_1–depth$(G/H)\le 1$. Let X and α: $S^k\to X$ be as in Theorem 1.2. Suppose that $\mathrm{Conn}(G/H)\ge\max\{k+1,\dim X\}$. Then the fibration

$$\Omega^{k+1}Y=\mathcal{F}_*(S^{k+1},G/H)\xrightarrow{j^\sharp}\mathcal{F}_*(X\cup_\alpha e^{k+1},G/H)\xrightarrow{i^\sharp}\mathcal{F}_*(X,G/H)$$

is rationally trivial if $\mathrm{bl}(\alpha)>1$.

Example 3.6 Recall from [4] that a simply-connected space Y is elliptic if $\dim\pi_*(Y)\otimes\mathbb{Q}<\infty$ and $\dim H_*(Y;\mathbb{Q})<\infty$. Let Y be an n–connected finite dimensional elliptic CW complex with a minimal model $(\wedge V,d)$. Let $\{v_i\}$ be a basis of V. If $v_{i_s}\in V_s-V_{s-1}$, then $\deg v_{i_s}\ge(s+1)n+1$ (see the Section 2 for the notation V_s). Put $m=d_1$–depth(Y) and let v be an element of V with the maximal degree. Then $\deg v$ is odd from Friedlander–Halperin [5, Theorem 1 and Lemma 2.5]. Therefore it follows from [5, Corollary 1.3(3)] that

$$(m+1)n+1\le\deg v_{i_m}\le\deg v\le\sum_{j:odd} j\cdot\dim V^j\le 2\dim Y-1$$

and hence $2\dim Y/n>m+1=d_1$–depth$(Y)+1$. Theorem 1.2 enable us to conclude that the fibration (1–1) is rationally trivial if $\mathrm{bl}(\alpha)+1\ge 2\dim Y/\mathrm{Conn}(Y)$.

We give examples which assert that the decomposition in Theorem 1.2 does not hold in general when $\mathrm{bl}(\alpha)\le\mathrm{WL}(Y)$. To this end, we here recall the result [8, Theorem 1.2] due to Kotani.

Let $(\wedge V, d)$ be a minimal model for a simply-connected space Y. Consider the decomposition $d = d_1 + d_2 + \cdots$ of the differential d as in Section 2. The *d–length* of Y, denoted d–length(Y), is the least integer m such that $d_i \equiv 0$ for $i < m-1$ and $d_{m-1} \not\equiv 0$. Observe that the d–length of Y is a topological invariant (see [8, Theorem 1.1]). As usual, we define the cup-length of a space X, $c(X)$, by the greatest integer n such that there are elements α_1, ..., α_n in $H^+(X;\mathbb{Q})$ for which $\alpha_1 \cup \cdots \cup \alpha_n \neq 0$. Then the main result in [8] is stated as follows.

Theorem 3.7 [8, Theorem 1.2] *Let X be a path connected, finite dimensional CW complex and Y a connected space with* $\mathrm{Conn}(Y) \geq \dim X$. *Suppose that X is formal. Then the cohomology algebra $H^*(\mathcal{F}_*(X,Y);\mathbb{Q})$ is a free algebra if and only if d*-length$(Y) > c(X)$.

Example 3.8 Consider the projective space $\mathbb{C}P^3 = \mathbb{C}P^2 \cup_\alpha e^6$. We observe that α is indecomposable in $\pi_*(\mathbb{C}P^2) \otimes \mathbb{Q}$. Since d-length$(Y) = 3 = c(\mathbb{C}P^3)$, it follows from Theorem 3.7 that $H^*(\mathcal{F}_*(\mathbb{C}P^3, Y);\mathbb{Q})$ is not free. Thus $\mathcal{F}_*(\mathbb{C}P^3, Y)$ is not rationally homotopy equivalent to the product $\mathcal{F}_*(\mathbb{C}P^2, Y) \times \Omega^6 Y$ because $H^*(\mathcal{F}_*(\mathbb{C}P^2, Y) \times \Omega^6 Y;\mathbb{Q})$ is free. Observe that $\mathrm{bl}(\alpha) = 0 = d_1$–depth$(Y)$ in this case.

Example 3.9 Let $(\wedge V, d) = (\wedge(x,y), d)$ be the minimal model for S^6, where $\deg x = 6$, $\deg y = 11$, $dx = 0$ and $dy = x^2$. Consider the fibration $\Omega^4 S^6 \xrightarrow{j^\sharp} \mathcal{F}_*(\mathbb{C}P^2, S^6) \xrightarrow{i^\sharp} \Omega^2 S^6$ which is induced from the cofibre sequence $S^2 \xrightarrow{i} \mathbb{C}P^2 = S^2 \cup_\gamma e^4 \xrightarrow{j} S^4$. Let ι be the generator in $\pi_2(S^2)\otimes\mathbb{Q}$. Observe that $\gamma = q[\iota,\iota]$ for some nonzero rational number q. We can choose $\mathbb{Q}[V \otimes C_*(\mathbb{L}_{\mathbb{Q}\{\tilde{\iota}, w_\gamma\}}, d)^+]$ as a Sullivan model for the function space $\mathcal{F}_*(\mathbb{C}P^2, S^6)$, where $\tilde{\iota}$ denotes the element in $\pi_1(\Omega S^2)\otimes\mathbb{Q}$ corresponding to ι via the connecting isomorphism $\pi_2(S^2)\otimes\mathbb{Q} \to \pi_1(\Omega S^2)\otimes\mathbb{Q}$. Put $v_4 = x \otimes s\tilde{\iota}$, $v_9 = y \otimes s\tilde{\iota}$, $v_2 = x \otimes (sw_\gamma - q(s\tilde{\iota} \wedge s\tilde{\iota}))$ and $v_7 = y \otimes (sw_\gamma - q(s\tilde{\iota}\wedge s\tilde{\iota}))$. Then a model for the above fibration is given by

$$(\wedge(v_4, v_9), 0) \to (\wedge(v_4, v_9, v_2, v_7), \overline{\delta}) \to (\wedge(v_2, v_7), 0)$$

where $\overline{\delta}(v_7) = -2qv_4{}^2$ and $\overline{\delta}(v_i) = 0$ for $i \neq 7$ (see the proof of Theorem 1.2 for the construction). Therefore the fibration is not rationally trivial. It is readily seen that $\mathrm{bl}([\iota,\iota]) = 1 = d_1$–depth$(S^6)$ in this case.

Example 3.10 Let Y be a 6–connected space whose minimal model has the trivial differential. Then the differentials of the minimal models for the spaces $\mathcal{F}_*(\mathbb{C}P^2, Y)$ and $\mathcal{F}_*(\mathbb{C}P^3, Y)$ are also trivial. Moreover we see that $\Omega^6 Y \xrightarrow{j^\sharp} \mathcal{F}_*(\mathbb{C}P^3, Y) \xrightarrow{i^\sharp}$

$\mathcal{F}_*(\mathbb{C}P^2, Y)$ is rationally trivial though $\mathrm{bl}(\alpha) = 0 = d_1$–depth$(Y)$. This fact implies that the converse assertion of Theorem 1.2 does not hold in general.

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4 Appendix

We prepare to prove Theorem 2.5. Let $(\wedge V, d)$ be the minimal model for a simply-connected space Y. Recall the graded Lie algebra L associated with a minimal model $(\wedge V, d)$ for Y (see [4, Section 21, (e)]). The graded vector space L is defined by $sL = \mathrm{Hom}(V, \mathbb{Q})$. We define a pairing $\langle\ ;\ \rangle\colon V \times sL \to \mathbb{Q}$ by $\langle v; sx\rangle = (-1)^{\deg v} sx(v)$. Moreover, using the pairing, define a trilinear map

$$\langle\ ;\ ,\ \rangle\colon \wedge^2 V \times sL \times sL \to \mathbb{Q}$$

by $\langle v \wedge w; sx, sy\rangle = \langle v; sx\rangle\langle w; sy\rangle + (-1)^{|v||w|}\langle w; sx\rangle\langle v; sy\rangle$. Then the Lie bracket $[\ ,\ ]$ in L is given by requiring that (4.1):

$$\langle v; s[x, y]\rangle = (-1)^{\deg y+1}\langle d_1 v; sx, sy\rangle$$

for $x, y \in L$ and $v \in V$. The result [4, Theorem 21.6] asserts that L is isomorphic to the homotopy Lie algebra L_Y. Therefore, in order to prove Theorem 2.5, it suffices to show that the d_1–depth of Y is equal to the integer $\mathrm{WL}(L)$, which is the greatest integer n such that $[L, L]^{(n)} \neq 0$. As in the proof of Theorem 1.2, we may write $x_{i_n,..,i_0}$ for the element $[x_{i_n}[x_{i_{n-1}}, ..., [x_{i_1}, x_{i_0}]]]$ in L.

Lemma 4.1 *For any $\alpha \in V_{n-1}$ and any $x_{i_n,..,i_0} \in [L, L]^{(n)}$, $\langle \alpha, sx_{i_n,...,i_0}\rangle = 0$.*

Proof We argue by induction on n. From the formula (4.1), we see that $\langle \alpha, sx_{i_1,i_0}\rangle = 0$ for any $\alpha \in V_0$. Suppose that $\langle \beta, sx_{i_{n-1},...,i_0}\rangle = 0$ for any $\beta \in V_{n-2}$. Let α be an element of V_{n-1}. Then we can write $d_1(\alpha) = \sum_j \beta_j \beta'_j$ with some elements β_j and β'_j of V_{n-2}. Thus it follows from the definition of the trilinear map $\langle\ ;\ ,\ \rangle$ that

$$\begin{aligned}\langle \alpha, sx_{i_n,...,i_0}\rangle &= \pm\langle d_1\alpha; sx_{i_n}, s[x_{i_{n-1}}, ...[x_{i_1}, x_{i_0}]]\rangle \\ &= \pm\langle \sum_j \beta_j\beta'_j; sx_{i_n}, s[x_{i_{n-1}}, ...[x_{i_1}, x_{i_0}]]\rangle \ = \ 0. \qquad \square\end{aligned}$$

Proposition 4.2 d_1–depth$(Y) \geq \mathrm{WL}(L)$.

Proof Suppose that $[L,L]^{(m)} \neq 0$. We choose a nonzero element $x_{i_m,\dots,i_0}$ of $[L,L]^{(m)}$. Let v_m be an element of V such that $\langle v_m, sx_{i_m,\dots,i_0}\rangle \neq 0$. Lemma 4.1 yields that $v_m \notin V_{m-1}$ and hence the d_1–depth$(Y) \geq m$. □

In order to complete the proof of Theorem 2.5, it remains to prove that d_1–depth(Y) is less than or equal to $\mathrm{WL}(L)$. To this end, we first characterize the vector space V_0 using the space S of indecomposable elements of L. One can express the vector space as $L = S \oplus [L,L]$.

Lemma 4.3 $sS = \mathrm{Hom}(V_0, \mathbb{Q})$.

Proof Let $\{x_i\}$ and $\{y_j\}$ be bases for S and $[L,L]$, respectively. Let $\{(sy_j)^*\} \cup \{(sx_i)^*\}$ be the basis of V which is the dual to the basis $\{sy_j\} \cup \{sx_i\}$ of sL. It suffices to prove that V_0 is the vector space spanned by $\{(sx_i)^*\}$. Since $\langle d(sx_i)^*; sx, sy\rangle = \langle (sx_i)^*; s[x,y]\rangle = 0$ for any $x, y \in V$, it follows that $(sx_i)^* \in V_0$. For any $v \in V_0$, we write $v = \sum_i \lambda_i (sx_i)^* + \sum_j \mu_j (sy_j)^*$ and $sy_j = \sum_{k_j} s[a_{k_j}, b_{k_j}]$ for some a_{k_j} and b_{k_j} in L. It follows that

$$\begin{aligned} 0 \;=\; \sum_{k_j} \langle dv; sa_{k_j}, sb_{k_j}\rangle &= \langle \sum_i \lambda_i (sx_i)^* + \sum_j \mu_j (sy_j)^*, \sum_{k_j} s[a_{k_j}, b_{k_j}]\rangle \\ &= \langle \sum_i \lambda_i (sx_i)^* + \sum_j \mu_j (sy_j)^*, sy_j\rangle \;=\; \mu_j. \end{aligned}$$

Thus we have $v = \sum_i \lambda_i (sx_i)^*$. □

We here study a fundamental property of the quadratic part of the differential d. Write $V_n = \overline{V_n} \oplus V_{n-1}$ and fix a basis $\{w_j\}$ for $\overline{V_n}$.

Lemma 4.4 *For any $u \in V_{n+1}$, there exist elements $e_j \in V_0$ and $f_s, g_s \in V_{n-1}$ such that*

$$d_1 u = \sum_j e_j w_j + \sum_s f_s g_s.$$

Proof The result for $n = 0$ is immediate. Let us assume that $n \geq 1$. We can write

$$d_1 u = \sum_{i \leq j} \lambda_{ij} w_i w_j + \sum_j e_j w_j + \sum_s f_s g_s$$

with some elements e_j, f_s, $g_s \in V_{n-1}$ and $\lambda_{ij} \in \mathbb{Q}$. By applying the differential d_1 to the equality, we have

$$\begin{aligned} 0 = d_1 d_1 u &= \sum_{i \le j} \lambda_{ij} d_1(w_i) w_j + \sum_{i \le j} (-1)^{|w_i|} \lambda_{ij} w_i d_1(w_j) + \sum_j d_1(e_j) w_j + Z \\ &= \sum_j \big(\sum_i \mu_{ij} d_1 w_i + d_1 e_j \big) w_j + Z \end{aligned}$$

in which $\mu_{ii} = 2\lambda_{ii}$, $\mu_{ij} = \lambda_{ij}$ for $i < j$, $\mu_{ij} = (-1)^{|w_j|+|w_j||d_1 w_i|} \lambda_{ij}$ for $i > j$ and Z is an appropriate element of $\wedge^{\ge 2} V_{n-1}$. Thus we see that $\sum_i \mu_{ij} d_1 w_i + d_1 e_j = 0$ for any j. Since $d_1 e_j \in \wedge V_{n-2}$, it follows that $\sum_i \mu_{ij} w_i$ is in V_{n-1} and hence $\sum_i \mu_{ij} w_i = 0$. The fact enables us to conclude that $\mu_{ij} = 0$ for any i, j and that e_j is in V_0. We have the result. □

Lemma 4.3 allows us to choose a basis $\{sx_k\}_{k \in J}$ for sS and its dual basis $\{e_k\}_{k \in J}$ for V_0. Let $\{w_m\}_{m \in M}$ be a basis for $\overline{V_1}$. We can write $d_1 w_m = \sum_{k_1, k_0} \lambda^{(m)}_{k_1, k_0} e_{k_1} e_{k_0}$, where $\lambda^{(m)}_{k_0, k_0} = 0$ if $|e_{k_0}|$ is odd.

Lemma 4.5 *Let $\{v^{(n)}_p\}_{1 \le p \le l_n}$ be a basis for $\overline{V_n}$, where $n \ge 1$. Then there exist rational numbers $\theta^{v^{(n)}_p}_{k_n, .., k_2, m}$ for all $k_n, .., k_2$ and m such that*

$$(-1)^{|s[x_{k_{n-1}}[\dots[x_{k_1}, x_{k_0}]\dots]]|} \langle v^{(n)}_p, s[x_{k_n}, [x_{k_{n-1}}[\dots, [x_{k_1}, x_{k_0}]\dots]]] \rangle = \sum_m \theta^{v^{(n)}_p}_{k_n, \dots, k_2, m} \lambda^{(m)}_{k_1, k_0}$$

and the matrix $\big(\theta^{v^{(n)}_p}_{k_n, .., k_2, m}\big)$ with l_n columns is of full rank; that is, the column vectors obtained from the matrix are linearly independent. Here, we regard the set $\{(k_n, .., k_2, m)\}$ as the ordered set $\{I_i\}$ by using the lexicographic order on elements $(k_n, .., k_2, m)$. Then the (i, p) component of the matrix $\big(\theta^{v^{(n)}_p}_{k_n, .., k_2, m}\big)$ is given by $\theta^{v^{(n)}_p}_{I_i}$.

Proof We argue by induction on n. In the case where $n = 1$, the result is immediate. We assume that $n \ge 2$ and that the assertion is true up to n. To simplify, we write v_p for $v^{(n+1)}_p$. Thanks to Lemma 4.4, we can express

$$d_1 v_p = \sum_{1 \le k \le q, 1 \le j \le r} \mu^{v_p}_{kj} e_k v^{(n)}_j + \sum_s f_s g_s$$

with some elements f_s and g_s in V_{n-1}, where $\mu^{v_p}_{kj} \in \mathbb{Q}$. Then it follows that

$$(-1)^{\varepsilon} \langle v_p, sx_{k_{n+1}, \dots, k_0} \rangle = \langle \sum_{k,j} \mu^{v_p}_{kj} e_k v^{(n)}_j + \sum_s f_s g_s \,;\, sx_{k_{n+1}}, sx_{k_n, \dots, k_0} \rangle =: \theta,$$

where $\varepsilon = |sx_{k_n,\dots,k_0}|$. Lemma 4.1 allows us to deduce that

$$\theta = \sum_{kj} \mu^{v_p}_{kj} \langle e_k, sx_{k_{n+1}} \rangle \langle v^{(n)}_j, sx_{k_n,\dots,k_0} \rangle$$
$$= \sum_j \mu^{v_p}_{k_{n+1}j} \Big(\sum_m \theta^{v^{(n)}_j}_{k_n,..,k_2,m} \lambda^{(m)}_{k_1,k_2} \Big) = \sum_m \Big(\sum_j \mu^{v_p}_{k_{n+1}j} \theta^{v^{(n)}_j}_{k_n,..,k_2,m} \Big) \lambda^{(m)}_{k_1,k_2}$$

We put $\theta^{v_p}_{k_{n+1},..,k_2,m} = \sum_j \mu^{v_p}_{k_{n+1}j} \theta^{v^{(n)}_j}_{k_n,..,k_2,m}$ and consider the matrix $\big(\theta^{v_p}_{k_{n+1},..,k_2,m}\big)$. Then, by definition, we see that the matrix is decomposed as

$$\Big(\theta^{v_p}_{k_{n+1},..,k_2,m}\Big) = \Big(\theta^{v_p}_{k_{n+1},I_i}\Big) = \begin{pmatrix} & \theta^{v_p}_{1I_1} & \\ & \vdots & \\ & \theta^{v_p}_{1I_s} & \\ \cdots & \theta^{v_p}_{2I_1} & \cdots \\ & \vdots & \\ & \vdots & \\ & \theta^{v_p}_{qI_s} & \end{pmatrix} = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} B,$$

where

$$A = \Big(\theta^{v^{(n)}_j}_{I_i}\Big) \quad \text{and} \quad B = \begin{pmatrix} & \mu^{v_p}_{11} & \\ & \vdots & \\ & \mu^{v_p}_{1r} & \\ \cdots & \mu^{v_p}_{21} & \cdots \\ & \vdots & \\ & \mu^{v_p}_{qr} & \end{pmatrix}.$$

Since the set $\{v_p\}$ is a basis for $\overline{V_{n+1}}$, it follows that the matrix B is of full rank. By assumption, A is of full rank and hence so is $\big(\theta^{v_p}_{k_{n+1},..,k_2,m}\big)$. This completes the proof. □

Theorem 2.5 follows from Proposition 4.2 and the following proposition.

Proposition 4.6 d_1–depth$(Y) \le \mathrm{WL}(L)$.

Proof Put $n = d_1$–depth(Y). It suffices to prove that the inequality holds in the case where $n \ge 1$. Let $\{v^{(n)}_p\}_{1 \le p \le l_n}$ be a basis for $\overline{V_n}$. We assume that

$$\langle v^{(n)}_p, s[x_{k_n}, [x_{k_{n-1}}[\dots, [x_{k_1}, x_{k_0}]\dots]]] \rangle = 0$$

for any $k_n, ..., k_1, k_0$. Then it is readily seen that $\sum_m \theta^{v_p^{(n)}}_{k_n,..,k_2,m}\lambda^{(m)}_{k_1,k_0} = 0$, where $\theta^{v_p^{(n)}}_{k_n,..,k_2,m}$ are rational numbers described in Lemma 4.5. Consider the linear combination $\sum_m \theta^{v_p^{(n)}}_{k_n,..,k_2,m} w_m$ with the basis $\{w_m\}$ for $\overline{V_1}$. We have

$$\begin{aligned} d_1(\sum_m \theta^{v_p^{(n)}}_{k_n,..,k_2,m} w_m) &= \sum_m \theta^{v_p^{(n)}}_{k_n,..,k_2,m} d_1(w_m) \\ &= \sum_m \theta^{v_p^{(n)}}_{k_n,..,k_2,m} \sum_{k_1,k_0} \lambda^{(m)}_{k_1,k_0} e_{k_1} e_{k_0} \\ &= \sum_{k_1,k_0} (\sum_m \theta^{v_p^{(n)}}_{k_n,..,k_2,m} \lambda^{(m)}_{k_1,k_0}) e_{k_1} e_{k_0} = 0. \end{aligned}$$

It follows that $\sum_m \theta^{v_p^{(n)}}_{k_n,..,k_2,m} w_m \in V_0$ and hence $\theta^{v_p^{(n)}}_{k_n,..,k_2,m} = 0$ for any m. Consequently, $\theta^{v_p^{(n)}}_{k_n,..,k_2,m}$ is zero for any $m, k_n, ..., k_2$, which is a contradiction. □

In the rest of this section, we shall prove Lemma 3.2. To this end, we first prepare a lemma.

Lemma 4.7 *The map $\eta\colon \mathbb{Q}[V \otimes C_*(\mathbb{L}_W)] \to \mathbb{Q}[V \otimes C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}})]$ in Lemma 3.2 is the inclusion of a relative Sullivan algebra.*

Proof We write $\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}} = \mathbb{L}_W \oplus Z$ with appropriate vector space Z. Then the $C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}})$ is decomposed as $C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}) = \wedge(s\mathbb{L}_W) \otimes \wedge(sZ) = \wedge(s\mathbb{L}_W) \otimes 1 \oplus \wedge(s\mathbb{L}_W) \otimes \wedge(sZ)^+$. We see that $V \otimes C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}) = V \otimes C_*(\mathbb{L}_W) \oplus V \otimes U$ and hence $\mathbb{Q}[V \otimes C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}})] = \mathbb{Q}[V \otimes C_*(\mathbb{L}_W)] \otimes \mathbb{Q}[V \otimes U]$, where $U = \wedge(s\mathbb{L}_W) \otimes \wedge(sZ)^+$. Let $U_{(j)}$ be the vector subspace of U consisting of elements with ordinary homology degree j, namely $U_{(j)} = (\wedge(s\mathbb{L}_W) \otimes \wedge(sZ)^+)_j$. Put $V(k) = \oplus_{i+j \le k} V_{ij}$, where $V_{ij} = V^i \otimes U_{(j)}$. It is readily seen that $\cup_k V(k) = V \otimes U$ and $\delta(V(k)) \subset \mathbb{Q}[V \otimes C_*(\mathbb{L}_W)] \otimes \mathbb{Q}[V(k-1)]$. Thus we have the result. □

Proof of Lemma 3.2 Let $i\colon X \to X \cup_\alpha e^{k+1}$ be the inclusion map and $l\colon C_*(\mathbb{L}_W) \to C_*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}})$ the DGC map induced by the natural inclusion $\mathbb{L} \to \mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}$. Then there exists a homotopy commutative diagram

$$\begin{tikzcd} A_{PL}(X) & A_{PL}(X \cup_\alpha e^{k+1}) \arrow[l, "A_{PL}(i)"'] \\ C^*(\mathbb{L}_W) \arrow[u, "\simeq"] & C^*(\mathbb{L}_{W \oplus \mathbb{Q}\{w_\alpha\}}), \arrow[l, "l^*"] \arrow[u, "\simeq"'] \end{tikzcd}$$

where two vertiacal arrows are quasi-isomorophisms and l^* denotes the dual map to l. By considering a Sullivan model $C^*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}}) \rightarrowtail D$ for l^* and applying Lifting lemma [3, Lemma 3.6], we have a commutative diagaram

$$\begin{tikzcd}
A_{PL}(X) & A_{PL}(X \cup_\alpha e^{k+1}) \arrow[l, "A_{PL}(i)"'] \\
D \arrow[u] \arrow[d] & C^*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}}) \arrow[l] \arrow[u] \arrow[d, "="] \\
C^*(\mathbb{L}_W) & C^*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}}) \arrow[l, "l^*"']
\end{tikzcd}$$

in which vertial arrows are quasi-isomorophisms. Thus from the naturality of the model due to Brown and Szczarba, we can construct a commutative diagram

$$\begin{tikzcd}
\mathbb{Q}[V] \arrow[r, "\simeq"'] \arrow[d, "m_1"'] & \bullet \arrow[d] & \bullet \arrow[l, "\simeq"'] \arrow[d] & \cdots & \bullet \arrow[r, "\simeq"'] \arrow[d] & \bullet \arrow[d] & A_{PL}(Y) \arrow[l, "\simeq"'] \arrow[d, "A_{PL}(ev_*)"] \\
\mathbb{Q}[V \otimes C_*(\mathbb{L}_W)] \arrow[r, "\simeq"'] \arrow[d, "\eta"'] & \bullet \arrow[d] & \bullet \arrow[l, "\simeq"'] \arrow[d] & \cdots & \bullet \arrow[r, "\simeq"'] \arrow[d] & \bullet \arrow[d] & A_{PL}(\mathcal{F}(X,Y)) \arrow[l, "\simeq"'] \arrow[d, "A_{PL}(i^\sharp)"] \\
\mathbb{Q}[V \otimes C_*(\mathbb{L}_{W\oplus\mathbb{Q}\{w_\alpha\}})] \arrow[r, "\simeq"'] & \bullet & \bullet \arrow[l, "\simeq"'] & \cdots & \bullet \arrow[r, "\simeq"'] & \bullet & A_{PL}(\mathcal{F}(X \cup_\alpha e^{k+1}, Y)) \arrow[l, "\simeq"']
\end{tikzcd}$$

in the category of DGA's in which all the horizontal arrows are quasi-isomorphisms (for the DGA's represented by dots, see [2] and also [9, Section 3], the previous and ensuring discussions). The reslts [9, Proposition 5.3] and Lemma 4.7 assert that m_1 and η are the inclusions of relative Sullivan algebras. Thus by applying Lifting lemma repeatedly, we have the two front commutative squares in Lemma 3.2. The commutativity of the back square follows from that of the two side triangles. This completes the proof. □

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