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Ordering the Reidemeister moves of a classical knot

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We show that any two diagrams of the same knot or link are connected by a sequence of Reidemeister moves which are sorted by type.

57M25; 57M27

It is one of the founding theorems of knot theory that any two diagrams of a given link may be changed from one into the other by a sequence of Reidemeister moves. One of the reasons why this result is so crucial to the subject is that it allows one to define a link invariant as an invariant of a diagram which is unchanged under Reidemeister moves.

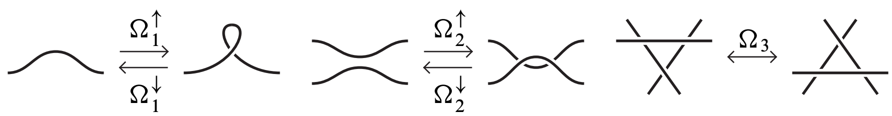

Figure 1: Reidemeister moves

Since Reidemeister's seminal paper on this topic in 1927 [2], there have been a number of steps taken to strengthen the original result in a variety of directions. In 1983, Bruce Trace [3] proved that type 1 moves may be omitted in the case where two knot diagrams have the same winding number and framing. Recent work by Joel Hass and Jeffrey Lagarias [1] has placed a bound on the number of moves required when one of the diagrams is the trivial unknot diagram.

In this paper we shall address the question of whether, given any two diagrams of a knot or link, there exists a sequence of Reidemeister moves between them which is sorted by type. We answer this question in the affirmative with the following theorem:

Theorem 1 *Given two diagrams D_1 and D_2 for a link L, D_1 may be turned into D_2 by a sequence of $\Omega_1^{\uparrow}$ moves, followed by a sequence of $\Omega_2^{\uparrow}$ moves, followed by a sequence of Ω_3 moves, followed by sequence of $\Omega_2^{\downarrow}$ moves.*

Furthermore, if D_1 and D_2 are diagrams of a link where the winding number and framing of each component is the same in each diagram, then D_1 may be turned into D_2 by a sequence of $\Omega_2^{\uparrow}$ moves, followed by a sequence of Ω_3 moves, followed by a sequence of $\Omega_2^{\downarrow}$ moves.

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In this paper, all link diagrams shall be regarded as 4–valent graphs embedded in $\mathbb{R}^2$ with signed intersections to denote overcrossings or undercrossings. All diagrams shall be oriented so as to represent an oriented link. $\Omega_1^{\uparrow}$, $\Omega_1^{\downarrow}$, $\Omega_2^{\uparrow}$, $\Omega_2^{\downarrow}$ and Ω_3 shall denote Reidemeister moves where the arrow indicates whether the move increases the number of crossings in the diagram, or decreases it, as shown in Figure 1. The winding number of a component of a link in a diagram is intuitively speaking the number of times that one must rotate anticlockwise when walking once around that component in the specified orientation. The framing (also known as the writhe) of a knot diagram is the number of crossings where the upper strand's orientation is 90 degrees clockwise from that of the lower strand, minus the number of crossings where the upper strand's orientation is 90 degrees anticlockwise from that of the lower strand. The framing of a component of a link diagram is obtained by taking the difference over only those crossings where both strands belong to the component in question. For more on these notions see Trace [3].

Returning to Theorem 1, the first part of the theorem in fact follows from the second part because of the following proposition:

Proposition 1 *Let D_1 and D_2 be two diagrams for a link L. Then we may apply $\Omega_1^{\uparrow}$ moves to D_1 so as to obtain a new diagram D_1' with the all the same winding numbers and framings as D_2.*

Proof We know that D_1 may be changed into D_2 by a sequence of Reidemeister moves. Note that only Ω_1 moves change the winding numbers and framings of a diagram, and that each Ω_1 move changes the winding number of the component on which it acts by ± 1 and the framing of the component on which it acts by ± 1. For each Ω_1 move in a sequence of Reidemeister moves from D_1 and D_2, we may carry out a $\Omega_1^{\uparrow}$ move on an edge in the same component in D_1 with the same effect on the winding number and framing of that component. After completing these moves we will have a new diagram D_1' with the same winding numbers and framings as D_2. □

We shall now turn our attention to the proof of Theorem 1. Our strategy will be to simulate each Ω_3 move with a sequence of $\Omega_2^{\uparrow}$ moves. In order to achieve this we will need to develop some new notation.

Definition 1 Let D be a link diagram in $\mathbb{R}^2$ and let $c\colon I \to \mathbb{R}^2$ be an embedded path such that $c(0)$ lies on the interior of an edge of D, $c(1)$ does not lie on D, and where $c(\text{int}(I))$ intersects D transversely in a finite number of points none of which are vertices of D, and which are given signings to denote whether the path crosses above or below D. Let C denote the image of this path so that $D \cup C$ is a graph which is 3–valent at one vertex, 1–valent at another, and 4–valent otherwise as shown in the left hand image of Figure 2.

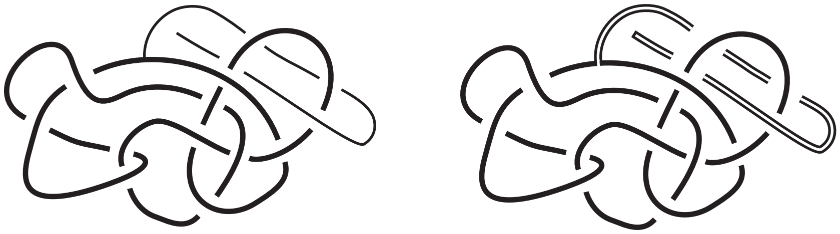


Figure 2: Adding a tail

Let $C \times [-\epsilon, \epsilon]$ denote a small product neighbourhood of C such that $(C \times [-\epsilon, \epsilon]) \cap D = (C \cap D) \times [-\epsilon, \epsilon]$. Let D' be the link diagram whose 4–valent graph is

$$D \cup \partial(C \times [-\epsilon, \epsilon]) \backslash (c(0) \times (-\epsilon, \epsilon))$$

and whose vertex signings are induced by those of $D \cup C$. The orientation of D' shall be that induced by D.

We shall say that D' *is obtained from* D *by adding a tail along* C. We shall call $\partial(C \times [-\epsilon, \epsilon]) \backslash (c(0) \times (-\epsilon, \epsilon))$ *the tail in* D' and we shall refer to C *as the core of this tail*. We shall write $D \rightsquigarrow D'$. Note that if $D \rightsquigarrow D'$ then D' may be obtained from D by a sequence of $\Omega_2^{\uparrow}$ moves. Note also that the core of a tail is an embedded arc, and not an immersed one.

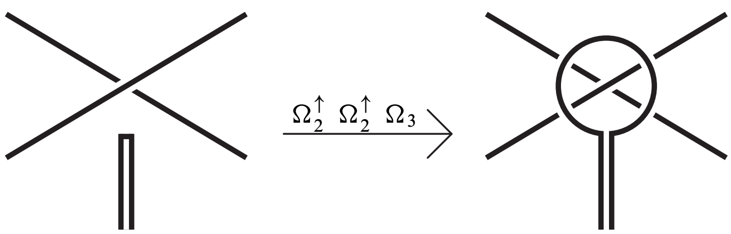


Figure 3: Turning a tail into a lollipop

Definition 2 Suppose that $D_1 \rightsquigarrow D_2$. Let D_3 be obtained from D_2 by performing two $\Omega_2^{\uparrow}$ moves and one Ω_3 move as shown in Figure 3. We shall then say that D_3 *is obtained from* D_1 *by adding a lollipop* and we shall write $D_1 \bigcirc\!\!\rightarrow D_3$.

Later on it will be important to distinguish between the part of the lollipop which circles the crossing and the part which consists of two parallel strands. We shall call these the *circle part* and the *tail part* of the lollipop respectively.

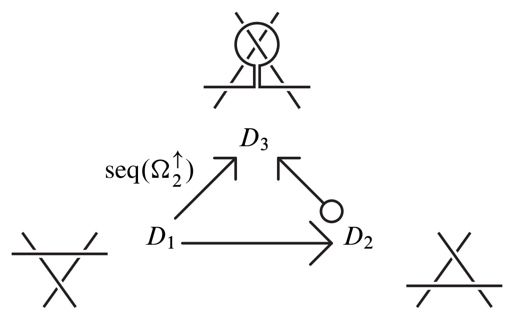
We are now in a position to say how we are going to simulate Ω_3 moves by means of $\Omega_2^{\uparrow}$ moves. This is captured in the following important lemma.

Lemma 1 *Suppose that D_2 is obtained from D_1 by means of an Ω_3 move. Then we may construct a diagram D_3 such that:*

(1) *D_3 may be obtained from D_1 by a sequence of $\Omega_2^{\uparrow}$ moves.*

(2) $D_2 \bigcirc\!\!\!\rightarrow D_3$

Proof Let D_3 be as shown below.



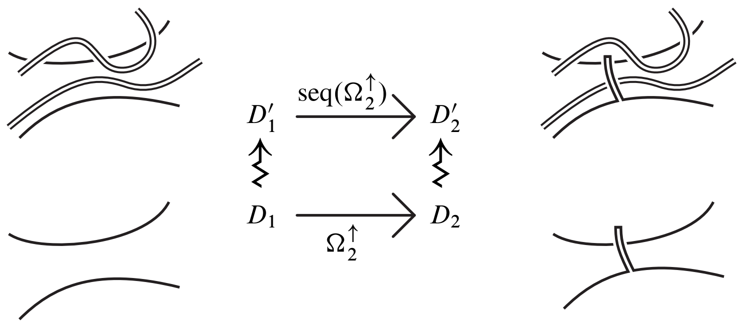
□

Lemma 2 *Let $D_1 \rightsquigarrow D_1'$. Suppose that D_2 may be obtained from D_1 by a Reidemeister move of type $\Omega_2^{\uparrow}$. It is possible to construct a diagram D_2' such that:*

(1) *D_2' may be obtained from D_1' by a sequence of $\Omega_2^{\uparrow}$ moves.*

(2) $D_2 \rightsquigarrow D_2'$

Proof Let C denote the core of the tail in D_1'. Let E_1 and E_2 be the edges (not necessarily distinct) in D_1 upon which our $\Omega_2^{\uparrow}$ move takes place. Note that E_1 and E_2 are incident to a face F of the diagram D_1. Let x_1 (resp. x_2) be a point on E_1 (resp. E_2) which does not lie in $C \times [-\epsilon, \epsilon]$. Let P be an embedded path from x_1 to x_2 whose interior lies entirely in F and which crosses $C \times [-\epsilon, \epsilon]$ transversely in a finite number of intervals. Let P' be a path obtained from P by extending it a small amount at x_2 into the neighbouring face. Then D_2' may be formed by adding a tail along P' to D_1' as shown below.



□

Corollary to Lemma 2 *Let $D_1 \rightsquigarrow D_1'$. Suppose that D_2 may be obtained from D_1 by means of a sequence of Reidemeister moves of type $\Omega_2^{\uparrow}$. It is possible to construct a diagram D_2' such that:*

(1) *D_2' may be obtained from D_1' by a sequence of $\Omega_2^{\uparrow}$ moves.*

(2) $D_2 \rightsquigarrow D_2'$

Proof Let $D_1 = E_1, \dots, E_n = D_2$ be a sequence of diagrams such that $E_i \xrightarrow{\Omega_2^{\uparrow}} E_{i+1}$. Thus we have:

$$\begin{array}{ccccccc}
D_1' = E_1' & \xrightarrow{\mathrm{seq}(\Omega_2^{\uparrow})} & E_2' & \xrightarrow{\mathrm{seq}(\Omega_2^{\uparrow})} & \cdots & \xrightarrow{\mathrm{seq}(\Omega_2^{\uparrow})} & E_n' = D_2' \\
\rotatebox{90}{$\rightsquigarrow$} & & \rotatebox{90}{$\rightsquigarrow$} & & & & \rotatebox{90}{$\rightsquigarrow$} \\
D_1 = E_1 & \xrightarrow[\Omega_2^{\uparrow}]{} & E_2 & \xrightarrow[\Omega_2^{\uparrow}]{} & \cdots & \xrightarrow[\Omega_2^{\uparrow}]{} & E_n = D_2
\end{array}$$

□

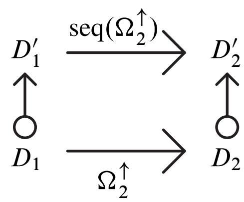
There is a similar pair of results for the adding of lollipops:

Lemma 3 *Let $D_1 \bigcirc\!\!\rightarrow D_1'$. Suppose that D_2 may be obtained from D_1 by a Reidemeister move of type $\Omega_2^{\uparrow}$. It is possible to construct a diagram D_2' such that:*

(1) *D_2' may be obtained from D_1' by a sequence of $\Omega_2^{\uparrow}$ moves.*

(2) $D_2 \bigcirc\!\!\rightarrow D_2'$

Proof In this case we proceed exactly as in the proof of Lemma 1 except that we insist that the path P avoids the circle part of of the lollipop.



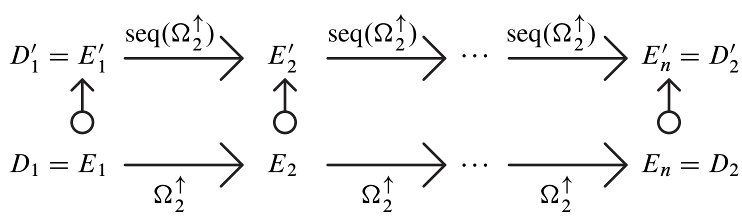
□

Corollary to Lemma 3 *Let $D_1 \bigcirc\!\!\!\to D_1'$. Suppose that D_2 may be obtained from D_1 by sequence of Reidemeister moves of type $\Omega_2^{\uparrow}$. It is possible to construct a diagram D_2' such that:*

(1) *D_2' may be obtained from D_1' by a sequence of $\Omega_2^{\uparrow}$ moves.*

(2) *$D_2 \bigcirc\!\!\!\to D_2'$*

Proof As before let $D_1 = E_1, \ldots, E_n = D_2$ be a sequence of diagrams such that $E_i \xrightarrow{\Omega_2^{\uparrow}} E_{i+1}$. In this case we have:



□

We need one more result before we can turn to the proof of Theorem 1.

Proposition 2 *Suppose that D_2 is obtained from D_1 by the addition of a sequence of tails and lollipops. Then there exists a diagram D_3 such that:*

(1) *D_3 may be obtained from D_2 by means of a sequence of $\Omega_2^{\uparrow}$ moves followed by a sequence of Ω_3 moves.*

(2) *D_3 may be obtained from D_1 by means of a sequence of $\Omega_2^{\uparrow}$ moves.*

Proof Our strategy will be to construct D_3 from D_2 in accordance with the first condition and then show that our new diagram D_3 satisfies the second condition.

Let $D_1 = E_1, \ldots, E_n = D_2$ be a sequence of diagrams such that for each $i \in \{1, \ldots, n-1\}$ either $E_i \rightsquigarrow E_{i+1}$ or $E_i \bigcirc\!\!\!\to E_{i+1}$. We shall start by performing moves on the

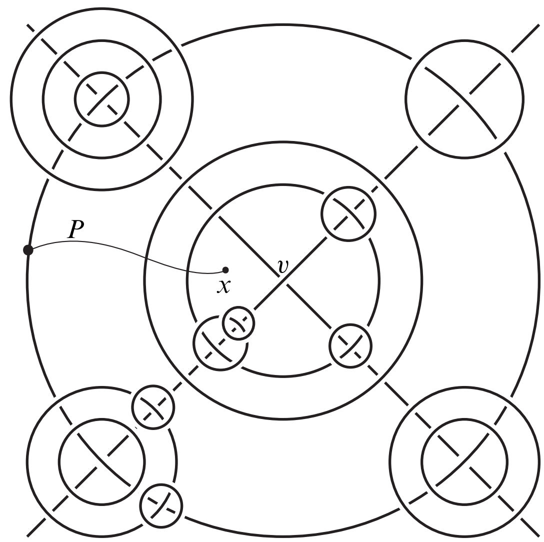


Figure 4: Diagram showing the circle part of a lollipop in D_2

circle parts of the lollipops in D_2. Note that each of these circle parts is associated with a particular vertex of D_2, namely the vertex around which the circle part was originally added, and furthermore that the circle parts associated to a particular vertex are disjoint and concentric. Consider all the circle parts around a vertex v. Let x be some point in a region R of D_2 which neighbours v. Let P be a path from x to a point on the outermost circle part C associated with v which avoids circle parts of other lollipops and avoids the tail part of C, as shown in Figure 4 which omits any tail parts of D_2 for the sake of clarity.

We may now undertake a sequence of type $\Omega_2^{\uparrow}$ moves in a neighbourhood of P as follows. It will be convenient to use the language of adding tails, but one should view this as a shorthand for describing a sequence of $\Omega_2^{\uparrow}$ moves. First add a tail to the innermost circle part associated to v along the part of P which is inside that circle part. Note that P will be disjoint from this tail in the resulting diagram apart from at x. Extend the tail a small amount so that P and the tail are now disjoint. Continue by adding tails to all the circle parts associated to v along P in the same way, working in order from the innermost circle part to the outermost circle part, C. This procedure is illustrated in Figure 5.

It is worth remembering the tail part of the diagram not shown in the figure, and observing that as long as we are just performing $\Omega_2^{\uparrow}$ moves then we can simply go over that part as required.

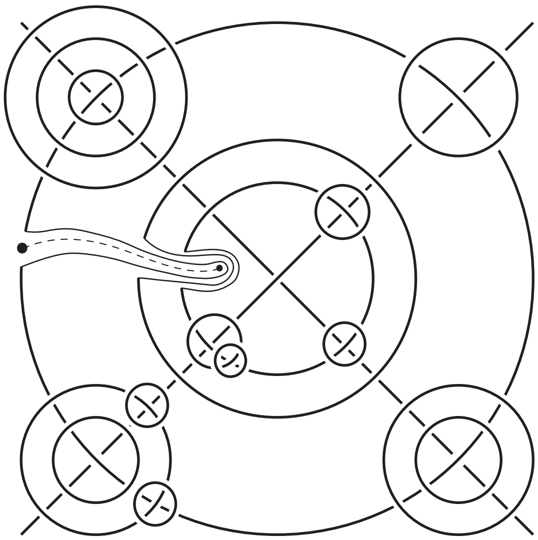


Figure 5: Add tails along P using $\Omega_2^{\uparrow}$ moves

Since x was chosen to lie in R, a region of D_2 which neighbours v, $\Omega_2^{\uparrow}$ moves may now be applied in turn to push the 'nested tails' which have just been added over the two edges of R which are adjacent to v, as shown in Figure 6.

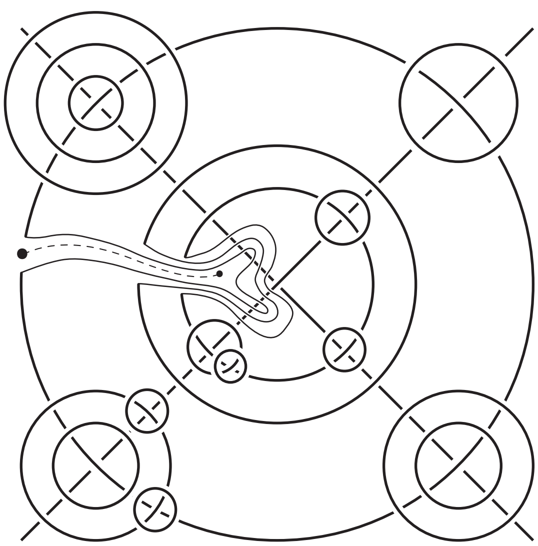


Figure 6: Perform some more $\Omega_2^{\uparrow}$ moves near each vertex v

Note that the procedure undertaken so far takes place inside C, the outermost circle part associated with v, but outside any circle parts associated to other vertices inside and on C. This is good news since it means that we may repeat this operation at all vertices with at least one circle part associated. After doing this, we are done with $\Omega_2^{\uparrow}$ moves. We now form D_3 by using Ω_3 moves to push all the circle parts associated to a particular vertex across that vertex, as in Figure 7, again observing that we may do this on each collection of circle parts independently.

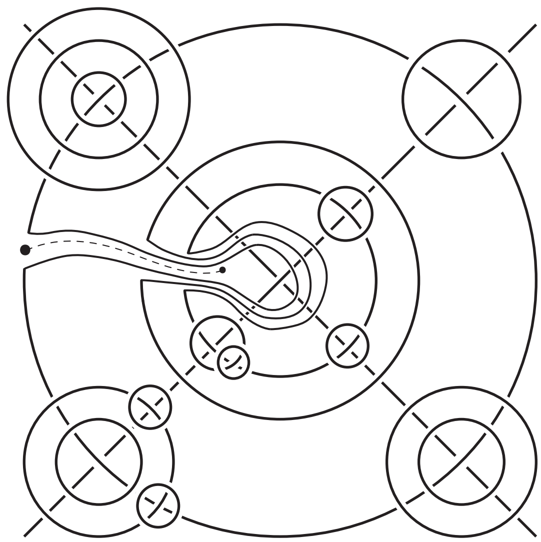


Figure 7: Perform these Ω_3 moves at each vertex

It is now time to show that D_3 may be obtained from D_1 by means of a sequence of $\Omega_2^{\uparrow}$ moves. Let us go back to the sequence $D_1 = E_1, \dots, E_n = D_2$ where $E_i \rightsquigarrow E_{i+1}$ or $E_i \bigcirc\!\!\rightarrow E_{i+1}$ for $i \in \{1, \dots, n-1\}$. Consider the part of D_2 which was added in the final step. If this was a tail, then it still is in D_3 and it may be removed by $\Omega_2^{\downarrow}$ moves. If it was a lollipop then it may also now be removed by $\Omega_2^{\downarrow}$ moves since the circle part is now as shown in Figure 8.

After removing the last tail or lollipop from D_3 we may now remove the second last in the same way. Repeating this process we will eventually reach D_1 by means of $\Omega_2^{\downarrow}$ moves. □

Theorem 1 *Given two diagrams D_1 and D_2 for a link L, D_1 may be turned into D_2 by a sequence of $\Omega_1^{\uparrow}$ moves, followed by a sequence of $\Omega_2^{\uparrow}$ moves, followed by a sequence of Ω_3 moves, followed by sequence of $\Omega_2^{\downarrow}$ moves.*

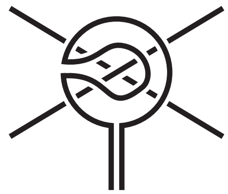


Figure 8: Circle part of the last lollipop

Furthermore, if D_1 and D_2 are diagrams of a link where the winding number and framing of each component is the same in each diagram, then D_1 may be turned into D_2 by a sequence of $\Omega_2^{\uparrow}$ moves, followed by a sequence of Ω_3 moves, followed by a sequence of $\Omega_2^{\downarrow}$ moves.

Proof By Proposition 1 it is enough to prove the second part of the theorem. Let D_1 and D_2 be diagrams of a link where the winding number and framing of each component is the same in each diagram. Bruce Trace proved in [3] that any two knot diagrams with the same winding number and framing may be turned from one into another by means of Ω_2 and Ω_3 moves. In fact his result may be readily generalised to link diagrams with the same hypotheses as we have made about D_1 and D_2. All one needs to do is to apply the method used in [3] to each component of the link.

We shall thus proceed by induction on $M(D_1, D_2)$, the minimum number of Reidemeister moves required to turn D_1 into D_2 with only Ω_2 and Ω_3 moves. The claim clearly holds for $M(D_1, D_2) = 1$. Let $D_1 = I_1, \dots, I_m = D_2$ be a sequence of link diagrams arising from a minimal length sequence of Ω_2 and Ω_3 moves connecting D_1 and D_2. Then $M(D_1, D_2) = M(I_2, D_2) + 1$. By the inductive hypothesis, I_2 may be turned into D_2 by a sequence of $\Omega_2^{\uparrow}$ moves, followed by a sequence of Ω_3 moves, followed by a sequence of $\Omega_2^{\downarrow}$ moves.

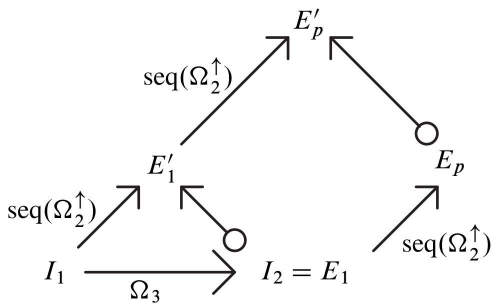
Let

$$I_2 = E_1, \dots, E_n = D_2$$

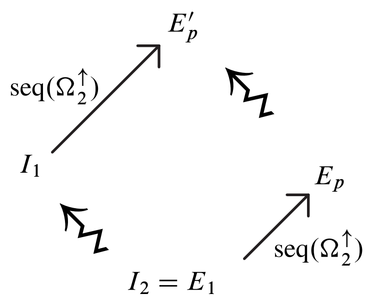
be a sequence of diagrams arising from such a sequence of Reidemeister moves. Let E_p, E_q $(1 \le p \le q \le n)$ be such that

$$E_1 \xrightarrow{\Omega_2^{\uparrow}}, \dots, \xrightarrow{\Omega_2^{\uparrow}} E_p \xrightarrow{\Omega_3}, \dots, \xrightarrow{\Omega_3} E_q \xrightarrow{\Omega_2^{\downarrow}}, \dots, \xrightarrow{\Omega_2^{\downarrow}} E_n.$$

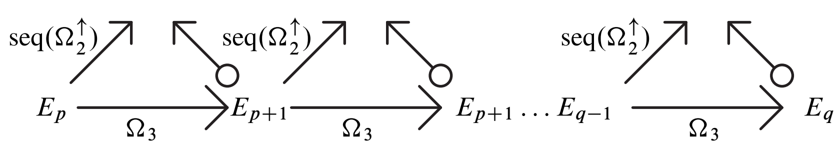
If the move from I_1 to I_2 is of type $\Omega_2^{\uparrow}$ then there is nothing to prove. The remaining cases to consider are if this move is of type Ω_3 or of type $\Omega_2^{\downarrow}$. In the former case apply Lemma 1 to this move and the Corollary to Lemma 3 to the sequence of $\Omega_2^{\uparrow}$ moves that follow it to obtain a diagram E'_p as shown:



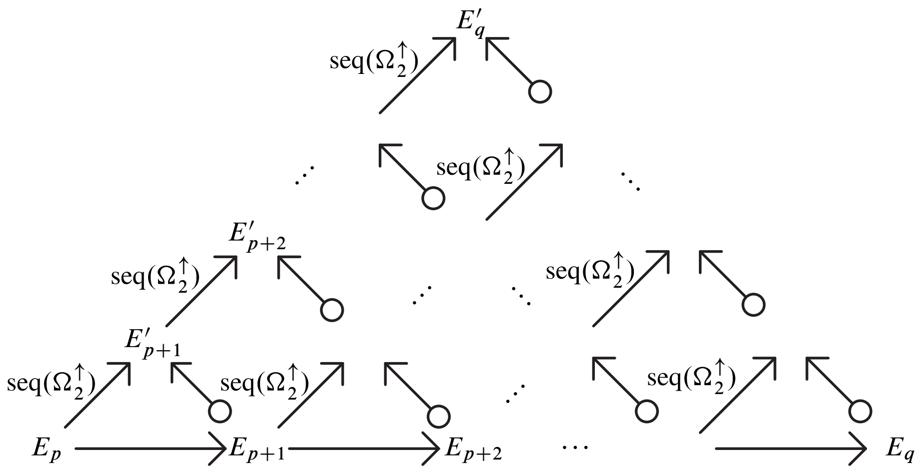
If the move from I_1 to I_2 is of type $\Omega_2^{\downarrow}$ then $I_2 \rightsquigarrow I_1$. Thus we may apply the Corollary to Lemma 2 to obtain E'_p as shown:



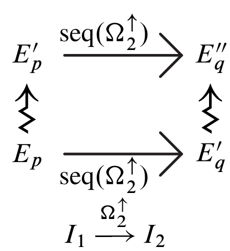
Thus in either case we may perform a sequence of $\Omega_2^{\uparrow}$ moves on I_1 to obtain a diagram E'_p such that $E_p \bigcirc\!\!\!\rightarrow E'_p$ or $E_p \rightsquigarrow E'_p$. Now, E_p and E_q are joined by a sequence of Ω_3 moves. Applying Lemma 1 to each of these we obtain the following:

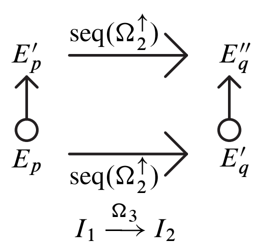


The stage is now set to apply the Corollary to Lemma 3 several times to obtain a diagram E'_q as shown in the next diagram:



Now, E'_p is a diagram with either $E_p \bigcirc\!\!\rightarrow E'_p$ or $E_p \rightsquigarrow E'_p$. If we apply the Corollary to Lemma 2 or the Corollary to Lemma 3 accordingly, then we get a diagram E''_q as shown:





Thus we have formed a diagram E''_q such that:

(1) E''_q is obtained from D_1 by means of a sequence of $\Omega_2^{\uparrow}$ moves.

(2) E''_q is obtained from E_q by the addition of a sequence of tails and lollipops.

We complete the proof by applying Proposition 2 to E''_q and E_q. □

We conclude this paper by noting that although the last theorem was proved by induction, we could have taken any sequence of Ω_2 and Ω_3 moves as our ingredients. In this way, one could obtain a (large) upper bound on the number of sorted moves required to pass from one diagram to the other in terms of the minimum number of unsorted moves.

References

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