Algebraic & Geometric Topology 6 (2006) 1025–1035

Intrinsic linking and knotting of graphs in arbitrary 3–manifolds

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We prove that a graph is intrinsically linked in an arbitrary 3–manifold M if and only if it is intrinsically linked in S^3. Also, assuming the Poincaré Conjecture, we prove that a graph is intrinsically knotted in M if and only if it is intrinsically knotted in S^3.

05C10, 57M25

1 Introduction

The study of intrinsic linking and knotting began in 1983 when Conway and Gordon [1] showed that every embedding of K_6 (the complete graph on six vertices) in S^3 contains a non-trivial link, and every embedding of K_7 in S^3 contains a non-trivial knot. Since the existence of such a non-trivial link or knot depends only on the graph and not on the particular embedding of the graph in S^3, we say that K_6 is *intrinsically linked* and K_7 is *intrinsically knotted*.

At roughly the same time as Conway and Gordon's result, Sachs [12; 11] independently proved that K_6 and $K_{3,3,1}$ are intrinsically linked, and used these two results to prove that any graph with a minor in the *Petersen family* (Figure 1) is intrinsically linked. Conversely, Sachs conjectured that any graph which is intrinsically linked contains a minor in the Petersen family. In 1995, Robertson, Seymour and Thomas [10] proved Sachs' conjecture, and thus completely classified intrinsically linked graphs.

Examples of intrinsically knotted graphs other than K_7 are now known, see Foisy [2], Kohara and Suzuki [3] and Shimabara [13]. Furthermore, a result of Robertson and Seymour [9] implies that there are only finitely many intrinsically knotted graphs that are minor-minimal with respect to intrinsic knottedness. However, as of yet, intrinsically knotted graphs have not been classified.

Published: 9 August 2006 DOI: 10.2140/agt.2006.6.1025

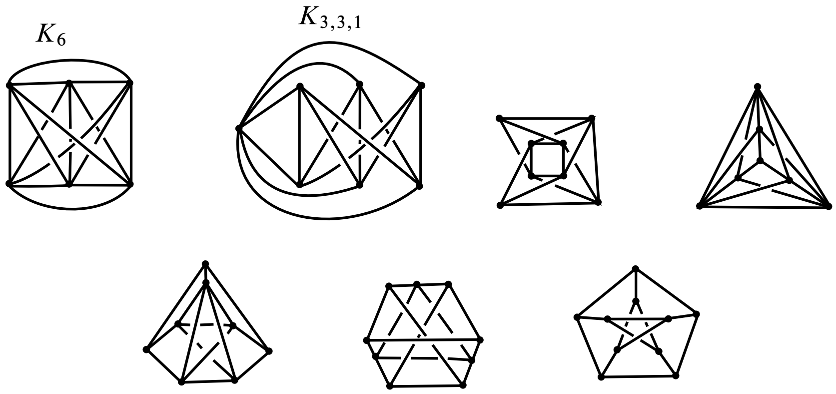

Figure 1: The Petersen family of graphs

In this paper we consider the properties of intrinsic linking and knotting in *arbitrary* 3–manifolds. We show that these properties are truly *intrinsic* to a graph in the sense that they do not depend on either the ambient 3–manifold or the particular embedding of the graph in the 3–manifold. Our proof in the case of intrinsic knotting assumes the Poincaré Conjecture.

We will use the following terminology. By a *graph* we shall mean a finite graph, possibly with loops and repeated edges. Manifolds may have boundary and do not have to be compact. All spaces are piecewise linear; in particular, we assume that the image of an *embedding* of a graph in a 3–manifold is a piecewise linear subset of the 3–manifold. An embedding of a graph G in a 3–manifold M is *unknotted* if every circuit in G bounds a disk in M; otherwise, the embedding is *knotted*. An embedding of a graph G in a 3–manifold M is *unlinked* if it is unknotted and every pair of disjoint circuits in G bounds disjoint disks in M; otherwise, the embedding is *linked*. A graph is *intrinsically linked* in M if every embedding of the graph in M is linked; and a graph is *intrinsically knotted* in M if every embedding of the graph in M is knotted. (So by definition an intrinsically knotted graph must be intrinsically linked, but not vice-versa.)

The main results of this paper are that a graph is intrinsically linked in an arbitrary 3–manifold if and only if it is intrinsically linked in S^3 (Theorem 1); and (assuming the Poincaré Conjecture) that a graph is intrinsically knotted in an arbitrary 3–manifold if and only if it is intrinsically knotted in S^3 (Theorem 2). We use Robertson, Seymour, and Thomas' classification of intrinsically linked graphs in S^3 for our proof of Theorem 1. However, because there is no analogous classification of intrinsically knotted graphs in S^3, we need to take a different approach to prove Theorem 2. In particular, the proof of Theorem 2 uses Proposition 2 (every compact subset of a simply connected 3–manifold is homeomorphic to a subset of S^3), whose proof in turn relies on the Poincaré

Conjecture. Our assumption of the Poincaré Conjecture seems reasonable, because Perelman [7; 8] has announced a proof of Thurston's Geometrization Conjecture, which implies the Poincaré Conjecture [4]. (See also Morgan and Tian [5].)

We would like to thank Waseda University, in Tokyo, for hosting the International Workshop on Knots and Links in a Spatial Graph, at which this paper was conceived. We also thank the Japan Society for the Promotion of Science for providing funding for the third author with a Grant-in-Aid for Scientific Research.

2 Intrinsically linked graphs

In this section, we prove that intrinsic linking is independent of the 3–manifold in which a graph is embedded. We begin by showing (Lemma 1) that any unlinked embedding of a graph G in a 3–manifold lifts to an unlinked embedding of G in the universal cover. In the universal cover, the linking number can be used to analyze intrinsic linking (Lemma 2), as in the proofs of Conway and Gordon [1] and Sachs [12; 11]. After we've shown that K_6 and $K_{3,3,1}$ are intrinsically linked in any 3–manifold (Proposition 1), we use the classification of intrinsically linked graphs in S^3, Robertson, Seymour, and Thomas [10], to conclude that any graph that is intrinsically linked in S^3 is intrinsically linked in every 3–manifold (Theorem 1).

We call a circuit of length 3 in a graph a *triangle* and a circuit of length 4 a *square*.

Lemma 1 *Any unlinked embedding of a graph G in a 3–manifold M lifts to an unlinked embedding of G in the universal cover $\widetilde{M}$.*

Proof Let $f\colon G \to M$ be an unlinked embedding. $\pi_1(G)$ is generated by the circuits of G (attached to a basepoint). Since $f(G)$ is unknotted, every cycle in $f(G)$ bounds a disk in M. So $f_*(\pi_1(G))$ is trivial in $\pi_1(M)$.

Thus, an unlinked embedding of G into M lifts to an embedding of G in the universal cover $\widetilde{M}$. Since the embedding into M is unlinked, cycles of G bound disks in M and pairs of disjoint cycles of G bound disjoint disks in M. All of these disks in M lift to disks in $\widetilde{M}$, so the embedding of the graph in $\widetilde{M}$ is also unlinked. □

Recall that if M is a 3–manifold with $H_1(M) = 0$, then disjoint oriented loops J and K in M have a well-defined linking number $\operatorname{lk}(J, K)$, which is the algebraic intersection number of J with any oriented surface bounded by K. Also, the linking number is symmetric: $\operatorname{lk}(J, K) = \operatorname{lk}(K, J)$.

It will be convenient to have a notation for the linking number modulo 2: Define $\omega(J,K) = \mathrm{lk}(J,K) \bmod 2$. Notice that $\omega(J,K)$ is defined for a pair of *unoriented* loops. Since linking number is symmetric, so is $\omega(J,K)$. If $J_1, \ldots, J_n$ are loops in an embedded graph such that in the list $J_1, \ldots, J_n$ every edge appears an even number of times, and if K is another loop, disjoint from the J_i, then $\sum \omega(J_i, K) = 0 \bmod 2$.

If G is a graph embedded in a simply connected 3–manifold, let

$$\omega(G) = \sum \omega(J,K) \bmod 2,$$

where the sum is taken over all *unordered* pairs (J,K) of disjoint circuits in G. Notice that if $\omega(G) \neq 0$, then the embedding is linked (but the converse is not true).

Lemma 2 *Let $\widetilde{M}$ be a simply connected 3–manifold, and let H be an embedding of K_6 or $K_{3,3,1}$ in $\widetilde{M}$. Let e be an edge of H, and let e' be an arc in $\widetilde{M}$ with the same endpoints as e, but otherwise disjoint from H. Let H' be the graph $(H-e)\cup e'$. Then $\omega(H') = \omega(H)$.*

Proof Let $D = e \cup e'$.

First consider the case that H is an embedding of K_6. We will count how many terms in the sum defining $\omega(H)$ change when e is replaced by e'. Let K_1, K_2, K_3 and K_4 be the four triangles in H disjoint from e (hence also disjoint from e' in H'), and for each i let J_i be the triangle complementary to K_i. The J_i all contain e. For each i, let $J_i' = (J_i - e) \cup e'$, and notice that

(1) $$\omega(J_i', K_i) = \omega(J_i, K_i) + \omega(D, K_i) \bmod 2.$$

Because each edge appears twice in the list K_1, K_2, K_3, K_4, we have $\omega(K_1, D) + \omega(K_2, D) + \omega(K_3, D) + \omega(K_4, D) = 0 \bmod 2$. Thus, $\omega(K_i, D)$ is nonzero for an even number of i. It follows from Equation (1) that there are an even number of i such that $\omega(J_i', K_i) \neq \omega(J_i, K_i)$. Thus, $\sum_{i=1}^4 \omega(J_i', K_i) = \sum_{i=1}^4 \omega(J_i, K_i) \bmod 2$, and

$$\begin{aligned}
\omega(H') &= \sum_{\substack{J,K \subseteq H' \\ \ni e' \notin J,K}} \omega(J,K) + \sum_{i=1}^4 \omega(J_i', K_i) \bmod 2 \\
&= \sum_{\substack{J,K \subseteq H \\ \ni e \notin J,K}} \omega(J,K) + \sum_{i=1}^4 \omega(J_i, K_i) \bmod 2 \\
&= \omega(H)
\end{aligned}$$

Next consider the case that H is an embedding of $K_{3,3,1}$. Let x be the vertex of valence six in H (and in H').

Case 1 e contains x. Then e is not in any square in H that has a complementary disjoint triangle. Let K_1, K_2 and K_3 be the three squares in H disjoint from e, and let J_1, J_2 and J_3 be the corresponding complementary triangles, all of which contain e. As in the K_6 case, let $J_i' = (J_i - e) \cup e'$ for each i; again we have Equation (1). Every edge in the list K_1, K_2, K_3 appears exactly twice, so $\omega(K_1, D) + \omega(K_2, D) + \omega(K_3, D) = 0 \bmod 2$. Thus, $\omega(K_i, D)$ is nonzero for an even number of i; and for an even number of i, $\omega(J_i', K_i) \neq \omega(J_i, K_i)$. The other pairs of circuits contributing to $\omega(H)$ do not involve e. As in the K_6 case, it follows that $\omega(H') = \omega(H)$.

Case 2 e doesn't contain x. Let J_0 be the triangle containing e, and let K_0 be the complementary square. Let J_1 through J_4 be the four squares that contain e, but not x (so that they have complementary triangles); and let K_1 through K_4 be the complementary triangles. With J_i' defined as in the other cases, we again have Equation (1). Every edge appears an even number of times in the list K_0, K_1, K_2, K_3, K_4, so $\sum_{i=0}^{4} \omega(K_i, D) = 0 \bmod 2$, and $\omega(K_i, D) \neq 0$ for an even number of i. As in the other cases, it follows that for an even number of i, $\omega(J_i', K_i) \neq \omega(J_i, K_i)$; and an even number of the terms in the sum defining $\omega(H)$ change when e is replaced by e'; and $\omega(H') = \omega(H)$. □

Proposition 1 *K_6 and $K_{3,3,1}$ are intrinsically linked in any 3–manifold M.*

Proof Let G be either K_6 or $K_{3,3,1}$, and let $f\colon G \to M$ be an embedding. Suppose for the sake of contradiction that $f(G)$ is unlinked. Let $\widetilde{M}$ be the universal cover of M. By Lemma 1, f lifts to an unlinked embedding $\tilde{f}\colon G \to \widetilde{M}$.

Let $\widetilde{G} = \tilde{f}(G) \subseteq \widetilde{M}$, and let $\widetilde{H}$ be a copy of G embedded in a ball in $\widetilde{M}$. Isotope $\widetilde{G}$ so that $\widetilde{H}$ and $\widetilde{G}$ have the same vertices, but do not otherwise intersect. Then $\widetilde{G}$ can be transformed into $\widetilde{H}$ by changing one edge at a time – replace an edge of $\widetilde{G}$ by the corresponding edge of $\widetilde{H}$, once for every edge. By repeated applications of Lemma 2, $\omega(\widetilde{G}) = \omega(\widetilde{H})$. Since $\widetilde{H}$ is inside a ball in $\widetilde{M}$, Conway and Gordon's proof [1], and Sachs' proof [12; 11], that K_6 and $K_{3,3,1}$ are intrinsically linked in S^3, show that $\omega(\widetilde{H}) = 1$.

Thus, $\omega(\widetilde{G}) = 1$, and there must be disjoint circuits J and K in $\widetilde{G}$ that do not bound disjoint disks in $\widetilde{M}$, contradicting that $\tilde{f}$ is an *unlinked* embedding. Thus, $f(G)$ is linked in M. □

Let G be a graph which contains a triangle Δ. Remove the three edges of Δ from G. Add three new edges, connecting the three vertices of Δ to a new vertex. The resulting graph, G', is said to have been obtained from G by a "$\Delta - Y$ move" (Figure 2). The seven graphs that can be obtained from K_6 and $K_{3,3,1}$ by $\Delta - Y$ moves are the *Petersen family* of graphs (Figure 1).

If a graph G' can be obtained from a graph G by repeatedly deleting edges and isolated vertices of G, and/or contracting edges of G, then G' is a *minor* of G.

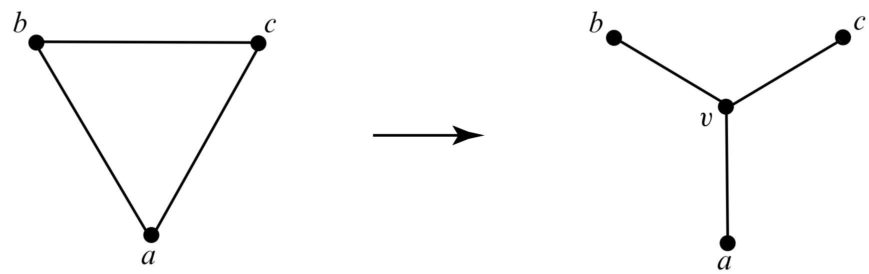

Figure 2: A $\Delta - Y$ Move

The following facts were first proved, in the S^3 case, by Motwani, Raghunathan and Saran [6]. Here we generalize the proofs to any 3–manifold M.

Fact 1 *If a graph G is intrinsically linked in M, and G' is obtained from G by a $\Delta - Y$ move, then G' is intrinsically linked in M.*

Proof Suppose to the contrary that G' has an unlinked embedding $f\colon G' \to M$. Let a, b, c and v be the embedded vertices of the Y illustrated in Figure 2. Let B denote a regular neighborhood of the embedded Y such that a, b and c are on the boundary of B, v is in the interior of B, and B is otherwise disjoint from $f(G')$. Now add edges ab, bc and ac in the boundary of B so that the resulting embedding of the K_4 with vertices a, b, c, and v is panelled in B (ie, every cycle bounds a disk in the complement of the graph). We now remove vertex v (and its incident edges) to get an embedding h of G such that if e is any edge of $G \cap G'$ then $h(e) = f(e)$ and the triangle abc is in ∂B.

Observe that if K is any circuit in $h(G)$ other than the triangle abc, then K is isotopic to a circuit in G'. The triangle abc bounds a disk in B, and since $f(G')$ is unknotted, every circuit in $f(G')$ bounds a disk in M. Thus $h(G)$ is unknotted. Also if J and K are disjoint circuits in $h(G)$ neither of which is abc, then $J \cup K$ is isotopic to a pair of disjoint circuits $J' \cup K'$ in $f(G')$. Since $f(G')$ is unlinked, J' and K' bound disjoint disks in M. Hence J and K also bound disjoint disks in M. Finally if K is a circuit in $h(G)$ which is disjoint from abc, then K is contained in $f(G')$. Since $f(G')$ is unknotted, K bounds a disk D in M. Furthermore, since B is a ball, we can isotope D to a disk which is disjoint from B. Now abc and K bound disjoint disks in M. So $h(G)$ is unlinked, contradicting the hypothesis that G is intrinsically linked in M. We conclude that G' is also intrinsically linked in M. □

Fact 2 *If a graph G has an unlinked embedding in M, then so does every minor of G.*

Proof The proof is identical to the proof for S^3. □

Theorem 1 *Let G be a graph, and let M be a 3–manifold. The following are equivalent:*

(1) *G is intrinsically linked in M,*

(2) *G is intrinsically linked in S^3,*

(3) *G has a minor in the Petersen family of graphs.*

Proof Robertson, Seymour and Thomas [10] proved that (2) and (3) are equivalent. We see as follows that (1) implies (2): Suppose there is an unlinked embedding of G in S^3. Then the embedded graph and its system of disks in S^3 are contained in a ball, which embeds in M.

We will complete the proof by checking that (3) implies (1). K_6 and $K_{3,3,1}$ are intrinsically linked in M by Proposition 1. Thus, by Fact 1, all the graphs in the Petersen family are intrinsically linked in M. Therefore, if G has a minor in the Petersen family, then it is intrinsically linked in M, by Fact 2. □

3 Compact subsets of a simply connected space

In this section, we assume the Poincaré Conjecture, and present some known results about 3–manifolds, which will be used in Section 4 to prove that intrinsic knotting is independent of the 3–manifold (Theorem 2).

Fact 3 *Assume that the Poincaré Conjecture is true. Let $\widetilde{M}$ be a simply connected 3–manifold, and suppose that $B \subseteq \widetilde{M}$ is a compact 3–manifold whose boundary is a disjoint union of spheres. Then B is a ball with holes (possibly zero holes).*

Proof By the Seifert–Van Kampen theorem, B itself is simply connected. Cap off each boundary component of B with a ball, and the result is a closed simply connected 3–manifold. By the Poincaré Conjecture, this must be the 3–sphere. □

Fact 4 *Let $\widetilde{M}$ be a simply connected 3–manifold, and suppose that $N \subseteq \widetilde{M}$ is a compact 3–manifold whose boundary is nonempty and not a union of spheres. Then there is a compression disk D in $\widetilde{M}$ for a component of ∂N such that $D \cap \partial N = \partial D$.*

Proof Since ∂N is nonempty, and not a union of spheres, there is a boundary component F with positive genus. Because $\widetilde{M}$ is simply connected, F is not incompressible in $\widetilde{M}$. Thus, F has a compression disk.

Among all compression disks for boundary components of N (intersecting ∂N transversely), let D be one such that $D \cap \partial N$ consists of the fewest circles. Suppose, for the sake of contradiction, that there is a circle of intersection in the interior of D. Let c be a circle of intersection which is innermost in D, bounding a disk D' in D. Either c is nontrivial in $\pi_1(\partial N)$, in which case D' is itself a compression disk; or c is trivial, bounding a disk on ∂N, which can be used to remove the circle c of intersection from $D \cap \partial N$. In either case, there is a compression disk for ∂N which has fewer intersections with ∂N than D has, contradicting minimality. Thus, $D \cap \partial N = \partial D$. □

We are now ready to prove the main result of this section. Because its proof uses Fact 3, it relies on the Poincaré Conjecture.

Proposition 2 *Assume that the Poincaré Conjecture is true. Then every compact subset K of a simply connected 3–manifold $\widetilde{M}$ is homeomorphic to a subset of S^3.*

Proof We may assume without loss of generality that K is connected. Let $N \subseteq \widetilde{M}$ be a closed regular neighborhood of K in $\widetilde{M}$. Then N is a compact connected 3–manifold with boundary. It suffices to show that N embeds in S^3.

Let $g(S)$ denote the genus of a connected closed orientable surface S. Define the complexity $c(S)$ of a closed orientable surface S to be the sum of the squares of the genera of the components S_i of S, so $c(S) = \sum_{S_i} g(S_i)^2$. Our proof will proceed by induction on $c(\partial N)$. We make two observations about the complexity function.

(1) $c(S) = 0$ if and only if S is a union of spheres.

(2) If S' is obtained from S by surgery along a non-trivial simple closed curve γ, then $c(S') < c(S)$.

We prove Observation (2) as follows. It is enough to consider the component S_0 of S containing γ. If γ separates S_0, then $S_0 = S_1 \# S_2$, where S_1 and S_2 are not spheres, and S' is the result of replacing S_0 by $S_1 \cup S_2$ in S. In this case, $c(S_0) = g(S_0)^2 = (g(S_1)+g(S_2))^2 = c(S_1)+c(S_2)+2g(S_1)g(S_2) > c(S_1)+c(S_2)$, since $g(S_1)$ and $g(S_2)$ are nonzero. On the other hand, if γ does not separate S_0, then surgery along γ reduces the genus of the surface. Then the square of the genus is also smaller, and hence again $c(S') < c(S)$.

If $c(\partial N) = 0$, then by Fact 3 N is a ball with holes, and so embeds in S^3, establishing our base case. If $c(\partial N) > 0$, then by Fact 4 there is a compression disk D for ∂N such that $D \cap \partial N = \partial D$. There are three cases to consider.

Case 1 $D \cap N = \partial D$. Let $N' = N \cup \text{nbd}(D)$. Since $\partial N'$ is the result of surgery on ∂N along a non-trivial simple closed curve, $c(\partial N') < c(\partial N)$, so by induction N' embeds in S^3. Hence N embeds in S^3.

Case 2 $D \cap N = D$, and D separates N. Then cutting N along D (ie removing $D \times (-1, 1)$) yields two connected manifolds N_1 and N_2, with $c(\partial N_1) < c(\partial N)$ and $c(\partial N_2) < c(\partial N)$. So N_1 and N_2 each embed in S^3. Consider two copies of S^3, one containing N_1 and the other containing N_2.

Let C_1 be the component of $S^3 - N_1$ whose boundary contains $D \times \{1\}$, and C_2 be the component of $S^3 - N_2$ whose boundary contains $D \times \{-1\}$. Remove small balls B_1 and B_2 from C_1 and C_2, respectively. Then glue together the balls $\text{cl}(S^3 - B_1)$ and $\text{cl}(S^3 - B_2)$ along their boundaries. The result is a 3–sphere containing both N_1 and N_2, in which $D \times \{1\}$ and $D \times \{-1\}$ lie in the boundary of the same component of $S^3 - (N_1 \cup N_2)$. So we can embed the arc $\{0\} \times (-1, 1)$ (the core of $D \times (-1, 1)$) in $S^3 - (N_1 \cup N_2)$, which means we can extend the embedding of $N_1 \cup N_2$ to an embedding of N.

Case 3 $D \cap N = D$, but D does not separate N. Then cutting N along D yields a new connected manifold N' with $c(\partial N') < c(\partial N)$, so N' embeds in S^3. As in the last case, we also need to embed the core γ of D. Suppose for the sake of contradiction that γ has endpoints on two different boundary components F_1 and F_2 of N'. Let β be a properly embedded arc in N' connecting F_1 and F_2. Then $\gamma \cup \beta$ is a loop in $\widetilde{M}$ that intersects the closed surface F_1 in exactly one point. But because $H_1(\widetilde{M}) = 0$, the algebraic intersection number of $\gamma \cup \beta$ with F_1 is zero. This is impossible since $\gamma \cup \beta$ meets F_1 in a single point. Thus, both endpoints of γ lie on the same boundary component of N', and so γ can be embedded in $S^3 - N'$. So the embedding of N' can be extended to an embedding of N in S^3. □

4 Intrinsically knotted graphs

In this section, we use Proposition 2 to prove that the property of a graph being intrinsically knotted is independent of the 3–manifold it is embedded in. Notice that since Proposition 2 relies on the Poincaré Conjecture, so does the intrinsic knotting result.

Theorem 2 *Assume that the Poincaré Conjecture is true. Let M be a 3–manifold. A graph is intrinsically knotted in M if and only if it is intrinsically knotted in S^3.*

Proof Suppose that a graph G is not intrinsically knotted in S^3. Then it embeds in S^3 in such a way that every circuit bounds a disk embedded in S^3. The union of the embedding of G with these disks is compact, hence is contained in a ball B in S^3. Any embedding of B in M yields an unknotted embedding of G in M.

Conversely, suppose there is an unknotted embedding $f\colon G \to M$. Let $\widetilde{M}$ be the universal cover of M. By using the same argument as in the proof of Lemma 1, we can lift f to an unknotted embedding $\widetilde{f} : G \to \widetilde{M}$. Let K be the union of $\widetilde{f}(G)$ with the disks bounded by its circuits. Then K is compact, so by Proposition 2, there is an embedding $g : K \to S^3$. Now $g \circ \widetilde{f}(G)$ is an embedding of G in S^3, in which every circuit bounds a disk. Hence $g \circ \widetilde{f}(G)$ is an unknotted embedding of G in S^3. □

Remark The proof of Theorem 2 can also be used, almost verbatim, to show that intrinsic *linking* is independent of the 3–manifold. Of course, this argument relies on the Poincaré Conjecture; so the proof given in Section 2 is more elementary.

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Received: 25 October 2005 Revised: 3 May 2006