The normaliser decomposition for p–local finite groups

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We construct an analogue of the normaliser decomposition for p–local finite groups $(S, \mathcal{F}, \mathcal{L})$ with respect to collections of $\mathcal{F}$–centric subgroups and collections of elementary abelian subgroups of S. This enables us to describe the classifying space of a p–local finite group, before p–completion, as the homotopy colimit of a diagram of classifying spaces of finite groups whose shape is a poset and all maps are induced by group monomorphisms.

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1 The main results

For finite groups Dwyer [5] defined three types of homology decompositions of classifying spaces of finite groups known as the "subgroup", "centraliser" and "normaliser" decompositions. These decompositions are functors $F\colon D \to \mathbf{Spaces}$, where D is a small category which is constructed using collections $\mathcal{H}$ of carefully chosen subgroups of G. The essential property of these functors is, that given a finite group G, the spaces $F(d)$ have the homotopy type of classifying spaces of subgroups of G. Moreover the category D is constructed using information about the conjugation in G of the subgroups in $\mathcal{H}$. We say that D depends on the fusion of the collection $\mathcal{H}$ of G.

The purpose of this note is to construct an analogue of the normaliser decomposition for p–local finite groups in certain important cases. Throughout this note we will freely use the terminology and notation that by now has become standard in the theory for p–local finite groups. The reader who is not familiar with the jargon is advised to read Section 2 prior to this section, and is also referred to [4] where p–local finite groups were initially defined.

It should be noted that the analogues of the "subgroup" and the "centraliser" decompositions for p–local finite groups was already known to Broto, Levi and Oliver [4, Section 2].

The normaliser decomposition which is introduced in this note enabled the author together with Antonio Viruel to analyze the nerve $|\mathcal{L}|$ of p–local finite groups $(S, \mathcal{F}, \mathcal{L})$ with small Sylow subgroups S. We prove that these are classifying spaces of, generally

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infinite, discrete groups [10]. The author also used normaliser decompositions to give an analysis of the spectra associated with the nerve, $|\mathcal{L}|$, of the linking systems due to Ruiz and Viruel in [15] and other "exotic" examples, see [9]. These results will appear separately as they involve techniques that have little to do with the actual construction of the normaliser decomposition.

We now describe the main results of this paper. Throughout we work simplicially, thus a space means a simplicial set. The category of simplicial sets is denoted by **Spaces**. The nerve of a small category $\mathbf{D}$ is denoted $\mathrm{Nr}(\mathbf{D})$ or $|\mathbf{D}|$. We obtain a functor $|-|\colon \mathbf{Cat} \to \mathbf{Spaces}$ where $\mathbf{Cat}$ is the category of small categories. A more detailed discussion can be found in Section 3

1.1 Definition Let $(S, \mathcal{F}, \mathcal{L})$ be a p–local finite group. A collection is a set $\mathcal{C}$ of subgroups of S which is closed under conjugacy in $\mathcal{F}$. That is if $P \leq S$ belongs to $\mathcal{C}$ then so do all the $\mathcal{F}$–conjugates of P. A collection $\mathcal{C}$ is called $\mathcal{F}$–centric if it consists of $\mathcal{F}$–centric subgroups of S.

1.2 Definition A k–simplex in a collection $\mathcal{C}$ is a sequence $\mathbf{P}$ of proper inclusions $P_0 < P_1 < \cdots < P_k$ of elements of $\mathcal{C}$. Two k–simplices $\mathbf{P}$ and $\mathbf{P}'$ are called conjugate if there exists an isomorphism $f \in \mathrm{Iso}_{\mathcal{F}}(P_k, P'_k)$ such that $f(P_i) = P'_i$ for all $i = 0, \dots, k$. The conjugacy class of $\mathbf{P}$ is denoted $[\mathbf{P}]$.

1.3 Definition The category $\overline{\mathrm{s}}\mathrm{d}\mathcal{C}$ is a poset whose objects are the conjugacy classes $[\mathbf{P}]$ of all the k–simplices in $\mathcal{C}$ where $k = 0, 1, 2, \dots$. A morphism $[\mathbf{P}] \to [\mathbf{P}']$ in $\overline{\mathrm{s}}\mathrm{d}\mathcal{C}$ exists if $\mathbf{P}'$ is conjugate to a subsimplex of $\mathbf{P}$.

Recall from 2.8 that in every p–local finite group it is possible to choose morphisms ι_P^Q in the linking system $\mathcal{L}$ which are lifts of inclusions $P \leq Q$ of $\mathcal{F}$–centric subgroups. The choice can be made in such a way that $\iota_Q^R \circ \iota_P^Q = \iota_P^R$ for inclusions $P \leq Q \leq R$.

1.4 Definition Let $\mathcal{C}$ be an $\mathcal{F}$–centric collection in $(S, \mathcal{F}, \mathcal{L})$ and let $\mathbf{P}$ be a k–simplex in $\mathcal{C}$. Define $\mathrm{Aut}_{\mathcal{L}}(\mathbf{P})$ as the subgroup of $\prod_{i=0}^k \mathrm{Aut}_{\mathcal{L}}(P_i)$ whose elements are the $(k{+}1)$–tuples $(\varphi_i)_{i=0}^k$ which render the following ladder commutative in $\mathcal{L}$

$$\begin{tikzcd}
P_0 \arrow[r, "\iota_{P_0}^{P_1}"] \arrow[d, "\varphi_0"'] & P_1 \arrow[r, "\iota_{P_1}^{P_2}"] \arrow[d, "\varphi_1"'] & \cdots \arrow[r, "\iota_{P_{k-1}}^{P_k}"] & P_k \arrow[d, "\varphi_k"] \\
P_0 \arrow[r, "\iota_{P_0}^{P_1}"'] & P_1 \arrow[r, "\iota_{P_1}^{P_2}"'] & \cdots \arrow[r, "\iota_{P_{k-1}}^{P_k}"'] & P_k
\end{tikzcd}$$

1.5 Proposition *The assignment $(\varphi_i)_{i=0}^k \mapsto \varphi_0$ gives rise to a canonical isomorphism of $\mathrm{Aut}_{\mathcal{L}}(\mathbf{P})$ with a subgroup of $\mathrm{Aut}_{\mathcal{L}}(P_0)$. More generally, if $\mathbf{P}'$ is a subsimplex of $\mathbf{P}$ in $\mathcal{C}$ then restriction induces a monomorphism of groups $\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) \to \mathrm{Aut}_{\mathcal{L}}(\mathbf{P}')$.*

Proof The second assertion follows immediately from Proposition 2.11. The first follows from the second by letting $\mathbf{P}'$ be the 1–simplex P_0. □

Notation $\mathcal{B}\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P})$ denotes the subcategory of $\mathcal{L}$ whose only object is P_0 and whose morphism set is $\mathrm{Aut}_{\mathcal{L}}(\mathbf{P})$.

1.6 Definition Given an $\mathcal{F}$–centric collection $\mathcal{C}$ in a p–local finite group $(S, \mathcal{F}, \mathcal{L})$, let $\mathcal{L}^{\mathcal{C}}$ denote the full subcategory of $\mathcal{L}$ generated by the objects set $\mathcal{C}$.

Frequently, the inclusion $\mathcal{L}^{\mathcal{C}} \subseteq \mathcal{L}$ induces a weak homotopy equivalence on nerves. For example, this happens when $\mathcal{C}$ contains all the $\mathcal{F}$–centric $\mathcal{F}$–radical subgroups of S. This fact is proved by Broto, Castellana, Grodal, Levi and Oliver [2, Theorem 3.5].

The following theorem applies to all $\mathcal{F}$–centric collections. The decomposition approximates $\mathcal{L}$ if the inclusion $\mathcal{L}^{\mathcal{C}} \subseteq \mathcal{L}$ induces an equivalence as explained above.

Theorem A *Fix an $\mathcal{F}$–centric collection $\mathcal{C}$ in a p–local finite group $(S, \mathcal{F}, \mathcal{L})$. Then there exists a functor $\delta_{\mathcal{C}}\colon \bar{\mathrm{s}}\mathrm{d}\mathcal{C} \to$ **Spaces** such that*

(1) *There is a natural weak homotopy equivalence*

$$\operatorname{hocolim}_{\bar{\mathrm{s}}\mathrm{d}\mathcal{C}} \delta_{\mathcal{C}} \xrightarrow{\simeq} |\mathcal{L}^{\mathcal{C}}|.$$

(2) *There is a natural weak homotopy equivalence $B\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) \xrightarrow{\simeq} \delta_{\mathcal{C}}([\mathbf{P}])$ for every k–simplex $\mathbf{P}$.*

(3) *The natural maps $\delta_{\mathcal{C}}([\mathbf{P}]) \to |\mathcal{L}^{\mathcal{C}}|$ are induced by the inclusion of categories $\mathcal{B}\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) \subseteq \mathcal{L}^{\mathcal{C}}$.*

(4) *If $\mathbf{P}'$ is a subsimplex of $\mathbf{P}$ then the equivalence (2) renders the following square commutative*

$$\begin{CD} B\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) @>{\simeq}>> \delta_{\mathcal{C}}([\mathbf{P}]) \\ @V{B\,\mathrm{res}^{\mathbf{P}}_{\mathbf{P}'}}VV @VVV \\ B\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}') @>{\simeq}>> \delta_{\mathcal{C}}([\mathbf{P}']) \end{CD}$$

Moreover if $\mathbf{P}$ *and* $\mathbf{P}'$ *are conjugate* k*–simplices and* $\psi \in \mathrm{Iso}_{\mathcal{L}}(P_0, P_0')$ *maps* $\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}')$ *onto* $\mathrm{Aut}_{\mathcal{L}}(\mathbf{P})$ *by conjugation then the following square commutes*

$$\begin{tikzcd} B\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}') \arrow[r, "\simeq"] \arrow[d, "Bc_\psi"'] & \delta_{\mathcal{C}}([\mathbf{P}']) \arrow[d, equal] \\ B\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) \arrow[r, "\simeq"'] & \delta_{\mathcal{C}}([\mathbf{P}]) \end{tikzcd}$$

Remark When $(S, \mathcal{F}, \mathcal{L})$ is associated with a finite group G one may consider the G–collection $\mathcal{H}$ consisting of all the subgroups of G which are conjugate to elements of the $\mathcal{F}$–collection $\mathcal{C}$. Dwyer [5, Section 3] constructs a poset $\overline{\mathrm{s}}\mathrm{d}\mathcal{H}$ and a functor $\delta_{\mathcal{H}}^{\mathrm{Dwyer}}\colon \overline{\mathrm{s}}\mathrm{d}\mathcal{H} \to \mathbf{Spaces}$ which he calls the normaliser decomposition. We will show in 5.2 that $\overline{\mathrm{s}}\mathrm{d}\mathcal{H} = \overline{\mathrm{s}}\mathrm{d}\mathcal{C}$ and that $\delta_{\mathcal{C}}$ and $\delta_{\mathcal{H}}^{\mathrm{Dwyer}}$ can be connected by a natural zigzag of mod–p equivalences. That is, a zigzag of natural transformations which at every object of $\overline{\mathrm{s}}\mathrm{d}\mathcal{C}$ give rise to an $H_*(-;\mathbb{Z}/p)$–isomorphism.

We now describe the second type of normaliser decomposition that we shall construct in this note. It is based on collections $\mathcal{E}$ of elementary abelian subgroups of S.

1.7 Definition For a k–simplex $\mathbf{E}$ in $\mathcal{E}$ define $\mathrm{Aut}_{\mathcal{F}}(\mathbf{E})$ as the subgroup of $\mathrm{Aut}_{\mathcal{F}}(E_k)$ consisting of the automorphisms f such that $f(E_i) = E_i$ for all $i = 0, \dots, k$.

Consider an $\mathcal{F}$–centric collection $\mathcal{C}$ in $(S, \mathcal{F}, \mathcal{L})$.

1.8 Definition Fix an elementary abelian subgroup E of S. The objects of the category $\overline{C}_{\mathcal{L}}(\mathcal{C}; E)$ are pairs (P, f) where $P \in \mathcal{C}$ and $f\colon E \to Z(P)$ is a morphism in $\mathcal{F}$. Morphisms $(P, f) \to (Q, g)$ in $\overline{C}_{\mathcal{L}}(\mathcal{C}; E)$ are morphisms $\psi \in \mathcal{L}(P, Q)$ such that $g = \pi(\psi) \circ f$ where $\pi\colon \mathcal{L} \to \mathcal{F}$ is the projection functor.

Observe that $\mathrm{Aut}_{\mathcal{F}}(E)$ acts on $\overline{C}_{\mathcal{L}^c}(E)$ by pre-composition. That is, every $h \in \mathrm{Aut}_{\mathcal{F}}(E)$ indices the assignment $(P, f) \mapsto (P, f \circ h)$.

1.9 Definition For a k–simplex $\mathbf{E}$ in $\mathcal{E}$ let $\check{N}_{\mathcal{L}}(\mathcal{C}; \mathbf{E})$ denote the subcategory of $\mathcal{L}$ whose objects are $P \in \mathcal{C}$ for which $E_k \le Z(P)$. A morphism $\varphi \in \mathcal{L}(P, Q)$ belongs to $\check{N}_{\mathcal{L}}(\mathcal{C}; \mathbf{E})$ if $\pi(\varphi)|_{E_k}$ is an element of $\mathrm{Aut}_{\mathcal{F}}(\mathbf{E})$.

Recall that the homotopy orbit space of a G–space X, ie the Borel construction $EG \times_G X$, is denoted by X_{hG}.

1.10 Proposition *Let $\mathbf{E}$ be a k–simplex in $\mathcal{E}$. There is a map*

$$\epsilon_{\mathbf{E}}\colon |\breve{N}_{\mathcal{L}}(\mathcal{C};\mathbf{E})| \to |\bar{C}_{\mathcal{L}}(\mathcal{C};E_k)|_{h\,\mathrm{Aut}_{\mathcal{F}}(\mathbf{E})}$$

which is a homotopy equivalence if E_k is fully $\mathcal{F}$–centralised. The map is natural with respect to inclusion of simplices.

Proof This is immediate from Proposition 5.3. □

A comment on the categories $\bar{C}_{\mathcal{L}}(\mathcal{C};E)$ is in place. If $\mathcal{C}$ is the collection of all the $\mathcal{F}$–centric subgroups of S and E is fully $\mathcal{F}$–centralised, then it is shown by Broto, Levi and Oliver [4, Theorem 2.6] that $|\bar{C}_{\mathcal{L}}(\mathcal{C};E)|$ has the homotopy type of the nerve of the centraliser linking system $|C_{\mathcal{L}}(E)|$. The categories $\breve{N}_{\mathcal{L}}(\mathcal{C};\mathbf{E})$ are more mysterious. Even when $\mathbf{E}$ is a 1–simplex E, the category $\breve{N}_{\mathcal{L}}(\mathcal{C};E)$ is in general only a subcategory of the normaliser linking system $N_{\mathcal{L}}(E)$ because the largest subgroup which appears as an object of $\breve{N}_{\mathcal{L}}(\mathcal{C};E)$ is $C_S(E)$ which in general is smaller than $N_S(E)$. When $C_S(E)=N_S(E)$ these categories are equal.

The next decomposition result, Theorem B, depends on a collection of elementary abelian groups $\mathcal{E}$ and a collection $\mathcal{C}$ of $\mathcal{F}$–centric subgroups of S. It approximates $\mathcal{L}$ if $\mathcal{C}$ contains, for example, all the $\mathcal{F}$–centric $\mathcal{F}$–radical subgroups of S. The collection $\mathcal{E}$ must be large enough as explicitly stated in the theorem. For example the collection of all the non-trivial elementary abelian subgroups will always be a valid choice.

1.11 Definition Given a group H and a prime p let $\Omega_p(H)$ denote the subgroup of H generated by all the elements of order p in H.

Theorem B *Consider a p–local finite group $(S,\mathcal{F},\mathcal{L})$, an $\mathcal{F}$–centric collection $\mathcal{C}$ and a collection $\mathcal{E}$ of elementary abelian subgroup of S which contains the subgroups $\Omega_p Z(P)$ for all $P\in\mathcal{C}$. Then there exists a functor $\delta_{\mathcal{E}}\colon \bar{\mathrm{s}}\mathrm{d}\mathcal{E}\to$ **Spaces** with the following properties.*

(1) *There is a natural weak homotopy equivalence* $\mathrm{hocolim}_{\bar{\mathrm{s}}\mathrm{d}\mathcal{E}}\ \delta_{\mathcal{E}} \xrightarrow{\simeq} |\mathcal{L}^{\mathcal{C}}|$.

(2) *For a k–simplex $\mathbf{E}$ in $\mathcal{E}$ there is a weak homotopy equivalence*

$$|\bar{C}_{\mathcal{L}}(\mathcal{C};E_k)|_{h\,\mathrm{Aut}_{\mathcal{F}}(\mathbf{E})} \xrightarrow{\simeq} \delta_{\mathcal{E}}([\mathbf{E}]).$$

(3) *Fix a k–simplex $\mathbf{E}$ where E_k is fully $\mathcal{F}$–centralised. The equivalences (1) and (2) give a natural map $\delta_{\mathcal{E}}([\mathbf{E}])\to|\mathcal{L}^{\mathcal{C}}|$ whose precomposition with $\epsilon_{\mathbf{E}}$ of Proposition 1.10 is induced by the realization of the inclusion of $\breve{N}_{\mathcal{L}}(\mathcal{C};\mathbf{E})$ in $\mathcal{L}^{\mathcal{C}}$.*

(4) *If* $\mathbf{E}'$ *is a* k*–subsimplex of an* n*–simplex* $\mathbf{E}$ *then the following square commutes up to homotopy*

$$\begin{array}{ccc} |\bar{C}_{\mathcal{L}}(\mathcal{C}; E_n)|_{h\,\mathrm{Aut}_{\mathcal{F}}(\mathbf{E})} & \xrightarrow{\simeq} & \delta_{\mathcal{E}}([\mathbf{E}]) \\ \downarrow & & \downarrow \\ |\bar{C}_{\mathcal{L}}(\mathcal{C}; E'_k)|_{h\,\mathrm{Aut}_{\mathcal{F}}(\mathbf{E}')} & \xrightarrow{\simeq} & \delta_{\mathcal{E}}([\mathbf{E}']) \end{array}$$

The homotopy is natural with respect to inclusion of simplices. In addition, the square commutes on the nose if $E'_k = E_n$.

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2 On p–local finite groups

The term p–local finite group was coined by Broto, Levi and Oliver [4]. It cropped up naturally in their attempt [3] to describe the space of self equivalences of a p–completed classifying space of a finite group G. They discovered that the relevant information needed to solve this problem lies in the fusion system of the p–subgroups of G and certain categories which they later on called “linking systems”. Historically, fusion systems were first introduced by Lluis Puig [13].

2.1 Definition Fix a prime p and let S be a finite p–group. A *fusion system* over S is a sub-category $\mathcal{F}$ of the category of groups whose objects are the subgroups of S and whose morphisms are group monomorphisms such that

(1) All the monomorphisms that are induced by conjugation by elements of S are in $\mathcal{F}$.

(2) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion of subgroups.

We say that two subgroups P, Q of S are $\mathcal{F}$*–conjugate* if they are isomorphic as objects of $\mathcal{F}$.

When g is an element of S and P, Q are subgroups of S such that $gPg^{-1} \le Q$, we let c_g denote the morphism $P \to Q$ defined by conjugation, namely $c_g(x) = gxg^{-1}$ for every $x \in P$.

We let $\mathrm{Hom}_S(P,Q)$ denote the set of all the morphisms $P\to Q$ in $\mathcal{F}$ that are induced by conjugation in S. Also notice that the factorization axiom (2) implies that all the $\mathcal{F}$–endomorphisms of a subgroup P are in fact automorphisms in $\mathcal{F}$. Thus we write $\mathrm{Aut}_{\mathcal{F}}(P)$ for the set of morphisms $\mathcal{F}(P,P)$.

2.2 Definition A subgroup P of S is called *fully $\mathcal{F}$–centralised* (resp. *fully $\mathcal{F}$–normalised*) if its S–centraliser $C_S(P)$ (resp. S–normaliser $N_S(P)$) has the maximal possible order in the $\mathcal{F}$–conjugacy class of P. That is, $|C_S(P)|\geq|C_S(P')|$ (resp. $|N_S(P)|\geq|N_S(P')|$) for every P' which is $\mathcal{F}$–conjugate to P.

2.3 Definition A fusion system $\mathcal{F}$ over a finite p–group S is called *saturated* if

I Every fully $\mathcal{F}$–normalised subgroup P of S is fully $\mathcal{F}$–centralised and moreover $\mathrm{Aut}_S(P)=N_S(P)/C_S(P)$ is a Sylow p–subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

II Every morphism $\varphi\colon P\to S$ in $\mathcal{F}$ whose image $\varphi(P)$ is fully $\mathcal{F}$–centralised can be extended to a morphism $\psi\colon N_\varphi\to S$ in $\mathcal{F}$ where

$$N_\varphi=\{g\in N_S(P):\varphi c_g\varphi^{-1}\in\mathrm{Aut}_S(P)\}.$$

2.4 Definition A subgroup P of S is called *$\mathcal{F}$–centric* if P and all of its $\mathcal{F}$–conjugates contain their S–centralisers, that is $C_S(P')=Z(P')$ for every subgroup P' of S which is $\mathcal{F}$–conjugate to P.

2.5 Definition A *centric linking system* associated to a saturated fusion system $\mathcal{F}$ over S consists of

(1) A small category $\mathcal{L}$ whose objects are the $\mathcal{F}$–centric subgroups of S,

(2) a functor $\pi\colon\mathcal{L}\to\mathcal{F}$ and

(3) group monomorphisms $\delta_P\colon P\to\mathrm{Aut}_{\mathcal{L}}(P)$ for every $\mathcal{F}$–centric subgroup P of S,

Such that the following axioms hold

(A) The functor π acts as the inclusion on object sets, that is $\pi(P)=P$ for every $\mathcal{F}$–centric subgroup P of S. For any two objects P,Q of $\mathcal{L}$, the group $Z(P)$ acts freely on the morphism set $\mathcal{L}(P,Q)$ via the restriction of $\delta_P\colon P\to\mathrm{Aut}_{\mathcal{L}}(P)$ to $Z(P)$. The induced map on morphisms sets

$$\pi\colon\mathcal{L}(P,Q)\to\mathcal{F}(P,Q)$$

identifies $\mathcal{F}(P,Q)$ with the quotient of $\mathcal{L}(P,Q)$ by the free action of $Z(P)$.

(B) For every $\mathcal{F}$–centric subgroup P of S the map $\pi\colon \mathrm{Aut}_{\mathcal{L}}(P) \to \mathrm{Aut}_{\mathcal{F}}(P)$ sends $\delta_P(g)$, where $g \in P$, to c_g.

(C) For every $f \in \mathcal{L}(P,Q)$ and every $g \in P$ there is a commutative square in $\mathcal{L}$

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\big\downarrow & & \big\downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow[f]{} & Q. \end{array}$$

Remark A morphism $f \in \mathcal{L}(P,Q)$ is called a *lift* of a morphism $\varphi \in \mathcal{F}(P,Q)$ if $\varphi = \pi(f)$.

2.6 Definition A *p–local finite group* is a triple $(S,\mathcal{F},\mathcal{L})$ where $\mathcal{F}$ is a saturated fusion system over the finite p–group S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of $(S,\mathcal{F},\mathcal{L})$ is the space $|\mathcal{L}|^\wedge_p$, that is the p–completion in the sense of Bousfield and Kan [1], of the realization of the small category $\mathcal{L}$.

2.7 When S is a Sylow p–subgroup of a finite group G, there is an associated p–local finite group denoted $(S,\mathcal{F}_S(G),\mathcal{L}_S(G))$. See [4, Proposition 1.3, remarks after Definition 1.8]. We shall write $\mathcal{F}$ for $\mathcal{F}_S(G)$ and $\mathcal{L}$ for $\mathcal{L}_S(G)$.

Morphism sets between $P,Q \le S$ are

$$\mathcal{F}(P,Q) = \mathrm{Hom}_G(P,Q) = N_G(P,Q)/C_G(P)$$

where $N_G(P,Q) = \{g \in G : gPg^{-1} \le Q\}$ and $C_G(P)$ acts on $N_G(P,Q)$ by right translation.

A subgroup P of S is, by [4, Proposition 1.3], $\mathcal{F}$–centric precisely when it is p–centric in the sense of [5, Section 1.19], that is, $Z(P)$ is a Sylow p–subgroup of $C_G(P)$. In this case $C_G(P) = Z(P) \times C'_G(P)$ where $C'_G(P)$ is the maximal subgroup of $C_G(P)$ of order prime to p. Morphism sets of $\mathcal{L} = \mathcal{L}_S(G)$ have, by definition, the form

$$\mathcal{L}(P,Q) = N(P,Q)/C'_G(P).$$

The functor $\pi\colon \mathcal{L}_S(G) \to \mathcal{F}_S(G)$ is the obvious projection functor. The monomorphism $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ is induced by the inclusion of P in $N_G(P)$.

It is shown by Broto, Levi and Oliver [4, after Definition 1.8] that $(S,\mathcal{F}_S(G),\mathcal{L}_S(G))$ is a p–local finite group and that $|\mathcal{L}_S(G)|^\wedge_p \simeq BG^\wedge_p$. It should also be remarked that there are examples of p–local finite groups that cannot be associated with any finite group. These are usually referred to as "exotic examples".

2.8 In every p–local finite group $(S,\mathcal{F},\mathcal{L})$ one can choose morphisms $\iota_P^Q \in \mathcal{L}(P,Q)$ for every inclusion of $\mathcal{F}$–centric subgroups $P \le Q$, in such a way that

(1) $\pi(\iota_P^Q)$ is the inclusion $P \le Q$,

(2) $\iota_Q^R \circ \iota_P^Q = \iota_P^R$ for every $\mathcal{F}$–centric subgroups $P \le Q \le R$ of S, and

(3) $\iota_P^P = \mathrm{id}$ for every $\mathcal{F}$–centric subgroup P of S.

This follows from [4, Proposition 1.11]. Using the notation there, one chooses $\iota_P^Q = \delta_{P,Q}(e)$ where e is the identity element in S. Whenever possible, in order to avoid cumbersome notation, we shall write ι for ι_P^Q.

2.9 From [4, Lemma 1.10(a)] it also follows that every morphism $\varphi\colon P \to Q$ in $\mathcal{L}$ factors uniquely as an isomorphism $\varphi'\colon P \to P'$ in $\mathcal{L}$ followed by the morphism $\iota\colon P' \to Q$. In fact $P' = \pi(\varphi)(P)$

2.10 It was observed by Broto, Levi and Oliver [4, remarks after Lemma 1.10] that every morphism in $\mathcal{L}$ is a monomorphism in the categorical sense. It was later observed by Broto, Castellana, Grodal, Levi and Oliver [2, Corollary 3.10] and independently by others, that every morphism in $\mathcal{L}$ is also an epimorphism. As an easy consequence we record for further use:

2.11 Proposition *Consider a p–local finite group $(S,\mathcal{F},\mathcal{L})$ and a commutative square in $\mathcal{F}$ on the left of the display below*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "incl"'] & P' \arrow[d, "incl"] \\ Q \arrow[r, "g"'] & Q' \end{tikzcd} \qquad\qquad \begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d, "\iota_P^Q"'] & P' \arrow[d, "\iota_{P'}^{Q'}"] \\ Q \arrow[r, "\tilde{g}"'] & Q' \end{tikzcd}$$

where P, P', Q and Q' are $\mathcal{F}$–centric subgroups of S. Then for every lift $\tilde{g}$ of g in $\mathcal{L}$ there exists a unique lift $\tilde{f}$ of f in $\mathcal{L}$ which render the square on the right commutative in $\mathcal{L}$. We denote $\tilde{f}$ by $\tilde{g}|_P$.

Given a lift $\tilde{f}$ for f, if there exists a lift $\tilde{g}$ for g rendering the square on the right commutative, then it is unique.

Proof The first assertion follows immediately from [4, Lemma 1.10(a)] by setting $\psi = \mathrm{incl}_{P'}^{Q'}$, $\tilde{\psi} = \iota_{P'}^{Q'}$ and $\tilde{\psi}\varphi = \tilde{g}\iota_P^Q$. The second assertion follows immediately from the fact that ι_P^Q is an epimorphism. $\square$

2.12 Fix a p–local finite group $(S, \mathcal{F}, \mathcal{L})$. Given a subgroup P of S, there are two important p–local finite groups associated with it: the centraliser of P when P is fully $\mathcal{F}$–centralised and the normaliser of P when P is fully $\mathcal{F}$–normalised. Both were defined by Broto Levi and Oliver in [4].

The centraliser fusion system $C_{\mathcal{F}}(P)$, where P is fully $\mathcal{F}$–centralised, is a subcategory of $\mathcal{F}$. As a fusion system it is defined over the S–centraliser of P denoted $C_S(P)$. Morphisms $Q \to Q'$ in $C_{\mathcal{F}}(P)$ are those morphisms $\varphi\colon Q \to Q'$ in $\mathcal{F}$ that can be extended to a morphism $\overline{\varphi}\colon PQ \to PQ'$ in $\mathcal{F}$ which induces the identity on P. The objects of the centric linking system $C_{\mathcal{L}}(P)$ associated to $C_{\mathcal{F}}(P)$ are the $C_{\mathcal{F}}(P)$–centric subgroups of $C_S(P)$. The set of morphisms $Q \to Q'$ in $C_{\mathcal{L}}(P)$ is a subset of $\mathcal{L}(PQ, PQ')$ consists of those morphisms $f\colon PQ \to PQ'$ such that $\pi(f)$ induces the identity on P and carries Q to Q'. It is shown in [4] that $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$ is a p–local finite group.

Now fix a subgroup $K \leq \mathrm{Aut}_{\mathcal{F}}(P)$ where P is fully normalised in $\mathcal{F}$. The K–normaliser fusion system $N_{\mathcal{F}}^K(P)$ is a subcategory of $\mathcal{F}$ defined over $N_S(P)$. The objects of $N_{\mathcal{F}}^K(P)$ are the subgroups of $N_S(P)$. A morphisms $\varphi \in \mathcal{F}(Q, Q')$ belongs to $N_{\mathcal{F}}^K(P)$ if it can be extended to a morphism $\overline{\varphi}\colon PQ \to PQ'$ in $\mathcal{F}$ which induces an automorphism from K on P. The fusion system $N_{\mathcal{F}}^K(P)$ is saturated. When $K = \mathrm{Aut}_{\mathcal{F}}(P)$ we denote this category by $N_{\mathcal{F}}(P)$ and call it the normaliser fusion system of P. The centric linking system $N_{\mathcal{L}}(P)$ associated to $N_{\mathcal{F}}(P)$ has the $N_{\mathcal{F}}(P)$–centric subgroups of $N_S(P)$ as its object set. The set of morphisms $Q \to Q'$ is the subset of $\mathcal{L}(PQ, PQ')$ consisting of those $f\colon PQ \to PQ'$ such that $\pi(f)$ carries Q to Q' and induces an automorphism on P.

3 The Grothendieck construction

Throughout this paper we work simplicially, namely a "space" means a simplicial set. For further details, the reader is referred to Bousfeld and Kan [1], May [12], Goerss and Jardine [7] and many other sources. In this section we collect several results from general simplicial homotopy theory that we shall use repeatedly in the rest of this note.

Homotopy colimits Fix a small category $\mathbf{K}$ and a functor $U\colon \mathbf{K} \to \mathbf{Spaces}$. The simplicial replacement of U is the simplicial space $\coprod_* U$ which has in simplicial dimension n the disjoint union of the spaces $U(K_0)$ for every chain

$$K_0 \to K_1 \to \cdots \to K_n$$

of n composable arrows in $\mathbf{K}$. The homotopy colimit of U denoted $\mathrm{hocolim}_{\mathbf{K}} U$ is the diagonal of $\coprod_* U$ regarded as a bisimplicial set. See Bousfield and Kan [1, Section XII.5].

Consider a functor $F\colon \mathbf{K}\to\mathbf{L}$ between small categories. For a functor $U\colon \mathbf{L}\to\mathbf{Spaces}$ there is an obvious natural map, cf [1, Section XI.9].

$$\operatorname{hocolim}_{\mathbf{K}} F^*U \to \operatorname{hocolim}_{\mathbf{L}} U.$$

For an object $L\in\mathbf{L}$, the comma category $(L\downarrow F)$ has the pairs $(K, L\xrightarrow{k\in\mathbf{K}} FK)$ as its objects. Morphisms $(K, L\xrightarrow{k} FK)\to(K', L\xrightarrow{k'} FK')$ are the morphisms $x\colon K\to K'$ such that $Fx\circ k=k'$. Similarly one defines the category $(F\downarrow L)$ whose object set consists of the pairs $(K, k\colon FK\to L)$. Compare MacLane [11].

3.1 Definition The functor $F\colon \mathbf{K}\to\mathbf{L}$ is called right-cofinal if for every object $L\in\mathbf{L}$ the category $(L\downarrow F)$ has a contractible nerve.

The following theorem was probably first proved by Quillen [14, Theorem A]. See also Hollender and Vogt [8, Section 4.4] and Bousfield and Kan [1, Section XI.9].

Cofinality Theorem *Let $F\colon \mathbf{K}\to\mathbf{L}$ be a right cofinal functor between small categories. Then for every functor $U\colon \mathbf{L}\to$ **Spaces** the natural map*

$$\operatorname{hocolim}_{\mathbf{K}} F^*U \to \operatorname{hocolim}_{\mathbf{L}} U$$

is a weak homotopy equivalence.

Associated with a functor $U\colon \mathbf{K}\to\mathbf{Spaces}$ there is a functor $F_*U\colon \mathbf{L}\to\mathbf{Spaces}$ called the homotopy left Kan extension of U along F. It is defined on every object $L\in\mathbf{L}$ by

$$F_*U(L)=\operatorname{hocolim}\Big((F\downarrow L)\xrightarrow{\text{proj}}\mathbf{K}\xrightarrow{F}\mathbf{Spaces}\Big).$$

See [8, Section 5], [6, Section 6]. The following theorem is originally due to Segal. See eg [8, Theorem 5.5].

Segal's Pushdown Theorem *Fix a functor $F\colon \mathbf{K}\to\mathbf{L}$ of small categories. Then for every functor $U\colon \mathbf{K}\to$ **Spaces** there is a natural weak homotopy equivalence*

$$\operatorname{hocolim}_{\mathbf{L}} F_*U\xrightarrow{\simeq}\operatorname{hocolim}_{\mathbf{K}} U.$$

The Grothendieck construction Recall that a small category $\mathbf{K}$ gives rise to a simplicial set $\mathrm{Nr}(\mathbf{K})$ called the *nerve* of $\mathbf{K}$. Its n–simplices are the chains of n composable arrows $K_0\to K_1\to\cdots\to K_n$ in $\mathbf{K}$. See, for example, Goerss and Jardine [7, Example 1.4] or Bousfield and Kan [1, Section XI.2]. We shall also use the notation $|\mathbf{K}|$ for the nerve of $\mathbf{K}$.

Given a functor $U\colon \mathbf{K} \to \mathbf{Cat}$ Thomason [17] defined the translation category $\mathbf{K} \int U$ associated to U as follows. The object set consists of pairs (K, X) where K is an object of $\mathbf{K}$ and X is an object of $U(K)$. Morphisms $(K_0, X_0) \to (K_1, X_1)$ are pairs (k, x) where $k\colon K_0 \to K_1$ is a morphism in $\mathbf{K}$ and $x\colon U(k)(X_0) \to X_1$ is a morphism in $U(K_1)$. Composition of $(K_0, X_0) \xrightarrow{(k_0,x_0)} (K_1, X_1)$ and $(K_1, X_1) \xrightarrow{(k_1,x_1)} (K_2, X_2)$ is given by

$$(k_1, x_1) \circ (x_0, k_0) = (k_1 \circ k_0, x_1 \circ U(k_1)(x_0)).$$

This category is also called the Grothendieck construction of U and the notation $\mathrm{Tr}_{\mathbf{K}} U$ is also used. Thomason [17] shows that there is a natural weak homotopy equivalence

$$\eta\colon \operatorname{hocolim}_{\mathbf{K}} |U| \xrightarrow{\simeq} |\mathrm{Tr}_{\mathbf{K}} U| \tag{3–1}$$

A natural transformation $U \Rightarrow U'$ gives rise to a canonical functor $\mathrm{Tr}_{\mathbf{K}} U \to \mathrm{Tr}_{\mathbf{K}} U'$. The induced map $|\mathrm{Tr}_{\mathbf{K}}(U)| \to |\mathrm{Tr}_{\mathbf{K}}(U')|$ corresponds via η (3–1) to the induced map $\operatorname{hocolim}_{\mathbf{K}} |U| \to \operatorname{hocolim}_{\mathbf{K}} |U'|$. Furthermore, for every object K in $\mathbf{K}$ the natural map

$$|U(K)| \to \operatorname{hocolim}_{\mathbf{K}} |U|$$

corresponds under (3–1) to the inclusion of categories

$$U(K) \to \mathrm{Tr}_{\mathbf{K}} U, \qquad \text{where } X \mapsto (K, X) \text{ and } x \mapsto (1_K, x). \tag{3–2}$$

Consider now a functor $F\colon \mathbf{K} \to \mathbf{L}$ of small categories. Given $U\colon \mathbf{L} \to \mathbf{Cat}$ there is a naturally defined functor

$$F_!\colon \mathrm{Tr}_{\mathbf{K}} F^* U \to \mathrm{Tr}_{\mathbf{L}} U, \qquad \text{where} \qquad \begin{cases} F_!(K, X \in F^*U(K)) = (FK, X) \\ F_!(k, x) = (Fk, x). \end{cases} \tag{3–3}$$

The functor $F_!$ is a model for the map $\operatorname{hocolim} F^*|U| \to \operatorname{hocolim} |U|$ in the sense that the following square commutes

$$\begin{array}{ccc} |\mathrm{Tr}_{\mathbf{K}} F^* U| & \xrightarrow{\eta} & \operatorname{hocolim}_{\mathbf{K}} F^*|U| \\ {\scriptstyle |F_!|}\downarrow & & \downarrow \\ |\mathrm{Tr}_{\mathbf{L}} U| & \xrightarrow{\eta} & \operatorname{hocolim}_{\mathbf{L}} |U| \end{array}$$

3.2 Definition For a functor $U\colon \mathbf{K} \to \mathbf{Cat}$ define $F_* U\colon \mathbf{L} \to \mathbf{Cat}$ by

$$F_*U(L) = \mathrm{Tr}\big((F \downarrow L) \xrightarrow{\text{proj}} \mathbf{K} \xrightarrow{U} \mathbf{Spaces}\big)$$

The maps η (3–1) provide a natural weak homotopy equivalence $|F_*U| \xrightarrow{\simeq} F_*|U|$. The equivalence in the pushdown theorem can be realized as the nerve of a functor between the transporter categories as follows.

3.3 Proposition *The functor $F_\#\colon Tr_{\mathbf{L}} F_* U \to Tr_{\mathbf{K}} U$ defined by*

$$F_\#\colon \big(L, (K, FK \to L), X \in UK\big) \mapsto (K, X)$$
$$F_\#\colon \big(L \xrightarrow{\ell} L', K \xrightarrow{k} K', U(k)(X) \xrightarrow{x} X'\big) \mapsto (k, x).$$

renders the following diagram commutative where the arrow at the top of the square is an equivalence by the pushdown theorem.

$$\begin{tikzcd} \operatorname{hocolim}_{\mathbf{L}} |F_* U| \arrow[drr, "\eta", "\simeq"'] & & \operatorname{hocolim}_{\mathbf{L}}\ F_*|U| \arrow[ll, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\sim"'] & \operatorname{hocolim}_{\mathbf{K}}\ |U| \arrow[d, "\sim"', "\eta"] \\ & & |Tr_{\mathbf{L}}(F_* U)| \arrow[r, "|F_\#|"'] & |Tr_{\mathbf{K}}(U)| \end{tikzcd} \tag{3–4}$$

It is useful to point out that if $\star\colon \mathbf{K} \to \mathbf{Cat}$ is the constant functor on the trivial category with one object and an identity morphism, then $\mathrm{Tr}_{\mathbf{K}}(\star) = \mathrm{Nr}(\mathbf{K})$.

4 EI categories

Fix an EI category $\mathcal{A}$, namely a category all of whose endomorphisms are isomorphisms. We shall assume that the category $\mathcal{A}$ is finite. We shall also assume that $\mathcal{A}$ is equipped with a height function, namely a function $h\colon \mathrm{Obj}(\mathcal{A}) \to \mathbb{N}$ such that $h(A) \le h(A')$ if there exists a morphism $A \to A'$ in $\mathcal{A}$ and equality holds if and only if $A \to A'$ is an isomorphism. Clearly, if $\mathcal{A}$ is an EI-category then so is $\mathcal{A}^{\mathrm{op}}$. The finiteness condition also implies that if $\mathcal{A}$ is heighted then so is $\mathcal{A}^{\mathrm{op}}$.

We can always choose a full subcategory $\mathcal{A}_{\mathrm{sk}}$ of $\mathcal{A}$ which contains one representative from each isomorphism class of objects in $\mathcal{A}$. We say that $\mathcal{A}_{\mathrm{sk}}$ is skeletal in $\mathcal{A}$. Clearly the inclusion $\mathcal{A}_{\mathrm{sk}} \subseteq \mathcal{A}$ is an equivalence of categories. In the language of Słomińska [16] $\mathcal{A}_{\mathrm{sk}}$ is an EIA category.

Throughout we let $\underline{k}$ denote the poset $\{0 \to 1 \to \cdots \to k\}$ considered as a small category.

4.1 Definition The subdivision category $s(\mathcal{A})$ is the category whose objects are height increasing functors $\mathbf{A}\colon \underline{k} \to \mathcal{A}$, namely $h(\mathbf{A}(i)) < h(\mathbf{A}(i+1))$ for all $i < k$. Morphisms $\mathbf{A} \to \mathbf{A}'$ in $s(\mathcal{A})$ are pairs (ϵ, φ) where $\epsilon\colon \underline{k}' \to \underline{k}$ is a strictly increasing function and $\varphi\colon \epsilon^*(\mathbf{A}) \to \mathbf{A}'$ is a natural isomorphism of functors $\underline{k}' \to \mathcal{A}$. Composition of $(\epsilon, \varphi)\colon \mathbf{A} \to \mathbf{A}'$ and $(\epsilon', \varphi')\colon \mathbf{A}' \to \mathbf{A}''$ is given by $(\epsilon \circ \epsilon', \epsilon'^*(\varphi) \circ \varphi')$.

Note that ϵ is determined by the heights of the values of $\mathbf{A}$ namely $\epsilon(i) = j$ if and only if $h(\mathbf{A}'(i)) = h(\mathbf{A}(j))$.

We shall further assume that $\mathcal{A}$ contains a subcategory $\mathcal{I}$ which is a poset with the property that every morphism $\varphi\colon A \to A'$ in $\mathcal{A}$ can be factored uniquely as $\varphi = \iota\varphi'$ where φ' is an isomorphism in $\mathcal{A}$ and ι is a morphism in $\mathcal{I}$. The ladder

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{A}(n-2) \arrow[r, "\varphi_{n-1}"] \arrow[d, "(\varphi_n'\circ\varphi_{n-1})'"', "\cong"] & \mathbf{A}(n-1) \arrow[r, "\varphi_n"] \arrow[d, "\varphi_n'"', "\cong"] & \mathbf{A}(n) \arrow[d, equal] \\
\cdots \arrow[r] & \mathbf{A}'(n-2) \arrow[r, "\iota"'] & \mathbf{A}'(n-1) \arrow[r, "\iota"'] & \mathbf{A}'(n)
\end{tikzcd}$$

shows that the the full subcategory $s_{\mathcal{I}}(\mathcal{A})$ of $s(\mathcal{A})$ consisting of the objects $\mathbf{A}$ in which all the arrows $\mathbf{A}(i) \to \mathbf{A}(i+1)$ belong to $\mathcal{I}$ is a skeletal subcategory of $s(\mathcal{A})$. We obtain two skeletal subcategories of $s(\mathcal{A})$

$$s(\mathcal{A}_{\mathrm{sk}}) \subseteq s(\mathcal{A}) \qquad \text{and} \qquad s_{\mathcal{I}}(\mathcal{A}) \subseteq s(\mathcal{A}). \tag{4–1}$$

We observe that $\mathrm{Hom}_{s(\mathcal{A})}(\mathbf{A}, \mathbf{A}')$ has a free action of $\mathrm{Aut}_{s(\mathcal{A})}(\mathbf{A}')$ with a single orbit. Also every $(\epsilon, \varphi)\colon \mathbf{A} \to \mathbf{A}'$ in $s(\mathcal{A})$ gives rise to a natural group homomorphism upon restriction and conjugation with the isomorphism $\varphi\colon \epsilon^*\mathbf{A} \approx \mathbf{A}'$

$$\varphi_*\colon \mathrm{Aut}_{s(\mathcal{A})}(\mathbf{A}) \to \mathrm{Aut}_{s(\mathcal{A})}(\mathbf{A}'). \tag{4–2}$$

4.2 Proposition *There is a right cofinal functor $p\colon s(\mathcal{A}) \to \mathcal{A}$ defined by*

$$p(\mathbf{A}) = \mathbf{A}(0), \qquad\qquad (\mathbf{A}\colon \underline{k} \to \mathcal{A}).$$

Proof Słomińska [16, Proposition 1.5] shows that the functor $p\colon s(\mathcal{A}_{\mathrm{sk}}) \to \mathcal{A}_{\mathrm{sk}}$ is right cofinal (Definition 3.1) hence so is $p\colon s(\mathcal{A}) \to \mathcal{A}$. □

4.3 Definition The category $\overline{s}(\mathcal{A})$ has the isomorphism classes $[\mathbf{A}]$ of the objects of $s(\mathcal{A})$ as its object set. There is a unique morphism $[\mathbf{A}] \to [\mathbf{A}']$ if there exists a morphism $\mathbf{A} \to \mathbf{A}'$ in $s(\mathcal{A})$. There is an obvious projection functor

$$\pi\colon s(\mathcal{A}) \to \overline{s}(\mathcal{A}), \qquad \mathbf{A} \mapsto [\mathbf{A}].$$

When $\mathcal{D}$ is a full subcategory of $s(\mathcal{A})$ one obtains a sub-poset $\overline{\mathcal{D}}$ of $\overline{s}(\mathcal{A})$ whose objects are the isomorphism classes of the objects of $\mathcal{D}$.

Clearly $\overline{s}(\mathcal{A})$ is a poset and it should be compared with Słomińska's construction of $s_0(\mathcal{A})$ in [16, Section 1]. Also note that $\overline{s}_{\mathcal{I}}(\mathcal{A}) = \overline{s}(\mathcal{A})$ because $s_{\mathcal{I}}(\mathcal{A})$ is skeletal in $s(\mathcal{A})$. Similarly $\overline{s}(\mathcal{A}_{\mathrm{sk}}) = \overline{s}(\mathcal{A})$.

4.4 Lemma *Let $J\colon \mathcal{C} \to \mathcal{D}$ be a functor of small categories with a left adjoint $L\colon \mathcal{D} \to \mathcal{C}$ such that $L \circ J = \mathrm{Id}$. Then J is right cofinal.*

Proof Fix an object $d \in \mathcal{D}$. We have to prove that the category $(d \downarrow J)$ has a contractible nerve. Let $(d \downarrow \mathcal{D})$ denote the category $(d \downarrow 1_{\mathcal{D}})$. It clearly has a contractible nerve because it has an initial object. The functors J and L induce obvious functors

$$J_*\colon (d \downarrow J) \to (d \downarrow \mathcal{D}), \qquad (c, d \xrightarrow{f} Jc) \mapsto (Jc, d \xrightarrow{f} Jc)$$
$$L_*\colon (d \downarrow \mathcal{D}) \to (d \downarrow J), \qquad (d', d \xrightarrow{f} d') \mapsto (Ld', d \xrightarrow{f} d' \xrightarrow{\eta} JLd').$$

It is obvious that $L_* \circ J_* = \mathrm{Id}$. Furthermore the unit $\eta\colon \mathrm{Id} \to LJ$ gives rise to a natural transformation $\mathrm{Id} \to J_* \circ L_*$. Therefore J_* induces a homotopy equivalence on nerves so $|(d \downarrow J)| \simeq |d \downarrow \mathcal{D})| \simeq *$. ☐

4.5 Proposition *For every functor $F\colon s(\mathcal{A}) \to$ **Cat** and every $\mathbf{A} \in s(\mathcal{A})$ there is a functor*

$$Tr_{\mathcal{B}\,\mathrm{Aut}(\mathbf{A})} F(\mathbf{A}) \to (\pi_* F)([\mathbf{A}])$$

which induces a weak homotopy equivalence on nerves. It is natural in the sense that every morphism $(\epsilon, \varphi)\colon \mathbf{A} \to \mathbf{A}'$ in $s(\mathcal{A})$, gives rise to a square

$$\begin{tikzcd}
Tr_{\mathrm{Aut}(\mathbf{A})} F(\mathbf{A}) \arrow[r] \arrow[d, "Tr_{\varphi_*} F(\varphi)"'] \arrow[dr, phantom, "\overset{\tau}{\Leftarrow}"] & (\pi_* F)([\mathbf{A}]) \arrow[d, "\pi_*([\varphi])"] \\
Tr_{\mathrm{Aut}(\mathbf{A}')} F(\mathbf{A}') \arrow[r] & (\pi_* F)([\mathbf{A}'])
\end{tikzcd}$$

which commutes up to a natural transformation τ which is functorial in (ϵ, φ). Here $\varphi_*\colon \mathrm{Aut}(\mathbf{A}) \to \mathrm{Aut}(\mathbf{A}')$ is the homomorphism induced by restriction and conjugation by $\varphi\colon \epsilon^*\mathbf{A} \approx \mathbf{A}'$ as we described in (4–2). The square commutes on the nose if $F(\varphi)\colon F(\mathbf{A}) \longrightarrow F(\mathbf{A}')$ is the identity.

Proof Fix an object $\mathbf{A}\colon \underline{k} \to \mathcal{A}$ in $s(\mathcal{A})$. Let $\Pi_{\mathbf{A}}$ be the full subcategory of $(\pi \downarrow [\mathbf{A}])$ consisting of the objects $(\mathbf{A}', [\mathbf{A}'] \xrightarrow{=} [\mathbf{A}])$. It is isomorphic to the full subcategory of $s(\mathcal{A})$ consisting of the isomorphism class of $\mathbf{A}$. The inclusion $J\colon \Pi_{\mathbf{A}} \to (\pi \downarrow [\mathbf{A}])$ has a left adjoint L where

$$L\colon (\mathbf{B}, [\mathbf{B}] \to [\mathbf{A}]) \mapsto \epsilon^*\mathbf{B}, \qquad \text{where } [\epsilon^*\mathbf{B}] = [\mathbf{A}] \text{ for } \epsilon\colon \underline{k}' \hookrightarrow \underline{k}.$$

Clearly ϵ is unique if it exists. There is a natural map $\mathbf{B} \to \epsilon^*\mathbf{B}$ induced by the identity on $\epsilon^*\mathbf{B}$ under the bijection $s(\mathcal{A})(\mathbf{B}, \epsilon^*\mathbf{B}) \approx s(\mathcal{A})(\epsilon^*\mathbf{B}, \epsilon^*\mathbf{B})$. We obtain a natural transformation $\mathrm{Id} \to JL$ which gives rise to bijections for every object $(\mathbf{A}', [\mathbf{A}'] \to [\mathbf{A}])$ in $(\pi \downarrow [\mathbf{A}])$

$$\mathrm{Hom}_{(\pi\downarrow[\mathbf{A}])}(\mathbf{B}, J\mathbf{A}') = \mathrm{Hom}_{s(\mathcal{A})}(\mathbf{B}, \mathbf{A}') \approx \mathrm{Hom}_{s(\mathcal{A})}(\epsilon^*\mathbf{B}, \mathbf{A}') = \mathrm{Hom}_{\Pi_\mathbf{A}}(L\mathbf{B}, \mathbf{A}').$$

Thus L is left adjoint to J and we apply Lemma 4.4. By definition $\Pi_\mathbf{A}$ is a connected groupoid with automorphism group $\mathcal{B}\,\mathrm{Aut}_{s(\mathcal{A})}(\mathbf{A})$. Therefore upon realization, the functor

$$\mathrm{Tr}\big(\mathcal{B}\,\mathrm{Aut}_{s(\mathcal{A})}(\mathbf{A}) \to s(\mathcal{A}) \xrightarrow{F} \mathbf{Cat}\big) \xrightarrow{\text{restriction}} \mathrm{Tr}\big((\pi\downarrow[\mathbf{A}]) \to s(\mathcal{A}) \xrightarrow{F} \mathbf{Cat}\big) = (\pi_* F)([\mathbf{A}])$$

induces a weak homotopy equivalence. Also, for a morphism $\varphi\colon \mathbf{A} \to \mathbf{A}'$ we get an obvious $\varphi_*\colon \Pi_\mathbf{A} \to \Pi'_\mathbf{A}$ by restriction and conjugation by the isomorphism $\varphi\colon \epsilon^*\mathbf{A} \to \mathbf{A}'$. It gives rise to the following diagram

$$\begin{tikzcd}
\mathcal{B}\,\mathrm{Aut}_{s(\mathcal{A})}(\mathbf{A}) \arrow[r, "\mathrm{incl}"] \arrow[d, "\varphi_*"] & \Pi_\mathbf{A} \arrow[r, "J"] \arrow[d, "\varphi_*"'] & (\pi \downarrow [\mathbf{A}]) \arrow[d, "{[\varphi]_*}"] \\
\mathcal{B}\,\mathrm{Aut}_{s(\mathcal{A})}(\mathbf{A}') \arrow[r, "\mathrm{incl}"] & \Pi_{\mathbf{A}'} \arrow[r, "J"] & (\pi \downarrow [\mathbf{A}'])
\end{tikzcd}$$

The morphism φ provides a canonical natural transformation $[\varphi]_* \circ J \circ \mathrm{incl} \to J \circ \mathrm{incl} \circ \varphi_*$. This provides the natural transformation τ in the statement of the proposition and its naturality with φ. If $F(\varphi)\colon F(\mathbf{A}) \longrightarrow F(\mathbf{A}')$ is the identity, then $F(\tau)$ becomes the identity and the square in the statement of the proposition commutes. $\square$

5 Proof of the main results

Fix a p–local finite group $(S, \mathcal{F}, \mathcal{L})$ and an $\mathcal{F}$–centric collection $\mathcal{C}$. Choose a subcategory $\mathcal{I} \subseteq \mathcal{L}^\mathcal{C}$ of distinguished inclusions, cf 2.8. Note that $\mathcal{L}^\mathcal{C}$ possesses a height function, see Section 4, by assigning to a subgroup P in $\mathcal{C}$ its order. Also every morphism in $\mathcal{L}^\mathcal{C}$ factors uniquely as an isomorphism followed by a morphism in $\mathcal{I}$.

We claim that (Definitions 1.3 and 4.3)

$$\overline{\mathrm{s}}\mathrm{d}\mathcal{C} = \overline{s}(\mathcal{L}^\mathcal{C}).$$

To see this recall that $s_\mathcal{I}(\mathcal{L}^\mathcal{C})$ is a skeletal subcategory of $s(\mathcal{L}^\mathcal{C})$, see (4–1), hence $\overline{s}(\mathcal{L}^\mathcal{C}) = \overline{s}_\mathcal{I}(\mathcal{L}^\mathcal{C})$. The functor $\pi\colon \mathcal{L} \to \mathcal{F}$ gives a functor $\overline{s}_\mathcal{I}(\mathcal{L}^\mathcal{C}) \to \overline{\mathrm{s}}\mathrm{d}\mathcal{C}$ because it maps the morphisms $\iota \in \mathcal{I}$ to inclusion of subgroups of S. It is an isomorphism of categories because conjugation 1.2 of two k–simplices $P_0 < \cdots < P_k$ and $P'_0 < \cdots < P'_k$ induced by an isomorphism $\varphi_k \in \mathrm{Iso}_\mathcal{F}(P_k, P'_k)$ can be lifted to an isomorphism of the corresponding objects in $s_{\mathcal{I}}(\mathcal{L}^{\mathcal{C}})$ by lifting the isomorphism $\varphi_k\colon P_k \to P'_k$ to $\mathcal{L}$ and using Proposition 2.11 to obtain the commutative ladder in $\mathcal{L}^{\mathcal{C}}$:

$$\begin{tikzcd} P_0 \arrow[r, "\iota"] \arrow[d, "\cong"'] & P_1 \arrow[r, "\iota"] \arrow[d, "\cong"'] & \cdots \arrow[r, "\iota"] & P_k \arrow[d, "\widetilde{\varphi_k}"] \\ P'_0 \arrow[r, "\iota"'] & P'_1 \arrow[r, "\iota"] & \cdots \arrow[r, "\iota"] & P'_k \end{tikzcd}$$

We remark that a k–simplex $P_0 < \cdots < P_k$ in $\mathcal{C}$ can be identified with the object $P_0 \xrightarrow{\iota} \cdots \xrightarrow{\iota} P_k$ of $s(\mathcal{L}^{\mathcal{C}})$. Under this identification we clearly have

$$\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}) = \operatorname{Aut}_{s(\mathcal{L}^{\mathcal{C}})}(\mathbf{P}).$$

When G is a discrete group we let $\mathcal{B}G$ denote the category with one object and G as its set of morphisms. For every k–simplex $\mathbf{P}$ in $\mathcal{C}$ we identify $\mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P})$ with the obvious subcategory of $\mathcal{B}\operatorname{Aut}_{\mathcal{L}}(P_0)$.

5.1 Theorem *Let $\mathcal{C}$ be an $\mathcal{F}$–centric collection in a p–local finite group $(S, \mathcal{F}, \mathcal{L})$. Then there exists a functor $\widetilde{\delta}_{\mathcal{C}}\colon \overline{\mathrm{s}}\mathrm{d}\mathcal{C} \to \mathbf{Cat}$ with the following properties*

(1) *There is a naturally defined functor $Tr_{\overline{\mathrm{s}}\mathrm{d}\mathcal{C}}(\widetilde{\delta}_{\mathcal{C}}) \to \mathcal{L}^{\mathcal{C}}$ which induces a weak homotopy equivalence on nerves.*

(2) *For every k–simplex $\mathbf{P}$ there is a canonical functor $\mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}) \to \widetilde{\delta}_{\mathcal{C}}([\mathbf{P}])$ which induces a weak homotopy equivalence on nerves. If $\mathbf{P}'$ is a subsimplex of $\mathbf{P}$ then the following square commutes*

$$\begin{tikzcd} \mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}) \arrow[r] \arrow[d, "\mathrm{res}^{\mathbf{P}}_{\mathbf{P}'}"'] & \widetilde{\delta}_{\mathcal{C}}([\mathbf{P}]) \arrow[d] \\ \mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}') \arrow[r] & \widetilde{\delta}_{\mathcal{C}}([\mathbf{P}']) \end{tikzcd}$$

(3) *The natural inclusion $\mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}) \subseteq \mathcal{L}^{\mathcal{C}}$ is equal to the composition*

$$\mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}) \to \widetilde{\delta}_{\mathcal{C}}([\mathbf{P}]) \subseteq Tr_{\overline{\mathrm{s}}\mathrm{d}\mathcal{C}}(\widetilde{\delta}_{\mathcal{C}}) \to \mathcal{L}^{\mathcal{C}}$$

(4) *An isomorphism of k–simplices $\psi\colon \mathbf{P}' \xrightarrow{\approx} \mathbf{P}$ in $s(\mathcal{L}^{\mathcal{C}})$ induces a commutative square*

$$\begin{tikzcd} \mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}) \arrow[r] \arrow[d, "c_\psi"'] & \widetilde{\delta}_{\mathcal{C}}([\mathbf{P}]) \arrow[d] \\ \mathcal{B}\operatorname{Aut}_{\mathcal{L}}(\mathbf{P}') \arrow[r] & \widetilde{\delta}_{\mathcal{C}}([\mathbf{P}']) \end{tikzcd}$$

Proof We have seen that $\overline{s}\mathrm{d}\mathcal{C} = \overline{s}(\mathcal{L}^{\mathcal{C}})$. Let $\star\colon \overline{s}(\mathcal{L}^{\mathcal{C}}) \to \mathbf{Cat}$ denote the constant functor on the trivial small category with one object and identity morphism. Use the projection functor $\pi\colon s(\mathcal{L}^{\mathcal{C}}) \to \overline{s}(\mathcal{L}^{\mathcal{C}})$ to define

$$\widetilde{\delta}_{\mathcal{C}} = \pi_*(\star)$$

According to Proposition 4.5 we have a canonical functor

$$\mathcal{B}\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) = \mathrm{Tr}_{\mathcal{B}\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P})}(\star) \to \pi_*(\star)([\mathbf{P}]) = \widetilde{\delta}_{\mathcal{C}}([\mathbf{P}])$$

which induces a weak homotopy equivalence. Since $\star$ is constant, the square in the statement of Proposition 4.5 commutes and we obtain the naturality assertions in point (2) and (4). The natural functor of (1) is defined using Proposition 3.3 and Proposition 4.2 by

$$\mathrm{Tr}_{\overline{s}\mathrm{d}\mathcal{C}}(\widetilde{\delta}_{\mathcal{C}}) = \mathrm{Tr}_{\overline{s}(\mathcal{L}^{\mathcal{C}})}(\pi_*(\star)) \xrightarrow{\pi_\#} \mathrm{Tr}_{s(\mathcal{L}^{\mathcal{C}})}(\star) = s(\mathcal{L}^{\mathcal{C}}) \xrightarrow{p} \mathcal{L}^{\mathcal{C}}.$$

It induces a weak homotopy equivalence by Segal's Pushdown Theorem, Proposition 3.3 and Cofinality Theorem. Whence point (1). Inspection of the functor $\pi_\#$, the inclusion $\mathcal{B}\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) \subseteq (\pi \downarrow [\mathbf{P}]) = \pi_*(\star)([\mathbf{P}])$ and Equation (3–2) yield point (3) □

Proof of Theorem A Apply Theorem 5.1 above and define $\delta_{\mathcal{C}} = |\widetilde{\delta}_{\mathcal{C}}|$. □

5.2 We now relate the construction in Theorem A to Dwyer's normaliser decomposition [5, Section 3]. We will show that the two functors are related by a zigzag of natural transformations which induce a mod–p equivalence.

Fix a finite group G and the p–local finite group $(S, \mathcal{F}, \mathcal{L})$ associated with it. A collection $\mathcal{C}$ of $\mathcal{F}$–centric subgroups of S gives rise to a G–collection $\mathcal{H}$ of p–centric subgroups of G (cf [5, Section 1.19], 2.7) by taking all the G–conjugates of the elements of $\mathcal{C}$. We let $\mathcal{T}^{\mathcal{H}}$ denote the transporter category of $\mathcal{H}$. That is, the object set of $\mathcal{T}^{\mathcal{H}}$ is $\mathcal{H}$ and the morphism set $\mathcal{T}^{\mathcal{H}}(H,K)$ is the set $N_G(H,K) = \{g \in G : g^{-1}Hg \le K\}$. We also let $\mathcal{T}^{\mathcal{C}}$ denote the full subcategory of $\mathcal{T}^{\mathcal{H}}$ having $\mathcal{C}$ as its object set. Almost by definition $\mathcal{T}^{\mathcal{C}}$ is skeletal in $\mathcal{T}^{\mathcal{H}}$. We also obtain a zigzag of functors (see 2.7)

$$\mathcal{T}^{\mathcal{H}} \leftarrow \mathcal{T}^{\mathcal{C}} \to \mathcal{L}^{\mathcal{C}}.$$

Dwyer [5, Section 3] defines a category $\overline{s}\mathrm{d}\mathcal{H}$ whose objects are the G–conjugacy classes $[\mathbf{H}]$ of the k–simplices $H_0 < \cdots < H_k$ in $\mathcal{H}$. There is a unique morphism $[\mathbf{H}] \to [\mathbf{H}']$ in $\overline{s}\mathrm{d}\mathcal{H}$ if and only if $\mathbf{H}'$ is conjugate in G to a subsimplex of $\mathbf{H}$. It follows directly from the definition of $\mathcal{H}$ as the smallest G–collection containing $\mathcal{C}$ and from the definition of $\mathcal{F} = \mathcal{F}_S(G)$ that $\overline{\mathrm{s}}\mathrm{d}\mathcal{H} = \overline{\mathrm{s}}\mathrm{d}\mathcal{C}$. We obtain a commutative diagram (see Definition 4.1)

$$\begin{tikzcd} s(\mathcal{T}^{\mathcal{H}}) \arrow[d, "\pi_2"'] & s(\mathcal{T}^{\mathcal{C}}) \arrow[l, "\supseteq"'] \arrow[r] \arrow[d, "\pi_1"'] & s(\mathcal{L}^{\mathcal{C}}) \arrow[d, "\pi"] \\ \overline{s}(\mathcal{T}^{\mathcal{H}}) \arrow[r, equal] & \overline{s}(\mathcal{T}^{\mathcal{C}}) \arrow[r, equal] & \overline{s}(\mathcal{L}^{\mathcal{C}}). \end{tikzcd}$$

Fix a k–simplex $\mathbf{P} = P_0 < \cdots < P_k$ in $\mathcal{C}$. Note that $(\pi_2 \downarrow [\mathbf{P}])$ is isomorphic to the subcategory of $\mathcal{T}^{\mathcal{H}}$ of the objects $\mathbf{P}'$ which admit a morphism to $\mathbf{P}$. It contains a full subcategory $\Pi_{\mathbf{P}}$ of the objects of $s(\mathcal{T}^{\mathcal{H}})$ that are isomorphic to $\mathbf{P}$; cf the proof of Proposition 4.5. By inspection $\Pi_{\mathbf{P}}$ is the translation category of the action of G on the orbit of $\mathbf{P}$, that is it is the transported category of the G–set $[\mathbf{P}]$ in $\mathcal{H}$ thought of as a functor $\mathcal{B}\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}) \to \mathbf{Sets}$, cf [5, Section 3.3]. Thus

$$\delta^{\mathrm{Dwyer}}_{\mathcal{H}}([\mathbf{P}]) = EG \times_G [\mathbf{P}] = \mathrm{Nr}(\Pi_{\mathbf{P}}).$$

The inclusion $J\colon \Pi_{\mathbf{P}} \hookrightarrow (\pi_2 \downarrow [\mathbf{P}])$ has a left adjoint $L\colon (\mathbf{P}', [\mathbf{P}'] \to [\mathbf{P}]) \mapsto \epsilon^*\mathbf{P}'$ where $(\epsilon, \varphi)\colon \mathbf{P}' \to \mathbf{P}$ is a morphism in $s(\mathcal{L}^{\mathcal{C}})$, see Definition 4.1. Compare the proof of Proposition 4.5. Lemma 4.4 implies that J is right cofinal. We obtain a zigzag of functors

$$\delta^{\mathrm{Dwyer}}_{\mathcal{H}} \xrightarrow[\simeq]{} |(\pi_2)_*(\star)| \xleftarrow[\simeq]{\mathrm{incl}} |(\pi_1)_*(\star)| \xrightarrow{\text{mod-}p} |\pi_*(\star)| = |\widetilde{\delta}_{\mathcal{C}}| = \delta_{\mathcal{C}}.$$

The third map induces a mod–p equivalence by the following argument. For any object $[\mathbf{P}]$ we obtain a map $(\pi_1)_*(\star)([\mathbf{P}]) \to \pi_*(\star)([\mathbf{P}])$ which by Proposition 4.5 is equivalent to the map

$$B\,\mathrm{Aut}_G(\mathbf{P}) \to B\,\mathrm{Aut}_{\mathcal{L}}(\mathbf{P}).$$

Since P_0 is p–centric then $C_G(P_0) = Z(P_0) \times C'_G(P_0)$ where $C'_G(P_0)$ is a characteristic p'–subgroup of $C_G(P_0)$ and $\mathrm{Aut}_{\mathcal{L}}(P_0) = N_G(P_0)/C'_G(P_0)$. Therefore $\mathrm{Aut}_G(\mathbf{P}) \to \mathrm{Aut}_G(\mathbf{P})/C'_G(P_0)$ induces a mod–p equivalence as needed.

We shall now prove Theorem B. Fix an $\mathcal{F}$–centric collection $\mathcal{C}$ and a collection $\mathcal{E}$ of elementary abelian subgroups in $(S, \mathcal{F}, \mathcal{L})$. Recall from Definition 1.8 and Definition 1.9 the definitions of $\overline{C}_{\mathcal{L}}(\mathcal{C}; E_k)$ and $\check{N}_{\mathcal{L}}(\mathcal{C}; \mathbf{E})$ where $\mathbf{E}$ is a k–simplex in $\mathcal{E}$.

5.3 Proposition *Fix a k–simplex $\mathbf{E}$ in $\mathcal{E}$, namely a functor $\mathbf{E}\colon \underline{k} \to \mathcal{F}^{\mathcal{E}}$. There is a functor*

$$\epsilon\colon \check{N}_{\mathcal{L}}(\mathcal{C}; \mathbf{E}) \to Tr_{\mathcal{B}\,\mathrm{Aut}_{\mathcal{F}}(\mathbf{E})}\Big(\overline{C}_{\mathcal{L}}(\mathcal{C}; \mathbf{E})\Big)$$

which is fully faithful and natural with respect to inclusion of simplices. If E_k is fully $\mathcal{F}$–centralised, its image is also skeletal and in particular induces homotopy equivalence $|\check{N}_{\mathcal{L}}(\mathcal{C}; \mathbf{E})| \xrightarrow{\simeq} |\overline{C}_{\mathcal{L}}(\mathcal{C}; E_k)|_{h\,\mathrm{Aut}_{\mathcal{F}}(\mathbf{E})}$.

Proof The objects of $\mathcal{H} := \mathcal{B}\operatorname{Aut}_{\mathcal{F}}(\mathbf{E}) \int \bar{C}_{\mathcal{L}}(\mathcal{C};\mathbf{E})$ are pairs (P,f) where $P \in \mathcal{C}$ and $f \in \mathcal{F}(E, Z(P))$. Morphisms are pairs (φ, g) where $\varphi \in \mathcal{L}(P,P')$ and $g \in \operatorname{Aut}_{\mathcal{F}}(\mathbf{E})$ such that $f' = \pi(\varphi) \circ f \circ g$ (see Section 3). Since f, f' are monomorphisms, g is determined by φ. Define $\epsilon \colon \check{N}_{\mathcal{L}}(\mathcal{C};\mathbf{E}) \to \mathcal{H}$ by

$$\begin{aligned} \epsilon(P) &= (P, E \xrightarrow{\text{incl}} Z(P) \le P) \\ \epsilon(P \xrightarrow{\varphi} P') &= (P \xrightarrow{\varphi} P', \pi(\varphi)|_{E_k}^{-1}). \end{aligned} \tag{5–1}$$

It is well defined and fully faithful by the definition of $\check{N}_{\mathcal{L}}(\mathcal{C};\mathbf{E})$. Naturality with respect to inclusion of simplices is readily verified. Consider an object $(P,f) \in \mathcal{H}$. Note that $f(E_k) \le Z(P)$, hence for $g := f^{-1} \in \operatorname{Iso}_{\mathcal{F}}(f(E_k), E_k)$ we must have (see Definition 2.3) $N_g \supseteq C_S(f(E_k)) \supseteq P$. By axiom II of Definition 2.3 we can extend g to an isomorphism $h \colon P \to P'$ in $\mathcal{F}$. Clearly P' is in $\mathcal{C}$ because the latter is a collection. Fix a lift $\tilde{h} \in \mathcal{L}(P,P')$ for h. We have $\pi(\tilde{h}) \circ f = h \circ \operatorname{incl}_{f(E_k)}^{P} \circ g^{-1} = \operatorname{incl}_{E_k}^{P'}$. Therefore $(\mathrm{id}_{E_k}, \tilde{h})$ is an isomorphism $(P,f) \cong \left(P', \operatorname{incl}_{E_k}^{P'}\right)$ in $\mathcal{H}$. This shows that ϵ embeds $\check{N}_{\mathcal{L}}(\mathcal{C};\mathbf{E})$ into a skeletal subcategory of $\mathcal{H}$ and the result follows. □

Proof of Theorem B For every $P \in \mathcal{C}$ define $\zeta(P) = \Omega_p Z(P)$, see Definition 1.11. Note that if $f \colon P \to P'$ is an isomorphism in $\mathcal{F}$ then $f^{-1} \colon \zeta(P') \to \zeta(P)$ is an isomorphism in $\mathcal{F}$. Also if $P \le P'$ in $\mathcal{C}$ then $Z(P) \le Z(P')$ because P and P' are $\mathcal{F}$–centric so their centres are equal to their S–centralisers. It easily follows that this assignment forms a functor

$$\zeta \colon \mathcal{L}^{\mathcal{C}} \to \mathcal{F}^{\mathcal{E}\,\mathrm{op}}.$$

Fix $E \in \mathcal{E}$. Since every homomorphism $f \colon E \to Z(P)$ factors through $\zeta(P)$ we see that $\bar{C}_{\mathcal{L}}(\mathcal{C};E) = (\zeta \downarrow E)$. In particular 3.2

$$\zeta_*(\star) \colon E \mapsto \bar{C}_{\mathcal{L}}(\mathcal{C};E). \tag{5–2}$$

We now observe that $\mathcal{F}^{\mathcal{E}}$ is an EI-category. The assignment $E \mapsto |E|$ gives rise to a height function in the sense of Section 4. Furthermore the set $\mathcal{I}$ of inclusions in $\mathcal{F}^{\mathcal{E}}$ forms a poset where every morphism in $\mathcal{F}^{\mathcal{E}}$ factors uniquely as an isomorphism followed by an element in $\mathcal{I}$. We conclude that

$$\bar{s}\mathcal{E} = \bar{s}_{\mathcal{I}}(\mathcal{F}^{\mathcal{E}}) \approx \bar{s}(\mathcal{F}^{\mathcal{E}}).$$

There is an isomorphism $\tau \colon s(\mathcal{F}^{\mathcal{E}\,\mathrm{op}}) \to s(\mathcal{F}^{\mathcal{E}})$ which is the identity on objects. On the morphism set between $\mathbf{E} \colon \underline{k} \to \mathcal{F}^{\mathcal{E}}$ and $\mathbf{E}' \colon \underline{n} \to \mathcal{F}^{\mathcal{E}}$ such that $\epsilon^*\mathbf{E} \approx \mathbf{E}'$ for some injective $\epsilon\colon \underline{n} \to \underline{k}$ (see Definition 4.1), τ has the effect

$$\mathrm{Hom}_{s(\mathcal{F}^{\mathcal{E}\,\mathrm{op}})}(\mathbf{E},\mathbf{E}') = \mathrm{Iso}_{\mathcal{F}^{\mathcal{E}\,\mathrm{op}}}(\epsilon^*\mathbf{E},\mathbf{E}') = \mathrm{Iso}_{\mathcal{F}^{\mathcal{E}}}(\mathbf{E}',\epsilon^*\mathbf{E}) \xrightarrow[\approx]{f\mapsto f^{-1}} \mathrm{Iso}_{\mathcal{F}^{\mathcal{E}}}(\epsilon^*\mathbf{E},\mathbf{E}') = \mathrm{Hom}_{s(\mathcal{F}^{\mathcal{E}})}(\mathbf{E},\mathbf{E}').$$

The functor $p\colon s(\mathcal{F}^{\mathcal{E}\,\mathrm{op}}) \to \mathcal{F}^{\mathcal{E}\,\mathrm{op}}$ of Proposition 4.2 fits into a commutative diagram

$$\begin{tikzcd} s(\mathcal{F}^{\mathcal{E}\,\mathrm{op}}) \arrow[r, "p"] \arrow[d, "\tau"', "\approx"] & \mathcal{F}^{\mathcal{E}\,\mathrm{op}} \arrow[d, equal] \\ s(\mathcal{F}^{\mathcal{E}}) \arrow[r, "\mu"] & \mathcal{F}^{\mathcal{E}\,\mathrm{op}} \end{tikzcd}$$

where $\mu(\mathbf{E}) = E_k$. Since τ is an isomorphism, μ is right cofinal. We obtain a zigzag of functors

$$\overline{\mathrm{s}}\mathrm{d}\mathcal{E} \xleftarrow{\pi} s(\mathcal{F}^{\mathcal{E}}) \xrightarrow{\mu} \mathcal{F}^{\mathcal{E}\,\mathrm{op}} \xleftarrow{\zeta} \mathcal{L}^{\mathcal{C}} \xrightarrow{\star} \mathbf{Cat}.$$

Define

$$\widetilde{\delta}_{\mathcal{E}} = \pi_* \circ \mu^* \circ \zeta_*(\star), \qquad \text{and} \qquad \delta_{\mathcal{E}} = |\widetilde{\delta}_{\mathcal{E}}|.$$

Since μ is right cofinal, the Cofinality Theorem and Segal's Pushdown Theorem imply a weak homotopy equivalence

$$\mathrm{hocolim}_{\overline{\mathrm{s}}\mathrm{d}\mathcal{E}}\ \delta_{\mathcal{E}} \xrightarrow{\simeq} |\mathcal{L}^{\mathcal{C}}|.$$

This is point (1) of the theorem. Inspection of (3–2), Proposition 3.3 and (3–3) shows that this equivalence is given as the realization of a functor $\mathrm{Tr}_{\overline{\mathrm{s}}\mathrm{d}\mathcal{E}}\widetilde{\delta}_{\mathcal{E}} \to \mathcal{L}^{\mathcal{C}}$ where

$$(\mathbf{E}, E_k \xrightarrow{f} P) \mapsto P$$
$$(\mathbf{E}\to\mathbf{E}', g \in \mathrm{Aut}_{\mathcal{F}}(\mathbf{E}'), P \xrightarrow{\varphi\in\mathcal{L}} P') \mapsto \varphi.$$

Point (2) follows from (5–2) and Proposition 4.5. Point (3) follows from Proposition 1.10, inspection of ϵ in Proposition 5.3, of (5–2) and the functor $\mathrm{Tr}_{\overline{\mathrm{s}}\mathrm{d}\mathcal{E}}(\widetilde{\delta}_{\mathcal{E}}) \to \mathcal{L}^{\mathcal{C}}$. Point (4) is a consequence of Proposition 4.5. □

References

[1] **A K Bousfield**, **D M Kan**, *Homotopy limits, completions and localizations*, Springer, Berlin (1972) MR0365573

[2] **C Broto**, **N Castellana**, **J Grodal**, **R Levi**, **B Oliver**, *Subgroup families controlling p–local groups*, J. London Math. Soc. to appear

[3] **C Broto**, **R Levi**, **B Oliver**, *Homotopy equivalences of p–completed classifying spaces of finite groups*, Invent. Math. 151 (2003) 611–664 MR1961340

[4] **C Broto**, **R Levi**, **B Oliver**, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. 16 (2003) 779–856 MR1992826

[5] **W G Dwyer**, *Homology decompositions for classifying spaces of finite groups*, Topology 36 (1997) 783–804 MR1432421

[6] **W G Dwyer**, **D M Kan**, *A classification theorem for diagrams of simplicial sets*, Topology 23 (1984) 139–155 MR744846

[7] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser Verlag, Basel (1999) MR1711612

[8] **J Hollender**, **R M Vogt**, *Modules of topological spaces, applications to homotopy limits and E_∞ structures*, Arch. Math. (Basel) 59 (1992) 115–129 MR1170635

[9] **A Libman**, *A Minami–Webb splitting of classifying spaces of finite groups and the exotic examples of Ruiz and Viruel*, preprint

[10] **A Libman**, **A Viruel**, *On the homotopy type of the non-completed classifying space of a p–local finite group*, preprint

[11] **S Mac Lane**, *Categories for the working mathematician*, Graduate Texts in Mathematics 5, Springer, New York (1998) MR1712872

[12] **J P May**, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies, No. 11, D. Van Nostrand Co.,, Princeton, N.J.-Toronto, Ont.-London (1967) MR0222892

[13] **L Puig**, *Unpublished notes*

[14] **D Quillen**, *Higher algebraic K–theory. I*, from: "Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)", Springer, Berlin (1973) 85–147. Lecture Notes in Math., Vol. 341 MR0338129

[15] **A Ruiz**, **A Viruel**, *The classification of p–local finite groups over the extraspecial group of order p^3 and exponent p*, Math. Z. 248 (2004) 45–65 MR2092721

[16] **J Słomińska**, *Homotopy colimits on E-I-categories*, from: "Algebraic topology Poznań 1989", Lecture Notes in Math. 1474, Springer, Berlin (1991) 273–294 MR1133907

[17] **R W Thomason**, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. 85 (1979) 91–109 MR510404

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