The universal Khovanov link homology theory

GAD NAOT

We determine the algebraic structure underlying the geometric complex associated to a link in Bar-Natan's geometric formalism of Khovanov's link homology theory ($n = 2$). We find an isomorphism of complexes which reduces the complex to one in a simpler category. This reduction enables us to specify exactly the amount of information held within the geometric complex and thus state precisely its universality properties for link homology theories. We also determine its strength as a link invariant relative to the different topological quantum field theories (TQFTs) used to create link homology. We identify the most general (universal) TQFT that can be used to create link homology and find that it is "smaller" than the TQFT previously reported by Khovanov as the universal link homology theory. We give a new method of extracting all other link homology theories (including Khovanov's universal TQFT) directly from the universal geometric complex, along with new homology theories that hold a controlled amount of information. We achieve these goals by making a classification of surfaces (with boundaries) modulo the 4TU/S/T relations, a process involving the introduction of genus generating operators. These operators enable us to explore the relation between the geometric complex and its algebraic structure.

57M25; 57M27

1 Introduction

During the recent few years, starting with [6], Khovanov type link homology theory has established itself as a dominant new field of research within link invariant theory. Creating a homology theory associated to each link, whose Euler characteristic is the Jones polynomial, has proven itself to be a stronger invariant with many advantages such as functorial properties regarding link cobordisms. Together with the development of the algebraic language used in the categorification process (the Khovanov type link homology theories) there emerged a geometric/topological formalism describing the entire process, due to Bar-Natan [5]. Initiating with the task of clarifying [6] and giving a visual geometric description for "standard" Khovanov link homology, the geometric formalism has evolved into a theory of its own. Using a fundamental geometric language it was shown how to create an underlying framework for Khovanov

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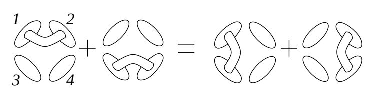
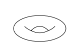
type link homology theories, which unifies, simplifies and in many ways generalizes, much of the work done before.

Though we assume familiarity with the main idea presented in [5], the basic notion is as follows. Given a link diagram D one builds the "cube of resolutions" from it (a cube built of all possible 0 and 1 smoothings of the crossings). The edges of the cube are then given certain surfaces (cobordisms) attached to them (with the appropriate signs). The entire cube is "summed" into a complex (in the appropriate geometric category) while taking care of some degree issues. We will call this complex the geometric complex throughout the paper. Figure 1 should serve the reader as a reminder of the process. The full description of it can be found in [5, Chapter 2].

The category in which one gets a link invariant is $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob_{/l}))$, the category of complexes, up to homotopy, built from columns and matrices of objects and morphisms (respectively) taken from $\mathcal{C}ob_{/l}$, which is the category of 2–dimensional (orientable) cobordisms between 1 dimensional objects (circles), where we allow formal sums of cobordisms over some ground ring, modulo the following local relations:

The 4TU relation: 

The S relation:  $= 0$

The T relation:  $= 2$

Using these relations a general theory was developed in which invariance proofs became easier and more general, and homology theories (TQFTs) became more natural — one gets them by applying "tautological" functors on the geometric complex. Though studied intensively in [5], the full scope of the geometric theory was not explored, and only various reduced cases (with extra geometrical relations put and ground ring adjusted) were used in connection with TQFTs and homology calculations. A full understanding of the interplay between TQFTs used to create different link homology theories and the underlying geometric complex was not achieved although the research on the TQFT side is considerably advanced, Khovanov [8].

The objectives of this paper are to explore the full geometric theory (working over $\mathbb{Z}$ with no extra relations imposed) in order to answer the following questions:

What is the algebraic *structure governing the full geometric theory? How does the category $\mathcal{C}ob_{/l}$ look, and what can we do with this information?*

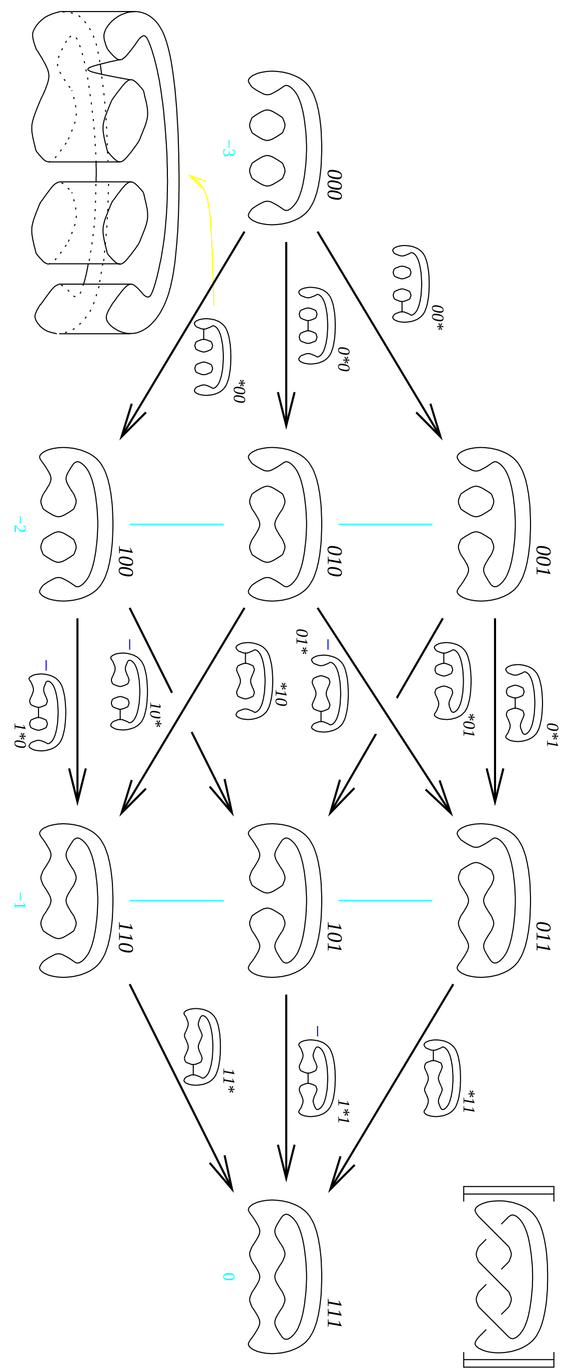


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Figure 1

In what sense exactly is the geometric theory universal? Does the full universal theory hold more information than the different Khovanov type link homology theories (TQFTs) applied to it?

What is the interplay between the different link homology theories (TQFTs) and their geometric "interpretation" (via the geometric complex)?

We start by classifying surfaces with boundary modulo the 4TU/S/T relations, and thus get hold of the way the underlying category of the geometric complex looks. The main tools for this purpose will be using *genus generating operators* in order to extend the ground ring. We prove a useful lemma regarding the free move of 2–handles between components of surfaces in $\mathcal{C}ob_{/l}$ and using the genus generating operators we get a simple classification of surfaces. We present a reduction formula for surfaces in terms of a family of free generators we identify. The classification introduces the topological/geometric motivation for the rest of the paper. Section 2 presents the classification when 2 is invertible in the ground ring and Section 4 presents the general case over $\mathbb{Z}$.

Then, we construct a reduction of the complex associated to a link. We build an isomorphism of complexes and find that the complex associated to a link is isomorphic to one that lives in a simpler category. This category has only one object and the entire complex is composed of columns of that single object. The complex maps are matrices with monomial entries in one variable (a genus generating operator). Thus the complex is equivalent to one built from free modules over a polynomial ring in one variable. Section 3 presents these results when 2 is invertible and Section 5 presents these results for the general case over $\mathbb{Z}$.

The isomorphism of complexes gives us an immediate result regarding the underlying algebraic structure of the universal geometric theory. It presents us with a "pre-TQFT" structure of purely topological/geometric nature. It turns out that the underlying structure of the full geometric complex associated to a link (over $\mathbb{Z}$) is the same as the one given by the *co-reduced* link homology theory using the following TQFT (over $\mathbb{Z}[H]$):

$$\Delta_1 : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ - H v_+ \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

$$m_1 : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto H v_- \end{cases}$$

In Section 6, armed with the complex reduction above and the underlying algebraic structure, we start exploring the most general TQFTs that can be applied to the geometric complex to get a link homology theory. This will result in a theorem which states that the above co-reduced TQFT structure is the universal TQFT as far as information in link homology is concerned.

We also get (Section 6) a new procedure of extracting other TQFTs directly from the geometric complex. This process is named *promotion* and can be used to get unfamiliar homology theories which contain the information coming only from surfaces up to a certain genus (in some sense a perturbation expansion in the genus), thus extrapolating between the standard Khovanov link homology theory and our universal one. This simplifies, completes and takes into a new direction some of the results of [8]. Our paper also generalizes some of the results of [5] and completes it in some respects. We finish with some comments and further discussion in Section 7).

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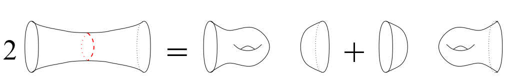
2 Classification of surfaces modulo the 4TU/S/T relations when 2 is invertible

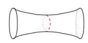
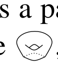
The geometric complex, an invariant of links and tangles, takes values in the category $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob_{/l}))$. We wish to study this category, as well as similar ones, in order to learn more about the invariant. Reducing the geometric complex into a simpler complex in a simpler category might give some further insight — computationally and theoretically. This section is devoted to the study of the underlying category $\mathcal{C}ob_{/l}$ of 2–dimensional orientable cobordisms between unions of circles in the simpler case when 2 is invertible in the ground ring that we work over ($\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{2}]$, for example). Specifically we classify all such surfaces modulo 4TU, S and T relations, which gives us the morphism groups of $\mathcal{C}ob_{/l}$.

It is known [5] that when 2 is invertible in the ground ring we work over, the 4TU relation is equivalent to *the neck cutting relation*:

NC relation: 2 

We plan on using this simpler relation in this section.

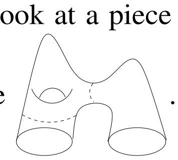
2.1 Notation

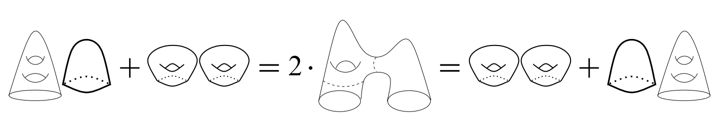
For the sake of clarity and due to the nature of the topic, we will try to give as many pictures and examples as possible. Still, one needs some formal description from time to time, thus we will need the use of some notation. Let $\Sigma_g(\alpha_1, \alpha_2, \cdots)$ denote a surface with genus g and boundary circles $\alpha_1, \alpha_2, \cdots$. A disconnected union of such surfaces will be denoted by $\Sigma_{g_1}(\alpha_1, \cdots)\Sigma_{g_2}(\beta_1, \cdots)$. If the genus or the boundary circles are not relevant for the argument at hand, they will be omitted. Whenever we have a piece of surface which looks like  we will call it *a neck*. If cutting a neck separates the component into 2 disconnected components then it will be a *separating neck*, if not then it is a *non-separating neck* which means it is a part of *a handle* on the surface. A handle on the surface always looks locally like , and by *2–handle* on the surface we mean a piece of surface which looks like . $\mathcal{C}ob_{/l}$ is a general notation for either embedded 2 dimensional cobordism in 3 dimensional space (say a cylinder, like in [5]) or abstract surfaces. One may choose whichever she likes, the theories (modulo the relations) are the same (see Section 7).

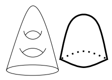
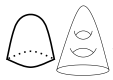
2.2 The 2–handle lemma

We start with proving a lemma that will become useful in classifying surfaces modulo the 4TU/S/T relations.

Lemma 2.1 *In $\mathcal{C}ob_{/l}$ 2–handles move freely between components of a surface. In other words, modulo the 4TU relation, a surface with a 2–handle on one of its connected components is equal to the same surface with the 2–handle removed and glued on a different component (see the picture below the proof for an example).*

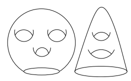
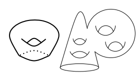
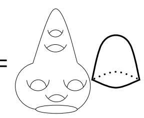
Proof The proof is an application of the neck cutting relation (NC), which follows from the 4TU relation (over any ground ring). We look at a piece of surface with a handle and a separating neck on it which looks like . The rest of the surface continues beneath the bottom circles and is not drawn. Applying the NC relation to the vertical dashed neck and to the horizontal dashed neck gives the following two equalities:



We get  $=$ . Since these are the top parts of *any* surface the lemma is proven. $\square$

Notice that this lemma applies to any ground ring we work over (in $\mathcal{C}ob_{/l}$), and does not require 2 to be invertible.

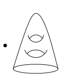
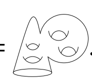
Example The following equality holds in $\mathcal{C}ob_{/l}$:

 $=$  $=$ .

The 2–handle lemma allows us to reduce the classification problem to a classification of surfaces with *at most* one handle in each connected component. This is done by extending the ground ring R to $R[T]$ with T a "global" operator (ie, acts *anywhere* on the surface) defined as follows:

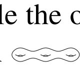
Definition 2.2 *The 2–handle operator*, denoted by T, is the operator that glues a 2–handle somewhere on a surface (anywhere).

Examples $T \cdot \Sigma_{g_1}(\alpha)\Sigma_{g_2}(\beta) = \Sigma_{g_1+2}(\alpha)\Sigma_{g_2}(\beta) = \Sigma_{g_1}(\alpha)\Sigma_{g_2+2}(\beta)$ or

$T \cdot$  $=$ .

The 2–handle operator is given degree -4 according to [5] degrees conventions, thus keeping all the elements of the theory graded. Observe that since the torus equals 2 in $\mathcal{C}ob_{/l}$ (the T relation), multiplying a surface Σ with $2T$ is equal to taking the union of Σ with the genus 3 surface:

$2T \cdot \Sigma = T \cdot \Sigma \cup$  $= \Sigma \cup$ 

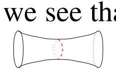
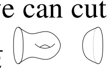
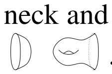
Note that the 2–handle operator does not operate on the empty cobordism. Since 2 is invertible the operation of the 2–handle operator can be naturally presented as the union with $/2$, thus allowing meaningful operation on the empty cobordism. We will adopt this presentation of T, and call it *the genus–3 presentation.*

2.3 Classification of surfaces modulo 4TU/S/T over $\mathbb{Z}[\frac{1}{2}]$

Proposition 2.3 *Over the extended ring $\mathbb{Z}[\frac{1}{2}, T]$, where T is the 2–handle operator in the genus–3 presentation, the morphism groups of $\mathcal{C}ob_{/l}$ (surfaces modulo S, T and 4TU relations) are generated freely by unions of surfaces with exactly one boundary component and genus 0 or 1 (**,* *), together with the empty cobordism.*

In other words, each morphism set from n circles to m circles is a free module of rank 2^{n+m} over $\mathbb{Z}[\frac{1}{2}, T]$, where T is the 2–handle operator in the genus–3 presentation.

Proof Given a surface Σ with (possibly) few connected components and with (possibly) few boundary circles we can apply the 2–handle lemma and reduce it to a surface that has at most genus 1 (i.e one handle) in each connected component. This is done by extending the ground ring we work over to $\mathbb{Z}[\frac{1}{2}, T]$, where T is the 2–handle operator.

When 2 is invertible the 4TU relation is equivalent to the NC relation, and by dividing the relation equation by 2 we see that we can cut any neck and replace it by the right hand side of the equation:  $= \frac{1}{2}$  $+ \frac{1}{2}$  . We will mod out this relation and reduce any surface into a combination of free generating surfaces. We must make sure that the reduction uses only the NC relation. Moreover the reduction must be well defined (take the NC relation to zero), thus securing the remaining generators from any relations.

Let us first look at the classification of surfaces with no boundary at all. These are easy to classify due to the S and T relations. Using the NC relation it is easy to show that $2\Sigma_{2g} = 0$ and with 2 invertible $\Sigma_{2g} = 0$. Thus we can consistently extend the ground ring with the operator T and get: $\Sigma_g = T^{\frac{g}{2}} \cdot$  $= 0$ for g even, and $\Sigma_g = T^{\frac{g-1}{2}} \cdot$  $= 2T^{\frac{g-1}{2}}$ for g odd. Thus every closed surface is reduced to a ground ring element. Allowing the empty cobordism ("empty surface"), every closed surface will reduce to it over $\mathbb{Z}[\frac{1}{2}, T]$ when we use the genus–3 presentation of T. The closed surfaces morphism set is thus isomorphic to $\mathbb{Z}[\frac{1}{2}, T]$ and through the genus–3 presentation acts on other morphism sets in the category, creating a module structure.

Now, when cutting a separating neck in a component, one reduces a component into two components at the expense of adding handles. Thus using only the NC relation, over $\mathbb{Z}[\frac{1}{2}, T]$, every surface reduces to a sum of surfaces whose components contain *exactly* one boundary circle and at most one handle.

The reduction can be put in a formula form, considered as a map Ψ taking a surface $\Sigma_g(\alpha_1, \ldots, \alpha_n)$ into a combination of generators (a sum of unions of surfaces with one boundary circle and genus 0 or 1). This map is an isomorphism of modules over the closed surfaces ring ($\mathbb{Z}[\frac{1}{2}, T]$–modules):

The Surface Reduction Formula Over $\mathbb{Z}[\frac{1}{2}, T]$

$$\begin{aligned}(1)\qquad &\Psi\colon \Sigma_g(\alpha_1,\dots,\alpha_n)\\ &\mapsto \frac{1}{2^{n-1}}\cdot\sum_{i_1,\dots,i_n=0,1}(g+z)\bmod 2\cdot T^{\lfloor\frac{g+z}{2}\rfloor}\cdot\Sigma_{i_1}(\alpha_1)\cdots\Sigma_{i_n}(\alpha_n)\end{aligned}$$

where z is the number of 0's among $\{i_1,\dots,i_n\}$. If $n=0$ it is understood that the sum is dropped and replaced by $g \bmod 2\cdot T^{\lfloor\frac{g}{2}\rfloor}$ times the empty surface. The formula extends naturally to unions of surfaces.

A close look at the definition will show that indeed this map uses only the NC relation to reduce the surface, in other words, the difference between the reduced surface and the original surface is only a combination of NC relations; (ie, $\Psi(\Sigma)-\Sigma=\sum NC$). To see this, one need to show that there is at least one way of using NC relations to get the formula. This can be done, for instance, by taking the boundary circles one by one, and each time cut the unique neck separating *only* this circle from the surface.

Last, we turn to the issue of possible relations between the generators. These can be created whenever we apply Ψ to two surfaces which are related by NC relation and get two different results. Thus the generators are free from relations if the reduction formula is well defined, ie, respects the NC relation. This is true, as the following argument proves $\Psi(NC)=0$. Look at the NC relation, $\Sigma_{g_1+1}(\alpha)\Sigma_{g_2}(\beta)+\Sigma_{g_1}(\alpha)\Sigma_{g_2+1}(\beta)-2\Sigma_{g_1+g_2}(\alpha,\beta)$, where $\alpha=\{\alpha_1,\dots,\alpha_n\}$ and $\beta=\{\beta_1,\dots,\beta_m\}$ denote families of boundary circles. Apply the formula everywhere:

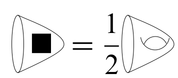
$$\begin{aligned}\frac{1}{2^{n+m-2}}\cdot[&(g_1+z_1+1 \bmod 2)(g_2+z_2 \bmod 2)T^{\lfloor\frac{g_1+z_1+1}{2}\rfloor}T^{\lfloor\frac{g_2+z_2}{2}\rfloor}\\ &+(g_1+z_1 \bmod 2)(g_2+z_2+1 \bmod 2)T^{\lfloor\frac{g_1+z_1}{2}\rfloor}T^{\lfloor\frac{g_2+z_2+1}{2}\rfloor}\\ &-(g_1+g_2+z_1+z_2 \bmod 2)T^{\lfloor\frac{g_1+z_1+g_2+z_2}{2}\rfloor}]\\ &\cdot\sum\Sigma_{i_1}(\alpha_1)\cdots\Sigma_{i_n}(\alpha_n)\Sigma_{j_1}(\beta_1)\cdots\Sigma_{j_m}(\beta_m)\end{aligned}$$

where z_1 is the number of 0's among $\{i_1,\dots,i_n\}$, z_2 is the number of 0's among $\{j_1,\dots,j_m\}$ and the sum is over these sets of indices as in formula (1). Going over all 4 options of (g_1+z_1, g_2+z_2) being (odd/even,odd/even) one can see the result is always 0.

Formula (1) gives a map from surfaces to combinations of generators. This map was shown to be well defined ($\Psi(NC) = 0$), and uses only the NC relations ($\Psi(\Sigma) - \Sigma = \sum NC$), thus completing the proof. $\square$

3 Reduction of the geometric complex associated to a link and the underlying algebraic structure over $\mathbb{Q}$

Armed with the classification of the morphisms of $\mathcal{C}ob_{/l}$ we are ready to try and simplify the complex associated to a link in the case where 2 is invertible. We introduce an isomorphism of objects in $\mathcal{C}ob_{/l}$, known as *delooping*, that extends to an isomorphism in $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathcal{C}ob_{/l})$. This will reduce the complex associated to a link into one taking values in a simpler category. One of the consequences is uncovering the underlying algebraic structure of the geometric complex. We will use the boxed surface notation where a boxed surface has the geometric meaning of half a handle:



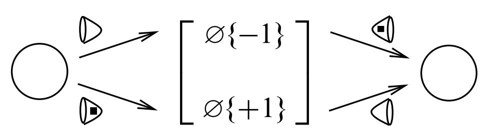
3.1 The reduction theorem over $\mathbb{Q}$

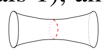
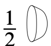
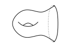
Theorem 1 *The complex associated to a link, over $\mathbb{Q}$ (or any ground ring R with 2 invertible), is equivalent to a complex built from the category of free $\mathbb{Q}[T]$–modules ($R[T]$–modules respectively).*

This theorem is a corollary of the following proposition, which we will prove first:

Proposition 3.1 *In $\mathrm{Mat}(\mathcal{C}ob_{/l})$, when 2 is invertible, the circle object is isomorphic to a column of two empty set objects. This isomorphism extends (taking degrees into account) to an isomorphism of complexes in $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathcal{C}ob_{/l})$, where the circle objects are replaced by empty sets together with the appropriate induced complex maps.*

Definition 3.2 The isomorphism that allows us to eliminate circles in exchange for a column of empty sets is called *delooping* and is given by:



Proof of Proposition 3.1 It is easy to check that the compositions of the morphisms above are the identity (thus making the circle isomorphic to the column of two empty sets). One direction (from empty sets to empty sets) just follows from the S and T relations (a boxed sphere equals 1), and the other direction (from circle to circle) just follows from the NC relation  $= \frac{1}{2}$   $+ \frac{1}{2}$   .

This proves that the circle object is isomorphic to a column of two empty set objects in $\mathrm{Mat}(\mathcal{C}ob_{/l})$. The extension to the category $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathcal{C}ob_{/l})$ is now straight forward by replacing every appearance of a circle by a column of two empty sets (with the degrees shifted as shown in the picture, maintaining the grading of the theory). The induced maps are the composed maps of the isomorphism above with the original maps of the complex. □

Proof of Theorem 1 The theorem follows from the proposition. The only objects that are left in the complex are the empty sets (in various degree shifts). The induced morphisms in the reduced complex are just the set of closed surfaces, isomorphic to $\mathbb{Q}[T]$ in the genus–3 presentation, thus making the complex equivalent to one in the category of free $\mathbb{Q}[T]$–modules. Note that the isomorphism in the proposition is between objects within the category $\mathrm{Mat}(\mathcal{C}ob_{/l})$ thus tautologically respects the 4TU/S/T relations. One can check directly that the 4TU relation is respected — since the 4TU relation is a local relation in the interior of the surface and the isomorphism is done on the boundary it obviously respects it. One needs to check that the functor from $\mathrm{Mat}(\mathcal{C}ob_{/l})$ to free $\mathbb{Q}[T]$–modules is well defined (respects 4TU on the morphism level) but this is obvious from the same reasons (it can be checked directly using the classification from the previous section). Thus the complex will look like columns of empty sets (the single object) and maps which are matrices with entries in $\mathbb{Q}[T]$. □

Remark One can show that the morphisms appearing in the complex associated to a link are further restricted to matrices with $\mathbb{Z}[\frac{1}{2}, T]$ entries. Furthermore, we can work with boxed surfaces and re-normalize the 2–handle operator by dividing it by 4, to get that the maps of the complex associated to a link are always $\mathbb{Z}[T]$ matrices. Then, degree considerations will tell us immediately that the entries are only monomials in T.

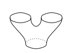
This theorem extends the simplification reported in [4; 3]. There, it was done in order to compute the "standard" Khovanov homology, which in the context of the geometric formalism means imposing one more relation which states that any surface having genus higher than 1 is set equal to zero (see also [5, Section 9]). In our context this means setting the action of the operator T to zero. When we set $T = 0$ the complex reduces to columns of empty sets with maps being integer matrices.

3.2 The algebraic structure underlying the complex over $\mathbb{Q}$

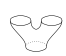
Now that we simplified the complex significantly it is interesting to see what can we learn about the underlying algebraic structure of the complex and about the complex maps in terms of the empty sets.

As a first consequence we can see that the circle (the basic object in the complex associated to a link) carries a structure composed of two copies of another basic object (the empty set). It has to be understood that this decomposition is a direct consequence of the 4TU/S/T relations and it is an intrinsic structure of the geometric complex that reflects these relations. This already suggests that any algebraic structure which respects 4TU/S/T and put on the circle (for instance a Frobenius algebra, to give a TQFT) will factor into a direct sum of two identical copies shifted by two degrees (two copies of the ground ring of the algebra, say). This shows, perhaps in a slightly more fundamental way, the result given in terms of Frobenius extensions in [8].

We denote $\varnothing\{-1\}$ by v_- (comparing to [5]) or X (comparing to [8]) and $\varnothing\{+1\}$ by v_+ or 1 (comparing accordingly). We also define a re-normalized 2–handle operator t to be equal the 2–handle operator T divided by 4 (now $t =$ $/8$ in the genus–3 presentation). Note that in the category we are working right now this is just a notation, the actual objects are the empty sets and we did not add any extra algebraic structure. We use the tensor product symbol to denote unions of empty sets and thus keeping track of degrees (remember that all elements of the theory are graded).

Take the pair of pants map  between $\bigcirc\bigcirc$ and $\bigcirc$, and look at the two complexes $\bigcirc\bigcirc \xrightarrow{\text{pants}} \bigcirc$ and $\bigcirc \xrightarrow{\text{pants}} \bigcirc\bigcirc$. Reduce these complexes using our theorem (ie, replace the circles with empty sets columns and replace the complex maps with the induced maps). We denote by Δ_2 the following composition:

$$\Delta_2 : \begin{bmatrix} v_- \\ v_+ \end{bmatrix} \xrightarrow{\text{isomorphism}} \bigcirc \longrightarrow \bigcirc\bigcirc \xrightarrow{\text{isomorphism}} \begin{bmatrix} v_- \otimes v_- \\ v_- \otimes v_+ \\ v_+ \otimes v_- \\ v_+ \otimes v_+ \end{bmatrix}$$



We denote by m_2 the composition in the reverse direction. By composing surfaces and using the re-normalized 2–handle operator it is not hard to see that the maps we get are:

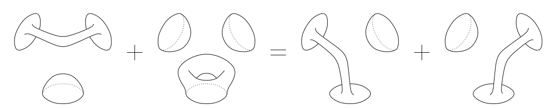
$$\Delta_2 : \begin{cases} v_+ \mapsto \begin{bmatrix} v_+ \otimes v_- \\ v_- \otimes v_+ \end{bmatrix} \\ v_- \mapsto \begin{bmatrix} v_- \otimes v_- \\ t v_+ \otimes v_+ \end{bmatrix} \end{cases} \qquad m_2 : \begin{cases} v_+ \otimes v_- \mapsto v_- \quad v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- \quad v_- \otimes v_- \mapsto t v_+ . \end{cases}$$

The maps have no linear structure, these are just 2×4 matrices of morphisms in the geometrical category. Nonetheless this is exactly the pre-algebraic structure of the generalized Lee TQFT. See [9] for the non generalized one (where $t=1$) and [5, section 9]. The above TQFT is denoted $\mathcal{F}_3$ in [8]. It is important to note that so far in our category we did not apply any TQFT or any other functor to get this structure, these maps are not multiplication or co-multiplication in an algebra. It is all done intrinsically within our category, and as we will see later it imposes restrictions on the most general type of functors one can apply to the geometric complex. One can say that the geometric structure over $\mathbb{Q}$ has the underlying structure of this specific *"pre-TQFT"*.

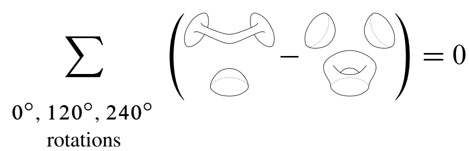
4 Classification of surfaces modulo 4TU/S/T relations over $\mathbb{Z}$

When 2 is not invertible the neck cutting relation is not equivalent to the 4TU relation and we need to make sure we use the "full version" of the 4TU relation in reducing surfaces into generators. Still there is a simpler equivalent version of the 4TU relation which involves only 3 sites on the surface:

$3S_1$:



$3S_2$:



The version that will be most convenient for us is the $3S1$ relation. We will use the 2–handle lemma (which does not depend on the invertibility of 2), extend the ground ring again, and identify a complete set of generators.

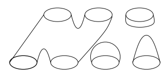
Definition 4.1 Assume a surface has at least one boundary circle. Choose one boundary circle. The component that contains the specially chosen boundary circle is called *the special component*. All other components will be called *non-special*.

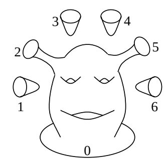
Definition 4.2 The *special 1–handle operator*, denoted H, is the operator that adds a handle to the special component of a surface.

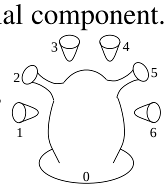
Note that $T=H^2$, though H acts "locally" (acts *only* on the special component) and T acts "globally" (anywhere on the surface). We extend our ground ring to $\mathbb{Z}[H]$. We remind the reader that the entire theory is still graded with H given degree -2.

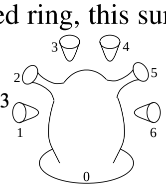
Remark 4.3 From now on we will assume that all surface has at least one boundary component, thus there is always a special component, and the action of H is well defined (by choosing a special component).

Proposition 4.4 *Over the extended ring $\mathbb{Z}[H]$ (after a choice of special boundary circle) the morphisms of $\mathcal{C}ob_{/l}$ with one of the source/target objects non empty (ie, surfaces modulo 4TU/S/T relations with at least one boundary circle) are generated freely by surfaces which are composed of a genus zero special component with any number of boundary circles on it and zero genus non-special components with exactly one boundary circle on them. H is the special 1–handle operator.*

An example for a generator would be  (the special circle is at the bottom left).

Another example would be the "Shrek surface": 

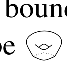
. This surface has a special boundary circle marked with the number zero (Shrek's neck), 6 other boundary circles (number 2 and 5 belong to the special component — the head) and 3 handles on the special component. Over the extended ring, this surface is generated by the "Shrek shadow" 

and equals to H^3 

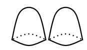
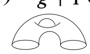
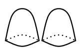
.

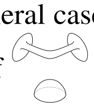
Remark 4.5 In order for the composition of surfaces (morphisms) to be H–linear, and thus respect the $\mathbb{Z}[H]$–module structure, one needs to restrict to a category where all the morphisms preserve the special component, ie, we always have the special boundary circles of the two composed morphisms in the same component. As we will see (next section) this happens naturally in the context of link homology.

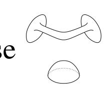
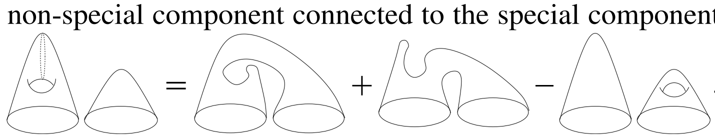
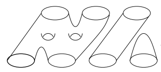
The rest of this section is devoted to the proof of the proposition above.

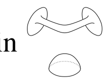
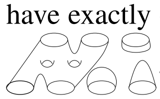
Proof We start with a couple of toy models. If the surface components involved in the $3S2$ (or $3S1$) relation has all together one boundary circle, then the $3S2$ (or $3S1$) relation is equivalent to NC relation, and further more up to the 2–handle lemma it is trivially satisfied. This makes the classification of surfaces with only one boundary component easy. Extend the ground ring to $\mathbb{Z}[T]$ and the generators will be  and

. We can now go one step further and use the "local" 1–handle operator H. Over $\mathbb{Z}[H]$ there is only one generator .

If the surface's components involved have only two boundary circles all together, then the $3S2$ (or $3S1$) relation is again equivalent to the NC relation. We can classify now all surfaces with 2 boundary circles modulo the 4TU relation over $\mathbb{Z}$. Let us extend the ground ring to $\mathbb{Z}[T]$ thus reducing to components with genus 1 at most. Then, we choose one of the boundary circles (call it α), and use the NC relation to move a handle from the component containing the other boundary circle (call it β) to the component containing α (ie, $\Sigma_1(\beta)\Sigma_g(\alpha) = 2\Sigma_g(\beta,\alpha) - \Sigma_0(\beta)\Sigma_{g+1}(\alpha)$). This leaves us with the following generators over $\mathbb{Z}[T]$: , ,  and  (α is drawn on the right). This asymmetry in the way the generating set looks like is caused by the symmetry braking in the way the NC relation is applied (we chose the special circle α, on the right). On the other hand it allows us again to replace the "global" 2–handle operator T, with the "local" 1–handle operator H, which operates only on the component containing α. Thus, over $\mathbb{Z}[H]$ we only have the following generators:  and . This set is symmetric again, and the asymmetry hides within the definition of H.

The place where the difference between the NC and $3S2$ (or $3S1$) relations comes into play is when three different components with boundary on each are involved. We would like to use, in the general case, the $3S1$ relation as a neck cutting relation for the neck in the upper part of  (by replacing it with the other 3 elements in the equation). The way the relation is applied is not symmetric (not even visually), while the upper 2 sites are interchangeable the lower site plays a special role. This hints that this scheme of classification would be easier when choosing a special component of the surface. This is done by choosing a special boundary circle as in definition 4.1.

Cutting non separating necks over $\mathbb{Z}$ Assume first that we only use  in the $3S1$ relation to cut a non-separating neck (the upper two sites remain in one connected component after the cut), and we take the lower site to be on the special component. Now, whenever we have a handle in a non-special component (dashed line on the left component in the picture below) we can move it using the $3S1$ relation to the special component (the component on the right) at the price of adding surfaces that has the non-special component connected to the special component, thus becoming special: . After this reduction we are left with surfaces involving handles only on the special component, and the rest of the components are of genus zero (with any amount of boundary circles on them). For example , where the special circle is at the bottom left.

Cutting separating necks over $\mathbb{Z}$ We reached a point where the surfaces left are unions of a special component with any number of handles on it and zero genus non-special components with any possible number of boundary circles. We can still use the $3S1$ relation to cut separating necks (the two upper sites in  will be on two disconnected components after the cut). We will use the relation with the upper two sites on a non-special component and the lower site on the special component. The result is separating the non-special component into two at the price of adding a handle to the special component and adding surfaces that have connecting necks between the special component and what used to be the non-special component (the other summands of $3S1$). Thus, whenever we have a non-special component that has more than one boundary circle, we can reduce it to a sum of components that are either special or have exactly one boundary circle. An example for a summand in a reduced surface is , where the special circle is at the bottom left.

Ground ring extension, surface reduction formula over $\mathbb{Z}$ We extend our ground ring to $\mathbb{Z}[H]$. We will put all the above discussion into one formula for reducing any surface into generators (considered as a map Ψ). Let α denote a family of boundary circles $\{\alpha_1,\dots,\alpha_n\}$ and let S denote the special boundary circle. We will use $\beta\in 2^\alpha$ for any subset $\{\alpha_{i_1},\dots,\alpha_{i_k}\}$ of α, and $\{\alpha_{i_{k+1}},\dots,\alpha_{i_n}\}$ for its complement in α.

The surface reduction formula over $\mathbb{Z}$

Special component reduction:

(2)
$$\Sigma_g(S,\dots)=H^g\cdot\Sigma_0(S,\dots)$$

Non-special component genus reduction ($g\geq 1$):

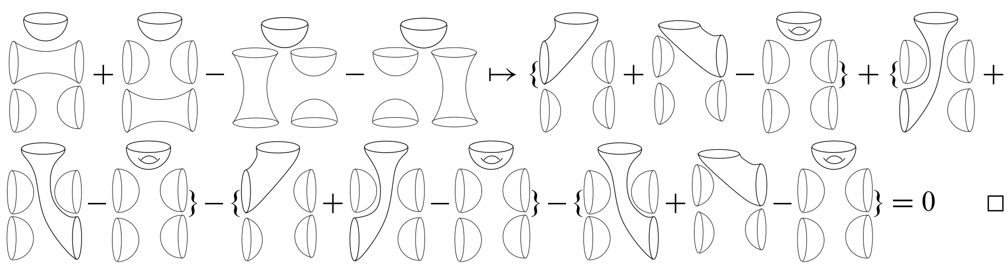
(3)
$$\Sigma_0(S,\dots)\Sigma_g(\alpha)=2\cdot(g \bmod 2)\cdot H^{g-1}\Sigma_0(S,\dots,\alpha)+(-1)^g H^g\Sigma_0(S,\dots)\Sigma_0(\alpha)$$

Non-special component neck reduction ($n\geq 2$):

(4)
$$\Sigma_0(S,\dots)\Sigma_0(\alpha)=\sum_{\beta\in 2^\alpha,\beta\neq\alpha}(-1)^{n-k-1}H^{n-k-1}\Sigma(S,\dots,\beta)\Sigma_0(\alpha_{i_{k+1}})\cdots\Sigma_0(\alpha_{i_n})$$

Formula (4) is plugged into the result of formula (3) in order to reduce any non-special component into combinations of generators. Extend the above formulas to unions of non-special components in a natural way (just iterate the use of the formula) and get a reduction of any surface into generators. The previous discussion shows that this formula uses the $3S1$ relation only (ie, $\Psi(\Sigma) - \Sigma = \sum 3S1$).

Finalizing the proof is done by taking the formula and checking that it is a map from surfaces to the generators (by definition) that uses only the 4TU ($3S1$) relations (proved by the discussion above) and that the generators are free (ie, it is well defined). The last part can be done by a direct computation (applying the formula to all sides of a 4TU relation to get zero) or by applying a TQFT to these surfaces. By choosing a TQFT that respects the 4TU relations (like the standard $X^2 = 0$ Khovanov TQFT) one can show that it separates these surfaces (ie, sends them to independent module maps). The details will not be described here. We do mention that the reason for $\Psi(4TU) = 0$ is coming from the fact that iterative application of the $3S1$ relation to a 4TU relation is zero as shown by the picture below (we draw only the pieces of the surfaces that differ and add the special component on top):



□

5 Reduction of the geometric complex associated to a link and the underlying algebraic structure over $\mathbb{Z}$

We would like to give a similar treatment for the general case over $\mathbb{Z}$ as we did over $\mathbb{Q}$. Recall that whenever working with surfaces over $\mathbb{Z}$ we need to pick a special boundary circle first. We mark the link at one point (anywhere). After doing so, we have a special circle at every appearance of an object of $\mathcal{C}ob_{/l}$ in the complex (the one containing the mark). For the sake of clarity we will always draw the special circle as *a line* (and call it the special line). This is actually a canonical choice when working with knots or 1–1 tangles. We defer further discussion on these issues to Section 7. We will use the dotted surface notation, where a dotted component means that a neck is connected between the dot and the special component.

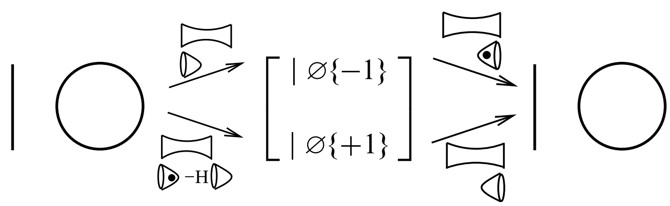
5.1 Reduction theorem over $\mathbb{Z}$

Theorem 2 *The complex associated to a link, over $\mathbb{Z}$, is equivalent to a complex built from the category of free $\mathbb{Z}[H]$–modules.*

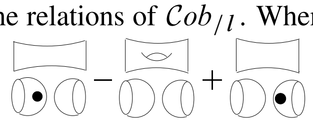
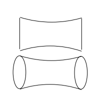
The theorem is a corollary of the following proposition, which we will prove first:

Proposition 5.1 *In $\mathrm{Mat}(\mathcal{C}ob_{/l})$ a non-special circle is isomorphic to a column of two empty set objects. This isomorphism extends (taking degrees into account) to an isomorphism of complexes in $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathcal{C}ob_{/l})$ where the non-special circle objects are replaced by empty sets together with the appropriate induced complex maps.*

Definition 5.2 The isomorphism from the proposition is called *delooping* and is given by the following diagram:



This diagram shows how to make a non-special circle disappear into a column of two empty sets in the presence of the special line. The special line (remember it is a notation for the special circle) functions as a "probe" which in his presence all the other circles can be isomorphed into empty sets, leaving us with nothing but the special line. Note that what used to be a tube between two special circles is now drawn as *a curtain* between two special lines, thus a dot will connect a neck to the curtain, and the special 1–handle operator H will add a handle to it.

Proof of Proposition 5.1 The proof is done by looking at the above diagram of morphisms. To prove that this is indeed an isomorphism we need to trace the arrows and use the relations of $\mathcal{C}ob_{/l}$. When composing the above diagram from $|\bigcirc$ to $|\bigcirc$ one gets:  (remember that H adds a handle to the special component). Modulo the $3S1$ relation this equals to  which is the identity cobordism. The other direction is easier, using trivially the S relation (exercise). The second part of the proposition is a natural extension of the above diagram to complexes, keeping track of degrees. Replace the appearances of non-special circles by columns of empty sets and compose the above isomorphism with the original complex maps to get the induced maps. ◻

Proof of Theorem 2 Given a link, one can mark it at one point and have a special circle (the one containing the marked point) in each appearance of an object from $\mathcal{C}ob_{/l}$ in the complex associated to the link. Using the proposition we can reduce this complex and replace all the non-special circles with columns of empty sets. The only thing left in the complex are columns with the special line (special circle) in its entries. The morphism set of that object are curtains (tubes) with any genus which is isomorphic to $\mathbb{Z}[H]$, where H is the special 1–handle operator. Thus we get a complex made of columns of free $\mathbb{Z}[H]$–modules and maps which are matrices with $\mathbb{Z}[H]$ entries. One would like to check that the functor from $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathcal{C}ob_{/l})$ to the category of $\mathbb{Z}[H]$–modules is indeed well defined (on the morphism level), but this is tautological from our definitions, as in the proof of the theorem over $\mathbb{Q}$ (it can still be shown directly by defining the functor only on generators of $\mathcal{C}ob_{/l}$ using our factorization from the previous section). □

Remark 5.3 Though surfaces with two boundary components in $\mathcal{C}ob_{/l}$ have two free generators over $\mathbb{Z}[H]$ only the connected generator (a curtain) appears in the complex associated to a link, therefore the morphism group above can be reduced from $\mathbb{Z}[H]\bigoplus\mathbb{Z}[H]$ to $\mathbb{Z}[H]$. As in the case over $\mathbb{Q}$ the appearance of H comes only in homogeneous form, ie, monomials entries, due to grading considerations.

5.2 The algebraic structure underlying the complex over $\mathbb{Z}$

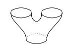
We follow the same trail as the reduction over $\mathbb{Q}$ to get some information on the underlying structure of the geometric complex over $\mathbb{Z}$.

The non-special circle, a basic object in the theory, decomposes into two copies of another fundamental object (the empty set) with relative degree shift 2. This restricts functors from the geometric category to any other category which might carry an algebraic structure of direct sums (TQFT for example, into the category of $\mathbb{Z}$–modules). The special line (special circle) stays as is, but as we will see below it actually carries an intrinsic “one dimensional” object. Later we will also see that the special line can be *promoted* to carry higher dimensional algebraic objects.

Denote $|\ \{-1\}$ by v_- (comparing to [5]) or X (comparing to [8]) and $|\ \{+1\}$ by v_+ or 1 (comparing accordingly). We use the tensor product symbol to denote unions of such empty sets (always with one special line only), thus keeping track of degree shifts.

Now, we take the pair of pants map  between $|\ \bigcirc\bigcirc$ and $|\ \bigcirc$. By this we mean the complexes $|\ \bigcirc\bigcirc \xrightarrow{\text{pants}} |\ \bigcirc$ and $|\ \bigcirc \xrightarrow{\text{pants}} |\ \bigcirc\bigcirc$ where the special lines are connected with a curtain. We reduce these complexes into empty sets complexes (replacing the non-special circles with empty sets columns and replacing the maps with the induced maps, as in the reduction theorem). We denote by Δ_1 the composition:

$$\Delta_1 : \begin{bmatrix} v_- \\ v_+ \end{bmatrix} \xrightarrow{\text{isomorphism}} |\ \bigcirc$$



$$\longrightarrow |\ \bigcirc\,\bigcirc \xrightarrow{\text{isomorphism}} \begin{bmatrix} v_- \otimes v_- \\ v_- \otimes v_+ \\ v_+ \otimes v_- \\ v_+ \otimes v_+ \end{bmatrix}$$

We denote by m_1 the composition in the reverse direction. By composing surfaces and using the 4TU/S/T relations the maps we get are (again, these are not linear maps, yet, but a matrix of cobordisms):

$$\Delta_1 : \begin{cases} v_+ \mapsto \begin{bmatrix} v_- \otimes v_+ \\ v_+ \otimes v_- \\ -Hv_+ \otimes v_+ \end{bmatrix} \\ v_- \mapsto v_- \otimes v_- \end{cases} \qquad m_1 : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto Hv_-. \end{cases}$$

Let us check what are the induced maps when the pair of pants  involve the special line. Denote by Φ the following composition:

$$\Phi : \begin{bmatrix} v_- \\ v_+ \end{bmatrix} \xrightarrow{\text{isomorphism}} |\ \bigcirc$$

$$\longrightarrow |$$

Denote by Ψ the composition in the reverse direction. Composing and reducing surfaces we get the following maps:

$$\Phi : \begin{cases} v_- \mapsto H \\ v_+ \mapsto 1 \end{cases} \qquad \Psi : \begin{cases} \varnothing \mapsto v_- \end{cases}$$

(Δ_1, m_1) is the type of algebraic structure one encounters in a TQFT (co-product and product of the Frobenius algebra). The appearance of a special line (special circle) is most natural in the construction of the *reduced knot homology*, introduced in [7]. In the setting of Khovanov homology theory, first, one views the entire chain complex as a complex of $\mathbb{A}$–modules ($\mathbb{A}$ is the Frobenius algebra underlying the TQFT used) through a natural action of $\mathbb{A}$ (here one has to mark the knot and encounter the special line). Then, one can take the kernel complex of multiplication by X to be the reduced complex. One can also take the *co-reduced complex*, which is the image complex of multiplication by X. The above (Δ_1, m_1) structure is exactly the TQFT denoted $\mathcal{F}_7$ in [8]. The $\bmod 2$ specialization of this theory appeared in [5, Section 9]. The additional structure coming from Ψ and Φ is exactly the structure of the *co-reduced* theory of the (Δ_1, m_1) TQFT, where the special line carries the "one dimensional" object generated by X. It is important to note again that this is an *intrinsic* information on the underlying structure, coming before any TQFT is even applied. This gives the above "pre-TQFT co-reduced structure" a unique place in the universal theory.

6 TQFTs and link homology theories put on the complex

The complex invariant is geometric in nature and one cannot form homology groups directly (kernels make no sense since the category is additive but not abelian). Thus we need to apply a functor into an algebraic category where one can form homology groups. We will classify such functors and present the universal link homology theory. We will also explore the relative strength and the information held within these functors. One of these types of functors is a TQFT which maps the category Cob into the category of modules over some ring. In the two dimensional case, which is the relevant case in link homology, such TQFT structures are equivalent to Frobenius systems and are classified by them [1; 8]. A priori, the link homology theory coming from a TQFT does not have to satisfy the 4TU/S/T relations of the complex. Section 6.3 shows why any TQFT (up to twisting and base change) used to create link homology can be put on the geometric complex, making the complex universal for link homology coming from TQFTs. We show that the universal link homology functor is actually a TQFT and every other link homology functor factorizes through it (including non TQFTs).

6.1 Two important TQFTs

Given the algebraic structures (Δ_1, m_1) and (Δ_2, m_2), introduced in the previous sections, one can add linearity and get the following algebraic structures (using the same notation):

$$\Delta_1 : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ - H v_+ \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

$$m_1 : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto H v_- \end{cases}$$

$$\Delta_2 : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- + T v_+ \otimes v_+ \end{cases} \qquad m_2 : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto T v_+ \end{cases}$$

Then, one can construct TQFTs (Frobenius systems) based on these algebraic structures. The first will be called $\mathcal{F}_H$ and is given by the algebra $A_H = R_H[X]/(X^2 - HX)$ over $R_H = R[H]$ (system $\mathcal{F}_7$ in [8]). The second system is $A_T = R_T[X]/(X^2 - T)$ over the ring $R_T = R[T]$ which will be called $\mathcal{F}_T$ (named $\mathcal{F}_3$ in [8]). Of course one can take the reduced or co-reduced homology structures, and we denote it by superscripts ($\mathcal{F}_H^{co}$ for example). Due to the fact that these are the underlying algebraic structures of the geometric complex it is not surprising that they dominate the information in link homology theory.

6.2 The universal link homology theories

We give the most general link homology theory that can be applied to the geometric complex.

Theorem 3 *Every functor used to create link homology theory from the geometric complex over $\mathbb{Q}$, or any ring with 2 invertible, factors through $\mathcal{F}_T$. Ie, the TQFT $\mathcal{F}_T$ is the universal link homology theory when 2 is invertible and holds the maximum amount of information.*

Proof Since we showed that in the geometric complex the morphism groups are $\mathbb{Q}[T]$–modules the target objects of the functor must be too (by functoriality). Since every object (circle) in the complex is isomorphic to a direct sum of two empty sets, the functor will be determined by choosing a single $\mathbb{Q}[T]$–module corresponding to the empty set. The universal choice would be $\mathbb{Q}[T]$ itself, and the complex coming from any other module will be obtained by tensoring the complex with it. □

Theorem 4 *Every functor used to create link homology theory from the geometric complex over $\mathbb{Z}$ factors through $\mathcal{F}_H^{co}$ which holds the maximum amount of information. Ie, the TQFT $\mathcal{F}_H^{co}$ is the universal functor for link homology.*

Proof The complex is equivalent to a free $\mathbb{Z}[H]$–modules complex, and the universal choice is $\mathbb{Z}[H]$. All other theories are just tensor products of the universal one. □

6.3 A "big" Universal TQFT by Khovanov

A priori, TQFTs can be used to create link homology theory without the geometric complex (meaning without satisfying the 4TU/S/T relations). In [8] Khovanov presents a universal rank 2 TQFT (Frobenius system) given by the following formula (denoted $\mathcal{F}_5$ there):

$$\Delta_{ht} : \begin{cases} 1 \mapsto 1 \otimes X + X \otimes 1 - h1 \otimes 1 \\ X \mapsto X \otimes X + t1 \otimes 1 \end{cases}$$

$$m_{ht}:\begin{cases}1\otimes X\mapsto X & 1\otimes 1\mapsto 1\\ X\otimes 1\mapsto X & X\otimes X\mapsto t1+hX.\end{cases}$$

This structure gives the Frobenius algebra $A_{ht}=R_{ht}[X]/(X^2-hX-t)$ over the ring $R_{ht}=\mathbb{Z}[h,t]$. Comparing to our notation (and these of [5]) is done by putting $v_-=X$ and $v_+=1$. We will denote this TQFT $\mathcal{F}_{ht}$.

Given a (tensorial) functor $\mathcal{F}$ from Cob to the category of R–modules (a TQFT), one can construct a link homology theory and ask whether it is (homotopy) invariant under Reidemeister moves. If it is invariant under the first Reidemeister move, then it is a Frobenius system of rank 2. Khovanov showed that every rank 2 Frobenius system can be *twisted* into a descended theory, $\mathcal{F}'$, which is a base change of $\mathcal{F}_{ht}$ (base change is just a unital ground ring homomorphism which induces a change in the algebra. The notion of twisting is explained in [8] and ref therein). The complexes associated to a link using $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic, thus all the information is still there after the twist. Since base change just tensors the chain complex with the appropriate new ring (over the old ring), system $\mathcal{F}_{ht}$ is universal for link homology theories coming from TQFTs. The fact that $\mathcal{F}_{ht}$ satisfies the 4TU/S/T relations allows one to apply it as a homology theory functor on the geometric complex and use [5] results (giving Proposition 6 in [8]). Base change does not change the fact that a theory satisfies the 4TU/S/T relations, and since $\mathcal{F}_{ht}$ satisfies these relations, we have that every tensorial functor $\mathcal{F}$ that is invariant under Reidemeister–1 move can be twisted into a functor $\mathcal{F}'$ that satisfies the relations of $\mathcal{C}ob_{/l}$ and thus can be put on the geometric complex without losing any homological information relative to $\mathcal{F}$. This gives the universality of the geometric complex for all link homology theories coming from TQFTs. The universal link homology functor presented in the theorems above captures all the information coming from TQFT constructions.

Over $\mathbb{Q}$ it is easy to see how our universal theory captures all the information of khovanov's "big" TQFT. We start with $\mathcal{F}_{ht}$ over $\mathbb{Q}$ (ie, $R_{ht}=\mathbb{Q}[h,t]$). By doing a change of basis: $\begin{cases}v_- - \frac{h}{2}v_+\mapsto v_-\\ v_+\mapsto v_+\end{cases}$ we get the theory given by:

$$\begin{cases}v_+\mapsto v_+\otimes v_- + v_-\otimes v_+\\ v_-\mapsto v_-\otimes v_- + (t+\frac{h^2}{4})v_+\otimes v_+\end{cases}\qquad \begin{cases}v_+\otimes v_-\mapsto v_- & v_+\otimes v_+\mapsto v_+\\ v_-\otimes v_+\mapsto v_- & v_-\otimes v_-\mapsto (t+\frac{h^2}{4})v_+\end{cases}$$

Re-normalizing $t+\frac{h^2}{4}=\frac{T}{4}:=\widetilde{T}$, we see that the above theory is just $\mathcal{F}_{\widetilde{T}}$ with the ground ring extended by another superficial variable h. Every Calculation done using $\mathcal{F}_{ht}$ to get link homology, is equal to the same calculation done with $\mathcal{F}_{\widetilde{T}}$ tensored with $\mathbb{Q}[h]$, and holds exactly the same information. Note that our complex reduction is performing this change of basis (to "kick out" one redundant variable) on the complex level, before even applying any TQFT. Doing first the complex reduction and then applying the TQFT $\mathcal{F}_{ht}$ will factorize the result through $\mathcal{F}_{\widetilde{T}}$.

In the general case, over $\mathbb{Z}$, one needs some further observations which we turn to now.

6.4 About the 1–handle and 2–handle operators in $\mathcal{F}_{ht}$

Given the TQFT $\mathcal{F}_{ht}$ we can ask what is the 1–handle operator H of this theory. Meaning, we want to know what is the operator that adds a handle to a cobordism, or put in other words, what is  viewed as a map $A \mapsto A$ between the two Frobenius algebras associated to the boundary circles. One can easily give the answer by looking at the multiplication and co-multiplication formulas: $H = 2X - h$. For the readers who know a bit about topological Landau–Ginzburg models, H is the Hessian of the potential, and indeed the results match . The 2–handle operator T follows immediately by computing $T = H^2$ and reducing modulo $X^2 - hX - t$ to get: $T = 4t + h^2$. The 2–handle operator is an element of the ground ring, as expected from our 2–handle lemma, and can be multiplied anywhere in a tensor product. The 1–handle operator is an element of A but not of the ground ring, and thus when operating on a tensor product one needs to specify the component to operate on, as we did by picking the special circle. Since A is two dimensional, picking the basis to be $\begin{pmatrix} 1 \\ X \end{pmatrix}$ would give a two dimensional representation of the operators above: $H = \begin{pmatrix} -h & 2t \\ 2 & h \end{pmatrix}$, $T = \begin{pmatrix} 4t + h^2 & 0 \\ 0 & 4t + h^2 \end{pmatrix}$.

6.5 The special line and H promotion

The main result of the previous section was a reduction of the complex associated to a link into a complex composed of columns of the special line and maps which are matrices with H monomials entries. Given such a complex one wants to create a homology theory out of it, ie, apply some functor to it that will put an H–module on the special line with the possibility of taking kernels (and forming homology groups). As we have seen, the intrinsic structure that the special line caries is the one dimensional module $\mathbb{Z}[H]$ generated by X. We can get a different structure by replacing H with any integer matrix of dimension n and the special line with direct sum of n copies of $\mathbb{Z}$. We call this type of process *a promotion* (realizing it is a fancy name for tensoring process). Another type of promotion is replacing H with a matrix (of dimension n) with polynomial entries in two variables (h and t, say) and then promoting the special line into the direct sum of n copies of $\mathbb{Z}[h,t]$. Note that some promotions lose information.

6.6 TQFTs via promotion and new type of link homology functors

First, we want to find a promotion of H that is equivalent to the theory $\mathcal{F}_{ht}$ and which does not lose any information on the complex level. From the above it is clear that such a promotion is $H = \begin{pmatrix} -h & 2t \\ 2 & h \end{pmatrix}$. The special line will be promoted into two copies of $\mathbb{Z}[h,t]$. Since each power of the promoted H adds a power of t and h to the entries, the power of the 1–handle operator H can be uniquely determined after the promotion and we lose no information. The algebraic complex one gets by applying this promotion is equal to the complex one gets by applying the TQFT $\mathcal{F}_{ht}$ before the complex reduction.

We can now easily get the other familiar TQFTs using the same complex reduction and H promotion technique:

$$\mathcal{F}_{\widetilde{T}} \leftarrow H = \begin{pmatrix} 0 & 2\widetilde{T} \\ 2 & 0 \end{pmatrix} \qquad \mathcal{F}_{H} \leftarrow H = \begin{pmatrix} -H & 0 \\ 2 & H \end{pmatrix}$$

One gets the standard Khovanov homology ($X^2 = 0$) by the promotion $H = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$. The special line is promoted to double copies of $\mathbb{Z}[\widetilde{T}]$, $\mathbb{Z}[H]$ and $\mathbb{Z}$ respectively. One can get the *reduced Khovanov homology* by focusing on the top left entry of the H promotion matrix. It is interesting to note that in the standard Khovanov homology case indeed we lose information and $H^2 = T = 0$. In the geometric language this means that in order to get the standard Khovanov homology one has to ignore surfaces with genus 2 and above. This was observed in [5]. Reduction to Lee's theory [9] is done by substituting $\widetilde{T} = 1$.

One can create other types of promotions that will enable us to control the order of H involved in the theory. We are able to cascade down from the most general theory, the one that involves all powers of H (ie, surfaces with any genera in the topological language), into a theory that involves only certain powers of H (ie, genera up to a certain number). For example, promote the special line to three copies of $\mathbb{Z}$, and H to the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. This theory involves only powers of H smaller or equal to 2, that is surfaces of genus up to 2. These promotions can be viewed as a family of theories extrapolating between the "standard" Khovanov TQFT and our universal theory (reminding of a perturbation expansion in string theory).

7 Further discussion

7.1 Comments on the question of homotopy classes versus homology theories

The geometric complex associated to a link is an invariant up to homotopy of complexes. Thus its fullest strength lies in the homotopy class of the complex itself, and have the potential of being a stronger invariant than any functor applied to the complex to produce a homology theory. The reduction given above might be the beginning of an approach to the following question: *classify all complexes associated to links up to homotopy*. The category $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathcal{C}ob_{/l})$ seemed at first too big and complicated for an answer, but complexes built on free modules over polynomial rings look more hopeful. The isomorphism of complexes reduce that question into the following question:

Question 1 *Classify all homotopy types of chain complexes with free $\mathbb{Z}[H]$–modules as chain groups and $\mathbb{Z}[H]$ matrices as maps (without loss of generality, with monomial entries).*

Another interesting question regarding the strength of the complex invariant is the following:

Do homology theories (functors from the topological category to an algebraic one) completely classify homotopy classes of complexes?

Combined with question 1 we get:

Question 2 *Does the geometric complex contain more information than all the possible algebraic complexes (homology theories) that can be put on it? In other words, are there two non-homotopic geometric complexes, associated to links in the geometric formalism, that are not separated by some functor to an algebraic category?*

Question 3 *If the answer is no, then do all the possible homology groups of these algebraic complexes classify the homotopy class of the geometric complex?.*

The answers will determine the relative strength of the geometric complex invariant. The complex reduction answers question 2.

Answer 2 *No*

Explanation The complex is built from a category equivalent to a category of free $\mathbb{Z}[H]$–modules. Thus, it has a faithful algebraic representation, ie, we have found a homology theory that represents faithfully the complexes in $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathcal{C}ob_{/l})$. Moreover, this homology theory is the co-reduced theory of the TQFT (Δ_1, m_1), which can be reached by applying a specific tautological functor. The geometric complex holds the same amount of information coming from the chain complex of this specific homology theory (TQFT).

We are left then with:

Question 3 (revised) *Do the homolgy groups of complexes in* $\mathrm{Kom}_{/h}\,\mathrm{Mat}(\mathbb{Z}[H])$*, associated to links, classify the homotopy type of the chain complexes?*

7.2 Comments on marking one of the boundary circles

We classified all the surfaces with at least one boundary circle over $\mathbb{Z}$. These are the ones relevant for link homology. In order to do that we picked up a special circle and marked it. The presentation of the generators depends on which circle we choose, but this choice has no importance for the classification itself and the topology of the generating surfaces. If the link is a knot, then in the context of knot homology and the geometric complex there is a natural way of determining a special circle in each and every appearance of an object of $\mathcal{C}ob_{/l}$ in the geometric complex. This is done by marking a point on the knot we start with — the special circle will be the circle with the mark on it. Marking the knot is not a new procedure in knot invariants theory and appears in many parts of quantum invariants theory. As far as this part of knot theory is concerned the marking of the knot has no effect on theory. One can look at this process as marking a point on the knot for cutting it open to a 1–1 tangle (or a "long knot") — the theories of knots and long knots (1–1 tangles) are "isomorphic" in our context. Once we have a 1–1 tangle, there is always a special line appearing naturally in the complex. When we deal with links, one might choose different components of the link to place the mark but the choice does not matter and gives isomorphic complexes (they might be presented differently though). This is obvious from the fact that the complex reduction is local, and thus in every appearance of an object of $\mathcal{C}ob_{/l}$ in the complex one can choose the special circle *independently* and apply the complex isomorphisms. Different choices are linked through a series of complex isomorphisms.

7.3 Comments on computations

As was shown in [4], for the case of the original standard Khovanov homology, fast computations can make one happy! At first sight the complex isomorphism presented above does not seem to reduce the geometric complex at all (it doubles the amount of objects in the complex). The surprising thing is that one can use this isomorphism (and some homology reduction technics) to create a very efficient crossing-by-crossing local algorithm to calculate the geometric complex and link homology. This was done for the standard Khovanov homology (high genera are set to zero) as reported in [4; 3], and more recently done for the universal case over $\mathbb{Z}[H]$, as reported in [2] (J Green implementing the algorithm by D Bar-Natan based on the work of the author presented here). It is important to mention that this is currently the fastest program to calculate Khovanov homology (and the complex itself), and that the isomorphism of the geometric complex (the delooping process) is a crucial component of it [3]. Computations of the universal complex (and related issues, like various promotions of that complex) will be treated in a future work.

7.4 Comments on tautological functors

In [5, Section 9] tautological functors where defined on the geometric complex associated to a link. One needs to fix an object in $\mathcal{C}ob_{/l}$, say $\mathcal{O}'$, and then the tautological functor is defined by $\mathcal{F}_{\mathcal{O}'}(\mathcal{O}) = \mathrm{Mor}(\mathcal{O}', \mathcal{O})$, taking morphisms to compositions of morphisms. Our classification allows us to state the following regarding tautological functors over the geometric complex:

Corollary 7.1 *The tautological functor $\mathcal{F}_{\bigcirc}(-)$ over $\mathbb{Z}[H]$ is the TQFT $\mathcal{F}_H$.*

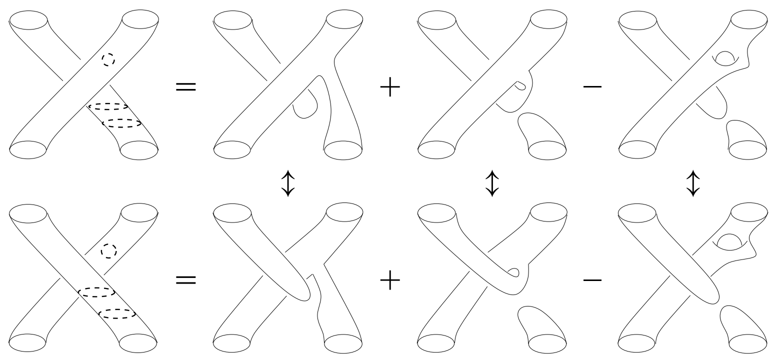
The tautological functor $\mathcal{F}_{\varnothing}(-)$ over $\mathbb{Q}[T]$ is the TQFT $\mathcal{F}_T$.

Proof Indeed this is a corollary to the surface classification. In the first case declare the special circle to be the source $\bigcirc$. □

Since every TQFT (after twisting) is factorized through $\mathcal{F}_H/\mathcal{F}_T$, tautological functors hold all the information one can get. Moreover, since the information held in the geometric complex is manifested in the theories $\mathcal{F}_H/\mathcal{F}_T$, it seems that asking about homology theories which are not tautological is not important. Also it seems, that there is no need in asking about non-tensorial functors (ie, functors for which $\mathcal{F}(\bigcirc\bigcirc) \ncong \mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc)$). For example, the tautological functor $\mathcal{F}_{\bigcirc^n}(-)$ is equivalent to $\mathcal{F}_{\bigcirc}(-) \otimes \mathcal{F}_{\bigcirc}(\bigcirc)^{\otimes n-1}$ and thus holds the same information as the the ones in the corollary, which can be considered as universal for that sake. The question whether every functor on $\mathcal{C}ob_{/l}$ can be represented as a tautological functor seems also less important due to the above.

7.5 Comments on embedded versus abstract surfaces

As noticed in [5, Section 11], when one looks at surfaces in $\mathcal{C}ob_{/l}$ over a ground ring with the number 2 invertible there is no difference between working with embedded surfaces (inside a cylinder say) or with abstract surfaces. This is true due to the fact that any knotting of the surface can be undone by cutting necks and pulling tubes to unknot the surface. In other words, by cutting and gluing back, using the NC relation (divided by 2) both ways, one can go from any knotted surface to the unknotted version of it embedded in 3 dimensional space. Our claim is that the same is true even when 2 is not invertible. The proof is a similar argument applied to any knotted surface using the $3S1$ relation:



The above picture shows that every crossing (a part of a knotted surface embedded in 3 dimensions) can be flipped using the $3S1$ relation twice. Apply the $3S1$ relation once on the dashed sites (going from top left), then smoothly change the surface (going down the arrows) and finally use the dashed sites for another application of the $3S1$ relation (reaching the bottom left). Every embedded surface can be unknotted this way, justifying previous comments about ignoring the issues of embedding (Section 2).

References

[1] **L Abrams**, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications 5 (1996) 569–587 MR1414088

[2] **D Bar-Natan**, *4TU-2:* a talk given at UQAM, Montreal (October 2005) Available at `http://www.math.toronto.edu/~drorbn/Talks/`

[3] **D Bar-Natan**, *Fast Khovanov Homology Computations* `arXiv:math.GT/0606318`

[4] **D Bar-Natan**, *I've Computed Kh(T(9,5)) and I'm Happy:* a talk given at George Washington University (February 2005) Available at `http://www.math.toronto.edu/~drorbn/Talks/`

[5] **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005) 1443–1499 MR2174270

[6] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR1740682

[7] **M Khovanov**, *Patterns in knot cohomology. I*, Experiment. Math. 12 (2003) 365–374 MR2034399

[8] **M Khovanov**, *Link homology and Frobenius extensions*, Fund. Math. 190 (2006) 179–190 MR2232858

[9] **E S Lee**, *On Khovanov invariant for alternating links* arXiv:math.GT/0210213

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