Twisted Alexander polynomials detect the unknot

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The group of a nontrivial knot admits a finite permutation representation such that the corresponding twisted Alexander polynomial is not a unit.

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1 Introduction

Twisted Alexander polynomials of knots in $\mathbb{S}^3$ were introduced by X-S Lin in [7]. They were defined more generally for any finitely presentable group with infinite abelianization by M Wada [11]. Many papers subsequently appeared on the topic. Notable among them is [5], by P Kirk and C Livingston, placing twisted Alexander polynomials of knots in the classical context of abelian invariants. A slightly more general approach by J Cha [1] permits coefficients in a Noetherian unique factorization domain.

In Hillman–Livingston–Naik [4] two examples are given of Alexander polynomial 1 hyperbolic knots for which twisted Alexander polynomials provide periodicity obstructions. In each case, a finite representation of the knot group is used to obtain a nontrivial twisted polynomial. Such examples motivate the question: Does the group of any nontrivial knot admit a finite representation such that the resulting twisted Alexander polynomial is not a unit (that is, not equal to $\pm t^i$)?

**Theorem** Let $k \subset \mathbb{S}^3$ be a nontrivial knot. There exists a finite permutation representation such that the corresponding twisted Alexander polynomial $\Delta_p(t)$ is not a unit.

A key ingredient of the proof of the theorem is a recent theorem of M Lackenby [6] which implies that some cyclic cover of $\mathbb{S}^3$ branched over $k$ has a fundamental group with arbitrarily large finite quotients. The quotient map pulls back to a representation of the knot group. A result of J Milnor [8] allows us to conclude that for sufficiently large quotients, the associated twisted Alexander polynomial is nontrivial.

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2 Preliminary material

2.1 Review of twisted Alexander polynomials

Let \( X \) be a finite CW complex. Its fundamental group \( \pi = \pi_1 X \) acts on the left of the universal cover \( \tilde{X} \) by covering transformations.

Assume that \( \epsilon \) is an epimorphism from \( \pi \) to an infinite cyclic group \( \langle t \rangle \). Given a Noetherian unique factorization domain \( R \), we identify the group ring \( R[\pi] \) with the ring of Laurent polynomials \( \Lambda = R[t, t^{-1}] \). (Here we will be concerned only with the case \( R = \mathbb{Z} \).

Assume further that \( \pi \) acts on the right of a free \( R \)-module \( V \) of finite rank via a representation \( \rho: \pi \to GL(V) \). Define a \( \Lambda - R[\pi] \) bimodule structure on \( \Lambda \otimes_R V \) by \( t^j (t^n \otimes v) = t^{n+j} \otimes v \) and \( (t^n \otimes v) g = t^{n+\epsilon(g)} \otimes v \rho(g) \) for \( v \in V \) and \( g \in \pi \). The groups of the cellular chain complex \( C_* (\tilde{X}; R) \) are left \( R[\pi] \)-modules. The twisted complex of \( X \) is defined to be the chain complex of left \( \Lambda \)-modules:

\[
C_* (X; \Lambda \otimes V) = (\Lambda \otimes V) \otimes_{R[\pi]} C_* (\tilde{X}; R).
\]

The twisted homology \( H_* (X; \Lambda \otimes V) \) is the homology of \( C_* (X; \Lambda \otimes V) \).

Since \( V \) is finitely generated and \( R \) is Noetherian, \( H_* (X; \Lambda \otimes V) \) is a finitely presentable \( \Lambda \)-module. Elementary ideals and characteristic polynomials are defined in the usual way. Begin with an \( n \times m \) presentation matrix corresponding to a presentation for \( H_1 (X; \Lambda \otimes V) \) with \( n \) generators and \( m \geq n \) relators. The ideal in \( \Lambda \) generated by the \( n \times n \) minors is an invariant of \( H_1 (X; \Lambda \otimes V) \). The greatest common divisor of the minors, the twisted Alexander polynomial of \( X \), is an invariant as well. It is well defined up to a unit of \( \Lambda \). Additional details can be found in [1]. An alternative, group-theoretical approach can be found in [9].

In what follows, \( X \) will denote the exterior of a nontrivial knot \( k \), that is, the closure of \( S^3 \) minus a regular neighborhood of \( k \).

2.2 Periodic representations

The knot group \( \pi \) is a semidirect product \( \langle x \rangle \rtimes \pi' \), where \( x \) is a meridional generator and \( \pi' \) denotes the commutator subgroup \( [\pi, \pi] \). Every element has a unique expression of the form \( x^j w \), where \( j \in \mathbb{Z} \) and \( w \in \pi' \).
For any positive integer $r$, the fundamental group of the $r$–fold cyclic cover $X_r$ of $X$ is isomorphic to $\langle x^r \mid \rangle \ltimes \pi'$. The fundamental group of the $r$–fold cyclic cover $M_r$ of $\Sigma^3$ branched over $k$ is the quotient group $\pi_1(X_r)/\langle \langle x^r \rangle \rangle$, where $\langle \langle \cdot \rangle \rangle$ denotes the normal closure. Consequently, $\pi_1 M_r = \pi'/[\pi', x^r]$.

**Definition 2.1** A representation $p: \pi' \to \Sigma$ is periodic with period $r$ if it factors through $\pi_1 M_r$. If $r_0$ is the smallest such positive number, then $p$ has least period $r_0$.

**Remark 2.2** The condition that $p$ factors through $\pi_1 M_r$ is equivalent to the condition that $p(x^{-r} ax^r) = p(a)$ for every $a \in \pi'$.

The following is a consequence of the fact that $M_1$ is $\Sigma^3$.

**Proposition 2.3** If $p: \pi' \to \Sigma$ has period 1, then $p$ is trivial.

Assume that $p: \pi' \to \Sigma$ is surjective and has least period $r_0$. We extend $p$ to a homomorphism $P: \pi \to \langle \xi \mid \xi^{r_0} \rangle \ltimes \Sigma^{r_0}$, mapping $x \mapsto \xi$ and elements $u \in \pi'$ to $(p(u), p(x^{-1}ux), \ldots, p(x^{-(r_0-1)}ux^{r_0-1})) \in \Sigma^{r_0}$. Conjugation by $\xi$ in the semidirect product induces $\theta: \Sigma^{r_0} \to \Sigma^{r_0}$ described by $(\alpha_1, \ldots, \alpha_{r_0}) \mapsto (\alpha_2, \ldots, \alpha_{r_0}, \alpha_1)$. The lemma below assures us that the image of $\pi'$ under $P$ has order no less than the order of $p(\pi')$.

**Lemma 2.4** $|P(\pi')| \geq |p(\pi')|$

**Proof** The image $P(\pi')$ is contained in $\Sigma^{r_0}$. First coordinate projection $\Sigma^{r_0} \to \Sigma$ obviously maps $P(\pi')$ onto $p(\pi')$. \(\square\)

In what follows we will assume that $\Sigma$ is finite. Hence $P(\pi)$ is also finite, and it is isomorphic to a group of permutations of a finite set acting transitively (that is, the the orbit of any element under $P(\pi)$ is the entire set.) We can ensure that the subgroup $P(\pi')$ also acts transitively, as the next lemma shows.

We denote the symmetric group on a set $\mathcal{A}$ by $S_{\mathcal{A}}$.

**Lemma 2.5** The group $P(\pi)$ embeds in the symmetric group $S_{P(\pi')}$ in such a way that $P(\pi')$ acts transitively.
Throughout this section, \( P(\pi') \) in \( S_{P(\pi')} \) via the right regular representation \( \psi: P(\pi') \rightarrow S_{P(\pi')} \). Given \( \beta = (\beta_1, \ldots, \beta_{r_0}) \in P(\pi') \), the permutation \( \psi(\beta) \) maps \( (\alpha_1, \ldots, \alpha_{r_0}) \in P(\pi') \) to \( (\alpha_1 \beta_1, \ldots, \alpha_{r_0} \beta_{r_0}) \). Extend \( \psi \) to \( \Psi: P(\pi) \rightarrow S_{P(\pi')} \) by assigning to \( \xi \) the permutation of \( P(\tau') \) given by \( (\alpha_1, \ldots, \alpha_{r_0}) \Psi(\xi) = (\alpha_2, \ldots, \alpha_{r_0}, \alpha_1) \). It is straightforward to check that \( \Psi \) respects the action \( \theta \) of the semidirect product, and hence is a well-defined homomorphism.

To see that \( \Psi \) is faithful, suppose that \( \Psi(\xi^i \beta) \) is trivial for some \( 1 \leq i < r_0, \beta \in P(\pi') \). Then \( \Psi(\xi^i) = \psi(\beta^{-1}) \). By considering the effect of the permutation on \( 1 = (1, \ldots, 1) \), we find that \( \beta \) must be \( 1 \) and hence the action of \( \Psi(\xi^i) \) is trivial. It follows that \( p \) has period \( i < r_0 \), contradicting the assumption that \( r_0 \) is the least period. \( \square \)

We summarize the above construction.

**Lemma 2.6** Given a finite representation \( p: \pi' \rightarrow \Sigma \) of period \( r \), there is a finite permutation representation \( P: \pi_1 X \rightarrow S_N \) such that \( P|_{\pi'} \) is \( r \)-periodic and transitive. Moreover, \( |P(\pi')| = N \geq |p(\pi')| \).

### 2.3 Twisted Alexander polynomials induced by periodic representations

Throughout this section, \( P: \pi \rightarrow S_N \) is assumed to be a permutation representation induced by a finite representation \( p: \pi' \rightarrow \Sigma \) of period \( r \), as in Lemma 2.6.

The representation \( P \) induces an action of \( \pi \) on the standard basis \( B = \{e_1, \ldots, e_N\} \) for \( V = \mathbb{Z}^N \). We obtain a representation \( \rho: \pi \rightarrow GL(V) \). Let \( \epsilon \) be the abelianization homomorphism \( \pi \rightarrow \langle t \mid \rangle \) mapping \( x \mapsto t \). A twisted chain complex \( C_*(X; \Lambda \otimes V) \) is defined as in Section 2.1.

The free \( \mathbb{Z}[\pi] \)-complex \( C_*(\tilde{X}) \) has a basis \( \{\tilde{z}\} \) consisting of a single lift of each cell \( z \) in \( X \). Then \( \{1 \otimes e_i \otimes \tilde{z}\} \) is a basis for the free \( \Lambda \)-complex \( C_*(X; \Lambda \otimes V) \) (cf page 640 of [5]).

We will use the following lemma from [10].

**Lemma 2.7** Suppose that \( A \) is a finitely generated \( \mathbb{Z}[t^{\pm 1}] \)-module admitting a square presentation matrix and has 0th characteristic polynomial \( \Delta(t) = c_0 \prod(t - \alpha_j) \). Let \( s = t^r \), for some positive integer \( r \). Then the 0th characteristic polynomial of \( A \), regarded as a \( \mathbb{Z}[s^{\pm 1}] \)-module, is \( \Delta(s) = c_0^r \prod(s - \alpha_j^r) \).

The map \( P: \pi \rightarrow S_N \) restricts to a representation of the fundamental group \( \pi' \) of the universal abelian cover \( X_\infty \). Let \( \widetilde{X}_\infty \) denote the induced \( N \)-fold cover. The \( \Lambda \)-modules \( H_1(\widetilde{X}_\infty) \) and \( H_1(X; \Lambda \otimes V) \) are isomorphic by two applications of Shapiro’s Lemma (see for example [4]).
Proposition 2.8 \( H_1(\hat{X}_\infty) \) is a finitely generated \( \mathbb{Z}[s^{\pm 1}] \)–module with a square presentation matrix, where \( s = t^r \).

**Proof** Construct \( X_\infty \) in the standard way, splitting \( X \) along the interior of Seifert surface \( S \) to obtain a relative cobordism \( (V; S', S'') \) bounding two copies \( S', S'' \) of \( S \). Then \( X_\infty \) is obtained by gluing countably many copies \( (V_j; S'_j, S''_j) \) end-to-end, identifying \( S''_j \) with \( S'_{j+1} \), for each \( j \in \mathbb{Z} \).

For each \( j \), let \( W_j = V_{jr} \cup \cdots \cup V_{jr+r-1} \) be the submanifold of \( X_\infty \) bounding \( S'_j \) and \( S''_{jr+r-1} \). Then \( X_\infty \) is the union of the \( W_j \)'s, glued end-to-end. After lifting powers of the meridian of \( k \), thereby constructing basepaths from \( S'_0 \) to each \( S'_{jr} \subset W_j \), we can then regard each \( \pi_1 W_j \) as a subgroup of \( \pi_1 X_\infty \cong \pi' \).

Conjugation by \( x \) in the knot group induces an automorphism of \( \pi' \), and the \( r \)th power maps \( \pi_1 W_j \) isomorphically to \( \pi_1 W_{j+1} \). Since \( p \) has period \( r \), we have \( p(x^{-t}ux^t) = p(u) \) for all \( u \in \pi' \). Hence \( P \) has the same image on each \( \pi_1 W_j \). By performing equivariant ambient 0–surgery in \( W_j \) to the lifted surfaces \( \hat{S}'_j \) (that is, adding appropriate hollow 1–handles to the surface), we can assume that the image of \( P(\pi_1 S'_j) \) acts transitively, and hence each preimage \( \hat{S}'_j \subset \hat{X}_\infty \) is connected.

The covering space \( \hat{X}_\infty \) is the union of countably many copies \( \hat{W}_j \) of the lift \( \hat{W}_0 \) glued end-to-end. The cobordism \( \hat{W}_0 \), which bounds two copies \( \hat{S}' \), \( \hat{S}'' \) of the surface \( \hat{S} \), can be constructed from \( \hat{S}' \times I \) by attaching 1– and 2–handles in equal numbers. Consequently, \( H_1 \hat{W}_0 \) is a finitely generated abelian group with a presentation of deficiency \( d \) (number of generators minus number of relators) equal to the rank of \( H_1 \hat{S}' \).

The \( r \)th powers of covering transformations of \( \hat{X}_\infty \) induce a \( \mathbb{Z}[s^{\pm 1}] \)–module structure on \( H_1 \hat{X}_\infty \). The Mayer–Vietoris theorem implies that the generators of \( H_1 \hat{W}_0 \) serve as generators for the module. Moreover, the relations of \( H_1 \hat{W}_0 \) together with \( d \) relations arising from the boundary identifications become an equal number of relations.

**Corollary 2.9** If \( \Delta_\rho(t) = 1 \), then \( H_1(\hat{X}_\infty) \) is trivial.

**Proof** Let \( s = t^r \), and regard \( H_1(\hat{X}_\infty) \) as a \( \mathbb{Z}[s^{\pm 1}] \)–module. Since the module has a square presentation matrix, its order ideal is principal, generated by \( \tilde{\Delta}_\rho(s) \). Lemma 2.7 implies that \( \tilde{\Delta}_\rho(s) = 1 \). Hence the order ideal coincides with the coefficient ring \( \mathbb{Z}[s, s^{-1}] \). However, the order ideal is contained in the annihilator of the module (see [2] or Theorem 3.1 of [3]). Thus \( H_1(\hat{X}_\infty) \) is trivial.

Since \( p \) factors through \( \pi_1 M_r \), so does \( P|_{\pi'} \). Let \( \hat{M}_r \) denote the corresponding \( N \)–fold cover.
Lemma 2.10 \( H_1 \hat{M}_r \) is a quotient of \( H_1 \hat{X}_\infty / (t^r - 1) H_1 \hat{X}_\infty \).

Proof Recall that \( \pi_1 M_r \cong \pi'/[\pi', x^r] \). Thus \( \pi_1 \hat{M}_r \cong \ker(P|_{\pi'})/[\pi', x^r] \), and by the Hurewicz theorem,
\[
H_1 \hat{M}_r \cong \ker(P|_{\pi'}) / \ker(P|_{\pi'})' \cdot [\pi', x^r].
\]
On the other hand, \( \pi_1 \hat{X}_\infty \) modulo the relations \( x^{-r} g x^r = g \) for all \( g \in \pi_1 \hat{X}_\infty \) is isomorphic to \( \ker(P|_{\pi'})/[\ker(P|_{\pi'}), x^r] \). Using the Hurewicz theorem again,
\[
H_1 \hat{X}_\infty / (t^r - 1) H_1 \hat{X}_\infty \cong \ker(P|_{\pi'}) / \ker(P|_{\pi'})' \cdot [\ker(P|_{\pi'}), x^r].
\]
The conclusion follows immediately.

Example 2.11 The group \( \pi \) of the trefoil has presentation \( \langle x, a | ax^2 a = xax \rangle \), where \( x \) represents a meridian, and \( a \) is in the commutator subgroup \( \pi' \). The Reidemeister–Schreier method yields the presentation
\[
\pi' = \langle a_j | a_j a_{j+2} = a_{j+1} \rangle,
\]
where \( a_j = x^{-j} ax^j \). Consider the homomorphism \( p: \pi' \to \Sigma = \langle \alpha | \alpha^3 \rangle \cong \mathbb{Z}_3 \) sending \( a_{2j} \mapsto \alpha \) and \( a_{2j+1} \mapsto \alpha^2 \). We extend \( p \) to \( P: \pi \to \hat{\Sigma} = \langle \xi | \xi^2 \rangle \ltimes \Sigma^2 \), sending \( x \mapsto \xi \). The image \( P(\pi') \) consists of the three elements \((1, 1), (\alpha, \alpha^2), (\alpha^2, \alpha)\); the image of \( \pi \) is isomorphic to the dihedral group \( D_3 \), which we regard as a subgroup of \( GL_3(\mathbb{Z}) \). Hence we have a representation \( \rho: \pi \to GL_3(\mathbb{Z}) \). Let \( \epsilon: \pi \to \langle t^\pm 1 \rangle \) be the abelianization homomorphism mapping \( x \mapsto t \). The product of \( \rho \) and \( \epsilon \) determines a tensor representation \( \rho \otimes \epsilon: \pi \to GL_3(\mathbb{Z}[t^\pm 1]) \) defined by \((\rho \otimes \epsilon)(g) = \rho(g)\epsilon(g)\), for \( g \in \pi \). We order our basis so that:
\[
(\rho \otimes \epsilon)(x) = \begin{pmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & t \end{pmatrix}, \quad (\rho \otimes \epsilon)(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]
We can assume that the CW structure on \( X \) contains a single 0–cell \( p \), 1–cells \( x, a \) and a single 2–cell \( r \).

The \( \rho \)–twisted cellular chain complex \( C_*(X; \Lambda \otimes V) \) has the form
\[
0 \to C_2 \cong \Lambda^3 \xrightarrow{\partial_2} C_1 \cong \Lambda^6 \xrightarrow{\partial_1} C_0 \cong \Lambda^3 \to 0.
\]
If we treat elements of \( \Lambda^3 \) and \( \Lambda^6 \) as row vectors, then the map \( \partial_2 \) is described by a \( 3 \times 6 \) matrix obtained in the usual way from the \( 1 \times 2 \) matrix of Fox free derivatives:
\[
\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial a} \end{pmatrix} = \begin{pmatrix} a + ax - 1 - xa & 1 + ax^2 - x \end{pmatrix}
\]
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replacing $x, a$ respectively with their images under $\rho \otimes \epsilon$. The result is:

$$\partial_2 = \begin{pmatrix} t-1 & 1 & -t & 1 & t^2-t & 0 \\ 0 & -t-1 & t+1 & -t & 1 & t^2 \\ 1-t & t & -t & 1 & 0 & 1-t \end{pmatrix}$$

The map $\partial_1$ is determined by $(\rho \otimes \epsilon)(x) - I$ and $(\rho \otimes \epsilon)(a) - I$:

$$\partial_1 = \begin{pmatrix} -1 & t & 0 \\ t & -1 & 0 \\ 0 & 0 & t-1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Dropping the first three columns of the matrix for $\partial_2$ produces a $3 \times 3$ matrix:

$$A = \begin{pmatrix} 1 & t^2-t & 0 \\ -t & 1 & t^2 \\ t^2 & 0 & 1-t \end{pmatrix}$$

Similarly, eliminating the last three rows of $\partial_1$ gives:

$$B = \begin{pmatrix} -1 & t & 0 \\ t & -1 & 0 \\ 0 & 0 & t-1 \end{pmatrix}$$

Theorem 4.1 of [5] implies that $\Delta_\rho(t)/\Delta_0(t) = \text{Det } A/\text{Det } B$, where $\Delta_0(t)$ is the 0th characteristic polynomial of $H_0\tilde{X}_\infty$. Since $\tilde{X}_\infty$ is connected, $\Delta_0(t) = t - 1$. Hence $\Delta_\rho(t) = (t^2 - t + 1)(t^2 - 1)$.

In this example, the cyclic resultant $\text{Res}(\Delta_\rho(t), t^2 - 1)$ vanishes, indicating that $H_1\tilde{X}_2$ is infinite. A direct calculation reveals that in fact $H_1\tilde{X}_2 \cong \mathbb{Z} \oplus \mathbb{Z}$.

**Remark 2.12** In the above example we see that the Alexander polynomial of the trefoil knot divides the twisted Alexander polynomial. Generally, the Alexander polynomial divides any twisted Alexander polynomial arising from a finite permutation representation of the knot group. A standard argument using the transfer homomorphism and the fact that $H_1X_\infty$ has no $\mathbb{Z}$-torsion shows that $H_1X_\infty$ embeds as a submodule in $H_1\tilde{X}_\infty$. Hence $\Delta(t)$, which is the 0th characteristic polynomial of $H_1X_\infty$, divides $\Delta_\rho(t)$, the 0th characteristic polynomial of $H_1\tilde{X}_\infty$. 

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3 Proof of the Theorem

Alexander polynomials are a special case of twisted Alexander polynomials corresponding to the trivial representation. Hence it suffices to consider an arbitrary nontrivial knot \( k \) with unit Alexander polynomial \( \Delta(t) \).

A complete list of those finite groups that can act freely on a homology 3–sphere is given in \([8]\). The only nontrivial such group that is perfect (that is, has trivial abelianization) is the binary icosahedral group \( A_5^* \), with order 120.

Since \( \Delta(t) \) annihilates \( H_1 X_\infty \), the condition that \( \Delta(t) = 1 \) implies that \( H_1 X_\infty \) is trivial or equivalently, that \( \pi' \) is perfect. Hence each branched cover \( M_r \) has perfect fundamental group and so is a homology sphere. Theorem 3.7 of \([6]\) implies that for some integer \( r > 2 \), the group \( \pi_1 M_r \) is “large” in the sense that it contains a finite-index subgroup with a free nonabelian quotient.

Any large group has normal subgroups of arbitrarily large finite index. Hence \( \pi_1 M_r \) contains a normal subgroup \( Q \) of index \( N_0 \) exceeding 120. Composing the canonical projection \( \pi' \to \pi_1 M_r \) with the quotient map \( \pi_1 M_r \to \pi_1 M_r / Q = \Sigma \), we obtain a surjective homomorphism \( p: \pi' \to \Sigma \) of least period \( r_0 \) dividing \( r \). By Proposition 2.3, we have \( r_0 > 1 \). Let \( P \) be the extension to \( \pi \), as in Lemma 2.6. By that lemma, the order \( N \) of \( P(\pi') \) is no less than \( N_0 = |p(\pi')| \).

As in section 2, realize \( P(\pi) \) as a group of permutation matrices in \( GL_N(\mathbb{Z}) \) acting transitively on the standard basis of \( \mathbb{Z}^N \). Let \( \tilde{M}_{r_0} \) be the cover of \( M_{r_0} \) induced by the representation \( P: \pi \to S_N \). The group of covering transformations acts freely on \( \tilde{M}_{r_0} \) and transitively on any point-preimage of the projection \( \tilde{M}_{r_0} \to M_{r_0} \). Its cardinality is equal \( N \) and so cannot be the binary icosahedral group. Hence \( \tilde{M}_{r_0} \) has nontrivial homology.

Lemma 2.10 and Corollary 2.9 imply that \( \Delta_\rho(t) \neq 1 \). \( \square \)

References


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