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Conjugacy of 2–spherical subgroups of Coxeter groups and parallel walls

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Let (W, S) be a Coxeter system of finite rank (ie $|S|$ is finite) and let $\mathfrak{A}$ be the associated Coxeter (or Davis) complex. We study chains of pairwise parallel walls in $\mathfrak{A}$ using Tits' bilinear form associated to the standard root system of (W, S). As an application, we prove the strong parallel wall conjecture of G Niblo and L Reeves [18]. This allows to prove finiteness of the number of conjugacy classes of certain one-ended subgroups of W, which yields in turn the determination of all co-Hopfian Coxeter groups of 2–spherical type.

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1 Introduction

1.1 Conjugacy of 2–spherical subgroups

A group Γ is called 2*–spherical* if it possesses a finite generating set T such that any pair of elements of T generates a finite subgroup. By Serre [21, Section 6.5, Corollaire 2], a 2–spherical group enjoys property (FA); in particular, it follows from Stalling's theorem that it is one-ended.

In the literature, a Coxeter group W is called 2–spherical if it has a Coxeter generating set S with the property that any pair of elements of S generates a finite subgroup. If W has a Coxeter generating set S such that some pair of elements of S generates an infinite subgroup, then it is easy to see that W splits non-trivially as an amalgamated product of standard parabolic subgroups, and hence W does not have Serre's property (FA). This shows that for Coxeter groups, the usual notion of 2–sphericity coincides with the notion introduced above.

A theorem stated by M Gromov [12, 5.3, C$'$] and proved by E Rips and Z Sela [19, Theorem 7.1] asserts that a hyperbolic group contains only finitely many conjugacy classes of subgroups isomorphic to a given finitely generated torsion-free one-ended group Λ. Another related result, stated in [12] and proved by T Delzant [7], is that a torsion-free hyperbolic group Γ has finitely many conjugacy classes of one-ended

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two-generated subgroups. Furthermore, it was shown by I Kapovich and R Weidmann [15, Corollary 1.5] that if Γ is locally quasiconvex, then Γ has finitely many conjugacy classes of one-ended l–generated subgroups for each $l \geq 1$. Results of this kind corroborate somehow the abundance of free subgroups in hyperbolic groups. Comparing Coxeter groups, the abundance of free subgroups is also an established fact (more precisely: Coxeter groups are either virtually abelian or virtually map onto non-abelian free groups by Margulis and Vinberg [17, Corollary 2]) and it is desirable to complement this fact with a precise finiteness property. In this direction, we obtain the following:

Theorem A

(i) *W contains finitely many conjugacy classes of 2–spherical subgroups if and only if W has no parabolic subgroup of irreducible affine type and rank ≥ 3.*

(ii) *W contains finitely many conjugacy classes of 2–spherical reflection subgroups Γ such that Γ has no nontrivial free abelian normal subgroup (equivalently: Γ has no direct component of affine type).*

(iii) *W contains finitely many conjugacy classes of 2–spherical subgroups Γ such that Γ has no nontrivial free abelian normal subgroup and $\mathbb{Z}\text{-rk}(\Gamma) \geq \mathbb{Z}\text{-rk}(W) - 1$.*

(iv) *Assume that the centralizer of every irreducible affine parabolic subgroup of rank ≥ 3 is finite. Then W contains finitely many conjugacy classes of 2–spherical subgroups Γ such that Γ is not infinite virtually abelian.*

By a reflection subgroup, we mean a subgroup generated by reflections; see Section 2.5 below for more information on reflection subgroups. Here $\mathbb{Z}\text{-rk}(\Gamma)$ denotes the maximal rank of a free abelian subgroup of Γ. Note that the condition on the centralizer of an affine parabolic subgroup in (iv) is immediate to check in the Coxeter diagram of (W, S) (see Lemma 22 below): the condition holds whenever for each irreducible affine subdiagram D of rank ≥ 3, the subdiagram induced on the set $D^{\perp}$ consisting of those vertices which are not connected to D is of spherical type.

The existence of homotheties in Euclidean space implies that any Coxeter group of affine type is isomorphic to a proper reflection subgroup. From this fact, it is easy to deduce that if W has a reflection subgroup W_0 of affine type, then W has infinitely many conjugacy classes of reflection subgroups of the same type as W_0. This explains why the number of conjugacy classes of 2–spherical subgroups of W necessarily depends on affine parabolic subgroups of W, as it appears in Theorem A. However, affine parabolic subgroups are not the only source of free abelian subgroups in Coxeter groups; in particular, many non-hyperbolic Coxeter groups possess no parabolic subgroup of irreducible affine type and rank ≥ 3.

A group which is not isomorphic to any of its proper subgroups (resp. quotients) is called *co-Hopfian* (resp. *Hopfian*). By a theorem of Mal'cev, any finitely generated residually finite group is Hopfian; hence Coxeter groups are Hopfian. By Rips and Zela [19, Theorem 3.1], rigid hyperbolic groups are co-Hopfian. We have just seen that affine Coxeter groups are not co-Hopfian. As a consequence of Theorem A, one obtains the following:

Corollary B *Suppose that W is 2–spherical. Then W is co-Hopfian if and only if W has no nontrivial free abelian normal subgroup or, equivalently, if the Coxeter diagram of (W, S) has no connected component of affine type.*

The proof of Theorem A makes an essential use of the $\mathsf{CAT}(0)$ cube complex $\mathfrak{X}$ constructed by G Niblo and L Reeves [18]. More precisely, one associates a cube of $\mathfrak{X}$ to every 2–spherical subgroup of W in such a way that the problem of counting conjugacy classes of 2–spherical subgroups in W becomes a matter of counting orbits of cubes in $\mathfrak{X}$. In particular, if W acts co-compactly on $\mathfrak{X}$, then there are finitely many orbits of cubes and this implies immediately that W possesses finitely many conjugacy classes of 2–spherical subgroups. However, it is known that the W–action on $\mathfrak{X}$ is not always co-compact: as shown in Caprace and Mühlherr [5], it is co-compact if and only if W has no parabolic subgroup of irreducible affine type and rank ≥ 3. In fact, Theorem A(i) can be viewed as a significant generalization of [5, Theorem 1.1], but the somewhat lengthy case-by-case discussions of [5] are here completely avoided. The problem of counting orbits of cubes in the general case (ie when $\mathfrak{X}/W$ is not compact) is settled here by Theorem 27 below, whose proof leads naturally to consider nested sequences of half-spaces of $\mathfrak{A}$. Such sequences are submitted to a strong alternative described in Theorem D below.

1.2 Separation of parallel walls

A well-known result on Coxeter groups is the so-called *parallel wall theorem*: it asserts that any point of the Davis complex, which is sufficiently far apart from a given wall, is separated from this wall by another wall. This was first proved by B Brink and R Howlett [3, Theorem 2.8]. Here, we obtain the following:

Theorem C *There exists a constant $Q = Q(W, S)$, depending only on the Coxeter system (W, S), such that the following holds. Given two walls μ_1, μ_2 of $\mathfrak{A}$ such that the distance from μ_1 to μ_2 is at least Q, there exists a wall m which separates μ_1 from μ_2.*

This was stated in [18] as the *strong parallel wall conjecture*. As noted by G Niblo and L Reeves, it implies the existence of a universal bound on the size of the link of a vertex in the $\mathsf{CAT}(0)$ cube complex $\mathfrak{X}$ constructed in [loc. cit.]. In fact, the latter corollary can be viewed as an immediate consequence of the more general fact, due to F Haglund and D Wise [13], that the cube complex of any finitely generated Coxeter group embeds virtually equivariantly in the Davis complex of a right-angled Coxeter group; the proof of this result relies in an essential way on Theorem C.

The proof of this theorem can be outlined as follows. Any two points of $\mathfrak{A}$ which are far apart, are separated by a large set of pairwise parallel walls. In particular, two parallel walls which are far apart yield a large set of pairwise parallel walls. Each of these walls is a separator-candidate. If it turns out that none of these candidates indeed separates the original pair of walls, then one constructs a configuration of nested triangles of walls. The existence of such a configuration is severely restricted by Theorem 8 below.

1.3 Chains of roots

As mentioned in the preceding subsections, a common feature of the proofs of Theorem A and Theorem C is that they both lead one to consider at some point sequences of pairwise parallel walls. In order to study these sequences, we use the standard root system $\Phi = \Phi(W, S)$. Recall that the set of roots Φ is a discrete subset of some finite-dimensional real vector space V, which is endowed with a W–action leaving Φ invariant. This action also preserves a bilinear form $(\cdot, \cdot)$, called *Tits bilinear form* (see Bourbaki [2] and Section 2.1 below).

This root system is a useful supplement of the Davis complex $\mathfrak{A}$, which is a complete $\mathsf{CAT}(0)$–space [6], on which W acts properly discontinuously and co-compactly. This complex is a thickened version of the Cayley graph of (W, S); in fact this Cayley graph is nothing but the 1–skeleton of $\mathfrak{A}$. Walls of $\mathfrak{A}$ are fixed point sets of reflections of W. Every wall cuts $\mathfrak{A}$ into two open convex subsets, the closure of which are called half-spaces. The set of all half-spaces (resp. walls) is denoted by $\Phi(\mathfrak{A})$ (resp. $\mathfrak{M}(\mathfrak{A})$).

Although different in nature, the root system and the Davis complex are closely related as follows. Once the identity $1 \in W$ has been chosen as a base vertex of $\mathfrak{A}$, one has a canonical W–equivariant bijection $\Phi \to \Phi(\mathfrak{A})$, which maps Π to the set of those half-spaces containing the vertex 1 but not its neighbors (see Lemma 3 below for more details). For the rest of this introduction, we identify Φ and $\Phi(\mathfrak{A})$ by means of this bijection; thus the words 'root' and 'half-space' become synonyms.

Theorem D *There exists a non-decreasing sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers, tending to $+\infty$ with n, such that $r_1 > 1$, which depends only on the Coxeter system*

(W,S) and such that the following property is satisfied. Given any chain of half-spaces $\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_n$ with $n>0$, exactly one of the following alternatives holds:

(1) $(\alpha_0,\alpha_n) \geq r_n$.

(2) $(\alpha_0,\alpha_n)=1$, *the group $\langle r_{\alpha_i} \mid i=0,\dots,n\rangle$ is infinite dihedral and it is contained in a parabolic subgroup of irreducible affine type of W.*

Theorem D may be viewed as summing up the technical heart of this paper. We will first establish a weak version of this theorem (Proposition 7), which is sufficient to obtain Theorem C. The latter result is then used to deduce Theorem D in its full strength (Section 6). This in turn is an essential tool in the proof of Theorem A.

2 Preliminaries

2.1 Root bases

A *root basis* is a triple $\mathfrak{E}=(V,(\cdot,\cdot),\Pi)$ consisting of a real vector space V, a symmetric bilinear form $(\cdot,\cdot)$ on V and a set $\Pi \subset V$ which satisfies the following conditions:

(RB1) For all $\alpha \in \Pi$, one has $(\alpha,\alpha)=1$.

(RB2) For all $\alpha \neq \beta \in \Pi$, one has

$$(\alpha,\beta) \in \{-\cos(\frac{\pi}{m}) \mid m \in \mathbb{Z}_{\geq 2}\} \cup (-\infty,-1].$$

(RB3) There exists a linear form $x \in V^*$ such that $x(\alpha)>0$ for all $\alpha \in \Pi$.

The most important example is the *standard root basis* of a Coxeter matrix $M=(m_{ij})_{i,j\in I}$. Recall from [2] that this root basis $\mathfrak{E}_M := (V_M,(\cdot,\cdot)_M,\Pi_M)$ is constructed as follows: set $V_M := \bigoplus_{i\in I}\mathbb{R}e_i$, $\Pi_M := \{e_i \mid i \in I\}$ and for all $i,j \in I$, set $(e_i,e_j)_M := -\cos(\frac{\pi}{m_{ij}})$ $(=-1$ if $m_{ij}=\infty)$.

Let now $\mathfrak{E}=(V,(\cdot,\cdot),\Pi)$ be any root basis. For each $\alpha \in V$, define $r_\alpha \in \mathrm{GL}(V)$ by

$$r_\alpha : x \mapsto x-2(x,\alpha)\alpha.$$

We make the following definitions:

- $S(\mathfrak{E}) := \{r_\alpha \mid \alpha \in \Pi\}$,
- $W(\mathfrak{E}) := \langle S(\mathfrak{E})\rangle \subset \mathrm{GL}(V)$,
- $\Phi(\mathfrak{E}) := \{w.\alpha \mid w \in W(\mathfrak{E}),\ \alpha \in \Pi\}$,
- $\Phi(\mathfrak{E})^+ := \{\phi \in \Phi(\mathfrak{E}) \mid \phi \in \sum_{\pi\in\Pi}\mathbb{R}^+\pi\}$.

The elements of $\Phi(\mathfrak{S})$ are called *roots*; the roots contained in $\Phi(\mathfrak{S})^+$ are called *positive*. The following lemma collects the basic facts on root bases which we will need in the sequel:

Lemma 1

(i) *Given $\mathfrak{S} = (V, (\cdot,\cdot), \Pi)$ a root basis, the pair $(W(\mathfrak{S}), S(\mathfrak{S}))$ is a Coxeter system and $\Phi(\mathfrak{S})$ is a discrete subset of V.*

(ii) *Conversely, if $\mathfrak{S}$ is the standard root basis associated with a given a Coxeter system (W, S), then there is a canonical isomorphism $W \to W(\mathfrak{S})$ mapping S onto $S(\mathfrak{S})$.*

(iii) *For all $w \in W(\mathfrak{S})$ and all $\alpha \in \Phi(\mathfrak{S})$, one has $w r_\alpha w^{-1} = r_{w.\alpha}$.*

(iv) *For all $\phi \in \Phi(\mathfrak{S})$, either $\phi \in \Phi(\mathfrak{S})^+$ or $-\phi \in \Phi(\mathfrak{S})^+$.*

Proof See [2, Theorem IV.1.1]. The proofs given in [2] deal only with standard root bases, but they apply without modification to any root basis. □

The *type* of the root basis $\mathfrak{S}$ is the type of the Coxeter system $(W(\mathfrak{S}), S(\mathfrak{S}))$, ie, the Coxeter matrix $M(\mathfrak{S}) := (m_{\alpha\beta})_{\alpha,\beta\in\Pi}$, where $m_{\alpha\beta}$ is the order of $r_\alpha r_\beta \in W(\mathfrak{S})$. The following lemma, whose proof is straightforward, recalls the relationship between two root bases of the same type:

Lemma 2 *Let $M = (m_{ij})_{i,j\in I}$ be a Coxeter matrix, $\mathfrak{S}_M = (V_M, (\cdot,\cdot)_M, \{e_i \mid i \in I\})$ be the standard root basis of type M and $\mathfrak{S} = (V, (\cdot,\cdot), \{\alpha_i \mid i \in I\})$ be any root basis of type M such that $(\alpha_i, \alpha_j) = -\cos(\pi/m_{ij})$ for all $i, j \in I$ with $m_{ij} < \infty$. Let $\varphi : W(\mathfrak{S}_M) \to W(\mathfrak{S})$ be the unique isomorphism such that $\varphi : r_{e_i} \mapsto r_{\alpha_i}$ for all $i \in I$. We have the following:*

(i) *There is a unique φ–equivariant bijection $f\colon \Phi(\mathfrak{S}_M) \to \Phi(\mathfrak{S})$ such that $f\colon e_i \mapsto \alpha_i$ for all $i \in I$.*

(ii) *If M is 2–spherical (ie $m_{ij} < \infty$ for all $i, j \in I$), then one has $(\alpha, \beta)_M = (f(\alpha), f(\beta))$ for all $\alpha, \beta \in \Phi(\mathfrak{S}_M)$.* □

2.2 Convention

From now on and until the end of the paper, we fix a root basis $\mathfrak{S} = (V, (\cdot,\cdot), \Pi)$ and we set $(W, S) := (W(\mathfrak{S}), S(\mathfrak{S}))$. Moreover, given $s \in S$, we denote the unique element α of Π such that $s = r_\alpha$ by α_s.

2.3 The Davis complex

Suppose that the set S is finite. The *Davis complex* $\mathfrak{A}$ associated with (W,S) is a piecewise Euclidean $\mathsf{CAT}(0)$ cell complex whose 1–skeleton is the Cayley graph of W with respect to the generating set S. The action of W on this Cayley graph induces naturally an action on $\mathfrak{A}$; this action is properly discontinuous and cocompact. By definition, a *wall* of $\mathfrak{A}$ is the fixed point set of a reflection of W. Hence a wall is a closed convex subset of $\mathfrak{A}$. A fundamental fact is that every wall cuts $\mathfrak{A}$ into two convex open subsets, whose respective closures are called *half-spaces*. Hence the boundary ∂h of a half-space h is a wall. The set of all half-spaces is denoted by $\Phi(\mathfrak{A})$.

Given a point $x \in \mathfrak{A}$ which does not lie on any wall (e.g. x is a vertex of $\mathfrak{A}$), the intersection $C(x) \subset \mathfrak{A}$ of all half-spaces containing x is compact. The set $C(x)$ is called a *chamber*. Every chamber contains exactly one vertex of $\mathfrak{A}$. Hence the W–action on $\mathfrak{A}$ is simply transitive on the chambers. Given two chambers C_1, C_2, we define the *numerical distance* from C_1 to C_2 as the number of walls which separate C_1 from C_2. If $w \in W$ is the unique element such that $w.C_1 = C_2$, then this distance equals the word length $\ell(w)$ of w.

Since the 1–skeleton of $\mathfrak{A}$ is the Cayley graph of (W,S), the edges of $\mathfrak{A}$ are labelled by the elements of S. Given $s \in S$, two chambers are called *s–adjacent* if they contain vertices which are joined by an edge labelled by s.

Lemma 3 *Let $C \subset \mathfrak{A}$ be a chamber. For each $s \in S$, let h_s be the half-spaces containing C but not the chamber of $\mathfrak{A}$ different from C and s–adjacent to C. Then one has the following:*

(i) *There exists a unique W–equivariant bijection $\zeta_C\colon \Phi(\mathfrak{E}) \to \Phi(\mathfrak{A})$ which maps α_s to h_s for all $s \in S$. The positive roots are mapped onto the half-spaces containing C.*

(ii) *Let $\alpha, \beta \in \Phi(\mathfrak{E})$. If $|(\alpha,\beta)| < 1$ then the walls $\partial\zeta_C(\alpha)$ and $\partial\zeta_C(\beta)$ meet. Conversely, if the walls $\partial\zeta_C(\alpha)$ and $\partial\zeta_C(\beta)$ meet, then $|(\alpha,\beta)| \leq 1$ and equality occurs if and only if $\alpha = \pm\beta$.*

(iii) *For all $\alpha, \beta \in \Phi(\mathfrak{E})$, one has $(\alpha,\beta) \geq 1$ if and only if $\zeta_C(\alpha) \subset \zeta_C(\beta)$ or $\zeta_C(\beta) \subset \zeta_C(\alpha)$.*

(iv) *For all $\alpha, \beta \in \Phi(\mathfrak{E})$, one has $(\alpha,\beta) \leq 0$ if and only if $\zeta_C(\alpha) \cap \zeta_C(\beta) \subset \zeta_C(r_\alpha(\beta)) \cap \zeta_C(r_\beta(\alpha))$.*

Proof Assertion (i) follows from the fact that the Cayley graph of (W,S) (and even the whole Davis complex) can be embedded in the Tits cone of the root basis $\mathfrak{E}$, see [16, Appendix B.4] for details. For (ii) and (iii), see [16, Proposition 1.4.7]. Assertion (iv) follows also by considering the Tits cone. □

2.4 The cube complex of G Niblo and L Reeves

Maintain the assumption that S is finite. In [18], G Niblo and L Reeves used the structure of wall space of the Cayley graph (or the Davis complex) of (W, S) to construct a $\mathsf{CAT}(0)$ cube complex $\mathfrak{X}$ endowed with a properly discontinuous W–action. We briefly recall here the construction and basic properties of $\mathfrak{X}$ for later reference.

Vertices of this cube complex are mappings $v\colon \mathfrak{M}(\mathfrak{A}) \to \Phi(\mathfrak{A})$ which satisfy the following two conditions:

- For all $m \in \mathfrak{M}(\mathfrak{A})$, we have $\partial(v(m)) = m$.
- For all $m, m' \in \mathfrak{M}(\mathfrak{A})$, if m and m' are parallel then either $v(m) \subset v(m')$ or $v(m') \subset v(m)$.

By definition, two distinct vertices are adjacent if and only if the subset of $\mathfrak{M}(\mathfrak{A})$ on which they differ is a singleton. Note that the Cayley graph of (W, S) is a subgraph of the so-obtained graph. By definition, the 1–skeleton of $\mathfrak{X}$ is the connected component of this graph which contains the Cayley graph of (W, S). The cubes are defined by 'filling in' all the cubical subgraphs of the 1–skeleton. It is shown in [18] that the cube complex $\mathfrak{X}$ is finite-dimensional, locally finite, and the canonical W–action is properly discontinuous, but not always co-compact. In fact [5], the W–action on $\mathfrak{X}$ is co-compact if and only if W has no parabolic subgroup of irreducible affine type and rank ≥ 3. We refer to Section 7 for more precise information on the W–orbits of cubes in $\mathfrak{X}$.

As it is the case for any $\mathsf{CAT}(0)$ cube complex (see [20]), the space $\mathfrak{X}$ is endowed with a collection $\mathfrak{M}(\mathfrak{X})$ of *walls* (resp. a collection $\Phi(\mathfrak{X})$ of *half-spaces*), which is by construction in canonical one-to-one correspondence with $\mathfrak{M}(\mathfrak{A})$ (resp. $\Phi(\mathfrak{A})$). More precisely, a wall is an equivalence class of edges, for the equivalence relation defined as the transitive closure of the relation of being opposite edges in some square. Thus every edge defines a wall which separates its two extremities. Given an equivalence class of edges, the corresponding wall can be realized geometrically as the convex closure of the set of midpoints of edges in this class. In this way, every wall becomes a closed convex subset of $\mathfrak{X}$ which separates $\mathfrak{X}$ into two convex subsets, called half-spaces. Note that a wall is itself a $\mathsf{CAT}(0)$ cube complex, which is a subcomplex of the first barycentric subdivision of $\mathfrak{X}$.

2.5 Reflection subgroups

A *reflection subgroup* of W is a subgroup generated by some set of reflections. The following basic fact is well-known:

Lemma 4 *Let H be a subgroup of W generated by a set R of reflections. Then there exists a unique set $\Pi' \subset \Phi(\mathfrak{E})^+$ such that $\mathfrak{E}' = (V, (\cdot,\cdot), \Pi')$ is a root basis and $W(\mathfrak{E}') = H$. Moreover one has $|\Pi'| \leq |R|$.*

Proof See [9] or [10]. □

By definition, the *type* (resp. *rank*) of the reflection group H is the type (resp. rank) of the Coxeter system $(H, S(\mathfrak{E}'))$ (see Lemma 1(i)). The reflection group H is called *standard parabolic* if $S(\mathfrak{E}') \subset S$. Let $S(\mathfrak{E}') = S_1 \cup \cdots \cup S_k$ be the finest partition of $S(\mathfrak{E}')$ into non-empty mutually centralizing subsets. The subgroups $\langle S_i \rangle \subset H$, $i = 1, \ldots, k$, are called the *direct components* of H.

3 Chains of roots

Throughout this section, we fix a base chamber $C \subset \mathfrak{A}$.

3.1 A partial ordering on the set of roots

Transforming the relation of inclusion $\subset$ on $\Phi(\mathfrak{A})$ by the bijection ζ_C of Lemma 3(i), one obtains a partial ordering on $\Phi(\mathfrak{E})$ which we also denote by $\subset$. By Lemma 3, two roots α, β are orderable by $\subset$ if and only if $(\alpha, \beta) \geq 1$.

Before stating the next lemma, we need to introduce a constant κ which is defined as follows:

$$\kappa = \sup\{|(\alpha,\beta)| \,:\, \alpha, \beta \in \Phi(\mathfrak{E}), |(\alpha,\beta)| < 1\}.$$

By Lemma 3(ii), the condition $|(\alpha,\beta)| < 1$ implies that the group $\langle r_\alpha, r_\beta \rangle$ is finite. Since W has finitely many conjugacy classes of finite subgroups, it follows in particular that $\kappa < 1$. Important to us will be the following:

Lemma 5 *Let $\alpha, \beta, \gamma \in \Phi(\mathfrak{E})$ be roots such that $\alpha \subset \beta \subset \gamma$. Then the following holds:*

(i) *One has*

$$(\alpha,\gamma) \geq \max\{(\alpha,\beta), (\beta,\gamma)\}.$$

(ii) *If moreover $(r_\beta(\alpha), \gamma) > -1$, then*

$$(\alpha,\gamma) \geq 2\,(\alpha,\beta) - \kappa.$$

(iii) *If $(r_\beta(\alpha), \gamma) \leq -1$, then $\beta \subset -r_\beta(\alpha) \subset \gamma$ or $\alpha \subset -r_\beta(\gamma) \subset \beta$.*

Proof For (i), see [16, Corollary 6.2.3]. Since $\beta \subset \gamma$, Lemma 3(iii) yields $(\beta,\gamma) \geq 1$. Therefore, one has:

$$\begin{aligned}(\alpha,\gamma) - 2\,(\alpha,\beta) &\geq (\alpha,\gamma) - 2\,(\alpha,\beta)(\beta,\gamma)\\ &= (\alpha - 2\,(\alpha,\beta)\beta,\gamma)\\ &= (r_\beta(\alpha),\gamma).\end{aligned}$$

Moreover, if $(r_\beta(\alpha),\gamma) > -1$, then $(r_\beta(\alpha),\gamma) \geq -\kappa$ by definition. This implies (ii). Assertion (iii) is a consequence of Lemma 3(ii) and (iii). □

3.2 An 'affine versus non-affine' alternative for chains of roots

In this section, we establish Proposition 7, which is a first approximation of Theorem D.

Let $\alpha_1,\alpha_2 \in \Phi(\mathfrak{S})$ be roots and let $h_i = \zeta_C(\alpha_i)$ for $i = 1,2$. Suppose that $\alpha_1 \subset \alpha_2$. The set

$$\Phi(\alpha_1;\alpha_2) := \{\beta \in \Phi(\mathfrak{S}) \mid \alpha_1 \subset \beta \subset \alpha_2\}$$

is finite. Indeed, its cardinality is bounded by the combinatorial distance from a vertex contained in h_2 to a vertex contained in the complement of h_1.

A set of roots $\Phi \subset \Phi(\mathfrak{S})$ is called *convex* if for all $\alpha_1,\alpha_2 \in \Phi$ and all $\beta \in \Phi(\mathfrak{S})$, one has $\beta \in \Phi$ whenever $\alpha_1 \subset \beta \subset \alpha_2$. A set of roots $\Phi \subset \Phi(\mathfrak{S})$ is called a *chain* if it is totally ordered by $\subset$. A chain is called *convex* if any two consecutive elements form a convex pair. In view of the preceding paragraph, it is easy to see that *any chain is contained in a convex chain*. This convex chain need not be unique, and it is in general properly contained in the convex closure of the initial chain. A chain of roots $\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_n$ is called *maximally convex* if for all chain $\beta_0 \subsetneq \beta_1 \subsetneq \cdots \subsetneq \beta_k$ such that $\beta_0 = \alpha_0$ and $\beta_k = \alpha_n$, one has $k \leq n$. Note that a maximally convex chain is convex.

As a first consequence of Lemma 5, we have:

Lemma 6 *Let $\alpha,\beta \in \Phi(\mathfrak{S})$ be such that $\alpha \subsetneq \beta$. If $(\alpha,\beta) = 1$, then the group $\langle r_\phi \mid \phi \in \Phi(\alpha;\beta)\rangle$ is infinite dihedral; in particular $\Phi(\alpha;\beta)$ is a chain.*

Proof Recall from Lemma 3(iii) that for all $\phi,\psi \in \Phi(\mathfrak{S})$, if $\phi \subset \psi$ then $(\phi,\psi) \geq 1$.

Choose a chain $\alpha = \alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_k = \beta$ of maximal possible length; this is possible since $\Phi(\alpha;\beta)$ is finite. Moreover, the maximality of the chain (α_i) implies that this chain is convex. By Lemma 5(i) we have $(\alpha_i,\alpha_j) = 1$ for all $i,j = 0,1,\ldots,k$. By Lemma 5(ii), we have $(r_{\alpha_{i+1}}(\alpha_i),\alpha_{i+2}) \leq -1$ since $(\alpha_i,\alpha_{i+2}) = 1$ for each i. Hence, by Lemma 5(iii), one has $\alpha_{i+1} \subset -r_{\alpha_{i+1}}(\alpha_i) \subset \alpha_{i+2}$ or $\alpha_i \subset -r_{\alpha_{i+1}}(\alpha_{i+2}) \subset \alpha_{i+1}$. By the convexity of the chain (α_i), we deduce in both cases that $r_{\alpha_{i+1}}(\alpha_i) = -\alpha_{i+2}$ since $\alpha_i \neq \alpha_{i+1} \neq \alpha_{i+2}$. Since i was arbitrary, it follows from Lemma 1(iii) that the group $\langle r_{\alpha_i} \mid i = 0, 1, \dots, k\rangle$ is actually generated by the pair $\{r_{\alpha_0}, r_{\alpha_1}\}$; in particular this group is infinite dihedral.

Let now ϕ be any element of $\Phi(\alpha;\beta)$. We have $-r_\alpha(\beta) \subsetneq -r_\alpha(\alpha_1) \subsetneq \alpha_0 = \alpha \subset \phi \subset \beta$. Since $(\alpha,\beta) = 1$, it follows that $(-r_\alpha(\beta), \beta) = 1$. Hence, by Lemma 5(i) we have $(-r_\alpha(\alpha_1), \phi) = 1$. Therefore, the restriction of the bilinear form $(\cdot,\cdot)$ to the subspace spanned by $\{-r_\alpha(\alpha_1), \alpha, \phi\}$ is positive semi-definite and that its radical is of codimension 1. By Lemma 4 and [2, Chapter VI, Section 4.3, Theorem 4], this implies that the reflection subgroup generated by $\{r_\alpha, r_{\alpha_1}, r_\phi\}$ is infinite dihedral. By the above, the latter group contains r_{α_i} for each i. Therefore the wall $\partial\zeta_C(\phi)$ is parallel to $\partial\zeta_C(\alpha_i)$ for each i. By the maximality of the chain $(\alpha_i)_{i\leq k}$, this implies that $\phi = \alpha_j$ for some $j \in \{0, 1, \dots, k\}$. In other words, we have $\Phi(\alpha;\beta) = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$, which completes the proof. □

A posteriori, the preceding lemma can be viewed as a consequence of Theorem D; however, the proof of the latter relies on Lemma 6 in an essential way.

The constant κ appearing in the next proposition was defined in Section 3.1.

Proposition 7 *Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_n$ be a maximally convex chain of roots and let $j \in \{1, \dots, n\}$. We have the following:*

(i) *Assume that $(\alpha_0, \alpha_{j-1}) = 1$. Then either $(\alpha_0, \alpha_j) = 1$ or $(\alpha_0, \alpha_j) \geq j(1-\kappa)$.*

(ii) *Assume that $(\alpha_0, \alpha_j) = 1 + \varepsilon$ for some $\varepsilon > 0$. Then $(\alpha_0, \alpha_n) > 1 + \frac{n}{2j}\varepsilon$.*

Proof (i) If $j = 1$, there is nothing to prove. Thus we assume $j > 1$. By Lemma 6 and by convexity of the chain (α_i), the group $\langle r_{\alpha_i} \mid i \in \{0, 1, \dots, j-1\}\rangle$ is infinite dihedral, generated by the pair $\{r_{\alpha_{j-2}}, r_{\alpha_{j-1}}\}$. Let $\beta = r_{\alpha_{j-1}}(\alpha_{j-2})$.

Assume first that $(\beta, \alpha_j) \leq -1$. Then Lemma 5(iii) implies $\beta = -\alpha_j$ and, hence, $r_{\alpha_j} \in \langle r_{\alpha_i} \mid i \in \{0, 1, \dots, j-1\}\rangle$. An easy computation using Lemma 1(iii) yields $(\alpha_0, \alpha_j) = 1$.

Assume now that $(\beta, \alpha_j) > -1$. Clearly this implies $(\beta, \alpha_j) \geq -\kappa$. Using Lemma 1(iii), one easily computes that $(\beta, \alpha_{j-1}) = -1$ and that $\alpha_0 = j.\alpha_{j-1} + (j-1).\beta$. Therefore, we deduce

$$\begin{aligned}(\alpha_0, \alpha_j) &= j.(\alpha_{j-1}, \alpha_j) + (j-1).(\beta, \alpha_j)\\ &\geq j - (j-1)\kappa\\ &= j(1-\kappa) + \kappa.\end{aligned}$$

(ii) By Lemma 5(i), we have $(\alpha_0, \alpha_n) \geq (\alpha_0, \alpha_j)$. Thus we may assume $n \geq 2j$, otherwise there is nothing to prove. Clearly $\alpha_j \subsetneq -r_{\alpha_j}(\alpha_0)$. Let

$$k = \max\{i \mid \alpha_i \subsetneq -r_{\alpha_j}(\alpha_0)\}.$$

Thus $k \geq j$.

Assume that $k \geq 2j$, namely that $\alpha_{2j} \subsetneq -r_{\alpha_j}(\alpha_0)$. Then one would have a chain

$$\alpha_0 \subsetneq -r_{\alpha_j}(\alpha_{2j}) \subsetneq -r_{\alpha_j}(\alpha_{2j-1}) \subsetneq \cdots \subsetneq -r_{\alpha_j}(\alpha_{j+1}) \subsetneq \alpha_j$$

contradicting the fact that the sequence $(\alpha_i)_{i \leq n}$, and hence $(\alpha_i)_{i \leq j}$, is maximally convex. Thus $k < 2j$. Let us now consider the ordered triple $\alpha_0 \subsetneq \alpha_j \subsetneq \alpha_{k+1}$.

Suppose first that $(r_{\alpha_j}(\alpha_0), \alpha_{k+1}) \leq -1$. Then Lemma 5(iii) implies that $-r_{\alpha_j}(\alpha_0) \subset \alpha_{k+1}$ or $\alpha_{k+1} \subset -r_{\alpha_j}(\alpha_0)$. On the other hand, the definition of k implies that $\alpha_k \subsetneq -r_{\alpha_j}(\alpha_0)$ and that α_{k+1} is not properly contained in $-r_{\alpha_j}(\alpha_0)$. Therefore, by the convexity of the chain $(\alpha_i)_{i \leq n}$, we have $r_{\alpha_j}(\alpha_0) = -\alpha_{k+1}$ in this case. We deduce that

$$\begin{aligned} (\alpha_0, \alpha_{k+1}) &= (\alpha_0, -r_{\alpha_j}(\alpha_0)) \\ &= 2(\alpha_0, \alpha_j)^2 - 1 \\ &= 1 + 4\varepsilon + 2\varepsilon^2 \\ &> 1 + 2\varepsilon. \end{aligned}$$

Suppose now that $(r_{\alpha_j}(\alpha_0), \alpha_{k+1}) > -1$. Then Lemma 5(ii) implies that $(\alpha_0, \alpha_{k+1}) \geq 2(\alpha_0, \alpha_j) - \kappa > 1 + 2\varepsilon$.

In both cases, we obtain $(\alpha_0, \alpha_{2j}) > 1 + 2\varepsilon$ by Lemma 5(i) because $k + 1 \leq 2j$. An immediate induction now yields $(\alpha_0, \alpha_{2^x j}) > 1 + 2^x\varepsilon$ for all positive integer x such that $2^x j \leq n$. Since the maximal such integer is $\lfloor \log_2(\frac{n}{j}) \rfloor$, we deduce, again from Lemma 5(i), that $(\alpha_0, \alpha_n) > 1 + 2^x\varepsilon$ with $x = \lfloor \log_2(\frac{n}{j}) \rfloor$. The desired inequality follows because $\lfloor \log_2(\frac{n}{j}) \rfloor > \log_2(\frac{n}{j}) - 1 = \log_2(\frac{n}{2j})$. □

4 Nested triangles of walls

When studying Coxeter groups, it is often useful to relate the combinatorics of walls in the Davis complex with the algebraic properties of the subgroup generated by the corresponding reflections. A typical well-known result of this kind is the basic fact that two walls meet if and only if the corresponding reflections generate a finite group. The purpose of this section is prove the following:

Theorem 8 *There exists a constant $L = L(W, S)$, depending only on the Coxeter system (W, S), such that the following property holds. Let μ, μ', $m_0, m_1, \ldots, m_n$ be walls of the Davis complex $\mathfrak{A}$ such that:*

(1) $\varnothing \neq \mu \cap \mu' \subset m_0$;

(2) *For all* $0 \le i < j < k \le n$, *the wall* m_j *separates* m_i *from* m_k;

(3) *For each* $i = 1, \dots, n$, *the wall* m_i *meets both* μ *and* μ'.

If $n > L$, *then the group generated by the reflections through the walls* μ, μ', m_0, $m_1, \dots, m_n$ *is isomorphic to a Euclidean triangle group. Moreover, it is contained in a parabolic subgroup of irreducible affine type.*

By a *Euclidean triangle group*, we mean an affine Coxeter group of rank 3, or equivalently, the automorphism group of one of the three (types of) regular tessellations of the Euclidean plane by triangles.

Theorem 8 has also proved essential in studying Euclidean flats isometrically embedded in the Davis complex [4]. In this paper, it is an essential ingredient in the proof of Theorem C.

4.1 Nested Euclidean triangles

Given a set of walls M, we denote by $W(M)$ the reflection subgroup of W generated by all reflections associated to elements of M.

Lemma 9 *Let* $\mu, \mu', m_0, m_1, \dots, m_n$ *be walls of the Davis complex* $\mathfrak{A}$ *which satisfy conditions (1), (2) and (3) of Theorem 8. Assume that the group* $W(\{\mu, \mu', m_0, m_n\})$ *is a Euclidean triangle subgroup. Then we have the following:*

(i) *The group* $W(\{\mu, \mu', m_0, m_1, \dots, m_n\})$ *is a Euclidean triangle subgroup, which is contained in a parabolic subgroup of affine type of* W; *in particular* $W(\{m_0, m_1, \dots, m_n\})$ *is infinite dihedral.*

(ii) *For any point* $x \in \mu' \cap m_n$, *there exist* $k = \lfloor \frac{n}{2} \rfloor$ *pairwise parallel walls* $m'_1, m'_2, \dots, m'_k$ *which separate* x *from* μ, *and such that*

$$W(\{m'_1, m'_2, \dots, m'_k\}) \subset W(\{\mu, \mu', m_0, m_n\});$$

in particular $W(\{\mu, m'_1, m'_2, \dots, m'_k\})$ *is infinite dihedral.*

Proof (i) By a theorem of D Krammer which is recalled in Proposition 16 below, the group $W(\{\mu, \mu', m_0, m_n\})$ is contained in a parabolic subgroup W_0 of irreducible affine type of W. The Davis complex of this parabolic subgroup is contained in $\mathfrak{A}$ as a residue ρ_0; in other words, there is a chamber C_0 such that the union $\rho_0 = \bigcup_{w \in W_0} w.C_0$ is a closed convex subset of $\mathfrak{A}$, whose stabilizer in W coincides with W_0. Since the reflection r_{m_0} belongs to $W(\mu, \mu')$ by condition (1), it follows that r_{m_0} and r_{m_n}

both stabilize ρ_0. In particular, the walls m_0 and m_n both meet ρ_0. Since ρ_0 is convex, it follows that every wall which separates m_0 from m_n meets ρ_0. Since ρ_0 is a residue, every reflection associated to a wall which cuts ρ_0 must stabilize ρ_0. It follows that $r_{m_i} \in \mathrm{Stab}_W(\rho_0) = W_0$ for all $i = 1, \dots, n$, as desired. Finally, it is an easy observation that any subgroup of an affine Coxeter group generated by reflections through pairwise parallel walls is infinite dihedral.

(ii) We have just seen that any wall m which separates m_0 from m_n meets ρ_0 and, hence, belongs to W_0. Such a wall is parallel to m_i for each i because parallelism of walls in an affine Coxeter group is an equivalence relation. This shows that we may assume, without loss of generality, that every wall which separates m_0 from m_n is one of the m_i's.

By assumption (1) the reflection r_μ and $r_{\mu'}$ do not commute. Therefore, by considering each of the three types of affine triangle groups separately, it is easily seen that for each $i = 1, \dots, k = \lfloor \frac{n}{2} \rfloor$, there exists a wall m'_i which is parallel to μ and such that $m_{2i} \cap \mu' \subset m'_i$. Choose any point y on $\mu \cap \mu'$ and consider a geodesic path joining x to y. This path is completely contained in μ' by convexity, and meets each m_i by assumption (2). Since $m_{2i} \cap \mu' \subset m'_i$, it follows that this path crosses m'_i for each $i = 1, \dots, k$. Since m'_i is parallel to μ, this means precisely that μ'_i separates x from μ. □

4.2 Critical bounds for hyperbolic triangles

Before stating the next lemma, we introduce an additional constant λ_{fin} which is defined as follows:

$$\lambda_{\text{fin}} = \sup\{x \in \mathbb{R} \mid \alpha = x.\phi + y.\psi,\ \alpha, \phi, \psi \in \Phi(\mathfrak{S}),\ |(\phi, \psi)| < 1\}.$$

(By convention, we set $\lambda_{\text{fin}} = 1$ if $|(\phi, \psi)| \geq 1$ for all $\phi, \psi \in \Phi$.) Note that the conditions $\alpha = x.\phi + y.\psi$ and $|(\phi, \psi)| < 1$ imply that the group $\langle r_\alpha, r_\phi, r_\psi \rangle$ is a finite dihedral group. Since W has finitely many conjugacy classes of finite subgroups, it follows that the constant λ_{fin} is finite; thus λ_{fin} is a positive real number.

Lemma 10 *Let $\phi, \phi', \alpha_0, \alpha_1 \in \Phi(\mathfrak{S})$ be roots such that $\alpha_0 \subsetneq \alpha_1$ and that the walls $\mu = \partial\phi$, $\mu' = \partial\phi'$, $m_0 = \partial\alpha_0$ and $m_1 = \partial\alpha_1$ satisfy conditions (1) and (3) of Theorem 8. If $W(\mu, \mu', m_0, m_1)$ is not a Euclidean triangle subgroup, then one has*

$$1 + \varepsilon \leq (\alpha_0, \alpha_1) \leq 2\kappa\lambda_{\text{fin}},$$

where $\varepsilon = \varepsilon(W, S) > 0$ is a positive constant which depends only on (W, S) and κ is the constant defined in Section 3.1.

It is well-known that, in the situation of the preceding lemma, if $W(\mu,\mu',m_1)$ is a Euclidean triangle subgroup, then $(\alpha_0,\alpha_1)=1$ (see the first lines of the proof of Proposition 14(ii) below).

Proof Condition (1) implies that $W(\mu,\mu',m_0)$ is a finite dihedral group. In particular we have $\alpha_0=\lambda_0.\phi+\mu_0.\phi'$ for some $\lambda_0,\mu_0\in\mathbb{R}$. We have

$$(\alpha_0,\alpha_1)=\lambda_0.(\phi,\alpha_1)+\mu_0.(\phi',\alpha_1)\leq\kappa(\lambda_0+\mu_0)\leq 2\kappa\lambda_{\text{fin}}.$$

Clearly the group $W(\mu,\mu',m_0,m_1)$ is infinite because the walls m_0 and m_1 are parallel. Since $W(\mu,\mu',m_0)$ is a finite dihedral group, it follows from Lemma 4 that the reflection group $W(\mu,\mu',m_0,m_1)$ is infinite of rank 3. Since it is not of affine type by hypothesis, it must be a hyperbolic triangle group, namely it is isomorphic to the automorphism group of a regular tessellation of the hyperbolic plane by triangles. Furthermore condition (3) implies that $W(\mu,\mu',m_0,m_1)$ is compact hyperbolic, ie, the tiles of the above tessellation are compact triangles. We view this tessellation as a geometric realization of the Coxeter complex associated to $W(\mu,\mu',m_0,m_1)$.

In this realization, the walls $\partial\alpha_0$ and $\partial\alpha_1$ are realized by parallel geodesic lines. It is known [1, Corollary A.5.8] that, if one writes $(\alpha_0,\alpha_1)=\frac{1}{2}(x+x^{-1})$ for some $x\geq 1$, then the hyperbolic distance between these lines in $\mathbb{H}^2$ is $\log(x)$ (see also Lemma 2(ii)). In view of this formula, we have $(\alpha_0,\alpha_1)>1$ because parallel walls do not meet in $\mathbb{H}^2\cup\partial\mathbb{H}^2$ in a regular tessellation by compact triangles.

It remains to show that (α_0,α_1) stays bounded away from 1 when ϕ,ϕ',α_0 and α_1 vary in $\Phi(\mathfrak{S})$. First, we note that, by the formula above and the fact that $W(\mu,\mu',m_0,m_1)$ is transitive on pairs of parallel walls at small distances (see Lemma 15 below), the scalar (α_0,α_1) stays bounded away from 1 when ϕ,ϕ',α_0 and α_1 vary in such a way that the group $W(\mu,\mu',m_0,m_1)$ remains in the same isomorphism class. Now, the desired result follow because $W(\mu,\mu',m_0,m_1)$ is a 2–spherical reflection subgroup of rank 3 of W, and W has finitely many types of such reflection subgroups: indeed, the type of a such a subgroup is a Coxeter matrix of size 3, all of whose entries divide some entry of the Coxeter matrix of (W,S). This concludes the proof. □

Remark 1 Proposition 14(ii) below, which relies on the preceding lemma through Theorems 8 and C, may be viewed as a generalization of the first inequality of Lemma 10.

Remark 2 It turns out that the positive constant ε appearing in Lemma 10 can be made 'universal', ie independent of (W,S). This is done by expressing the minimal hyperbolic distance between two walls of a regular tessellation of $\mathbb{H}^2$ by compact triangles, as a monotonic function of the area of the fundamental tile T. This tile is a triangle whose angles are of the form $(\frac{\pi}{k}, \frac{\pi}{l}, \frac{\pi}{m})$ for some $k, l, m \in \mathbb{N}$. Thus the area of T is $\pi - \frac{\pi}{k} - \frac{\pi}{l} - \frac{\pi}{m}$, and it is not difficult to compute that this area has a global minimum, which is reached for $(k, l, m) = (2, 3, 7)$. However, the universality of ε is not relevant to our purposes.

4.3 Proof of Theorem 8

We will show that the constant L can be defined by

$$L = \max \left\{ 2,\ \frac{2\kappa\lambda_{\text{fin}}}{1-\kappa},\ \frac{8\kappa^2\lambda_{\text{fin}}^2 - 4\kappa\lambda_{\text{fin}}}{\varepsilon(1-\kappa)} \right\}.$$

Let $\phi, \phi', \alpha_0, \alpha_1, \dots, \alpha_n \in \Phi(\mathfrak{S})$ be roots such that $\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_n$ and that the walls $\mu = \partial\phi$, $\mu' = \partial\phi'$ and $m_i = \partial\alpha_i$ for $i = 0, 1, \dots, n$ satisfy conditions (1), (2) and (3) of Theorem 8. Suppose moreover that $n > L$.

By Lemma 9, we may assume that $W(\mu, \mu', m_0, m_n)$ is not a Euclidean triangle subgroup, otherwise we are done. Therefore, Lemma 10 yields $1+\varepsilon \le (\alpha_0, \alpha_n) \le 2\kappa\lambda_{\text{fin}}$.

We now make another estimate of the value of (α_0, α_n). To this end, note first that every wall which separates $m_0 = \partial\alpha_0$ from $m_n = \partial\alpha_n$ must meet $\mu = \partial\phi$ and $\mu' = \partial\phi'$ because μ and μ' are convex, as are all walls. Thus, after replacing the sequence $(\alpha_i)_{i \le n}$ by a maximally convex chain of roots whose extremities are α_0 and α_n, one obtains a new set of roots which satisfies again conditions (1), (2) and (3) of the theorem. We henceforth assume without loss of generality that the sequence $(\alpha_i)_{i \le n}$ is maximally convex.

Let $j = \min\{i \mid (\alpha_0, \alpha_i) > 1\}$. Thus $(\alpha_0, \alpha_{j-1}) = 1$ and Proposition 7(i) yields $(\alpha_0, \alpha_j) \ge j(1-\kappa)$. By Lemma 5(i), we have $(\alpha_0, \alpha_j) \le (\alpha_0, \alpha_n)$ and, hence, we deduce that $j \le \frac{2\kappa\lambda_{\text{fin}}}{1-\kappa}$.

We now apply Proposition 7(ii). This yields $(\alpha_0, \alpha_n) > 1 + \frac{n}{2j}\varepsilon$. Since $n > L$ we have

$$\frac{n}{2j} \ge \frac{n(1-\kappa)}{4\kappa\lambda_{\text{fin}}} > \frac{2\kappa\lambda_{\text{fin}} - 1}{\varepsilon}.$$

We deduce that $(\alpha_0, \alpha_n) > 2\kappa\lambda_{\text{fin}}$, a contradiction. This shows that the group $W(\mu, \mu', m_0, m_n)$ is a Euclidean triangle subgroup and, hence, the desired result follows from Lemma 9. $\square$

5 Separation of parallel walls

5.1 On the walls of an infinite dihedral subgroup

Note that if a set M of walls is such that the group $W(M)$ is infinite dihedral, then the elements of M are pairwise parallel.

Lemma 11 *Let M be a set of walls such that the group $W(M)$ is infinite dihedral. Let m be a wall which meets at least 8 elements of M in $\mathfrak{A}$. Then m meets all elements of M and either the reflection r_m through m centralizes $W(M)$ or $W(M \cup \{m\})$ is a Euclidean triangle subgroup.*

Proof By Lemma 4, we may – and shall – assume, without loss of generality, that $W = W(M \cup \{m\})$. Since W is infinite, it is of rank ≥ 2. Moreover W is not infinite dihedral otherwise the walls in $M \cup \{m\}$ would be pairwise parallel, in contradiction with the hypotheses. Thus W is of rank > 2. On the other hand, it is generated by a reflection together with a dihedral reflection group; hence it has a generating set consisting of 3 reflections. By Lemma 4, this shows that W is of rank 3.

If W is of reducible type, then the reflection r_m through m must centralize $W(M)$ and, hence, the wall m meets every element of M. Thus we are done in this case, and we assume from now on that W is of irreducible type. A Coxeter group of irreducible type and rank 3 is either a Euclidean triangle subgroup, or a hyperbolic triangle subgroup, ie, a group isomorphic to the automorphism group of a regular tessellation of the hyperbolic plane $\mathbb{H}^2$ by triangles.

If W is a Euclidean triangle subgroup, then m meets every element of M, since parallelism of walls is an equivalence relation in Euclidean geometry. Thus we are done in this case as well, and it remains to deal with the case when W is a hyperbolic triangle subgroup. In particular W is Gromov-hyperbolic. We consider the regular tessellation of $\mathbb{H}^2$ by triangles whose automorphism group is isomorphic to W, and view this tessellation as a geometric realization of (W, S). In particular the walls are geodesic lines in $\mathbb{H}^2$.

Up to enlarging M if necessary, we may assume that for each wall μ, if $r_\mu \in W(M)$ then $\mu \in M$. We first prove that m meets only finitely many elements of M. Suppose the contrary in order to obtain a contradiction. Choose a chain of half-spaces $\dots \subsetneq h_{-1} \subsetneq h_0 \subsetneq h_1 \subsetneq \dots$ such that $M = \{\partial h_i \mid i \in \mathbb{Z}\}$ and that for each $i \in \mathbb{Z}$, the set $h_i \cap m$ contains a geodesic ray. This is possible because m crosses infinitely many ∂h_i's. Note that the intersection $\bigcap_{i \in \mathbb{Z}} h_i$ is a single point ξ of the visual boundary $\partial\mathbb{H}^2$, which must be an endpoint of m by the above. Thus ξ is fixed by r_m and by the translation subgroup of $W(M)$. But the stabilizer of ξ in $\mathrm{Isom}(\mathbb{H}^2)$ is solvable. Hence, given a nontrivial translation $t \in W(M)$, the group $\langle t, r_m t r_m \rangle$ is solvable, whence virtually abelian of $\mathbb{Z}$–rank 1. This shows that $r_m t r_m$ fixes both of the two points at infinity fixed by t, hence so does r_m. Furthermore, the pair consisting of these two points is stabilized by $W(M)$, which finally shows that $W = W(M \cup \{m\})$ stabilizes a pair of points of $\partial\mathbb{H}^2$. This is absurd because W is not virtually solvable. Thus m meets finitely many elements of M.

The next step is to show that r_m commutes with at most one reflection in $W(M)$. Indeed, if μ and μ' are two distinct elements of M such that r_m centralizes $\langle r_\mu, r_{\mu'} \rangle$ then m meets every translate of μ under $\langle r_\mu, r_{\mu'} \rangle$, but there are infinitely many such translates, which all belong to M. This is absurd, because we have just seen that m meets finitely many elements of M.

We now consider again a chain of half-spaces $\cdots \subsetneq h_{-1} \subsetneq h_0 \subsetneq h_1 \subsetneq \dots$ such that $M = \{\partial h_i \mid i \in \mathbb{Z}\}$. We assume moreover that the numbering of the h_i's is such that the set of all those j's such that m meets ∂h_j is $\{-k', \dots, 0, \dots, k\}$ for some integers k, k' with $k' + 1 \geq k \geq k' \geq 0$.

We have seen that the angle between m and ∂h_j equals $\frac{\pi}{2}$ for at most one $j \in \mathbb{Z}$. By the choice of numbering of the h_i's made above, this means in particular that the only possible j such that m is perpendicular to ∂h_j is $j = 0$. In particular the wall $m_1 = r_{\partial h_1}(m)$ is different from m. Since $m \cap \partial h_j \neq \varnothing$ for all $j \in \{-k', \dots, 0, \dots, k\}$ and since $r_{\partial h_1}(\partial h_i) = h_{2-i}$ for all i, it follows that $m_1 \cap \partial h_j \neq \varnothing$ for all $j \in \{2-k, \dots, 2+k'\}$. This means that the walls m, m_1 and $\partial h_{k''}$ form a compact geodesic triangle T in $\mathbb{H}^2$, where $k'' = \min\{k,\ 2+k'\}$. Since $k \leq k'+1$, we have $k'' = k$. Furthermore, the triangle T is cut by ∂h_j for all $j \in \{2, \dots, k-1\}$. Now, the triangular tessellation of $\mathbb{H}^2$ by chambers induces a Coxeter decomposition of T by triangles, namely a tessellation of T by triangles such that two triangles sharing an edge are switched by the orthogonal reflection through that edge. But all Coxeter decompositions of hyperbolic triangles are classified [11, Section 5.1]. Using this classification, together with the fact that the angle between m and ∂h_j is $< \frac{\pi}{2}$ for all $j \in \{1, \dots, k\}$, one deduces easily that $k \leq 3$. In particular, we have also $k' \leq 3$. This shows that m meets at most 7 elements of M. □

5.2 On walls which separate a vertex from its projection

Lemma 12 *Let $\phi \in \Phi(\mathfrak{A})$ be a half-space, $x \in \mathfrak{A}$ be a vertex not contained in ϕ and y be a vertex contained in ϕ and at minimal combinatorial distance from x. Let ψ be a half-space containing x but not y, such that $\partial\phi \neq \partial\psi$. Then we have the following:*

(i) $\phi \cap \psi \subset r_\phi(\psi) \neq \psi$.

(ii) $x, y \notin r_\phi(\psi)$.

(iii) *If moreover* $\partial\phi$ *meets* $\partial\psi$ *then* $\partial r_\phi(\psi)$ *meets every wall which separates* x *from* $\partial\psi$ *and which meets* $\partial\phi$.

Proof For any half-space h we denote by $-h$ the other half-space bounded by ∂h.

If $\phi\cap\psi=\varnothing$, then the walls $\partial\phi$ and $\partial\psi$ are parallel and $r_\phi(\psi)$ is properly contained in ϕ. We deduce that $x\notin r_\phi(\psi)$. By the minimality assumption on y, it follows that $-\phi$ contains a vertex y' neighboring y. Note that $y'=r_\phi(y)$. Since $\partial\phi$ is the only wall separating y from y' we have $y'\in-\psi$, which yields $y=r_\phi(y')\in r_\phi(-\psi)=-r_\phi(\psi)$, or equivalently $y\notin r_\phi(\psi)$.

Thus we may assume that $\phi\cap\psi$ is nonempty. This implies that the walls $\partial\phi$ and $\partial\psi$ meet, otherwise ϕ would be contained in ψ which is impossible since $y\in\phi\cap(-\psi)$ by assumption. Therefore $\langle r_\phi, r_\psi\rangle$ is a finite dihedral group.

For any two half-spaces α,β such that $\langle r_\alpha, r_\beta\rangle$ is a finite dihedral group, it is straightforward to check the following observations (see also Lemma 3):

(1) $\alpha\cap\beta\subset r_\alpha(\beta)\Leftrightarrow\alpha\cap\beta\subset r_\beta(\alpha)$.

(2) $\alpha\cap\beta\subset r_\alpha(\beta)\Leftrightarrow(-\alpha)\cap(-\beta)\subset -r_\alpha(\beta)$.

(3) $\alpha\cap\beta\not\subset r_\alpha(\beta)\Rightarrow(-\alpha)\cap\beta\subset r_\alpha(\beta)$.

Assume now that $\phi\cap\psi\not\subset r_\phi(\psi)$ in order to obtain a contradiction. By (3), this yields $(-\phi)\cap\psi\subset r_\phi(\psi)$. Since $x\in(-\phi)\cap\psi$, we deduce from (1) that $x\in -r_\psi(\phi)$; similarly, since $y\in\phi\cap(-\psi)$, we deduce from (1) and (2) that $y\in r_\psi(\phi)$.

Consider now a minimal combinatorial path γ: $x=x_0,x_1,\dots,x_n=y$ joining x to y in the 1–skeleton of $\mathfrak{A}$ (see Figure 1). We first show that the reflections r_ϕ and r_ψ do not commute. If they commuted, we would have $r_\psi(\phi)=\phi$ and hence, the vertex $r_\psi(y)$ would belong to ϕ. On the other hand, the path γ crosses $\partial\psi$, namely there exists $i\in\{1,\dots,n\}$ such that ψ contains x_{i-1} but not x_i. Thus the path $x_0,\dots,x_{i-1}=r_\psi(x_i),r_\psi(x_{i+1}),\dots,r_\psi(y)$, which is of length $n-1$, joins x to a vertex of ϕ, which contradicts the minimality assumption on y. This shows that r_ϕ and r_ψ do not commute and, hence, that $\partial r_\psi(\phi)\neq\partial\phi$.

Note that, by the minimality assumption on y, we must have $x_{n-1}\in-\phi$. We have seen that both $\partial\psi$ and $\partial r_\psi(\phi)$ separate x from y. Since $\partial r_\psi(\phi)\neq\partial\phi$ and since $\partial\phi$ is the only wall which separates x_{n-1} from y, it follows that the restricted path γ': $x_0,x_1,\dots,x_{n-1}$ crosses both $\partial\psi$ and $\partial r_\psi(\phi)$, but it does not cross $\partial\phi$. On the other hand we have $(-\phi)\cap\psi\subset -r_\psi(\phi)$ by (1), which implies that γ' crosses first

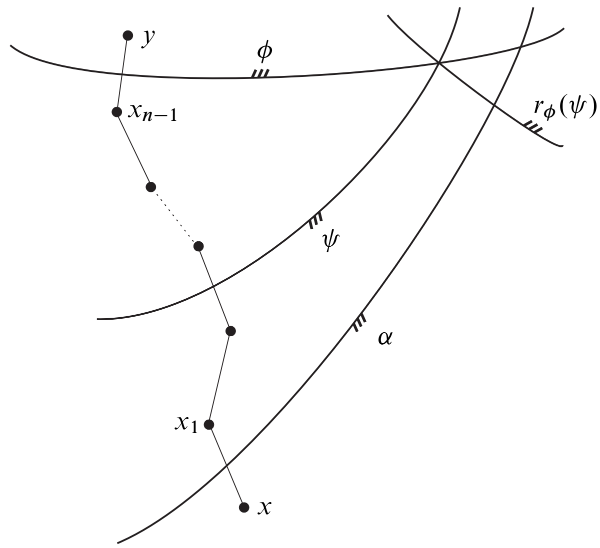

Figure 1: Proof of Lemma 12

$\partial\psi$ and then $\partial r_\psi(\phi)$. In other words, there exist $i < j \in \{1, \dots, n-1\}$ such that ψ contains x_{i-1} but not x_i and $r_\psi(\phi)$ contains x_j but not x_{j-1}. Now, the path $x_0, \dots, x_{i-1} = r_\psi(x_i), r_\psi(x_{i+1}), \dots, r_\psi(x_{j-1}), r_\psi(x_j)$ is of length $j < n$ and joins $x = x_0$ to $r_\psi(x_j) \in \phi$. Again, this contradicts the minimality assumption on y. This proves (i).

Note that $x_{n-1} \in (-\phi) \cap (-\psi) \subset -r_\phi(\psi)$ by (2). Since $\partial\phi$ is the only wall which separates y from x_{n-1} and since $\partial\phi \neq \partial r_\phi(\psi)$, we deduce $y \notin r_\phi(\psi)$. Assume in order to obtain a contradiction that $x \in r_\phi(\psi)$. We may then apply (i) to the half-space $r_\phi(\psi)$, which yields $\phi \cap r_\phi(\psi) \subset \psi$. Transforming by r_ϕ yields $(-\phi) \cap \psi \subset r_\phi(\psi)$. By (i), we have also $\phi \cap \psi \subset r_\phi(\psi)$ and we deduce $\psi = (\phi \cap \psi) \cup ((-\phi) \cap \psi) \subset r_\phi(\psi)$, whence $\psi = r_\phi(\psi)$. This means that r_ϕ and r_ψ commute, and we have seen above that it is not the case. This shows that $x, y \notin r_\phi(\psi)$, whence (ii).

It remains to prove (iii). Let thus α be a half-space containing x and entirely contained in ψ, and suppose that $\partial\alpha$ meets $\partial\phi$. We must prove that $\partial\alpha$ meets $\partial r_\phi(\psi)$. Let m be the midpoint of the edge joining y to x_{n-1}; thus m lies on $\partial\phi$. By hypothesis, the half-space α contains x but not y; therefore, the geodesic path joining x to m must meet the wall $\partial\alpha$ in a point a. Let also a' be a point lying on $\partial\phi \cap \partial\alpha$. Consider the geodesic triangle whose vertices are m, a, a'. Since $\partial\psi$ separates m from a' and since $\partial\psi \cap \partial\phi = \partial\phi \cap \partial r_\phi(\psi)$, it follows that $\partial r_\phi(\psi)$ separates m from a'. By (ii), it follows that $\partial r_\phi(\psi)$ does not separate m from a. Therefore, the wall $\partial r_\phi(\psi)$ separates a from a'. Since both a and a' lie on $\partial\alpha$ and since walls are convex, we deduce that $\partial\alpha$ meets $\partial r_\phi(\psi)$. This finishes the proof. □

5.3 Existence of pairs of parallel walls

Lemma 13 *Suppose that S is finite. Then there exists a constant $N = N(W, S)$ such that any set of more than N walls contains a pair of parallel walls.*

Proof See [18, Lemma 3]. □

5.4 Proof of Theorem C

Note that $\mathfrak{A}$ is quasi-isometric to its 1–skeleton, which is the Cayley graph of (W, S). Therefore, if suffices to show the existence of a constant Q such that any pair of parallel walls, at *combinatorial* distance greater than Q from one another, is separated by another wall. This is what we prove now; for convenience we denote by $V(\mathfrak{A})$ the vertex set of $\mathfrak{A}$ and by d the combinatorial distance on $V(\mathfrak{A})$.

By Lemma 13 combined with Ramsey's theorem, there exists a constant Q such that any set of more than Q walls contains a subset of $4l+1$ pairwise parallel walls, where $l = \inf\{n \in \mathbb{N} \mid n > L, n \geq 8\}$ and L is the constant of Theorem 8.

Let $\alpha, \beta \in \Phi(\mathfrak{A})$ be half-spaces such that $\alpha \cap \beta = \varnothing$ and let

$$d(\alpha, \beta) = \min\{d(x, y) \mid x, y \in V(\mathfrak{A}), x \in \alpha, y \in \beta\}.$$

Assume that $d(\alpha, \beta) > Q$ and let $x, y \in V(\mathfrak{A})$, $x \in \alpha$, $y \in \beta$ such that $d(x, y) = d(\alpha, \beta)$. We must prove that there is a wall m which separates $\partial\alpha$ from $\partial\beta$; equivalently m must separate x from y, and be parallel to, but distinct from, $\partial\alpha$ and $\partial\beta$.

The set $\mathfrak{M}(x, y)$ consisting of all walls which separate x from y is of cardinality $d(x, y)$. Thus it contains a subset M of cardinality $4l+1$ consisting of pairwise parallel walls. If $\partial\alpha$ and $\partial\beta$ both belong to M then we are done. Hence we may assume without loss of generality that $\partial\beta \notin M$. There are two cases to consider: either $\partial\alpha \notin M$ or $\partial\alpha \in M$. If $\partial\alpha$ is parallel to all elements of M, we may replace M by $M \cup \{\partial\alpha\}$ and we are reduced to the case when $\partial\alpha \in M$.

We henceforth assume that $\partial\alpha \notin M$ and that some element of M meets $\partial\alpha$. In fact, we may assume that each element of M meets $\partial\alpha$ or $\partial\beta$, otherwise we are done. Up to exchanging α and β, we may therefore assume that at least $2l+1$ elements of M meet $\partial\alpha$. Let $\phi_0 \subsetneq \phi_1 \subsetneq \cdots \subsetneq \phi_{2l}$ be half-spaces containing y but not x, such that $\partial\phi_i \in M$

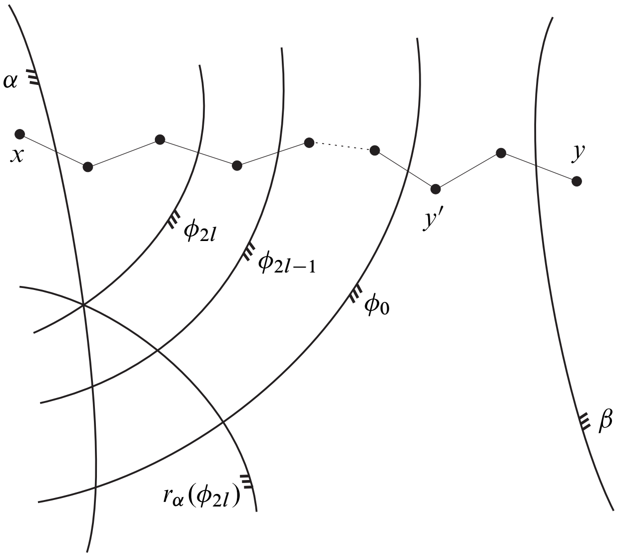

Figure 2: Proof of Theorem C

meets $\partial\alpha$ for each i, see Figure 2. We have $\varnothing \neq \partial\alpha \cap \partial\phi_{2l} = \partial\alpha \cap \partial r_\alpha(\phi_{2l}) \subset \partial\phi_{2l}$. We apply Lemma 12(iii) with the half-spaces α and ϕ_{2l}: this shows that $\partial r_\alpha(\phi_{2l})$ meets $\partial\phi_i$ for each $i = 0, 1, \dots, 2l$. Therefore, the hypotheses of Theorem 8 are satisfied; this shows that $\langle r_\alpha, r_{\phi_i} \mid i = 0, 1, \dots, 2l\rangle$ is a Euclidean triangle subgroup. Hence, we may apply Lemma 9(ii) which yields a set M' of l pairwise parallel walls which separate $\partial\alpha$ from any point on $\partial\phi_0 \cap \partial r_\alpha(\phi_{2l})$; furthermore the group $W(M' \cup \{\partial\alpha\})$ is infinite dihedral.

We now show that $M' \subset \mathfrak{M}(x, y)$. To this end, let y' be the last vertex belonging to ϕ_0, on a geodesic path in $V(\mathfrak{A})$ joining y to x, and let x' be any point on $\partial\alpha \cap \partial\phi_0$, see Figure 2. By Lemma 12(ii), the points x and y' lie on the same side of $\partial r_\alpha(\phi_{2l})$. Moreover, since $\partial\phi_{2l}$ separates x from x' and since $\partial\alpha \cap \partial\phi_{2l} = \partial\alpha \cap \partial r_\alpha(\phi_{2l})$, it follows that $\partial r_\alpha(\phi_{2l})$ separates x from x'. Therefore $\partial r_\alpha(\phi_{2l})$ also separates x' from y'. Hence, the geodesic path joining x' to the $\mathsf{CAT}(0)$ projection of y' on $\partial\phi_0$ meets $\partial r_\alpha(\phi_{2l})$ in a point z. We have seen above that the elements of M' separate z from $\partial\alpha$; in particular they separate z from x and, hence, they separate y' from x because any geodesic segment crosses every wall at most once. Since $\mathfrak{M}(x, y') \subset \mathfrak{M}(x, y)$, it follows that $M' \subset \mathfrak{M}(x, y)$ as desired.

Thus all elements of M' are parallel to $\partial\alpha$ and separate x from y. If any element of M' is parallel to $\partial\beta$, then we are done. We henceforth assume that all elements of

M' meet $\partial\beta$. Since M' is of cardinality $l \geq 8$ and since $W(M' \cup \{\partial\alpha\})$ is infinite dihedral, Lemma 11 implies that $\partial\alpha$ meets $\partial\beta$, which is absurd.

It remains to deal with the case when $\partial\alpha \in M$. If any element of M distinct from $\partial\alpha$ is parallel to β, then we are done. Thus we may assume that all other elements of M meet $\partial\beta$. Repeating the same arguments as above appealing to Theorem 8 and Lemma 9(ii), we obtain a set $M'' \subset \mathfrak{M}(x, y)$, of cardinality $2l$, consisting of pairwise parallel walls, which all separate $\partial\beta$ from x and such that $W(M'' \cup \{\partial\beta\})$ is infinite dihedral. Once again, if any element of M'' is parallel to $\partial\alpha$ then we are done. Otherwise, we deduce from Lemma 11 that $\partial\beta$ meets $\partial\alpha$, a contradiction. □

6 The parabolic closure of a dihedral reflection subgroup

A basic fact on Coxeter groups is that any intersection of parabolic subgroups is itself a parabolic subgroup. This allows to define the *parabolic closure* $\mathrm{Pc}(R)$ of a subset $R \subset W$: it is the smallest parabolic subgroup of W containing R.

Given two roots $\alpha, \beta \in \Phi(\mathfrak{E})$ such that $\alpha \neq \pm\beta$, it a well-known consequence of Lemma 3 that the dihedral group $\langle r_\alpha, r_\beta \rangle$ is infinite if and only if $|(\alpha, \beta)| \geq 1$. The main result of this section is the following:

Proposition 14 *Assume that $\mathfrak{E}$ is the standard root basis. Let $\alpha, \beta \in \Phi(\mathfrak{E})$ be such that $\alpha \neq \pm\beta$. Then there exists a constant $\varepsilon > 0$, depending only on the Coxeter system (W, S), such that the following assertions hold:*

(i) *If $|(\alpha, \beta)| = 1$, then the parabolic closure of $\langle r_\alpha, r_\beta \rangle$ is of irreducible affine type; in particular, it is virtually abelian.*

(ii) *If $|(\alpha, \beta)| > 1$, then $|(\alpha, \beta)| \geq 1 + \varepsilon$ and the parabolic closure of $\langle r_\alpha, r_\beta \rangle$ is not of affine type; in particular, it contains a free subgroup of rank 2.*

The proof of Theorem D of the introduction, which is given in Section 6.4 below, is a combination of Propositions 7 and 14.

6.1 Orbits of pairs of walls

Since W has finitely many conjugacy classes of reflections, it follows that W has finitely many orbits of walls in $\mathfrak{A}$. More generally, we have the following:

Lemma 15 *For any $r \in \mathbb{R}_+$, the group W has finitely many orbits of pairs of walls $\{m, m'\}$ such that $d(m, m') \leq r$.*

Proof If two walls meet, then the corresponding reflections generate a finite subgroup. Since W has finitely many conjugacy classes of finite subgroups, it follows immediately that W has finitely many orbits of pairs of intersecting walls. Therefore, it suffices to consider orbits of pairs of parallel walls, or equivalently, pairs of disjoint half-spaces.

As in the proof of Theorem 8, we will consider only the combinatorial distance on $V(\mathfrak{A})$, which we denote again by d. To every pair of disjoint half-spaces $\{\alpha, \beta\}$, we associate an oriented combinatorial path $\gamma(\alpha;\beta)$ in the 1–skeleton $\mathfrak{A}_0$ as follows: Choose $x, y \in V(\mathfrak{A})$ such that $x \in \alpha$, $y \in \beta$ and $d(x, y)$. Since α and β are disjoint, the walls $\partial\alpha$ and $\partial\beta$ both belong to $\mathfrak{M}(x, y)$ and, hence, there exists a geodesic path joining x to y in $\mathfrak{A}_0$, whose first edge crosses $\partial\alpha$ and whose last edge crosses $\partial\beta$. We define $\gamma(\alpha;\beta)$ to be any such geodesic path. Clearly the length of $\gamma(\alpha;\beta)$ equals $d(\alpha, \beta)$ and moreover, if $\{\alpha', \beta'\}$ is a pair of disjoint roots such that $w.\gamma(\alpha';\beta') = \gamma(\alpha;\beta)$ for some $w \in W$, then $w.\alpha' = \alpha$ and $w.\beta' = \beta$. Therefore, in order to prove the desired finiteness property, it suffices to show that W has finitely many orbits on oriented combinatorial paths of length $\leq r$. This is true because W is transitive on $V(\mathfrak{A})$ and $\mathfrak{A}$ is locally finite. □

6.2 On the parabolic closure

By definition of the Davis complex $\mathfrak{A}$, the stabilizer of any point of $\mathfrak{A}$ in W is a finite parabolic subgroup. Since any finite subgroup of W fixes a point of $\mathfrak{A}$, it follows that any finite subgroup is contained in a finite parabolic subgroup; in other words, if a subgroup of W is finite, then so is its parabolic closure. This is well-known. A fundamental result, due to D Krammer, is that this is also true for affine reflection subgroups:

Proposition 16 *Given any reflection subgroup R, if R is of irreducible affine type and rank ≥ 3, then so is its parabolic closure.*

Proof Notice first that if a reflection subgroup is of irreducible type, then so is its parabolic closure. Therefore, the desired assertion follows from [5, Theorem 3.3], which builds upon the results of [16, Chapter 6]. □

The following lemma provides useful criterions ensuring that a given reflection belongs to a certain parabolic closure:

Lemma 17 *Let M be a set of cardinality at least 2, consisting of pairwise parallel walls and let m be any wall. Assume that any of the following conditions hold:*

(1) *m separates two elements of M.*

(2) *$W(M \cup \{m\})$ is infinite dihedral.*

(3) *$W(M \cup \{m\})$ is a Euclidean triangle subgroup.*

Then the reflection r_m belongs to the parabolic closure $\mathrm{Pc}(W(M))$.

Proof Point (1) follows from the same convexity arguments as in the proof of Lemma 9(i).

(2) The hypotheses imply that $W(M)$ is infinite dihedral. Let t be a nontrivial translation of $W(M)$ and let $\mu \in M$. Since $W(M \cup \{m\})$ is infinite dihedral, there exists a nonzero integer $n \in \mathbb{Z}$ such that m separates μ from $t^n.\mu$. By (1), this yields $r_m \in \mathrm{Pc}(W(\mu, t^n.\mu))$. On the other hand, we have clearly $W(\mu, t^n.\mu) \subset W(M)$, whence $\mathrm{Pc}(W(\mu, t^n.\mu)) \subset \mathrm{Pc}(W(M))$.

(3) By Proposition 16, the parabolic subgroup $\mathrm{Pc}(W(M \cup \{m\}))$ is of irreducible affine type. On the other hand $W(M) \subset W(M \cup \{m\})$ and hence $\mathrm{Pc}(W(M)) \subset \mathrm{Pc}(W(M \cup \{m\}))$. Since any proper parabolic subgroup of a parabolic of irreducible affine type is finite while $W(M)$ is infinite, we deduce that $\mathrm{Pc}(W(M)) = \mathrm{Pc}(W(M \cup \{m\}))$. □

6.3 Proof of Proposition 14

In order to simplify notation, we assume that a base chamber $C \subset \mathfrak{A}$ has been fixed and we identify $\Phi(\mathfrak{S})$ with $\Phi(\mathfrak{A})$ be means of the bijection ζ_C of Lemma 3(i), and we note $\Phi = \Phi(\mathfrak{S}) = \Phi(\mathfrak{A})$.

Proof of Proposition 14(i) Let $\alpha, \beta \in \Phi$ be roots such that $\alpha \neq \pm\beta$ and $|(\alpha, \beta)| = 1$. We must prove that $\mathrm{Pc}(r_\alpha, r_\beta)$ is of irreducible affine type. Since $\langle r_\alpha, r_\beta \rangle$ is infinite and since every proper parabolic subgroup of a Coxeter group of irreducible affine type is finite, it suffices to prove that r_α and r_β are contained in a common parabolic subgroup of irreducible affine type.

Up to replacing α or β by its opposite, we may assume without loss of generality that $\alpha \subset \beta$, whence $(\alpha, \beta) = 1$ by Lemma 3(iii).

Let $x, y \in V(\mathfrak{A})$ be vertices such that $x \in \alpha$, $y \in -\beta$ and $d(x, y) = d(\alpha, -\beta)$. Let $\Phi(x, y)$ be the set of all half-spaces containing x but not y. By Lemma 6, the set $\Phi(\alpha; \beta)$ is a chain. We denote it by

$$\alpha = \alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_k = \beta.$$

Moreover, Lemma 6 implies that the group $W(\Phi(\alpha; \beta)) = \langle r_{\alpha_i} \mid i = 0, \ldots, k \rangle$ is infinite dihedral, generated by $\{r_{\alpha_0}, r_{\alpha_1}\}$.

Let $t = r_\alpha r_\beta$. A straightforward computation shows that $(\alpha, t^n.\beta) = 1$ for all $n \in \mathbb{Z}$. Moreover, we have $r_\beta \in \mathrm{Pc}(\langle r_\alpha, r_{t^n.\beta}\rangle)$ for all $n \neq 0$ by Lemma 17. Therefore, up to replacing β by $t^n.\beta$ with n sufficiently large, we may assume without loss of generality that $k \geq 8$.

Assume first that $\Phi(\alpha;\beta) = \Phi(x, y)$. Then, considering a geodesic path joining x to y in the 1–skeleton of $\mathfrak{A}$, we see that there is a combinatorial path of length 2 crossing successively the walls $\partial\alpha_0$ and $\partial\alpha_1$. This means that the infinite dihedral group $W(\Phi(\alpha;\beta)) = \langle r_{\alpha_0}, r_{\alpha_1}\rangle$ is a parabolic subgroup of rank 2. Thus it is of irreducible affine type and it contains r_β since $\beta \in \Phi(\alpha;\beta)$. Hence we are done in this case.

Assume now that $\Phi(\alpha;\beta)$ is properly contained in $\Phi(x, y)$ and consider an element $\gamma \in \Phi(x, y)\backslash\Phi(\alpha;\beta)$. By the definition of $\Phi(\alpha;\beta)$, it follows that $\partial\gamma$ must meet $\partial\alpha$ or $\partial\beta$. Without loss of generality, we assume that $\partial\gamma$ meets $\partial\beta$. Let

$$\Phi_0(\gamma) = \{\phi \in \Phi(\alpha;\beta) \mid \partial\phi \text{ meets } \partial\gamma\}.$$

Since walls are convex, it follows that if $\alpha_i, \alpha_{i'} \in \Phi_0(\gamma)$, then $\alpha_j \in \Phi_0(\gamma)$ for all $i \leq j \leq i'$.

Note that for each $\phi \in \Phi(x, y)$, the reflections r_β and r_ϕ do not commute, otherwise we would have $x \in \phi = r_\beta(\phi)$, in contradiction with Lemma 12(ii).

Suppose that $|\Phi_0(\gamma)| > 7$; this happens whenever $\alpha \in \Phi_0(\gamma)$. Since $W(\Phi(\alpha;\beta))$ is infinite dihedral, this implies by Lemma 11 that either r_γ centralizes the group $W(\Phi(\alpha;\beta))$ or that $W(\Phi(\alpha;\beta))$ is contained in a Euclidean triangle subgroup. We have just seen r_γ does not centralize $r_\beta \in W(\Phi(\alpha;\beta))$, so $W(\Phi(\alpha;\beta))$ is contained in a Euclidean triangle subgroup. By Proposition 16, a Euclidean triangle subgroup is contained in a parabolic subgroup of irreducible affine type, hence we are done in this case.

Similarly, if the wall $\partial r_\beta(\gamma)$ meets the boundary of more than 7 elements of $\Phi(\alpha;\beta)$, it follows also that $W(\Phi(\alpha;\beta))$ is contained in a Euclidean triangle group, because $r_{r_\beta(\gamma)} = r_\beta r_\gamma r_\beta$ does not commute with r_β since r_γ does not. Hence we are done in this case as well.

It remains to consider the case when $\partial\alpha$ meets neither $\partial\gamma$ nor $\partial r_\beta(\gamma)$. This yields $\alpha \subset \gamma$ and $\alpha \subset -r_\beta(\gamma)$, whence $(\alpha,\gamma) \geq 1$ and $(\alpha, r_\beta(\gamma)) \leq -1$ by Lemma 3(iii). Therefore we obtain

$$2 \leq 1 + (\alpha,\gamma) \leq 2(\alpha,\beta)(\beta,\gamma).$$

By Lemma 3(iv) and Lemma 12(i), we have $(\beta,\gamma) > 0$. Thus we obtain

$$(\alpha,\beta) \geq \frac{1}{(\beta,\gamma)}.$$

On the other hand $(\beta, \gamma) < 1$ by Lemma 3(ii), because $\partial\beta$ meets $\partial\gamma$. This contradicts the fact that $(\alpha, \beta) = 1$, thereby proving that this last case does not occur. □

Proof of Proposition 14(ii) If (W, S) is of irreducible affine type, then Tits bilinear form is positive semi-definite [2, Chapter VI, Section 4.3, Theorem 4]. Therefore, for all $\alpha, \beta \in \Phi$ we have $|(\alpha, \beta)| \leq \sqrt{(\alpha, \alpha)(\beta, \beta)} = 1$ by Cauchy–Schwarz. This shows that if (W, S) is arbitrary and if $\mathrm{Pc}(r_\alpha, r_\beta)$ is of irreducible affine type, then $|(\alpha, \beta)| \leq 1$.

It remains to prove that $\varepsilon > 0$, where

$$\varepsilon = -1 + \inf\{(\alpha, \beta) \mid \alpha, \beta \in \Phi,\ (\alpha, \beta) > 1\}.$$

To this end, we define P_0 to be the set of all ordered pairs $(\alpha; \beta) \in \Phi \times \Phi$ such that

(1) $\alpha \subset \beta$,

(2) $(\alpha, \beta) > 1$,

(3) For all $\gamma \in \Phi$ such that $\alpha \subset \gamma \subset \beta$, one has $\gamma \in \{\alpha, \beta\}$.

We also define P_1 to be the set of all ordered pairs $(\alpha; \beta) \in \Phi \times \Phi$ satisfying (1) and (2) but not (3). In view of Lemma 3(iii), we have

$$1 + \varepsilon = \inf\{(\alpha, \beta) \mid (\alpha; \beta) \in P_0 \cup P_1\}.$$

In other words, we have $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$, where $1 + \varepsilon_i = \inf\{(\alpha, \beta) \mid (\alpha; \beta) \in P_i\}$ for $i = 0, 1$.

Condition (3) means that the walls $\partial\alpha$ and $\partial\beta$, which are parallel, are not separated by any wall. By Theorem C, this implies that the distance from $\partial\alpha$ to $\partial\beta$ is at most Q. Therefore, the group W has finitely many orbits in P_0 by Lemma 15. In particular the set $\{(\alpha, \beta) \mid (\alpha; \beta) \in P_0\}$ is finite and, hence, we have $\varepsilon_0 > 0$.

Let now $(\alpha; \beta) \in P_1$. The set $\Phi(\alpha; \beta) = \{\phi \in \Phi \mid \alpha \subset \phi \subset \beta\}$ is finite. Since $(\alpha; \beta) \notin P_0$, there exists $\gamma \in \Phi(\alpha; \beta)$ distinct from α and β. Among all such γ's, we choose one which is minimal for $\subset$. In particular, the pair $(\alpha; \gamma)$ satisfies condition (3). Since $\alpha \subset \gamma$ we have $(\alpha, \gamma) \geq 1$ by Lemma 3(iii). There are two cases.

First, assume that $(\alpha, \gamma) > 1$. In that case we have $(\alpha; \gamma) \in P_0$ whence $(\alpha, \gamma) \geq 1 + \varepsilon_0$. By Lemma 15(i), this yields $(\alpha, \beta) \geq 1 + \varepsilon_0$.

Assume now that $(\alpha, \gamma) = 1$. Choose $\gamma' \in \Phi(\alpha; \beta)$ such that $(\alpha, \gamma') = 1$ and such that γ' is maximal (for $\subset$) with respect to these properties. By Lemma 6, the set $\Phi(\alpha; \gamma')$, which is contained in $\Phi(\alpha; \beta)$, is a chain. Let γ'' be the maximal element of $\Phi(\alpha; \gamma') \backslash \{\gamma'\}$ and consider $r_{\gamma'}(\gamma'')$. Using Lemma 6, it is immediate to compute that

$(\alpha, r_{\gamma'}(\gamma'')) = -1$; in particular, by maximality of γ' we have $-r_{\gamma'}(\gamma'') \notin \Phi(\alpha;\beta)$, or in other words $-r_{\gamma'}(\gamma'') \not\subset \beta$. Similarly, if $\beta \subset -r_{\gamma'}(\gamma'')$, then Lemma 5(i) yields $(\alpha,\beta) \leq (\alpha, -r_{\gamma'}(\gamma'')) = 1$, contradicting (2). Therefore, Lemma 5(iii) implies that $(r_{\gamma'}(\gamma''),\beta) > -1$. Using Lemma 5(i) and (ii), this yields successively

$$(\alpha,\beta) \;\geq\; (\gamma'',\beta) \;\geq\; 2(\gamma',\beta)-\kappa \;\geq\; 2-\kappa.$$

The preceding two paragraphs show that for all $(\alpha;\beta) \in P_1$, we have $(\alpha,\beta) \geq \max\{1+\varepsilon_0, 2-\kappa\}$, whence $\varepsilon_1 = \min\{\varepsilon_0, 1-\kappa\}$. Recall from Section 3.1 that $\kappa < 1$; furthermore, we have seen above that $\varepsilon_0 > 0$. This shows finally that $\varepsilon > 0$, as desired. □

6.4 Proof of Theorem D

Let $\alpha_0 \subsetneq \alpha_1 \subsetneq \cdots \subsetneq \alpha_n$ be a sequence of roots. If $(\alpha_0,\alpha_n) = 1$, then one is in the alternative (2) of the theorem by Lemmas 6, 17 and Proposition 14(i). Thus we may assume that $(\alpha_0,\alpha_n) > 1$ and we must show that (α_0,α_n) is bounded from below by a non-decreasing function of n.

In order to estimate (α_0,α_n), we choose a maximally convex chain $\beta_0 \subsetneq \cdots \subsetneq \beta_m$ such that $\beta_0 = \alpha_0$ and $\beta_m = \alpha_n$. By definition, we have $m \geq n$. Let $j = \min\{i \geq 0 \mid (\beta_0,\beta_i) > 1\}$. Thus $j > 0$. By Proposition 7, we have $(\beta_0,\beta_m) > \max\{j(1-\kappa),\ 1+\frac{m}{2j}\varepsilon\}$, where ε is the constant of Proposition 14(ii). One easily computes that

$$j(1-\kappa) \geq 1 + \frac{m}{2j}\varepsilon \qquad \Leftrightarrow \qquad j \geq \frac{-1+\sqrt{1+2\varepsilon(1-\kappa)m}}{1-\kappa}.$$

Therefore, one has

$$(\beta_0,\beta_m) > \sqrt{1+2\varepsilon(1-\kappa)m}-1 \qquad \text{if} \qquad j \geq \frac{-1+\sqrt{1+2\varepsilon(1-\kappa)m}}{1-\kappa}$$

and

$$(\beta_0,\beta_m) > 1 + \frac{\varepsilon(1-\kappa)m}{-2+2\sqrt{1+2\varepsilon(1-\kappa)m}} \qquad \text{otherwise.}$$

Set $r_0 = 1$ and for all positive integer k, define

$$r_k = \min\Big\{-1+\sqrt{1+2\varepsilon(1-\kappa)k},\ 1+\frac{\varepsilon(1-\kappa)k}{-2+2\sqrt{1+2\varepsilon(1-\kappa)k}}\Big\}.$$

Note that $(r_k)_{k\geq 1}$ is a non-decreasing sequence with $r_1 > 1$, which tends to $+\infty$ with k. We have seen above that $(\beta_0,\beta_m) \geq r_m$. Since $\beta_0 = \alpha_0$ and $\beta_m = \alpha_n$ and $m \geq n$, this yields $(\alpha_0,\alpha_n) \geq r_n$, and we are in the alternative (1) of the theorem. This finishes the proof. □

7 Conjugacy of 2–spherical subgroups

The purpose of this section is to prove Theorem A and its corollary. As mentioned in the introduction, this first requires to study in details some aspects of the CAT(0) cube complex $\mathfrak{X}$ of G Niblo and L Reeves; see Proposition 24 and Theorem 27 below for specific statements.

7.1 Pairwise intersecting walls

Any n–dimensional cube c of a CAT(0) cube complex determines a unique n–tuple of walls, denoted by $M(c)$, consisting of the walls which contain the center of that cube. The following basic fact is an important property:

Lemma 18 *Let c, c' be cubes of a CAT(0) cube complex such that every wall in $M(c)$ meets every wall in $M(c')$. Let d be the combinatorial distance from c to c'. Then there exists a cube c'' at combinatorial distance at most d from c, such that $M(c'') = M(c) \cup M(c')$.*

Proof See [20, Theorem 4.14]. The estimate of the distances follows from an easy induction. □

The only cube complex we will need to consider here is the CAT(0) complex $\mathfrak{X}$. We have seen in Section 2.4 that the walls and hyperplanes of $\mathfrak{X}$ are in canonical one-to-one correspondence with those of $\mathfrak{A}$. In order to simplify notation, we identify $\mathfrak{M}(\mathfrak{A})$ with $\mathfrak{M}(\mathfrak{X})$ (resp. $\Phi(\mathfrak{A})$ with $\Phi(\mathfrak{X})$). A fundamental difference between $\mathfrak{A}$ and $\mathfrak{X}$ is that a reflection of W fixes a wall of $\mathfrak{A}$ pointwise, while in $\mathfrak{X}$ a reflection acts mostly non-trivially on the wall it stabilizes. This explains why the property of Lemma 18 cannot hold in $\mathfrak{A}$, unless (W, S) is right-angled in which case $\mathfrak{A} = \mathfrak{X}$. However, pairs of walls behave always similarly in $\mathfrak{A}$ and in $\mathfrak{X}$:

Lemma 19 *Two walls meet in $\mathfrak{A}$ if and only if they meet in $\mathfrak{X}$. Two half-spaces are nested (resp. have empty intersection) in $\mathfrak{A}$ if only if they are nested (resp. have empty intersection) in $\mathfrak{X}$.*

Proof This is a straightforward consequence of the construction of $\mathfrak{X}$, see Section 2.4 and [18] for more details. □

Note that the corresponding statement fails for triples of walls. Indeed, every triple of pairwise intersecting walls has nonempty intersection in $\mathfrak{X}$ by Lemma 18, but this is obviously false in $\mathfrak{A}$: any Euclidean or hyperbolic triangle group provides a counterexample.

As above, for any set of walls M, we denote by $W(M)$ the subgroup of W generated by all reflections through elements of M.

Lemma 20 *Let M be a set of pairwise intersecting walls. Then $W(M)$ is of 2–spherical type.*

Proof By Lemma 19, if two walls meet in $\mathfrak{X}$, then they meet in $\mathfrak{A}$ and, hence, the corresponding reflections generate a finite subgroup. This shows that $W(M)$ is 2–spherical in the sense of Section 1.1. We have seen in this this section that 2–spherical Coxeter groups are precisely those Coxeter groups of 2–spherical type, in the usual sense. This means that $W(M)$ is of 2–spherical type as a reflection subgroup. □

7.2 On standard parabolic subgroups

Recall that a *standard parabolic subgroup* of W is a subgroup generated by some subset of S, and a subgroup is *parabolic* if it is conjugate to a standard parabolic subgroup. In some situations, it is useful to keep track of an element of W which conjugates a given parabolic subgroup to a standard parabolic one. This motivates the following definition: given a vertex v_0 (or a chamber) of $\mathfrak{A}$ and a set M_0 of walls such that each element of M_0 separates v_0 from a neighboring vertex, we say that the parabolic subgroup $W(M_0)$ is *standard with respect to* v_0. Thus a standard parabolic subgroup is a parabolic subgroup which is standard with respect to the base chamber of $\mathfrak{A}$ (ie, the chamber which corresponds to the identity $1 \in W$ in the Cayley graph).

Lemma 21 *Let M be a set of pairwise parallel walls, of cardinality > 7, such that the parabolic closure $P = \mathrm{Pc}(W(M))$ is of affine type. Let v_0 be a vertex of $\mathfrak{A}$. Assume that, given any wall m, if m separates v_0 from some wall in M, then $m \in M$. Then the parabolic subgroup P is standard with respect to v_0.*

Proof Given a parabolic subgroup W_0 of W and a chamber c of $\mathfrak{A}$ such that W_0 is standard with respect to c, then the union of the W_0–orbit of c is a closed convex subset of $\mathfrak{A}$, which we call a W_0*–residue*. Clearly, the parabolic subgroup W_0 is standard with respect to a given chamber if and only if this chamber is contained in some W_0–residue.

Let c_0 be the unique chamber of $\mathfrak{A}$ containing v_0 and let ρ_0 be the W_0–residue at minimal combinatorial distance from c_0. We must prove that $c_0 \subset \rho_0$. Assume the contrary in order to obtain a contradiction. Let c' be the chamber of ρ_0 at minimal combinatorial distance from c_0, let c'' be a chamber adjacent to c' and closer to c_0 than c'. Finally, let m be the wall which separates c' from c''.

The reflection r_m does not belong to P. Indeed, if $r_m \in P$, then r_m would stabilize ρ_0 which would imply that $c'' = r_m(c')$ be contained in ρ_0, in contradiction with the definition of c'. It follows in particular that the wall m separates c_0 from ρ_0: otherwise m would separate two adjacent chambers contained in ρ_0 and hence r_m would swap these chambers, but, on the other hand, the only element of W swapping these chambers belongs to P.

Since $r_m \notin P$ we have $m \notin M$. Moreover, any wall $m' \in M$ meets ρ_0 because the reflection $r_{m'} \in P$ stabilizes ρ_0 which is closed and convex. Therefore, the wall m meets m' otherwise m would separate v_0 from m' which is excluded by hypothesis. Thus m meets each element of M.

Since M has at least 8 elements, it follows from Lemma 11 that either r_m centralizes $W(M)$ or that $W(M \cup \{m\})$ is a Euclidean triangle subgroup. The second case is impossible by Lemma 17(3) because $r_m \notin P = \mathrm{Pc}(W(M))$. Thus r_m centralizes $W(M)$ and, hence, normalizes the parabolic closure P of $W(M)$.

The set $r_m(\rho_0)$ is a $(r_m P r_m)$–residue and, hence, a P–residue by the preceding paragraph. Since $r_m(\rho_0)$ contains the chamber $c'' = r_m(c')$, we obtain a contradiction with the minimality assumption we made on ρ_0. This finishes the proof. □

7.3 The normalizer of an affine parabolic subgroup

The following fact is well-known; it is more generally true for any infinite parabolic subgroup of irreducible type.

Lemma 22 *Let $P \subset W$ be a parabolic subgroup of irreducible affine type. Then the normalizer of P in W splits as a direct product: $N_W(P) = P \times C_W(P)$. In particular, any reflection which normalizes P either belongs to P or centralizes P.*

Proof See [8, Proposition 5.5]. □

7.4 Free abelian normal subgroups in Coxeter groups

The following statement of independent interest is a consequence of the work of Daan Krammer [16, Section 6.8] on free abelian subgroups of Coxeter groups:

Lemma 23 *The group W possesses a nontrivial free abelian normal subgroup if and only if the Coxeter diagram of (W,S) has a connected component of irreducible affine type.*

Proof If (W,S) has a connected component of affine type, then the translation subgroup of the corresponding affine parabolic subgroup is a nontrivial free abelian normal subgroup. Thus the 'if' part is clear. Conversely, let H be a nontrivial free abelian normal subgroup of W. Let $W = W_1 \times \dots W_k$ be the decomposition of W into its direct components and let pr_i be the canonical projection of W onto W_i. By Selberg's lemma W has a finite index torsion free subgroup. In particular H has a finite index subgroup H' such that $\mathrm{pr}_i(H')$ is torsion free for all i. Since H is free abelian, we may assume without loss of generality that H' is normalized by W. In particular $\mathrm{pr}_i(H')$ is a free abelian normal subgroup of W_i for each i. This shows that, in order to finish the proof, it suffices to show that an *irreducible* Coxeter group which possesses a nontrivial free abelian normal subgroup must be of affine type.

We assume henceforth that (W,S) is irreducible, but not of affine type otherwise we are done. Since H is normal in W, so is its parabolic closure $\mathrm{Pc}(H)$. Since a parabolic subgroup is normal if and only if it is a direct component, it follows that $\mathrm{Pc}(H) = W$ since (W,S) is irreducible. This is true even after replacing H by a finite index subgroup normalized by W. Therefore, [16, Theorem 6.8.3] implies that H is of rank one because (W,S) is not of affine type. In particular the centralizer of H in W is of index at most 2. On the other hand H is of finite index in its centralizer by [16, Corollary 6.3.10]. This shows that H is of finite index in W. Since W is not of affine type, it follows that W must be finite and, hence, that H is trivial. This is a contradiction. □

7.5 The cubical chamber

Recall from Section 2.4 that the Cayley graph of (W,S) is equivariantly embedded in the 1–skeleton $\mathcal{X}^{(1)}$ of $\mathcal{X}$. We denote this subgraph by $\mathcal{X}_0$. Given a vertex $v \in \mathcal{X}_0$, let $\Psi(v)$ be the set of those half-spaces which contain v but not its neighbors in $\mathcal{X}_0$. The set $\bigcap_{\psi\in\Psi(v)} \psi$, viewed as a subset of $\mathcal{X}$, is called the *cubical chamber* containing v. Given two points $x, y \in \mathcal{X}$ we denote by $\mathfrak{M}(x,y)$ the set of walls which separate x from y.

Proposition 24 *Let v_0 be a vertex of $\mathcal{X}_0$ and let x be a vertex of $\mathcal{X}$ belonging to the cubical chamber containing v_0. Let $M \subset \mathfrak{M}(v_0,x)$ be a set of pairwise parallel walls. There exists a constant $K = K(W,S)$ such that if M has more than K elements, then $W(M)$ is infinite dihedral and its parabolic closure $\mathrm{Pc}(W(M))$ is of irreducible affine type and rank ≥ 3.*

Proof Define a constant $\lambda_{\max}$ as follows:

$$\lambda_{\max} = \sup\{\lambda \in \mathbb{R}_+ \mid \lambda \text{ is a coefficient of some } \phi \in \Phi(\mathfrak{S})^+_{\min} \text{ in the basis } \Pi\},$$

where

$$\Phi(\mathfrak{S})^+_{\min} = \{\alpha \in \Phi(\mathfrak{S})^+ \mid \phi \in \Phi(\mathfrak{S})^+, \phi \subset \alpha \Rightarrow \phi = \alpha\}.$$

It follows from the parallel wall theorem [3, Theorem 2.8] that the set $\Phi(\mathfrak{S})^+_{\min}$ is finite. Therefore, the constant $\lambda_{\max}$ is a well-defined positive real number.

Let also $r = |S|$ be the rank of (W, S) and κ be the constant defined in Section 3.1. We will show that the desired constant K can be defined as $K = \min\{n \in \mathbb{N} \mid r_n > r\kappa\lambda_{\max}\}$, where $(r_n)_{n\in\mathbb{N}}$ is the sequence of Theorem D.

Let $\Phi(x, v_0)$ be the set of those half-spaces which contain x but not v_0. The set $\Phi(x, v_0)$ contains a nested sequence $\phi_0 \subsetneq \phi_1 \subsetneq \cdots \subsetneq \phi_l$ of half-spaces such that $M \subset \{\partial\phi_i \mid i = 0, 1, \ldots, l\}$. Without loss of generality, we may – and shall – assume that $(\phi_i)_{i\le l}$ is a maximal nested sequence contained in $\Phi(x, v_0)$: since $\Phi(x, v_0)$ is finite, any nested sequence contained in $\Phi(x, v_0)$ can be completed in order to obtain a maximal nested sequence. Note that if $(\phi_i)_{i\le l}$ is maximal, then no wall separates v_0 from $\partial\phi_l$, because if such a wall existed, we could lengthen the nested sequence $(\phi_i)_{i\le l}$ of one unit by adding the half-space containing x and determined by this extra wall.

Let Ψ be the set of those half-spaces which contain v_0 but not its neighbors in $\mathcal{X}_0$ and let $\Psi_0 = \{\psi \in \Psi \mid \phi_0 \not\subset \psi\}$. Note that every half-space contains a vertex of $\mathcal{X}_0$. Furthermore, it follows from the definition of Ψ that the only vertex of $\mathcal{X}_0$ which is contained in $\bigcap_{\psi\in\Psi} \psi$ is v_0. Therefore, we deduce that for each $\phi \in \Phi(x, v_0)$, there exists $\psi \in \Psi$ such that $\phi \not\subset \psi$. In particular, the set Ψ_0 is nonempty. Since the sequence $(\phi_i)_{i\le l}$ is nested, we deduce that $\phi_i \not\subset \psi$ for all $i = 0, 1, \ldots, l$ and all $\psi \in \Psi_0$.

Note that for all $i \in \{0, 1, \ldots, l\}$ and all $\psi \in \Psi_0$, the vertex x is contained in $\phi_i \cap \psi$ while the vertex v_0 is contained in ψ but not in ϕ_i. Since $\phi_i \not\subset \psi$ and since v_0 belongs to an edge crossed by $\partial\psi$, it then follows that the walls $\partial\phi_i$ and $\partial\psi$ meet for all $i \in \{0, 1, \ldots, l\}$ and all $\psi \in \Psi_0$.

Let us choose as base chamber $C \subset \mathfrak{A}$ the unique chamber containing the vertex v_0. Once this chamber has been fixed, we know by Lemma 3(i) that the sets $\Phi(\mathfrak{S})$ and $\Phi(\mathfrak{A})$ are in canonical W–equivariant bijection. In order to simplify notation, we omit to write the function ζ_C which realizes this bijection and identify thereby the sets $\Phi(\mathfrak{S})$ and $\Phi(\mathfrak{A})$. In this way, the set Π of the root basis $\mathfrak{S}$ is identified with Ψ, the set $\Phi(\mathfrak{S})^+$ is identified with the set of half-spaces containing v_0 and the set $\Phi(\mathfrak{S})^+_{\min}$

with the set of those half-spaces h which contain v_0 and such that no wall separates v_0 from ∂h.

We claim that $(\phi_0, \phi_l) = 1$. In order to prove the claim, we will apply Proposition 7. It follows from the above that $-\phi_l \in \Phi(\mathfrak{S})^+_{\min}$. Write $-\phi_l = \sum_{\psi\in\Psi} \lambda_\psi \psi$ with $\lambda_\psi \geq 0$. Note that by the definition of Ψ_0, we have $\phi_0 \subset \psi$, whence $(\phi_0, \psi) \geq 1$ by Lemma 3(iii), for all $\psi \in \Psi\backslash\Psi_0$. Moreover, we have seen that $\partial\phi_0$ meets $\partial\psi$, whence $|(\phi_0, \psi)| \leq \kappa$, for each $\psi \in \Psi_0$. Therefore, we have:

$$\begin{aligned}(\phi_0, \phi_l) &= (-\phi_0, -\phi_l)\\ &= \textstyle\sum_{\psi\in\Psi} \lambda_\psi(-\phi_0, \psi)\\ &\leq \textstyle\sum_{\psi\in\Psi_0} \lambda_\psi(-\phi_0, \psi)\\ &\leq \textstyle\sum_{\psi\in\Psi_0} \kappa\lambda_{\max}\\ &= |\Psi_0|\kappa\lambda_{\max}\\ &\leq r\kappa\lambda_{\max}.\end{aligned}$$

Since $l \geq K$, we have $r_l \geq r_K > r\kappa\lambda_{\max}$ and, therefore, we deduce from Theorem D that $(\phi_0, \phi_l) = 1$. Moreover, the group $\langle r_{\phi_i} \mid i = 0, \dots, l\rangle$ is infinite dihedral and its parabolic closure P is of irreducible affine type. By Lemma 17, we have $P = \mathrm{Pc}(W(M))$.

It remains to show that P is of rank at least 3. By Lemmas 3(iv) and 12(i), there exists $\psi \in \Psi$ such that $(\psi, \phi_0) < 0$. We have seen above that if $\psi \notin \Psi_0$ then $(\psi, \phi_0) \geq 1$. Thus there exists $\psi \in \Psi_0$ such that $(\psi, \phi_0) < 0$. We have seen that $\partial\psi$ meets $\partial\phi_i$ for each $i = 0, 1, \dots, l$. Since $l \geq 8$, we deduce from Lemma 11 that either the reflection r_ψ centralizes $\langle r_{\phi_i} \mid i = 0, 1, \dots, l\rangle$ or $\langle r_\psi, r_{\phi_i} \mid i = 0, 1, \dots, l\rangle$ is a Euclidean triangle subgroup. But r_ψ does not commute with r_{ϕ_0}. Thus by Lemma 17 P contains a Euclidean triangle subgroup and, hence, it is of rank ≥ 3. □

Remark Note that a cubical chamber contains finitely many vertices if and only if the W–action on $\mathcal{X}$ is co-compact. Thus Proposition 24 implies that if the W–action is not co-compact then W possesses a parabolic subgroup of irreducible affine type and rank ≥ 3. Conversely, if W has such a parabolic subgroup, then it is easily seen that some, whence any, cubical chamber contains infinitely many vertices. Therefore, we recover the characterization of all those Coxeter groups acting co-compactly on $\mathcal{X}$; this was first established in [5].

Combining the preceding proposition with Lemma 21, one obtains the following useful precision:

Corollary 25 *Let v_0 be a vertex of $\mathcal{X}_0$ and let x be a vertex of $\mathcal{X}$ belonging to the cubical chamber containing v_0. Let $M \subset \mathfrak{M}(v_0, x)$ be a set of pairwise parallel walls* *of cardinality greater than* $K+8$, *where* K *is the constant of Proposition 24. Then the parabolic subgroup* $\mathrm{Pc}(W(M))$ *is standard with respect to* v_0.

Proof Up to enlarging M is necessary, we may – and shall – assume that M is a maximal subset of $\mathfrak{M}(v_0, x)$ consisting of pairwise parallel walls. By Proposition 24, the group $W(M)$ is infinite dihedral and its parabolic closure P is of irreducible affine type (note that enlarging M does not change P).

Consider a wall m which separates v_0 from some $m' \in M$. Let M_0 be the subset of all those elements of M_0 which meet m. By Lemma 11, the set M_0 has at most 7 elements. Therefore, the set $M \backslash M_0 \cup \{m\}$ is a set of pairwise parallel walls contained in $\mathfrak{M}(v_0, x)$, to which we may apply Proposition 24. In particular the group $W(M \backslash M_0 \cup \{m\})$ is infinite dihedral.

Since the set $M \backslash M_0$ contains at least two elements, it follows that $W(M \backslash M_0)$ is infinite. Therefore, we have $\mathrm{Pc}(W(M \backslash M_0)) = P$ because $W(M \backslash M_0) \subset P$ and any proper parabolic subgroup of a parabolic subgroup of irreducible affine type is finite. In view of the preceding paragraph, we deduce from Lemma 17(2) that $r_m \in P$. Since P is of affine type and since m is parallel to some element of M, it follows that m is parallel to all elements of M. By the maximality of M, this yields $m \in M$. The desired assertion now follows from Lemma 21. □

Knowing that a parabolic subgroup is standard with respect to v_0 will be relevant for the following reason:

Lemma 26 *Let* P *be a parabolic subgroup of* W, *which is standard with respect to some vertex* v_0 *of the Cayley graph. Let* M_0 *be a set of walls such that* $W(M_0)$ *is a finite subgroup of* P. *Then there exists a cube* $c_0 \subset \mathfrak{X}$ *and an element* $w \in P$ *such that* $w.c_0$ *contains* v_0 *and that* $M(c_0) = M_0$.

Proof Since $W(M_0)$ is finite, so is its parabolic closure $P_0 = \mathrm{Pc}(W(M_0))$. Clearly, it suffices to prove the lemma for the set M_0 consisting of all those walls m such that $r_m \in P_0$.

Since $P_0 \subset P$ and P is standard with respect to v_0, there exists $w \in P$ such that wP_0w^{-1} is standard with respect to v_0. Let M be the set of those walls m such that $r_m \in wP_0w^{-1}$. Since P_0 is finite, the elements of M meet pairwise by Lemma 19. For each $n \in \mathbb{N}$, let M_n be the subset of M consisting of the walls at combinatorial distance n from v_0 in the Cayley graph $\mathfrak{X}_0$; by convention a vertex is at distance 1 from a wall if that wall separates the vertex from one of its neighbors. Note that $W(M_1) = wP_0w^{-1}$ because wP_0w^{-1} is standard with respect to v_0. In particular M_1 is nonempty. By

Lemma 18, there exists a cube c_1 containing v_0 such that $M(c_1) = M_1$. Applying Lemma 18 inductively, one obtains a nested sequence of cubes $c_1 \subset c_2 \subset \dots$ such that $M(c_n) = \bigcup_{i=1}^n M_i$. Since M is finite, the set M_n is empty for n large enough. Therefore the union $c'_0 = \bigcup_n c_n$ is a cube containing v_0 and such that $M(c'_0) = M$. Now the cube $c_0 = w^{-1}.c'_0$ has the desired property. □

7.6 Tuples of walls which meet far away from the Cayley graph

Theorem A of the introduction will be deduced from the following result:

Theorem 27 *There exists a constant $A = A(W, S)$ such that the following property holds. Let M be a set of walls such that the intersection $\bigcap_{m \in M} m$ is nonempty in $\mathfrak{X}$. If the distance from $\mathfrak{X}_0$ to $\bigcap_{m \in M} m$ is at least A, then $W(M)$ has a direct component of affine type and rank ≥ 3; in particular, it has a free abelian normal subgroup of rank ≥ 2.*

Proof By Lemma 13 combined with Ramsey's theorem, there exists a constant K' such that any set of at least K' walls contains a subset of more than $K + 8$ pairwise parallel walls, where K is the constant of Proposition 24. We choose $A \in \mathbb{R}_+$ large enough so that the ball of combinatorial radius K' centered at some vertex of $\mathfrak{X}_0$ is properly contained in the ball of radius A centered at that same vertex. Since W acts transitively on the vertices of $\mathfrak{X}_0$, the so-defined constant A does not depend on the chosen vertex.

By Lemma 18 there exists a cube $c \subset \mathfrak{X}$ such that $M(c) = M$. In order to prove the theorem, it suffices to show that if every such cube is at combinatorial distance at least K' from $\mathfrak{X}_0$, then $W(M)$ has a direct component of affine type and rank ≥ 3.

Let thus c be a cube such that $M(c) = M$ and that c is at minimal combinatorial distance from $\mathfrak{X}_0$. Let d be this distance, let x be a vertex of c and v_0 be a vertex of $\mathfrak{X}_0$ at combinatorial distance d from x and assume that $d > K'$.

Let Ψ be the set of those half-spaces which contain v_0 but not its neighbors in $\mathfrak{X}_0$. Assume that that $x \notin \psi$ for some $\psi \in \Psi$. Given any minimal path from x to v_0, this path crosses the wall $\partial\psi$. Since $\partial\psi$ separates v_0 from one of its neighbors, say v, it follows that there exists a minimal path from x to v_0 whose last edge crosses $\partial\psi$. Thus the distance from x to v is one less than the distance from x to v_0. This contradicts the definition of v_0 since, by the definition of the elements of Ψ, the vertex v belongs to $\mathfrak{X}_0$. This shows that x belongs to the cubical chamber containing v_0.

The cardinality of $\mathfrak{M}(x, v_0)$ coincides with the combinatorial distance from x to v_0. By the definition of K', it follows that $\mathfrak{M}(x, v_0)$ contains a subset M' of pairwise parallel walls and of cardinality greater than $K+8$. Up to enlarging M' is necessary, we may – and shall – assume that M' is a maximal subset of pairwise parallel walls contained in $\mathfrak{M}(v_0, x)$. By Proposition 24, the group $W(M')$ is infinite dihedral and its parabolic closure, which we denote by P, is of irreducible affine type.

Let x' be a vertex of the cube c neighboring x and let $m \in M$ be the wall which separates x from x'. By definition, the combinatorial distance from x' to $\mathcal{X}_0$ is at least d. This implies that the combinatorial distance from v_0 to x' is $d+1$ and, hence, that $\mathfrak{M}(v_0, x') = \mathfrak{M}(v_0, x) \cup \{m\}$. We now show that the reflection r_m normalizes the parabolic subgroup P.

Assume first that $m = \partial\psi$ for some $\psi \in \Psi$. In that case, the wall m must meet every element of M' otherwise m would be separated from $\partial\psi$ by some element of M', which is absurd since $m = \partial\psi$. By Lemma 11, this implies that either r_m centralizes $W(M')$ or that $W(M' \cup \{m\})$ is a Euclidean triangle subgroup. In the first case, the reflection r_m normalizes P; in the second one, we have $r_m \in P$ by Lemma 17.

Assume now that $m \notin \{\partial\psi \mid \psi \in \Psi\}$. Equivalently, this means that x' belongs to the cubical chamber containing v_0.

Suppose that m does not meet any element of M'. Then $M' \cup \{m\}$ is a set of pairwise parallel walls to which we may apply Proposition 24. This proves that $W(M' \cup \{m\})$ is infinite dihedral and, hence, the reflection r_m through m belongs to P by Lemma 17.

Suppose now that the subset $M'(m)$ of those elements of M' which meet m is nonempty. Suppose first that $M'(m)$ contains less than 8 elements. Then

$$M' \backslash \big(M'(m)\big) \cup \{m\} \subset \mathfrak{M}(v_0, x')$$

is a set of pairwise parallel walls to which we may apply Proposition 24. As in the preceding paragraph, we obtain that r_m belongs to P. Suppose now that $M'(m)$ contains at least 8 elements. By Lemma 11, this implies that either r_m centralizes $W(M')$ or that $W(M' \cup \{m\})$ is a Euclidean triangle subgroup. In the first case, the reflection r_m normalizes P; in the second one, we have $r_m \in P$ by Lemma 17.

In all cases, we have seen that r_m normalizes P. Since every element of M separates x from one of its neighboring vertices in c, it follows that the group $W(M)$ is contained in the normalizer $N_W(P)$ of P in W. Therefore, by Lemma 22, each direct component of $W(M)$ is either contained in P or centralizes P.

Assume that every wall of M meets every wall of M'. By the maximality of M', the set M' possesses an element m' such that m' is not separated from x by any wall. By the definition of $\mathcal{X}$, it follows that x belongs to an edge which is cut by the wall

m'. Since every wall in M meets m', we deduce from Lemma 18 that c is a face of a $(n+1)$–cube c'' whose center is contained in m'. Let c' be the n–cube which is opposite c in c''. Thus c and c' are separated by m' and $M(c) = M(c') = M$. By construction, the combinatorial distance from v_0 to c' is strictly smaller than the combinatorial distance from v_0 to c, which contradicts the definition of c.

Therefore M contains an element m_P which does not meet all elements of M'. By the same arguments as above, we see that r_{m_P} belongs to P. Since P is a Coxeter group of affine type in which the set M' corresponds to one direction of hyperplanes, it follows that m_P does not meet any element of M'. If $m \in M$ is another element of M which does not meet all elements of M', then we obtain similarly that $r_m \in P$ and that m is parallel to all elements of $M' \cup \{m_P\}$. Since m meets m_P because the elements of M meet pairwise, it follows that $m = m_P$. This proves that, with the exception of m_P which meets no element of M', all other elements of M meet all elements of M'.

Let P_0 be the subgroup generated by the direct components of $W(M)$ which are contained in P. Thus $r_{m_P} \in P_0 \subset P$. Note that any reflection subgroup of an affine Coxeter group is either finite or of affine type. Therefore, in order to finish the proof, it suffices to prove that P_0 is infinite because P_0 is of 2–spherical type by Lemma 20 and, hence, if it is infinite, then its rank is at least 3.

Let $M_0 = \{m \in M \mid r_m \in P_0\}$ and let $M_1 = M \backslash M_0$. Thus $W(M_1)$ centralizes P and every element of M_1 meets every element of M'. Recall that d denotes the combinatorial distance from v_0 to x. By Lemma 18 there is a cube c_1', containing x, such that $M(c_1') = M_1 \cup \{m'\}$. Let c_1 be the face of that cube such that $x \notin c_1$ and $M(c_1) = M_1$. In particular, the combinatorial distance from c_1 to v_0 equals $d-1$.

Assume now that $P_0 = W(M_0)$ is finite in order to obtain a contradiction. By Lemma 26, there exists an element $w \in P$ and a cube $c_0 \subset \mathfrak{X}$ such that $M(c_0) = M_0$ and that $w.c_0$ contains v_0. Note that $M(w.c_0) = w(M(c_0))$ and hence $W(M(w.c_0)) = wW(M_0)w^{-1} \subset P$. Since P, and hence w, centralizes $W(M_1)$ it follows that every wall in M_1 meets every wall in $M(w.c_0)$. Note that if c, c' are cubes of $\mathfrak{X}$ at combinatorial distance r from one another and such that $M(c) \subset M(c')$, then every vertex of c is at combinatorial distance r from c'. Therefore, by Lemma 18, there is a cube c_2, at combinatorial distance at most $d-1$ from v_0, such that $M(c_2) = M(w.c_0) \cup M(c_1)$. Since w centralizes $W(M_1) = W(M(c_1))$, if follows that $w(M(c_1)) = M(c_1)$. Therefore, the cube $w^{-1}.c_2$ is at combinatorial at most $d-1$ from the Cayley graph $\mathfrak{X}_0$ and we have $M(w^{-1}.c_2) = M = M(c)$. This contradicts the definition of c, which finishes the proof. □

7.7 Proof of Theorem A

Let $\mathfrak{T}(W)$ be the set of all subsets $T \subset W\backslash\{1\}$ such that each pair of elements of T generates a finite group. Let also $\mathfrak{C}(\mathfrak{X})$ denote the union of all cubes of $\mathfrak{X}$. We define a map $\sigma\colon \mathfrak{T}(W) \to \mathfrak{C}(\mathfrak{X})$ as follows.

Let $t \in T$. Thus t is of finite order and, hence, its parabolic closure $\mathrm{Pc}(t)$ is finite. Define $M(t) := \{m \in \mathfrak{M}(\mathfrak{A}) \mid r_m \in \mathrm{Pc}(t)\}$. Thus for all $m, m' \in M(t)$, the group $\langle r_m, r'_m\rangle$ is finite and, hence, the wall m meets m'. Let now $t, t' \in T$. Since $\langle t, t'\rangle$ is finite, so is its parabolic closure $\mathrm{Pc}(\langle t, t'\rangle)$, which contains $\mathrm{Pc}(t)$ and $\mathrm{Pc}(t')$ by definition. Therefore, for any $m \in M(t)$ and $m' \in M(t')$, the group $\langle r_m, r_{m'}\rangle$ is finite and, hence, the wall m meets m'. This shows that the elements of $M(T) = \bigcup_{t\in T} M(t)$ meet pairwise.

By Lemma 18, there is a cube c in $\mathfrak{X}$ such that $M(c) = M(T)$. Among all such cubes, choose one which is at minimal combinatorial distance from the Cayley graph $\mathfrak{X}_0$; we define $\sigma(T)$ to be that cube. Note that the group W acts on $\mathfrak{T}(W)$ by conjugation and on $\mathfrak{C}(\mathfrak{X})$ via its action on $\mathfrak{X}$, but the map σ is *not* W–equivariant because the definition of σ depends on an arbitrary choice.

Now we associate to each cube $c \in \mathfrak{C}(\mathfrak{X})$ a finite subset $\mathfrak{T}(c) \subset \mathfrak{T}(W)$ as follows. Consider a k–tuple of subsets $M_1, M_2, \dots M_k \subset M(c)$ which satisfy the following conditions:

- $M(c) = \bigcup_{i=1}^{k} M_i$;
- For all $i, j \in \{1, \dots, k\}$, the group $W(M_i \cup M_j)$ is finite.

Given such a k–tuple, choose a nontrivial element $t_i \in W(M_i)$ for each $i = 1, \dots, k$. Clearly we have $T = \{t_i \mid i = 1, \dots, k\} \in \mathfrak{T}(W)$. We denote by $\mathfrak{T}(c)$ the set consisting of all those elements of $\mathfrak{T}(W)$ which are obtained from c in this manner. Note that the construction of T depends on some choices, but each choice has to be made between a finite number of possibilities. Therefore $\mathfrak{T}(c)$ is a finite subset of $\mathfrak{T}(W)$. Note also that the map $\mathfrak{T}\colon \mathfrak{C}(\mathfrak{X}) \to 2^{\mathfrak{T}(W)}$ is W–equivariant: for all $w \in W$ we have $\mathfrak{T}(w.c) = \{wTw^{-1} \mid T \in \mathfrak{T}(c)\}$.

Let $\mathfrak{C}_0(\mathfrak{X})$ be the set of all those cubes c such that $W(M(c))$ has no direct component of affine type and that c is at minimal combinatorial distance from $\mathfrak{X}_0$ among all cubes c' such that $M(c') = M(c)$. Clearly the W–action on $\mathfrak{X}$ preserves $\mathfrak{C}_0(\mathfrak{X})$. By Theorem 27, the distance from $\mathfrak{X}_0$ to any element of $\mathfrak{C}_0(\mathfrak{X})$ is uniformly bounded. Since $\mathfrak{X}$ is locally finite and W is transitive on the vertices of $\mathfrak{X}_0$, it follows that W has finitely many orbits in $\mathfrak{C}_0(\mathfrak{X})$.

Let now $\mathfrak{T}_0(W) \subset \mathfrak{T}(W)$ be the subset consisting of all those T such that $W(M(T))$ has no direct component of affine type. Note that W acts on $\mathfrak{T}_0(W)$ by conjugation. We claim that W has finitely many orbits in $\mathfrak{T}_0(W)$. Let $\{c_1, \dots, c_k\} \subset \mathfrak{C}_0(\mathfrak{X})$ be a set of representatives of the W–orbits and let $T \in \mathfrak{T}(W)$. By definition, we have $\sigma(T) \in \mathfrak{C}_0(\mathfrak{X})$ and $T \in \mathfrak{T}(\sigma(T))$. Let $w \in W$ such that $w.\sigma(T) = c_i$ for some i. Thus $wTw^{-1} \in \mathfrak{T}(c_i)$. This shows that the finite subset $\bigcup_{i=1}^k \mathfrak{T}(c_i) \subset \mathfrak{T}(W)$ contains a representative of each W–orbit in $\mathfrak{T}_0(W)$. This proves the claim.

Let $\mathfrak{G}$ be any set of subgroups of W invariant under conjugation. Assume that each $\Gamma \in \mathfrak{G}$ possesses a generating set which belongs to $\mathfrak{T}_0(W)$. Then, by the claim, the set $\mathfrak{G}$ is a *finite* union of conjugacy classes. Using this observation, we now prove the desired assertions successively.

(i) The 'only if' part is clear. Suppose that W has a parabolic subgroup of irreducible affine type and rank ≥ 3. Then W has no reflection subgroup of irreducible affine type and rank ≥ 3 by Proposition 16 and it follows that $\mathfrak{T}_0(W) = \mathfrak{T}(W)$. By the above, it follows that W has finitely many conjugacy classes of 2–spherical subgroups.

(ii) Let $\mathfrak{G}_1$ be the set of all 2–spherical reflection subgroups with no direct component of irreducible affine type. By definition, every $\Gamma \in \mathfrak{G}_1$ has a generating set $T \in \mathfrak{T}(W)$ consisting of reflections, and such that $\Gamma = W(M(T))$. Therefore $T \in \mathfrak{T}_0(W)$ and the desired finiteness property follows.

(iii) Let $\mathfrak{G}_2$ be the set of all 2–spherical subgroups Γ such that Γ has no nontrivial free abelian normal subgroup and $\mathbb{Z}\text{-rk}(\Gamma) \geq \mathbb{Z}\text{-rk}(W) - 1$. Let $\Gamma \in \mathfrak{G}_2$ and $T \in \mathfrak{T}(W)$ be a generating set of Γ. Clearly $\Gamma \subset W(M(T))$. Let $W(M(T)) = W_1 \times \cdots \times W_l$ be the decomposition of $W(M(T))$ into its direct components, and assume that W_1 is of affine type. Let T_1 be the translation subgroup W_1. We have $T_1 \cap \Gamma = \{1\}$ because $T_1 \cap \Gamma$ is a free abelian normal subgroup of Γ. On the other hand, it follows from the direct decomposition above that any free abelian subgroup of Γ has a finite index subgroup which centralizes T_1. Moreover $W(M(T))$ is 2–spherical by Lemma 20; therefore, as an infinite 2–spherical Coxeter group, W_1 is of rank ≥ 3 and hence $\mathbb{Z}\text{-rk}(T_1) \geq 2$. This shows that $\mathbb{Z}\text{-rk}(\langle T_1 \cup \Gamma \rangle) \geq \mathbb{Z}\text{-rk}(\Gamma) + 2 > \mathbb{Z}\text{-rk}(W)$, a contradiction. Therefore $W(M(T))$ has no direct component of affine type. In other words, we have $T \in \mathfrak{T}_0(W)$ and the desired finiteness property follows.

(iv) Let $\mathfrak{G}_3$ be the set of all 2–spherical subgroups Γ such that Γ is not infinite virtually abelian. Let $\Gamma \in \mathfrak{G}_3$ and $T \in \mathfrak{T}(W)$ be a generating set of Γ. Let $W(M(T)) = W_1 \times \cdots \times W_l$ be the decomposition of $W(M(T))$ into its direct components, and assume that W_i is of affine type for each $i \leq j$ but W_i is not for $i > j$. Thus for all $i \leq j < i'$, the group $W_{i'}$ centralizes W_i and, hence, normalizes its parabolic closure $\mathrm{Pc}(W_i)$, which is of affine type by Proposition 16. Therefore, by Lemma 22, either

$W_{i'}$ centralizes $\mathrm{Pc}(W_i)$ or $W_{i'} \subset \mathrm{Pc}(W_i)$. Since the centralizer of $\mathrm{Pc}(W_i)$ is finite by hypothesis and since $W_{i'}$ is not affine, it follows in both cases that $W_{i'}$ is finite. This shows that if $W(M(T))$ has a component of affine type, then each direct component of $W(M(T))$ is either affine or finite. In that case $W(M(T))$ would be virtually abelian, which is impossible since $\Gamma \subset W(M(T))$. Therefore $W(M(T))$ has no component of affine type and $T \in \mathfrak{T}_0(W)$. □

7.8 Co-Hopfian Coxeter groups : proof of Corollary B

Let $W = W_1 \times \cdots \times W_l$ be the decomposition of W into its direct components.

If W_1 is of affine type, then there exists a monomorphism $\phi_1\colon W_1 \to W_1$ which is not surjective. Then the unique homomorphism ϕ whose restriction on W_1 (resp. W_i) is ϕ_1 (resp. the identity for $i > 1$) is a monomorphism which is not surjective. Thus W is not co-Hopfian.

Assume now that for each i the group W_i is not affine. By Lemma 23, this implies that W has no nontrivial free abelian normal subgroup. Therefore, the group W contains finitely many conjugacy classes of subgroups isomorphic to W by Theorem A(iii). By the main result of [14], the outer automorphism group of W is finite. We deduce that W admits only finitely many monomorphisms into itself up to conjugation. The rest of the proof is similar to [19, Proof of Theorem 3.1]; for convenience, we reproduce the details.

Let $\varphi\colon W \to W$ be a monomorphism and assume that φ is not surjective in order to obtain a contradiction. For each n, the centralizer of $\varphi^n(W)$ in W is finite, otherwise it would contain an element of infinite order γ (by Selberg's lemma, W has a torsion free subgroup of finite index) and the group $\langle \{\gamma\} \cup \varphi^n(W) \rangle$ would be of $\mathbb{Z}$–rank strictly greater than $\mathbb{Z}\text{-rk}(W)$. Since W has finitely many conjugacy classes of finite subgroups, there is a finite subgroup A which is the centralizer of $\varphi^n(W)$ for all sufficiently large n. In particular $\varphi(A) = A$.

Since W admits only finitely many monomorphisms into itself up to conjugation, there exist arbitrarily large integers k, l and an element $g \in W$ such that for all $w \in W$ we have

$$g\varphi^k(w)g^{-1} = \varphi^{k+l}(w).$$

Therefore, conjugating $\varphi^k(w)$ by g is equivalent to transform it by φ^l, so we have:

$$g\varphi^{k+l}(w)g^{-1} = \varphi^l(\varphi^{k+l}(w)) = \varphi^l(g\varphi^k(w)g^{-1}) = \varphi^l(g)\varphi^{k+l}(w)\varphi^l(g^{-1}).$$

In particular $g^{-1}\varphi^l(g) \in A$, so $\varphi^l(g) = ga$ for some $a \in A$. Since $\varphi(A) = A$, for each $m \geq 1$ we have $\varphi^{ml}(g) = ga_m$ for some $a_m \in A$. By the pigeonhole principle, we obtain $\varphi^{m_1 l}(g) = \varphi^{m_2 l}(g)$ for some $m_1 < m_2$. But this shows that $\varphi^{m_1 l}(g) \in \varphi^n(W)$ for all integers n. As for all $w \in W$ we have:

$$\varphi^{k+l+m_1 l}(w) = \varphi^{m_1 l}(g)\varphi^{k+m_1 l}(w)\varphi^{m_1 l}(g^{-1})$$

and since moreover $\varphi^{m_1 l}(g) \in \varphi^{k+m_1 l}(W)$, it follows that

$$\varphi^{k+l+m_1 l}(W) = \varphi^{k+m_1 l}(W),$$

a contradiction. □

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