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Knots with unknotting number 1 and essential Conway spheres

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For a knot K in S^3, let $\mathbf{T}(K)$ be the characteristic toric sub-orbifold of the orbifold (S^3, K) as defined by Bonahon–Siebenmann. If K has unknotting number one, we show that an unknotting arc for K can always be found which is disjoint from $\mathbf{T}(K)$, unless either K is an EM–knot (of Eudave-Muñoz) or (S^3, K) contains an EM–tangle after cutting along $\mathbf{T}(K)$. As a consequence, we describe exactly which large algebraic knots (ie, algebraic in the sense of Conway and containing an essential Conway sphere) have unknotting number one and give a practical procedure for deciding this (as well as determining an unknotting crossing). Among the knots up to 11 crossings in Conway's table which are obviously large algebraic by virtue of their description in the Conway notation, we determine which have unknotting number one. Combined with the work of Ozsváth–Szabó, this determines the knots with 10 or fewer crossings that have unknotting number one. We show that an alternating, large algebraic knot with unknotting number one can always be unknotted in an alternating diagram.

As part of the above work, we determine the hyperbolic knots in a solid torus which admit a non-integral, toroidal Dehn surgery. Finally, we show that having unknotting number one is invariant under mutation.

57N10; 57M25

1 Introduction

Montesinos showed [24] that if a knot K has unknotting number 1 then its double branched cover M can be obtained by a half-integral Dehn surgery on some knot K^* in S^3. Consequently, theorems about Dehn surgery can sometimes be used to give necessary conditions for a knot K to have unknotting number 1. For instance, $H_1(M)$ must be cyclic, and the $\mathbb{Q}/\mathbb{Z}$–valued linking form on $H_1(M)$ must have a particular form (Lickorish [19]). If K is a 2–bridge knot, then M is a lens space, and hence, by the Cyclic Surgery Theorem (Culler, Gordon, Luecke and Shalen [5]), K^* must be a torus knot. In this way the 2–bridge knots with unknotting number 1

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have been completely determined (Kanenobu and Murakami [15]). Another example is Scharlemann's theorem that unknotting number 1 knots are prime [28]; this can be deduced from the fact (proved later, however by Gordon and Luecke [9]) that only integral Dehn surgeries can give reducible manifolds (see Zhang [38]). Finally, we mention the recent work of Ozsváth and Szabó [27], in which the Heegaard Floer homology of M is used to give strong restrictions on when K can have unknotting number 1, especially if K is alternating.

The present paper explores another example of this connection. Here, the Dehn surgery theorem is the result of Gordon and Luecke [11] that the hyperbolic knots with non-integral toroidal Dehn surgeries are precisely the Eudave-Muñoz knots $k(\ell, m, n, p)$ [6]; this gives information about when a knot K whose double branched cover M is toroidal can have unknotting number 1.

First we clarify the extension of the main result of [11] to knots in solid tori that is described in the Appendix of [11]. In Section 3, we define a family of hyperbolic knots $J_\varepsilon(\ell, m)$ in a solid torus, $\varepsilon \in \{1, 2\}$ and ℓ, m integers, each of which admits a half-integer surgery yielding a toroidal manifold. (The knots $J_\varepsilon(\ell, m)$ in the solid torus are the analogs of the knots $k(\ell, m, n, p)$ in the 3–sphere.) We then use [11] to show that these are the only such:

Theorem 4.2 *Let J be a knot in a solid torus whose exterior is irreducible and atoroidal. Let μ be the meridian of J and suppose that $J(\gamma)$ contains an essential torus for some γ with $\Delta(\gamma, \mu) \geq 2$. Then $\Delta(\gamma, \mu) = 2$ and $J = J_\varepsilon(\ell, m)$ for some ε, ℓ, m.*

This theorem along with the main result of [11] then allows us to describe in Theorem 5.2 the relationship between the torus decomposition of the exterior of a knot K in S^3 and the torus decompositon of any non-integral surgery on K. In particular, Theorem 5.2 says that the canonical tori of the exterior of K and of the Dehn surgery will be the same unless K is a cable knot (in which case an essential torus of the knot exterior can become compressible), or K is a $k(\ell, m, n, p)$, or K is a satellite with pattern $J_\varepsilon(\ell, m)$ (in the latter cases a new essential torus is created).

We apply these theorems about non-integral Dehn surgeries to address questions about unknotting number. The knots $k(\ell,m,n,p)$ ($J_\varepsilon(\ell, m)$) are strongly invertible. Their quotients under the involutions give rise to *EM–knots*, $K(\ell,m,n,p)$ (*EM–tangles*, $\mathcal{A}_\varepsilon(\ell, m)$ resp.) which have essential Conway spheres and yet can be unknotted (trivialized, resp.) by a single crossing change. Theorem 6.2 descibes when a knot with an essential Conway sphere or 2–torus can have unknotting number 1. This is naturally stated in the context of the characteristic decomposition of a knot along toric 2–suborbifolds given by Bonahon and Siebenmann in [3]. The characteristic torus decomposition of the double branched cover of a knot K corresponds to the characteristic decomposition of the orbifold $O(K)$, where $O(K)$ refers to S^3 thought of as an orbifold with singular set K and cone angle π (see [3]). This decomposition of $O(K)$ is along Conway spheres and along tori disjoint from K, the collection of which is denoted $\mathbf{T}(K)$. When $O(K)$ is cut along $\mathbf{T}(K)$, $\mathrm{Seif}(K)$ denotes the components corresponding to Seifert-fibered components of the canonical torus decomposition in the double branched cover. An *unknotting arc*, $(a, \partial a)$, for K is an arc such that $a \cap K = \partial a$ that guides a crossing move that unknots K. Under the correspondence between crossing changes and Dehn surgeries in the double branched cover, Theorem 5.2 becomes

Theorem 6.2 *Let K be a knot with unknotting number 1. Then one of the following three possibilities holds.*

(1) (a) *Any unknotting arc $(a, \partial a)$ for K can be isotoped in (S^3, K) so that $a \cap \mathbf{T}(K) = \varnothing$.*

(b) *If $\mathbf{T}(K) \neq \varnothing$ and K has an unknotting arc $(a, \partial a)$ in $\mathrm{Seif}(K)$ then $(a, \partial a)$ is isotopic to an (r, s)–cable of an exceptional fiber of $\mathrm{Seif}(K)$, for some $s \geq 1$.*

(2) (a) *K is an EM–knot $K(\ell, m, n, p)$.*

(b) *$O(K)$ has a unique connected incompressible 2–sided toric 2–suborbifold S, a Conway sphere, K has an unknotting arc $(a, \partial a)$ with $|a \cap S| = 1$ (the standard unknotting arc for $K(\ell, m, n, p)$), and K has no unknotting arc disjoint from S.*

(3) *K is the union of essential tangles $\mathcal{P} \cup \mathcal{P}_0$, where $\mathcal{P}_0$ is an EM–tangle $\mathcal{A}_\varepsilon(\ell, m)$ and $\partial\mathcal{P}_0$ is in $\mathbf{T}(K)$. Any unknotting arc for K can be isotoped into $\mathcal{P}_0$. The standard unknotting arc for $\mathcal{A}_\varepsilon(\ell, m)$ is an unknotting arc for K.*

Scharlemann and Thompson proved [30; 31] that if a satellite knot has unknotting number one, then an unknotting arc can be isotoped off any companion 2–torus. (This follows from Corollary 3.2 of [30] when the genus of K, $g(K)$, is ≥ 2. When $g(K) = 1$, it follows from the proof of Corollary 3.2 of [31], or from Corollary 1 of Kobayashi [17], which say that a knot K has $u(K) = g(K) = 1$ if and only if it is a Whitehead double.) The following corollary of Theorem 6.2 can be thought of as a generalization of this result.

Corollary 6.3 *Let K be a knot with unknotting number 1, that is neither an EM–knot nor a knot with an EM–tangle summand with essential boundary. Let F be an incompressible 2–sided toric 2–suborbifold of $O(K)$. Then any unknotting arc $(a, \partial a)$ for K can be isotoped in (S^3, K) so that $a \cap F = \varnothing$.*

When a knot or link contains an essential Conway sphere, one can perform mutations along that sphere. Boileau asked [16, Problem 1.69(c)] if the unknotting number of a link is a mutation invariant. We prove that it is at least true for knots with unknotting number one.

Theorem 7.1 *Having unknotting number 1 is invariant under mutation.*

We would like to thank Alan Reid for suggesting that we consider mutation.

We apply our results to the knots that are algebraic in the sense of Conway [4] (see also Thistlethwaite [33]), and which have an essential Conway sphere. We call such a knot K a *large algebraic* knot. Note that the double branched cover of K is a graph manifold (ie, the union of Seifert fiber spaces identified along their boundaries). Theorem 6.2 gives particularly strong constraints on unknotting arcs for knots in this class.

Theorem 8.2 *Let K be a large algebraic knot with unknotting number 1. Then either*

(1) *any unknotting arc for K can be isotoped into either*

  (a) *one of the rational tangles $\mathcal{R}(p/q)$ in an elementary tangle of type I; or*

  (b) *the rational tangle $\mathcal{R}(p/q)$ in an elementary tangle of type II.*

  In case (a), the crossing move transforms $\mathcal{R}(p/q)$ to $\mathcal{R}(k/1)$ for some k, and $p/q = \frac{2s^2}{2rs\pm 1} + k$, where $s \geq 1$ and $(r,s) = 1$.

  In case (b), the crossing move transforms $\mathcal{R}(p/q)$ to $\mathcal{R}(1/0)$, and $p/q = \frac{2rs\pm 1}{2s^2}$, where $s \geq 1$ and $(r,s) = 1$.

(2) (a) *K is an EM–knot $K(\ell, m, n, p)$.*

  (b) *$O(K)$ has a unique connected incompressible 2–sided toric 2–suborbifold S, a Conway sphere, K has an unknotting arc a with $|a \cap S| = 1$ (the standard unknotting arc for $K(\ell, m, n, p)$), and K has no unknotting arc disjoint from S.*

(3) *K is the union of essential tangles $\mathcal{P} \cup \mathcal{P}_0$, where $\mathcal{P}_0$ is an EM–tangle $\mathcal{A}_\varepsilon(\ell, m)$ and $\partial\mathcal{P}_0$ is in $T(K)$. Any unknotting arc for K can be isotoped into $\mathcal{P}_0$. The standard unknotting arc for $\mathcal{A}_\varepsilon(\ell, m)$ is an unknotting arc for K.*

In Section 10 we apply Theorem 8.2 to the knots in Conway's tables [4] of knots up to 11 crossings that can be immediately seen to be large algebraic by virtue of their description in terms of Conway's notation. There are 174 such knots, and we show that exactly 24 of them have unknotting number 1. In particular, combining our results with those of Ozsváth and Szabó, the knots with 10 or fewer crossings that have unknotting number 1 are now completely determined (see [27]).

It follows from Theorem 8.2 that the unknotting number 1 question is decidable for large algebraic knots.

Theorem 11.2 *There is an algorithm to decide whether or not a given large algebraic knot K, described as a union of elementary marked tangles (Figure 8.1) and 4–braids in $S^2 \times [0,1]$, has unknotting number 1, and, if so, to identify an unknotting crossing move.*

We remark that the algorithm in Theorem 11.2 is straightforward to carry out in practice.

Finally, in Section 12, we consider large algebraic knots which are alternating and show

Theorem 12.5 *Let K be an alternating large algebraic knot with unknotting number 1. Then K can be unknotted by a crossing change in any alternating diagram of K.*

The authors would like to thank Mario Eudave-Munõz for pointing out a gap in the original proof of Lemma 2.2 in the case that the double branched cover of K is a Seifert fiber space. Also, the first named author wishes to acknowledge partial support for this work by the National Science Foundation (grant DMS-0305846).

2 Preliminaries

For us, a *tangle* will be a pair (B, A) where B is S^3 with the interiors of a finite number (≥ 1) of disjoint 3–balls removed, and A is a disjoint union of properly embedded arcs in B such that A meets each component of ∂B in four points. Two tangles (B_1, A_1) and (B_2, A_2) are *homeomorphic* if there is a homeomorphism of pairs $h\colon (B_1, A_1) \to (B_2, A_2)$.

A *marking* of a tangle (B, A) is an identification of each pair $(S, S \cap A)$, where S is a component of ∂B, with $(S^2, Q = \{NE, NW, SW, SE\})$. A *marked* tangle is a tangle together with a marking. Two marked tangles are *equivalent* if they are homeomorphic by an orientation-preserving homeomorphism that preserves the markings.

A tangle (B^3, A) in the 3–ball is *essential* if $S^2 - A$ is incompressible in $B^3 - A$.

Let $\mathcal{T} = (B, A)$ be a knot in S^3 or a tangle. A *Conway sphere* in $\mathcal{T}$ is a 2–sphere $S \subset \operatorname{int} B$ such that S meets A transversely in four points. S is *essential* if $S - A$ is incompressible in $B - A$ and $(S, S \cap A)$ is not pairwise parallel in (B, A) to $(S_0, S_0 \cap A)$ for any component S_0 of ∂B.

A *rational* tangle is a marked tangle that is homeomorphic to the trivial tangle in the 3–ball, $(D^2, 2 \text{ points}) \times I$. As marked tangles, rational tangles are parametrized by $\mathbb{Q} \cup \{1/0\}$. We denote the rational tangle corresponding to $p/q \in \mathbb{Q} \cup \{1/0\}$ by $\mathcal{R}(p/q)$. We will adopt the convention of Eudave-Muñoz [7] for continued fractions. Thus $[a_1, a_2, \dots, a_n]$ will denote the rational number $\frac{p}{q} = a_n + \frac{1}{a_{n-1} + \frac{1}{\dots + \frac{1}{a_1}}}$. We will sometimes write $\mathcal{R}(p/q)$ as $\mathcal{R}(a_1, \dots, a_n)$.

Let $\mathcal{T} = (B^3, A)$ be a tangle in the 3–ball. A *slope* of $\mathcal{T}$ is the isotopy class (rel ∂) of an embedded arc τ in ∂B^3 such that $\partial\tau \subset A \cap \partial B^3$. Given a marking on $\mathcal{T}$, the slopes of $\mathcal{T}$ are in 1–1 correspondence with $\mathbb{Q} \cup \{1/0\}$ (via the double branched cover, $S^1 \times S^1$, of ∂B^3 along $A \cap \partial B^3$). If $\mathcal{T}$ is rational, then A defines a slope on ∂B^3. The rational number corresponding to this slope is that assigned to $\mathcal{T}$ in the preceding paragraph, p/q. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ are slopes on some tangle $\mathcal{T}$, then the *distance* between these slopes, denoted $\Delta(\frac{p_1}{q_1}, \frac{p_2}{q_2})$, is $|p_1 q_2 - p_2 q_1|$, and is the minimal intersection number between the corresponding isotopy classes in the double branched cover of ∂B^3 along $A \cap \partial B^3$.

Definition An alternating diagram of a marked tangle in B^3 is said to be *positive* (*negative*, resp.) if the first crossings encountered from the boundary (with pictured marking) are as shown in Figure 2.1. An alternating diagram of a marked tangle in $S^2 \times I$ is said to be *positive* (*negative*, resp.) if filling it with $\mathcal{R}(1/0)$ gives a positive (negative, resp.) diagram of a tangle in B^3.

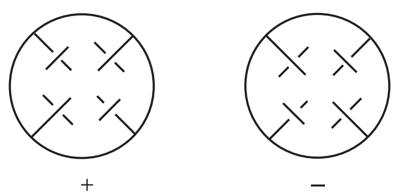


Figure 2.1

By our conventions, then, $\mathcal{R}(p/q)$ has a positive alternating diagram when $p/q > 0$.

Let $\mathcal{M}(*, *)$ be the tangle in the thrice-punctured 3–sphere illustrated in Figure 2.2. If $\alpha, \beta \in \mathbb{Q} \cup \{1/0\}$, then $\mathcal{M}(\alpha, \beta)$ will denote the tangle in the 3–ball obtained by inserting rational tangles $\mathcal{R}(\alpha)$, $\mathcal{R}(\beta)$ into A, B respectively (with respect to the markings of ∂A and ∂B given by Figure 2.2). Similarly, $\mathcal{M}(\alpha, *)$ (resp. $\mathcal{M}(*, \beta)$) will denote the tangle in $S^2 \times I$ obtained by inserting $\mathcal{R}(\alpha)$ (resp. $\mathcal{R}(\beta)$) into A (resp. B).

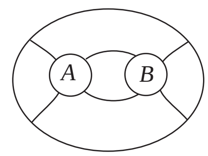


Figure 2.2

If $\alpha, \beta \in \mathbb{Q} - \mathbb{Z}$ then $\mathcal{M}(\alpha, \beta)$ is a *Montesinos tangle of length 2*. Note that transferring horizontal twists between A and B shows that $\mathcal{M}(\alpha + m, \beta - m) = \mathcal{M}(\alpha, \beta)$ for all $m \in \mathbb{Z}$.

In general we will denote the double branched cover of a tangle $\mathcal{T}$ by $\widetilde{\mathcal{T}}$. However, we will denote $\widetilde{\mathcal{M}}(*, *)$ by $D^2(*, *)$; it is homeomorphic to $P \times S^1$, where P is a pair of pants. Similarly, denote the double branched cover of $\mathcal{M}(p/q, *)$ by $D^2(p/q, *)$. If $q > 1$ this is a Seifert fiber space over the annulus with one exceptional fiber of multiplicity q. Finally, the double branched cover of $\mathcal{M}(p_1/q_1, p_2/q_2)$ is $D^2(p_1/q_1, p_2/q_2)$; if $q_1, q_2 > 1$ this is a Seifert fiber space over the disk with two exceptional fibers of multiplicities q_1 and q_2.

Let $\mathcal{S}(*, *; *, *)$, the *square* tangle, be the marked tangle shown in Figure 2.3; it is the union of two copies of $\mathcal{M}(*, *)$. If $\alpha, \beta, \gamma, \delta \in \mathbb{Q} \cup \{\infty\}$, then $\mathcal{S}(\alpha, \beta; \gamma, \delta)$ is the knot or link obtained by inserting the corresponding rational tangle into A, B, C, D respectively.

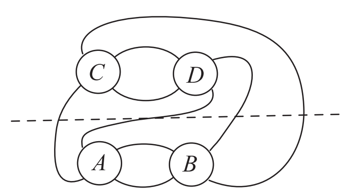


Figure 2.3: $\mathcal{S}(*, *; *, *)$

Lemma 2.1 $\mathcal{S}(\alpha, \beta; *, *) \cong \mathcal{S}(\beta, \alpha; *, *)$ *by a homeomorphism whose restriction to* ∂C (∂D) *is rotation through* $180°$ *about the horizontal axis.*

Proof This follows by rotating Figure 2.4 through $180°$ about the vertical axis shown, using the fact that a rational tangle is unchanged by rotation through $180°$ about the vertical axis. □

To state the next lemma, let D_8 be the order 8 dihedral group of all permutations of $\{\alpha, \beta, \gamma, \delta\}$ that preserve the partition $\{\{\alpha, \beta\}, \{\gamma, \delta\}\}$.

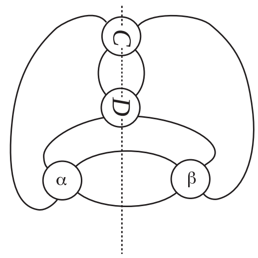


Figure 2.4: $\mathcal{S}(\alpha,\beta;*,*)$

Lemma 2.2

(1) $\mathcal{S}(\alpha,\beta;\gamma,\delta)=\mathcal{S}(\alpha+m,\beta-m;\gamma+n,\delta-n)$ *for all* $m,n\in\mathbb{Z}$.

(2) $\mathcal{S}(\alpha,\beta;\gamma,\delta)=\mathcal{S}(\pi(\alpha),\pi(\beta);\pi(\gamma),\pi(\delta))$ *for all* $\pi\in D_8$.

(3) $\mathcal{S}(-\alpha,-\beta;-\gamma,-\delta)=-\mathcal{S}(\alpha,\beta;\gamma,\delta)$.

If $\alpha,\beta,\gamma,\delta,\alpha',\beta',\gamma',\delta'\in\mathbb{Q}-\mathbb{Z}$ *then*

(4) $\mathcal{S}(\alpha,\beta;\gamma,\delta)=\mathcal{S}(\alpha',\beta';\gamma',\delta')$ *(resp.* $\pm\mathcal{S}(\alpha',\beta';\gamma',\delta')$*) if and only if* $(\alpha,\beta;\gamma,\delta)$ *and* $(\alpha',\beta';\gamma',\delta')$ *are related by a composition of the transformations in (1) and (2) (resp. the transformations in (1), (2) and (3)).*

Proof (1) follows from the property of $\mathcal{M}(\alpha,\beta)$ noted earlier.

To prove (2), observe that rotating Figure 2.3 through $180°$ about an axis perpendicular to the plane of the paper shows that $\mathcal{S}(\alpha,\beta;\gamma,\delta)=\mathcal{S}(\delta,\gamma;\beta,\alpha)$. (A rational tangle is unchanged by rotation through $180°$ about any of the three co-ordinate axes.) Also, by Lemma 2.1, $\mathcal{S}(\alpha,\beta;\gamma,\delta)=\mathcal{S}(\beta,\alpha;\gamma,\delta)$. The group generated by these two permutations is the dihedral group D_8.

(3) follows by changing all the crossings in the diagrams of $\mathcal{R}(\alpha)$, $\mathcal{R}(\beta)$, $\mathcal{R}(\gamma)$, $\mathcal{R}(\delta)$.

To prove (4), let $K=\mathcal{S}(\alpha,\beta;\gamma,\delta)$, $K'=\mathcal{S}(\alpha',\beta';\gamma',\delta')$, and let M,M' be the double branched covers of K and K' respectively; thus $M=D^2(\alpha,\beta)\cup D^2(\gamma,\delta)$, and similarly for M'. Parametrize slopes on the torus T, the double branched cover of $S=\partial\mathcal{M}(\alpha,\beta)$, by the parametrization of slopes on S coming from the marking of $\mathcal{M}(\alpha,\beta)$ in Figure 2.4. Thus $1/0$ is the slope of the Seifert fiber ϕ of $D^2(\alpha,\beta)$, $0/1$ is the slope of the Seifert fiber ψ of $D^2(\gamma,\delta)$, and similarly for ϕ',ψ'.

Suppose $K=K'$. Then there is an orientation-preserving homeomorphism $h\colon M\to M'$. Since T is, up to orientation-preserving homeomorphism, the unique separating, incompressible torus in M, and similarly for T', we may suppose that $h(T)=$

T'. We may assume further, by interchanging $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ if necessary, that $h(D^2(\alpha, \beta)) = D^2(\alpha', \beta')$, and, since the Seifert fiberings of $D^2(\alpha, \beta)$ etc. are unique, that $h(\phi) = \phi'$, $h(\psi) = \psi'$.

Recall that if N is a Seifert fiber space over D^2 with two exceptional fibers, then to describe N as $D^2(\mu, \nu)$ $(\mu, \nu \in \mathbb{Q} - \mathbb{Z})$, we remove disjoint Seifert fibered neighborhoods of the exceptional fibers, getting $P \times S^1$, where P is a pair of pants, and choose a section $s\colon P \to P \times S^1$. In identifying $\widetilde{\mathcal{M}}(\alpha, \beta)$ with $D^2(\alpha, \beta)$ we use the section that takes the boundary components of P to curves of slope $0/1$ with respect to the markings in Figure 2.4 of S and the boundaries of the rational tangles $\mathcal{R}(\alpha)$ and $\mathcal{R}(\beta)$.

Since $h\colon D^2(\alpha, \beta) \to D^2(\alpha', \beta')$ is an orientation-preserving homeomorphism which preserves the slopes $1/0$ and $0/1$ on S, the descriptions $D^2(\alpha, \beta)$ and $D^2(\alpha', \beta')$ differ only in the possible re-ordering of the two exceptional fibers and the choice of section s, subject to $s(\partial_0 P)$ having slope $0/1$, where $\partial_0 P$ is the boundary component of P that corresponds to the outer (unfilled) boundary component of $\mathcal{M}(*, *)$ in Figure 2.2. This choice corresponds to twisting a given section along an annulus $a \times S^1 \subset P \times S^1$, where a is an arc in P with one endpoint in each component of $\partial P - \partial_0 P$. This in turn corresponds to replacing (α, β) by $(\alpha + m, \beta - m)$ for some $m \in \mathbb{Z}$. Applying the same considerations to $D^2(\gamma, \delta)$ and $D^2(\gamma', \delta')$ gives the desired conclusion.

The parenthetical statement in (4) now follows from (3). □

By a *crossing move* on a knot K we mean the operation of passing one strand of K through another. More precisely, we take a 3–ball B_0 in S^3 such that $(B_0, B_0 \cap K) = \mathcal{T}_0$ is a trivial tangle, and replace it by the trivial tangle $\mathcal{T}_0'$ shown in Figure 2.5. This determines an arc $(a, \partial a) \subset (S^3, K)$ as shown in Figure 2.5. Note that $\mathcal{T}_0$ is a relative regular neighborhood of $(a, \partial a)$ in (S^3, K). Conversely, the arc a, together with a framing of a, determines the crossing move. If the resulting knot K' is the unknot, we say that a is an *unknotting arc* for K.

Note that we distinguish between a crossing *move* and a crossing *change*, reserving the latter term for a change of crossing in a knot diagram.

If K is a knot in S^3, then $u(K)$ is its *unknotting number*. That is, $u(K)$ is the smallest number of crossing moves required to unknot K.

Write $\mathcal{T} = (S^3, K) - \operatorname{int} \mathcal{T}_0$, and assume that K' is the unknot. Then taking double branched covers gives

$$M = B_2(K) = B_2(\mathcal{T}) \cup B_2(\mathcal{T}_0) = X \cup V_0 \,,$$
$$S^3 = B_2(K') = B_2(\mathcal{T}) \cup B_2(\mathcal{T}_0') = X \cup V_0' \,,$$

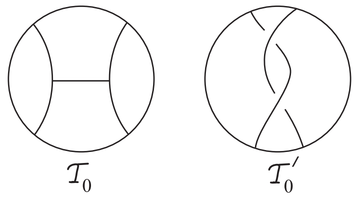


Figure 2.5

where V_0, V_0' are solid tori with meridians γ, μ, say, on ∂X, such that $\Delta(\gamma, \mu) = 2$. Thus the core of V_0' is a knot K^* in S^3, with exterior X and meridian μ, and $K^*(\gamma) = X(\gamma) \cong M$.

This connection between crossing moves and Dehn surgery in the double branched cover is due to Montesinos [24].

We now recall the characteristic toric orbifold decomposition of a knot, due to Bonahon and Siebenmann [3].

Let K be a prime knot in S^3. Regard S^3 as an orbifold $O(K)$ with singular set K, each point of K having isotropy group rotation of $\mathbb{R}^3$ about $\mathbb{R}^1$ through angle π. Since K is prime, the Characteristic Toric Orbifold Splitting Theorem of [3] asserts the existence of a collection $\mathbf{T}(O(K)) = \mathbf{T}(K)$ of disjoint incompressible 2–sided toric 2–suborbifolds, unique up to orbifold isotopy, such that (i) each component of $O(K)$ cut along $\mathbf{T}(K)$ is either atoroidal or S^1–fibered (as an orbifold), and (ii) $\mathbf{T}(K)$ is minimal with respect to this property. (See [3, Splitting Theorem 1].) Each component of $\mathbf{T}(K)$ is either a 2–torus disjoint from K or a Conway sphere.

$\mathbf{T}(K)$ may be described as follows; see Boileau and Zimmermann [2]. Let M be the double branched cover of (S^3, K), with covering involution h: $M \to M$. Let $\mathbf{T}(M)$ be the JSJ–decomposition of M. By Meeks and Scott [20], we may assume that $\mathbf{T}(M)$ is h–invariant. For each component T of $\mathbf{T}(M)$ such that $h(T) = T$ and h exchanges the sides of T, replace T by two parallel copies that are interchanged by h. Denote this new collection of tori by $\mathbf{T}^+(M)$. Then $\mathbf{T}(K)$ is the quotient $\mathbf{T}^+(M)/h$ in S^3.

3 EM–knots and EM–tangles

In [6] Eudave-Muñoz constructed an infinite family of knots $K = K(\ell, m, n, p)$ such that (1) K has unknotting number 1, (2) K has a (unique) essential Conway sphere S, and (3) no unknotting arc for K is disjoint from S. Passing to double branched covers these give rise (by [24]; see Section 2) to a family of hyperbolic knots $k(\ell, m, n, p)$ in

S^3, called the *Eudave-Muñoz* knots in [11], each of which has a half-integral toroidal surgery. To distinguish the $K(\ell, m, n, p)$'s from the $k(\ell, m, n, p)$'s we shall call the former *EM–knots*.

Definition The crossing move described in [6] that unknots $K(\ell, m, n, p)$ will be called the *standard crossing move* of $K(\ell, m, n, p)$.

Recall [6] that the parameters ℓ, m, n, p are restricted as follows: one of n, p is always 0; $|\ell| > 1$; if $p = 0$ then $m \neq 0$, $(\ell, m) \neq (2, 1)$ or $(-2, -1)$, and $(m, n) \neq (1, 0)$ or $(-1, 1)$; if $n = 0$ then $m \neq 0$ or 1, and $(\ell, m, p) \neq (-2, -1, 0)$ or $(2, 2, 1)$.

The EM–knots can be conveniently described in terms of the tangle $\mathcal{S}$ defined in Section 2 (see Figure 2.3 to describe $\mathcal{S}$ as a marked tangle).

Lemma 3.1 *The EM–knot* $K(\ell, m, n, p) = \mathcal{S}(\alpha, \beta; \gamma, \delta)$ *where* $\alpha, \beta, \gamma, \delta$ *are as follows:*

$$p = 0 : \quad \alpha = -\frac{1}{\ell}, \quad \beta = \frac{m}{\ell m - 1}, \quad \gamma = \frac{2mn + 1 - m - n}{4mn - 2m + 1}, \quad \delta = -\frac{1}{2}$$

$$n = 0 : \quad \alpha = -\frac{1}{\ell}, \quad \beta = \frac{2mp - m - p}{\ell(2mp - m - p) - 2p + 1}, \quad \gamma = \frac{m - 1}{2m - 1}, \quad \delta = -\frac{1}{2}.$$

Proof This follows immediately from [7, Proposition 5.4] (after allowing for sign errors). □

Lemma 3.1, together with the restrictions on the parameters ℓ, m, n, p, easily implies

Corollary 3.2 *Any EM–knot is of the form* $\mathcal{S}(\alpha, \beta; \gamma, \delta)$ *with* $\alpha, \beta, \gamma, \delta \in \mathbb{Q} - \mathbb{Z}$, $|\alpha|, |\beta|, |\gamma|, |\delta| < 1$, *and* $\alpha\beta < 0$, $\gamma\delta < 0$.

We will need to consider a collection of tangles in the 3–ball, the EM–tangles, closely related to the EM–knots. We describe them as two families, corresponding to the cases $p = 0$ and $n = 0$ of the EM–knots. More precisely, let $\mathcal{A}_1(\ell, m)$ be the tangle obtained from the knot $K(\ell, m, n, 0)$ by removing the "C"–tangle, and $\mathcal{A}_2(\ell, m)$ be the tangle obtained from $K(\ell, m, 0, p)$ by removing the "B"–tangle. Here ℓ and m are subject to the same restrictions as for $K(\ell, m, n, p)$, ie, $|\ell| > 1$ in both cases, and for $\mathcal{A}_1(\ell, m)$, $m \neq 0$, $(\ell, m) \neq (2, 1)$ or $(-2, -1)$, while for $\mathcal{A}_2(\ell, m)$, $m \neq 0$ or 1.

We therefore have the following:

Definition 3.3 The *EM–tangle* $\mathcal{A}_\varepsilon(\ell,m)$ is given by $\mathcal{A}_1(\ell,m) = \mathcal{S}(\alpha,\beta;*,\delta)$, $\mathcal{A}_2(\ell,m) = \mathcal{S}(\alpha,*;\gamma,\delta)$, where $\alpha,\beta,\gamma,\delta$ are as follows:

$$\varepsilon = 1\ : \alpha = -\frac{1}{\ell}\,,\quad \beta = \frac{m}{\ell m - 1}\,,\quad \delta = -\frac{1}{2}\,;\ |\ell| > 1,\ m \neq 0,$$
$$(\ell,m) \notin \{(2,1),(-2,-1)\}$$
$$\varepsilon = 2\ : \alpha = -\frac{1}{\ell}\,,\quad \gamma = \frac{m-1}{2m - 1}\,,\quad \delta = -\frac{1}{2}\,;\ |\ell| > 1,\ m \notin \{0,1\}.$$

(The $\mathcal{A}_\varepsilon(\ell,m)$ are pictured in Figure 4.3 where the twist boxes represent vertical twists.)

$K(\ell,m,n,p)$ contains an essential Conway sphere S, decomposing it into two Montesinos tangles: $K(\ell,m,n,p) = \mathcal{M}(\alpha,\beta)\cup_S \mathcal{M}(\gamma,\delta)$. This gives rise to a decomposition of the double branched cover of $K(\ell,m,n,p)$ as $N_1 \cup_T N_2$, where N_i is a Seifert fiber space over the disk with two exceptional fibers, $i=1,2$, and $T = \partial N_1 = \partial N_2 = \widetilde{S}$ is the double branched cover of S. Similarly, the essential Conway sphere S in $\mathcal{A}_\varepsilon(\ell,m)$ gives a decomposition of its double branched cover as $N_1 \cup_T N_2$, where N_2 is as above, and N_1 is a Seifert fiber space over the annulus with one exceptional fiber.

The remainder of this section is devoted to proving the following theorem which says that the EM–knots and the EM–tangles are determined by their double branched covers.

Theorem 3.4

(1) *Let K be a knot in S^3 whose double branched cover is homeomorphic to that of $K(\ell,m,n,p)$. Then $K = \pm K(\ell,m,n,p)$.*

(2) *Let $\mathcal{T}$ be a tangle in B^3 whose double branched cover is homeomorphic to that of $\mathcal{A}_\varepsilon(\ell,m)$. Then $\mathcal{T}$ and $\mathcal{A}_\varepsilon(\ell,m)$ are homeomorphic tangles.*

In order to prove Theorem 3.4, we first study involutions on the manifolds $D^2(*,*)$, $D^2(p/q,*)$ and $D^2(p_1/q_1, p_2/q_2)$. The definition of equivalence that is appropriate to our purposes is the following. Two homeomorphisms $f,g\colon X \to X$ are *strongly conjugate* if there is a homeomorphism $h\colon X\to X$ isotopic to the identity such that $f = h^{-1}gh$. If $X_0 \subset X$, then f and g are strongly conjugate *rel* X_0 if h can be chosen to be isotopic to the identity by an isotopy fixed on X_0. The set of fixed points of an involution τ will be denoted by $\mathrm{Fix}(\tau)$.

To define a standard model for a pair of pants P, let D^2 be the unit disk in $\mathbb{R}^2$, let D_1 and D_2 be disjoint round disks in $\operatorname{int} D^2$ with their centers on the x–axis, and let $P = D^2 - \operatorname{int}(D_1\cup D_2)$. The map $(x,y)\mapsto(x,-y)$ defines an orientation-reversing involution ρ_P of P, which we will call *reflection*.

Lemma 3.5 *Let τ be a non-trivial involution on a pair of pants P. If each boundary component of P is invariant under τ then τ is strongly conjugate to reflection.*

Proof Let $\alpha = P \cap (x\text{–axis})$, a disjoint union of three arcs properly embedded in P. Since the restriction of τ to each boundary component of P is either the identity, conjugate to rotation through π, or conjugate to reflection, we may assume that $\partial\alpha$ is invariant under τ. By analyzing the intersections of α and $\tau(\alpha)$, one can show that after conjugating τ, α can be taken to be invariant under τ. This implies that α is fixed by τ. The two disks of $P-\alpha$ are either exchanged or invariant. In the first case, τ is conjugate to reflection, and in the second τ must be the identity. Finally, since any homeomorphism of P is isotopic to one that commutes with $\rho_{\boldsymbol{P}}$, in the first case τ is strongly conjugate to $\rho_{\boldsymbol{P}}$. □

Recall (Section 2) that $D^2(*,*)$ (resp. $D^2(p/q,*)$, resp. $D^2(p_1/q_1, p_2/q_2)$) is the double branched cover of the tangle $\mathcal{M}(*,*)$ (resp. $\mathcal{M}(p/q,*)$, resp. $\mathcal{M}(p_1/q_1, p_2/q_2)$). The *standard involution* on $D^2(*,*)$, $D^2(p/q,*)$ or $D^2(p_1/q_1, p_2/q_2)$ is the non-trivial covering transformation corresponding to this double branched cover. Note that in particular, identifying $D^2(*,*)$ with $P \times S^1$, the standard involution on $D^2(*,*)$ is the map $(x,\theta) \mapsto (\rho_{\boldsymbol{P}}(x), -\theta)$.

Lemma 3.6 *Let τ be a non-trivial orientation-preserving involution on $P \times S^1$. If each component of the boundary is invariant under τ then τ is strongly conjugate to either the standard involution or a free involution that leaves each S^1–fiber invariant.*

Proof By Tollefson [35], there is a Seifert fibration of $P \times S^1$ that is invariant under τ. As the Seifert fibration is unique up to isotopy, we may therefore assume, after strongly conjugating τ, that τ preserves the product S^1–fibration of $P \times S^1$. Thus τ induces an involution $\tau_{\boldsymbol{P}}$ on P. By Lemma 3.5, $\tau_{\boldsymbol{P}}$ is either the identity or strongly conjugate to $\rho_{\boldsymbol{P}}$.

If $\tau_{\boldsymbol{P}}$ is the identity, then τ takes each S^1–fiber to itself by an orientation-preserving involution, hence by either the identity or a map conjugate to rotation through π. By continuity, the action is the same on each fiber. Therefore τ is either the identity or free.

We may suppose, then, that $\tau_{\boldsymbol{P}}$ is strongly conjugate to $\rho_{\boldsymbol{P}}$, and hence, by strongly conjugating τ, that $\tau_{\boldsymbol{P}} = \rho_{\boldsymbol{P}}$.

Let C be a boundary component of P. Note that $\tau_{\boldsymbol{P}}$ fixes two points in C. The restriction of τ to each of the two corresponding S^1–fibers is therefore conjugate to reflection $\theta \mapsto -\theta$. It follows that the restriction of τ to $C \times S^1$ is strongly conjugate to $-I$, given by $(\varphi, \theta) \mapsto (-\varphi, -\theta)$ (see Hartley [13]). So we may assume that $\tau = -I$ on each boundary component of $P \times S^1$. Since any two S^1–fibrations of $P \times S^1$ that agree on the boundary are isotopic rel ∂, we can still assume that τ preserves the product fibration and that $\tau_P = \rho_P$.

Let α_1, α_2 be the two arc components of $\mathrm{Fix}(\rho_P)$ shown in Figure 3.1, and let C_1, C_2 be the two boundary components of P indicated in the same figure.

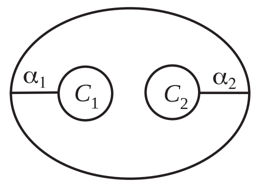


Figure 3.1

Let A_i be the vertical annulus $\alpha_i \times S^1$, and let T_i be the boundary torus $C_i \times S^1$, $i = 1, 2$. Note that A_i is invariant under τ, $i = 1, 2$. Then $\tau|(A_i = \alpha_i \times S^1)$ is conjugate to the involution $(x, \theta) \mapsto (x, -\theta)$, by a homeomorphism $g_i\colon A_i \to A_i$ that is isotopic rel ∂ to a power of a Dehn twist along the core of A_i. Hence, conjugating τ by the corresponding power of a vertical Dehn twist h_i along a torus in a collar neighborhood of T_i, we may assume that $\tau|A_i$ is $(x, \theta) \mapsto (x, -\theta)$. Since h_i is isotopic to the identity, the strong conjugacy class of τ is unchanged.

By a further isotopy rel ∂, we may assume that $A_i = A_i \times \{0\}$ has a neighborhood $A_i \times [-1, 1]$ on which τ acts by $(x, \theta, t) \mapsto (x, -\theta, -t)$. Removing $(A_1 \cup A_2) \times (-1, 1)$ from $P \times S^1$, τ induces an involution τ_0 on $D^2 \times S^1$ which is equal to $-I$ on the boundary. Hence τ_0 is strongly conjugate rel ∂ to the involution $((x, y), \theta) \mapsto ((x, -y), -\theta)$ [13]. Reattaching $(A_1 \cup A_2) \times [-1, 1]$ we get that τ is strongly conjugate to the standard involution. □

Lemma 3.7 *Let τ be a non-trivial orientation-preserving involution on $D^2(p_1/q_1, p_2/q_2)$, where $q_1 \neq q_2$.*

(1) *If $\mathrm{Fix}(\tau)$ has non-empty intersection with the boundary then τ is strongly conjugate to the standard involution.*

(2) *If τ acts freely on the boundary then the Seifert fibration of $D^2(p_1/q_1, p_2/q_2)$ may be isotoped so that τ leaves each Seifert fiber on the boundary invariant.*

Proof By [35], $D^2(p_1/q_1, p_2/q_2)$ has a Seifert fibration for which τ is fiber-preserving. Since the Seifert fibration is unique up to isotopy, and since $q_1 \neq q_2$, the exceptional fibers must be invariant, and hence they have disjoint invariant fibered neighborhoods V_1 and V_2, say. Thus τ restricts to an involution τ_0 on the complement of these neighborhoods, $D^2(*,*) = P \times S^1$. Note that τ_0 leaves each boundary component of $D^2(*,*)$ invariant.

(1) Here $\mathrm{Fix}(\tau_0) \neq \varnothing$, so by Lemma 3.6 τ_0 is strongly conjugate to the standard involution. Extending over V_1 and V_2, and using [13], we get that τ is strongly conjugate to the standard involution.

(2) Since τ_0 acts freely on at least one of the boundary components of $D^2(*,*)$, by Lemma 3.6 the (product) Seifert fibration of $D^2(*,*)$ may be isotoped so that τ leaves each fiber invariant. Now $\tau|\partial V_i$ can be extended to an involution τ_i of V_i that leaves each Seifert fiber invariant. Since τ_i and $\tau|V_i$ agree on ∂V_i, they are strongly conjugate rel ∂V_i [13]. The corresponding isotopy of V_i (rel ∂V_i) takes the Seifert fibration of V_i to one such that each fiber is invariant under τ. □

Lemma 3.8 *Let τ be a non-trivial orientation-preserving involution on $D^2(p/q,*)$ such that* $\mathrm{Fix}(\tau)$ *has non-empty intersection with the boundary. Then τ is strongly conjugate to the standard involution.*

Proof This is the same as the proof of Part (1) of Lemma 3.7. □

Proof of Theorem 3.4 Case (1) Write $K_0 = K(\ell,m,n,p)$. Then $(S^3,K_0) = (B_1,A_1) \cup_S (B_2,A_2)$, where S is an essential Conway sphere and (B_i,A_i) is a Montesinos tangle of length 2, $i=1,2$. The double branched cover of (S^3,K_0) is $N = N_1 \cup_{\widetilde{S}} N_2$, where N_i, the double branched cover of (B_i,A_i), is a Seifert fiber space over the disk with two exceptional fibers, $i=1,2$, and $\widetilde{S} = \partial N_1 = \partial N_2$ is the double branched cover of $(S, S\cap K_0)$. The Seifert fibers of N_1 and N_2 intersect once on $\widetilde{S}$. The covering involution $\sigma\colon N \to N$ restricts to the standard involution σ_i on N_i, $i=1,2$.

Now suppose K is a knot in S^3 whose double branched cover is homeomorphic to N. Let $\tau\colon N\to N$ be the corresponding covering involution. Since $\widetilde{S}$ is the unique incompressible torus in N, up to isotopy, by [20, Theorem 8.6] we may assume that $\widetilde{S}$ is invariant under τ.

Claim 1 *Each N_i is invariant under τ.*

Proof If τ interchanges N_1 and N_2 then $\mathrm{Fix}(\tau)$ is contained in $\widetilde{S}$. With respect to some parametrization of $\widetilde{S}$ as $S^1\times S^1$, $\mathrm{Fix}(\tau)$ is a (2,1)–curve and τ leaves each (0,1)–curve γ invariant, taking it to itself by reflection in the pair of points $\mathrm{Fix}(\tau)\cap\gamma$

(thus the quotient $\widetilde{S}/\tau$ is a Möbius band). Then $S^3 = N_1/(\tau|\widetilde{S})$ is homeomorphic to N_1 with a solid torus V attached so that γ bounds a meridian disk of V. Hence N_1 is a knot exterior with meridian γ. Applying the same argument to N_2, we see that in $N = N_1 \cup N_2$ the meridians of N_1 and N_2 are identified. But this is not true: when each side of N is the exterior of a knot in S^3, the argument in Lemma 1.3 of [6] (or Lemmas 3.1 and 9.5) shows that the meridian of one side is identified with the Seifert fiber of the other. □

Let τ_i be the restriction of τ to N_i, $i = 1, 2$.

Claim 2 *τ_i is strongly conjugate to the standard involution σ_i on N_i, $i = 1, 2$.*

Proof If $\mathrm{Fix}(\tau)$ meets $\widetilde{S}$ then the result follows from Lemma 3.7(1) and Lemma 3.9 below.

If $\mathrm{Fix}(\tau)$ is disjoint from $\widetilde{S}$, then by Lemma 3.7(2) and Lemma 3.9 the Seifert fibrations of N_1 and N_2 can be isotoped so that, on $\widetilde{S}$, each S^1–fiber of each fibration is invariant under τ. But since the fibers of the two fibrations intersect once on $\widetilde{S}$, this is clearly impossible. □

Write $S_i = \partial B_i$, $i = 1, 2$, and let $f\colon (S_1, S_1 \cap A_1) \to (S_2, S_2 \cap A_2)$ be the gluing homeomorphism that defines $(S^3, K_0) = (B_1, A_1) \cup_f (B_2, A_2)$. To compare K_0 and K, we need the notion of a mutation involution, which is defined at the beginning of Section 7.

Claim 3 *(S^3, K) is homeomorphic to $(B_1, A_1) \cup_{\mu f} (B_2, A_2)$ for some mutation involution μ of $(S_2, S_2 \cap A_2)$.*

Proof Let $\widetilde{f}\colon \partial N_1 \to \partial N_2$ be a lift of f, giving $N = N_1 \cup_{\widetilde{f}} N_2$. Note that $\widetilde{f}\sigma_1 = \sigma_2 \widetilde{f}$ and $\widetilde{f}\tau_1 = \tau_2 \widetilde{f}$. Also, $N/\tau = (N_1/\tau_1) \cup_g (N_2/\tau_2)$ for some $g\colon \partial(N_1/\tau_1) \to \partial(N_2/\tau_2)$ such that $\widetilde{f}$ is a lift of g.

By Claim 2, there is a homeomorphism $\widetilde{h}_i\colon N_i \to N_i$, isotopic to the identity, such that $\tau_i = \widetilde{h}_i^{-1}\sigma_i\widetilde{h}_i$, $i = 1, 2$. Then $\widetilde{h}_i$ induces a homeomorphism $h_i\colon N_i/\tau_i \to N_i/\sigma_i = (B_i, A_i)$, $i = 1, 2$. Let ∂h_i be the restriction of h_i to $\partial(N_i/\tau_i)$. Then $h_1 \cup h_2$ induces a homeomorphism $h\colon N/\tau = (N_1/\tau_1) \cup_g (N_2/\tau_2) \to (B_1, A_1) \cup_e (B_2, A_2)$, where $e = (\partial h_2)^{-1} g (\partial h_1)$. Then e lifts to $\widetilde{e} = (\partial \widetilde{h}_2)^{-1} \widetilde{f} (\partial \widetilde{h}_1)$, which is isotopic to $\widetilde{f}$.

Let μ be the mutation involution of $(S_2, S_2 \cap A_2)$ such that the composition μf agrees with e on some point of $S_1 \cap A_1$. Since $\widetilde{e}$ is isotopic to $\widetilde{f}$, e and f induce the same function from the set of (unoriented) isotopy classes of essential simple closed curves in $S_1-(S_1\cap A_1)$ to the set of those in $S_2-(S_2\cap A_2)$. Hence e and μf do also. Since e and μf agree on a point of $S_1\cap A_1$, they must be isotopic as maps of pairs. Therefore $(S^3,K)\cong(B_1,A_1)\cup_e(B_2,A_2)\cong(B_1,A_1)\cup_{\mu f}(B_2,A_2)$. □

By Claim 3, K is a mutation of K_0 along S. By [6] such mutations yield K_0 again. This completes the proof of the theorem in Case (1).

Case (2) We have $\mathcal{A}_\varepsilon(\ell,m)=(B_1,A_1)\cup_S(B_2,A_2)$, where B_1 is $S^2\times I$, $(B_1,A_1)=\mathcal{M}(p/q,*)$, and (B_2,A_2) is a Montesinos tangle of length 2 as in Case (1); see Definition 3.3. The double branched cover of $\mathcal{A}_\varepsilon(\ell,m)$ is then $N=N_1\cup_{\widetilde{S}}N_2$, where N_1 is now a Seifert fiber space over the annulus with one exceptional fiber. The covering involution σ restricts to the standard involution σ_i on N_i, $i=1,2$.

Suppose $\mathcal{T}$ is a tangle in B^3 whose double branched cover is homeomorphic to N, and let τ be the corresponding covering involution. Since $\widetilde{S}$ is the unique essential torus in N, we may assume $\tau(\widetilde{S})=\widetilde{S}$. Let τ_i be the restriction of τ to N_i, $i=1,2$. By Lemma 3.8, τ_1 is strongly conjugate to the standard involution on N_1. In particular $\mathrm{Fix}(\tau)\cap\widetilde{S}\neq\varnothing$, and so, by Lemmas 3.7(1) and 3.9, τ_2 is strongly conjugate to the standard involution on N_2. Now Claim 3 holds, exactly as in Case (1). Since $\mathcal{A}_\varepsilon(\ell,m)$ is unchanged by mutation along S, (by Lemma 2.1 and the fact that rotating a rational tangle through π about a co-ordinate axis does not change it), $\mathcal{T}$ is homeomorphic to $\mathcal{A}_\varepsilon(\ell,m)$. □

Lemma 3.9 *Let $N=N_1\cup N_2$ be the double branched cover of S^3 over $K(\ell,m,n,p)$ or $\mathcal{A}_\varepsilon(\ell,m)$, as in the discussion before the statement of Theorem 3.4. Let q_1,q_2 be the orders of the two exceptional fibers of N_i for some i. Then $q_1\neq q_2$.*

Proof When N is the cover of an EM–tangle, the lemma applies to N_2. By Definition 3.3 the exceptional fibers are of orders $(|\ell|,|1-\ell m|)$ with $|\ell|>1$, $m\neq 0$ or of orders $(2,|2m-1|)$ where $m\neq 0,1$. In either case, the lemma easily follows.

So assume N is the double branched cover of $K(\ell,m,n,p)$, and recall Lemma 3.1.

If $p=0$, the orders of the exceptional fibers are $(|\ell|,|1-\ell m|)$ for N_1 and $(2,|4mn-2m+1|)$ for N_2. In this case the lemma is clear.

So we assume $n=0$. In this case the exceptional fibers have orders $(2,|2m-1|)$ in N_1 and $(|\ell|,|2\ell mp-\ell m-\ell p-2p+1|)$ in N_2. We need to consider the solutions to

$$\begin{aligned}
&& \pm l &= 2\ell mp-\ell m-\ell p-2p+1\\
(1)\qquad &\iff & 2p-1 &= \ell(2mp-m-p\pm 1)\\
&\iff & 2(2p-1) &= \ell(4mp-2m-2p\pm 2)\\
(2)\qquad &\iff & 2(2p-1) &= \ell((2m-1)(2p-1)+a)
\end{aligned}$$

where a is -3 or 1. When $n=0$, we assume that $m \neq 0,1$. Thus $|2m-1| \geq 3$. Then

$$|(2m-1)(2p-1)+a| \geq |(2m-1)(2p-1)| - 3 \geq 2|2p-1| + |2p-1| - 3 \,.$$

Since $|\ell| \geq 2$, (1) implies that $|2p-1| \geq 3$. Thus (2) becomes

$$|2(2p-1)| \geq |\ell||2(2p-1)| \,,$$

a contradiction. □

4 Knots in solid tori

In this section we describe the hyperbolic knots in a solid torus that have a non-integral toroidal Dehn surgery. We gave a description of such knots in [11, Corollary A.2]; here we sharpen this to a complete characterization.

The exteriors of the knots are the double branched covers of certain tangles in $S^2 \times I$, which we now describe. Let $\mathcal{C}(A,B,C,D)$ be the tangle shown in Figure 4.1. (To

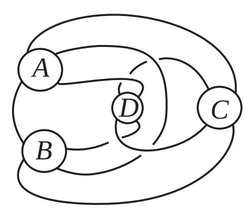


Figure 4.1: $\mathcal{C}(A,B,C,D)$

be consistent with the notation of [7], we here regard A,B,C,D as denoting rational tangles; a puncture that is not filled in will as usual be indicated by a $*$.) Note that in the terminology of [11], $\mathcal{C}(A,B,C,D) = \mathcal{B}(A,B,C) + R(D) = \mathcal{P}(A,B,C,\frac{1}{2},D)$. Define tangles $\mathcal{T}_1(\ell,m)$ and $\mathcal{T}_2(\ell,m)$ in $S^2 \times I$ as follows.

> $\mathcal{T}_1(\ell,m) = \mathcal{C}(A,B,*,*)$, where $A = \mathcal{R}(\ell)$, $B = \mathcal{R}(m,-\ell)$, and ℓ, m are integers such that $|\ell| > 1$, $m \neq 0$, and $(\ell,m) \neq (2,1)$ or $(-2,-1)$.
>
> $\mathcal{T}_2(\ell,m) = \mathcal{C}(A,*,C,*)$, where $A = \mathcal{R}(\ell)$, $C = \mathcal{R}(m-1,2,0)$, and ℓ, m are integers such that $|\ell| > 1$, $m \neq 0,1$.

See Figure 4.2 (the boxes correspond to vertical twists).

Note that $\mathcal{T}_1(\ell,m)$ is obtained by removing the "C–tangle" from $B(\ell,m,n,0)$ of [6], and similarly $\mathcal{T}_2(\ell,m)$ is obtained by removing the "B–tangle" from $B(\ell,m,0,p)$.

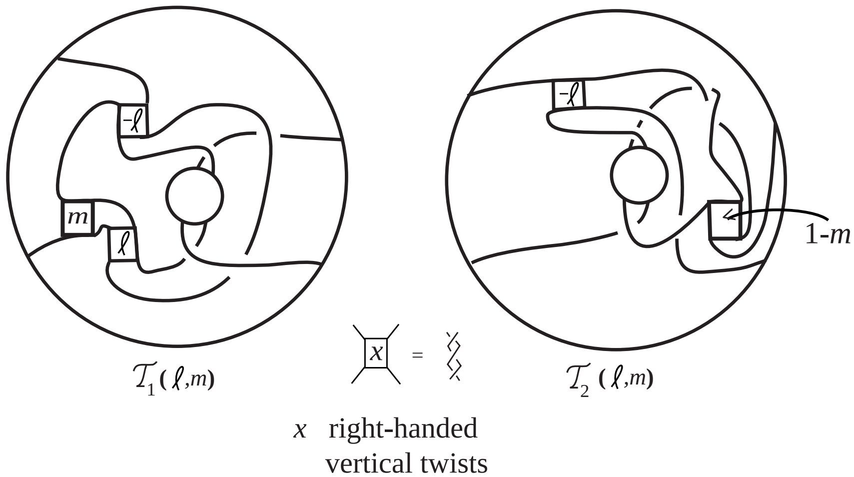


Figure 4.2

Filling the "D"–puncture of $\mathcal{T}_1(\ell,m)$ or $\mathcal{T}_2(\ell,m)$ with the $\frac{1}{0}$–tangle, ie, $\mathcal{C}(A,B,*,\frac{1}{0})$ or $\mathcal{C}(A,*,C,\frac{1}{0})$ with A,B,C as indicated above, gives a rational tangle. So, the corresponding $\frac{1}{0}$–Dehn filling of the double branched cover of $\mathcal{T}_\varepsilon(\ell,m)$, $\varepsilon\in\{1,2\}$, is a solid torus. Let $J_\varepsilon(\ell,m)$ denote the core of this Dehn filling, seen as a knot in this solid torus.

Note that the EM–tangle $\mathcal{A}_\varepsilon(\ell,m)$, $\varepsilon=1,2$, defined in Section 3, is the tangle in B^3 obtained by filling the "D"–puncture of $\mathcal{T}_\varepsilon(\ell,m)$ with the $\frac{1}{2}$–tangle. That is

$$\mathcal{A}_1(\ell,m)=\mathcal{C}\Big(A,B,*,\tfrac{1}{2}\Big)$$
$$\mathcal{A}_2(\ell,m)=\mathcal{C}\Big(A,*,C,\tfrac{1}{2}\Big)$$

(with A,B as for $\mathcal{T}_1(\ell,m)$, $\mathcal{T}_2(\ell,m)$). See Figure 4.3, which gives a marking to $\mathcal{A}_\varepsilon(\ell,m)$.

Definition Denote by $\mathcal{T}_\varepsilon(\ell,m)(\frac{p}{q})$ the tangle in the 3–ball gotten by filling the D–puncture of $\mathcal{T}_\ell(\ell,m)$ with the rational tangle $\mathcal{R}(p/q)$. The change in filling $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{2})$ to $\mathcal{T}(\ell,m)(\frac{1}{0})$ corresponds to a crossing move taking the toroidal tangle $\mathcal{A}_\varepsilon(\ell,m)$ to a rational tangle. We will refer to this as the *standard crossing move* on $\mathcal{A}_\varepsilon(\ell,m)$. The arc guiding this crossing change is the *standard unknotting arc* for $\mathcal{A}_\varepsilon(\ell,m)$.

$\mathcal{A}_\varepsilon(\ell,m)$ ($\varepsilon=1$ or 2) contains an essential Conway sphere S, which induces a decomposition of the double branched covering $\widetilde{\mathcal{A}}_\varepsilon(\ell,m)=N_1\cup_T M_2$, where N_1

is a Seifert fiber space over A^2 with one exceptional fiber, and M_2 is a Seifert fiber space over D^2 with two exceptional fibers. See Figure 4.3. Thus if $J = J_\varepsilon(\ell, m)$ for some ε, ℓ, m, then J is a knot in $S^1 \times D^2$, with meridian μ, say, and $J(\gamma) = \widetilde{\mathcal{A}}_\varepsilon(\ell, m) = N_1 \cup_T M_2$ contains an essential separating torus T, for some γ such that $\Delta(\gamma, \mu) = 2$.

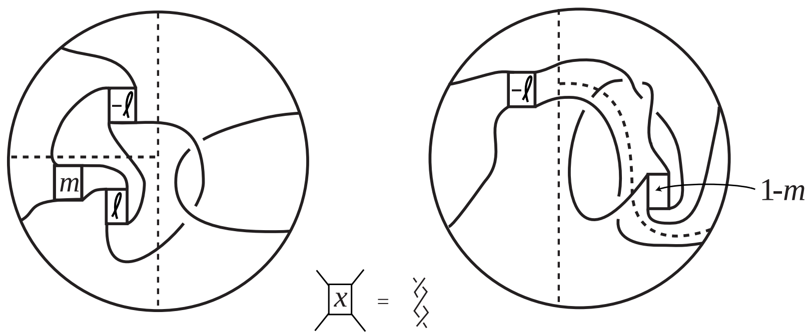


Figure 4.3

Theorem 4.1 *$J_\varepsilon(\ell, m)$ is a hyperbolic knot in the solid torus.*

We give the proof of Theorem 4.1 at the end of this section.

The following theorem says the $J_\varepsilon(\ell, m)$ are exactly the hyperbolic knots in solid tori which admit non-integral toroidal surgeries.

Theorem 4.2 *Let J be a knot in a solid torus whose exterior is irreducible and atoroidal. Let μ be the meridian of J and suppose that $J(\gamma)$ contains an essential torus for some γ with $\Delta(\gamma, \mu) \geq 2$. Then $\Delta(\gamma, \mu) = 2$ and $J = J_\varepsilon(\ell, m)$ for some ε, ℓ, m.*

Remark $J = J_\varepsilon(\ell, m)$ means there is a homeomorphism of the solid torus (possibly orientation-reversing) taking J to $J_\varepsilon(\ell, m)$.

The following is the corresponding statement about tangles. Theorem 3.4(2) says that $\mathcal{A}_\varepsilon(\ell, m)$ is determined by its double branched cover. At this point, it is not known if the same is true for $\mathcal{T}_\varepsilon(\ell, m)$. This complicates the statement of Theorem 4.3.

Theorem 4.3 *Let $\mathcal{T}(*,*)$ be a tangle in $S^2 \times I$ which is irreducible and atoroidal as a $\mathbb{Z}_2$–orbifold. If $\mathcal{T}(*,\alpha)$ is rational, and $\mathcal{T}(*,\beta)$ is orbifold-toroidal, where $\Delta(\alpha,\beta) \geq 2$, then the double branched cover of $\mathcal{T}(*,*)$ is homeomorphic to the double branched cover of $\mathcal{T}_\varepsilon(\ell,m)$ for some ε,ℓ,m. Under this homeomorphism, the slopes α,β on $\mathcal{T}$ correspond to slopes $1/0$, $1/2$ respectively on D of $\mathcal{C}$.*

Furthermore, there are tangle homeomorphisms $h_1\colon \mathcal{T}(*,\beta) \to \mathcal{A}_\varepsilon(\ell,m) = \mathcal{T}_\varepsilon(\ell,m)(\frac{1}{2})$ and $h_2\colon \mathcal{T}(*,\alpha) \to \mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$ such that $(h_2|\partial)(h_1|\partial)^{-1}$ is the identity (where $\partial\mathcal{T}(*,\beta) = \partial\mathcal{T}(*,\alpha) \subset \partial\mathcal{T}(*,*)$, and $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$, $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{2})$ are marked from Figure 4.2).

Addendum *$\mathcal{T}(*,\alpha)$ is rational and thus determines a slope, $\frac{p_1}{q_1}$, on its boundary. $\mathcal{T}(*,\beta)$ is the tangle $\mathcal{T}_\varepsilon(\ell,m)$ whose double branched cover contains a unique essential annulus. The boundary of this annulus determines a slope on the boundary of the cover, which determines a tangle slope, $\frac{p_2}{q_2}$, on the boundary of $\mathcal{T}(*,\beta)$. Then $\Delta(p_1/q_1, p_2/q_2) = |p_1q_2 - p_2q_1| > 1$.*

Proof of Theorem 4.3 Let $\mathcal{T}(*,*)$ be as in the theorem. Let X be its double branched cover. Let $X(\alpha), X(\beta)$ be the Dehn fillings of X corresponding to the double branched covers of $\mathcal{T}(*,\alpha)$, $\mathcal{T}(*,\beta)$ (resp.). Then by assumption, X is irreducible and atoroidal, $X(\alpha)$ is a solid torus and $X(\beta)$ is toroidal. Since $\Delta(\alpha,\beta) > 1$, Corollary A.2 of [11] proves that X is the double branched cover of $\mathcal{C}(A,B,C,*)$ where one of A,B,C is the empty tangle and the others are rational (using the fact proven there that $\frac{1}{2} \in \{\alpha',\beta',\gamma'\}$). Furthermore, under this identification, $\alpha = \frac{1}{0}$, $\beta = \frac{1}{2}$. That is, $\mathcal{C}(A,B,C,\frac{1}{0})$ is a rational tangle, and $\mathcal{C}(A,B,C,\frac{1}{2})$ is orbifold-toroidal. By symmetry we may assume that either B or C is the empty tangle above. Lemma 4.4 and Lemma 4.5 below show that A,B,C are as in the definition of $\mathcal{T}_\varepsilon(\ell,m)$. Now the double branched cover of $\mathcal{T}(*,\beta)$ is the same as the double branched cover of $\mathcal{C}(A,B,C,\frac{1}{2}) = \mathcal{A}_\varepsilon(\ell,m)$. By Theorem 3.4, $\mathcal{T}(*,\beta)$ and $\mathcal{A}_\varepsilon(\ell,m)$ are homeomorphic tangles. Such a homeomorphism h_1 determines a framing on $\partial X(\beta)$ ($\mathcal{A}_\varepsilon(\ell,m)$ is a marked tangle), hence on $\partial X(\alpha)$. The meridian disk of $X(\alpha)$ determines a rational number in this framing which corresponds to the rational tangle $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$. This implies that $\mathcal{T}(*,\alpha)$ is the same as the rational tangle $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$ under the marking determined by h_1. This is the second paragraph of Theorem 4.3. □

Proof of Addendum to Theorem 4.3 In the context of the proof of Theorem 4.3, $X(\alpha)$, $X(\beta)$ are the double branched covers of $\mathcal{C}(A,B,C,\frac{1}{0})$, $\mathcal{C}(A,B,C,\frac{1}{2})$ with A,B,C as in the definition of $\mathcal{T}_\varepsilon(\ell,m)$. The slope of the essential annulus of $X(\beta)$ corresponds to the tangle slope $\frac{1}{0}$ in Figure 4.2 (see Figure 4.3). The slope corresponding to the rational tangle $\mathcal{T}(*,\alpha)$ corresponds to the slope of the meridian disk in $X(\alpha)$. This in turn corresponds in Figure 4.2 to the $\frac{p}{q}$ of the rational tangle $\mathcal{C}(A,B,C,\frac{1}{0})$. We need to show then that $|q|>1$. For $\mathcal{T}_1(\ell,m)$, we see this by simply noting that $\mathcal{C}(A,B,C,\frac{1}{0})$ capped off by strands of slope $\frac{1}{0}$ is the unlink only when $m=0$. For $\mathcal{T}_2(\ell,m)$, $\mathcal{C}(A,*,C,\frac{1}{0})$ corresponds to the rational number $\frac{2m-1}{\ell(2m-1)-2}$. That is, $q=\ell(2m-1)-2$. The conditions that $|\ell|>1$, $m\neq 0,1$ imply that $|q|>1$. □

Proof of Theorem 4.2 This is the proof of Theorem 4.3 using the definition of $J_\varepsilon(\ell,m)$, and without Theorem 3.4. □

Lemma 4.4 *If $\mathcal{C}(A,*,C,\frac{1}{0})$ is a rational tangle and $\mathcal{C}(A,*,C,\frac{1}{2})$ is orbifold-toroidal, then $A=\mathcal{R}(s)$, $C=\mathcal{R}(t,2,0)$ for $s,t\in\mathbb{Z}$ with $|s|,|2t+1|>1$.*

Proof We follow the argument of Lemma 5.1 of [7]. Rewrite $\mathcal{C}(A,*,C,\frac{1}{0})$ as in Figure 4.4. From this we deduce that $C'=\mathcal{R}(\frac{1}{t'})$, $t'\in\mathbb{Z}$. Thus $\mathcal{C}(A,*,C,\frac{1}{0})$ is as in Figure 4.5. (Note that our convention in this paper is that twist boxes represent *vertical* twists).

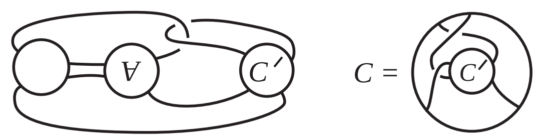


Figure 4.4

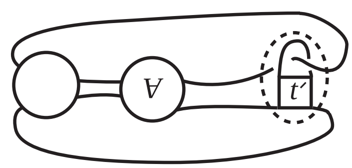


Figure 4.5

The tangle encapsulated in the Conway sphere in Figure 4.5 is $\mathcal{R}(2,t',0)=\mathcal{R}(\frac{2}{2t'+1})$. Thus either $t'=0,-1$ or $A=\mathcal{R}(s)$ for some $s\in\mathbb{Z}$. In either case, $C=\mathcal{R}(t',1,1,0)=\mathcal{R}(t,2,0)$ where $t=-(t'+1)$.

Figure 4.6 shows $\mathcal{C}(A,*,C,\frac{1}{2})$, where $C=\mathcal{R}(t,2,0)$ corresponds to the rational number $\frac{t}{2t+1}$. Thus $\mathcal{C}(A,*,C,\frac{1}{2})$ orbifold-toroidal implies that $\Delta(\frac{1}{0},\frac{t}{2t+1})>1$. Thus $|2t+1|>1$. Consequently, $t'=-(t+1)\neq -1,0$ and $A=\mathcal{R}(s)$. Again, that $\mathcal{C}(A,*,C,\frac{1}{2})$ is orbifold-toroidal guarantees that $|s|>1$. □

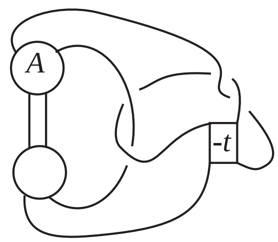


Figure 4.6

Lemma 4.5 *If $\mathcal{C}(A, B, *, \frac{1}{0})$ is a rational tangle then, up to symmetry exchanging A and B, $A = \mathcal{R}(s)$ and $B = \mathcal{R}(t, -s)$ for $s, t \in \mathbb{Z}$. If $\mathcal{C}(A, B, *, \frac{1}{2})$ is orbifold-toroidal then $|s| > 1$, $t \neq 0$ and $(s, t) \neq (2, 1), (-2, -1)$.*

Proof Isotoping $\mathcal{C}(A, B, *, \frac{1}{0})$ to Figure 4.7 we see that one of A or B must be an integral tangle. By symmetry we assume it is A, $A = \mathcal{R}(s)$. Thus we are as in Figure 4.8, from which we see that $B' = \mathcal{R}(1/t)$. Thus $B = \mathcal{R}(t, -s)$. Now assume $\mathcal{C}(A, B, *, \frac{1}{2})$ is orbifold-toroidal. See Figure 4.9. Then $|s| > 1$, and $B = \mathcal{R}(t, -s)$ corresponds to tangle slope $\frac{1-st}{t}$. Thus $\Delta(\frac{0}{1}, \frac{1-st}{t}) > 1$. That is, $|1 - st| > 1$. Thus $t \neq 0$ and $(s, t) \neq (2, 1), (-2, -1)$. $\square$

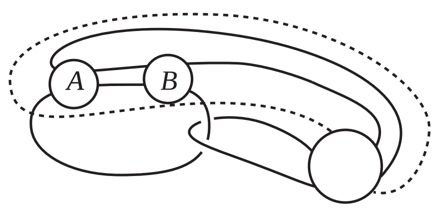


Figure 4.7

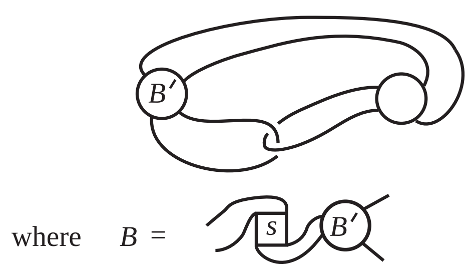


Figure 4.8

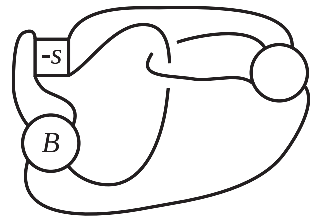


Figure 4.9

Proof of Theorem 4.1 We prove this for $J_1(\ell, m)$. The proof for $J_2(\ell, m)$ is similar. Recall that by attaching the appropriate "C"–tangle to $\mathcal{T}_1(\ell, m)$ we get the tangle $B(\ell, m, n, 0)$ of [6]. There are infinitely many such fillings corresponding to different values of n. Looking at double branched covers, this says that the corresponding Dehn fillings of X, the exterior of $J_1(\ell, m)$, are the exteriors of the hyperbolic knots $k(\ell, m, n, 0)$. Denote the two components of ∂X as $\partial_1 X$, $\partial_2 X$, where $\partial_1 X$ is the component along which these fillings are made (corresponding to the boundary of the ambient solid torus of $J_1(\ell, m)$). Because infinitely many fillings of X are hyperbolic, either X is hyperbolic or there is a cable space along $\partial_1 X$ (Theorem 2.4.4 of [5]). We assume the latter for contradiction. Then the slope of each of these Dehn fillings is distance 1 from a unique slope γ, and furthermore, Dehn filling X along γ has a lens space summand. The $\mathbb{Z}_2$–orbifold quotient of the γ–Dehn filling of X is pictured in Figure 4.10 (the C–tangle for this picture corresponds to replacing the n twist box of $B(\ell, m, n, 0)$ with $\mathcal{R}(1/0)$). But inserting $\mathcal{R}(1/0)$ into the "D"–tangle of Figure 4.10 gives the unlink of two components. That is, there is a filling of $\partial_2 X$ such that, along with the filling of $\partial_1 X$ along γ, gives $S^2 \times S^1$. But this contradicts the fact that first filling X along γ yields a lens space summand. □

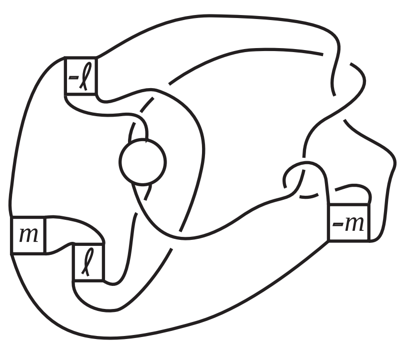


Figure 4.10

5 Non-integral surgery and the JSJ–decomposition

In this section we consider a non-integral Dehn filling $X(\gamma)$ on the exterior X of a knot in S^3, and analyze the relation between the JSJ–decompositions of X and $X(\gamma)$.

If M is an irreducible 3–manifold we shall denote by $\mathrm{Seif}(M)$ the disjoint union of the Seifert fibered pieces of the JSJ–decomposition of M. In the case of a knot exterior X, the possible components of $\mathrm{Seif}(X)$ have been described by Jaco and Shalen.

Lemma 5.1 [14, Lemma VI.3.4] *Let X be the exterior of a knot in S^3, and let W be a component of $\mathrm{Seif}(X)$. Then W is either a torus knot space, a cable space, or a composing space.*

The relation between $\mathbf{T}(X)$ and $\mathbf{T}(X(\gamma))$ is described in the following theorem and its addendum.

Theorem 5.2 *Let X be the exterior of a knot k in S^3, and suppose $\Delta(\gamma,\mu)\geq 2$ where μ is the meridian of k. Let W be the component of X cut along $\mathbf{T}(X)$ that contains ∂X. Then exactly one of the following four possibilities holds.*

(1) $\mathbf{T}(X(\gamma))=\mathbf{T}(X)$;

(2) $\mathbf{T}(X)=\varnothing$, *$X$ is hyperbolic, k is an Eudave-Muñoz knot $k(\ell,m,n,p)$, $\Delta(\gamma,\mu)=2$, $X(\gamma)=M_1\cup_T M_2$, where M_i is a Seifert fiber space over D^2 with two exceptional fibers, $i=1,2$, and $\mathbf{T}(X(\gamma))=T$;*

(3) *k is contained in a tubular neighborhood $N(k_0)$ of a non-trivial knot k_0 as a $J_\varepsilon(\ell,m)$–satellite of k_0, $\partial W=T_0\cup\partial X$, where $T_0=\partial N(k_0)$, $\Delta(\gamma,\mu)=2$, $W(\gamma)=N_1\cup_T M_2$, where N_1 is a Seifert fiber space over A^2 with one exceptional fiber and M_2 is a Seifert fiber space over D^2 with two exceptional fibers, and $\mathbf{T}(X(\gamma))=\mathbf{T}(X)\cup T$;*

(4) *k is a (p,q)–cable of a non-trivial knot k_0 with exterior X_0, $q\geq 2$, $\partial W=T_0\cup\partial X$ as in (3), $\gamma=\frac{npq\pm 1}{n}$, $n\geq 2$, and $\mathbf{T}(X(\gamma))=\mathbf{T}(X)-T_0$.*

We spell out more details about cases (1) and (4) of Theorem 5.2 in the following addendum, where k_γ denotes the core of the Dehn filling solid torus in $X(\gamma)$.

Addendum 5.3 *In case (1) of Theorem 5.2 we have*

(a) *if $W=X$ then $X(\gamma)$ is atoroidal;*

(b) *if $W\neq X$ then $W(\gamma)$ is hyperbolic if and only if W is hyperbolic;*

(c) *if $W \neq X$ then $W(\gamma)$ is Seifert fibered if and only if W is Seifert fibered. If W is a cable space then $W(\gamma)$ is a Seifert fiber space over D^2 with two exceptional fibers, and if W is a composing space with $(n+1)$ boundary components then $W(\gamma)$ is a Seifert fiber space over an n–punctured sphere with a single exceptional fiber of multiplicity $\Delta(\gamma,\mu)$. In both cases k_γ is isotopic to an exceptional fiber of $W(\gamma)$.*

In case (4) of Theorem 5.2 we have

(a) *W is a cable space and $W(\gamma)$ is a solid torus, with meridian γ_0, say, on T_0;*

(b) *conclusion (1) holds for $X_0(\gamma_0)$;*

(c) *k_γ is isotopic to an (r,s)–cable of the core of $W(\gamma)$, for some $s > 1$.*

Proof of Theorem 5.2

Case I $|\partial W| = 1$

Here $\mathbf{T}(X) = \varnothing$ and $W = X$, so X is either hyperbolic or Seifert fibered. In the first case, either (1) or (2) holds by [11]. In the second case, k is a torus knot, $X(\gamma)$ is a Seifert fiber space over S^2 with at most three exceptional fibers, and $\mathbf{T}(X(\gamma)) = \mathbf{T}(X) = \varnothing$.

Case II $|\partial W| = 2$

There are two subcases.

(A) W **hyperbolic**

$W(\mu)$ is a solid torus; therefore $W(\gamma)$ is irreducible and ∂–irreducible, by [29] and [5] respectively. If $W(\gamma)$ is hyperbolic then $\mathbf{T}(X(\gamma)) = \mathbf{T}(X)$, and (1) holds. By [23, Proposition 9], $W(\gamma)$ is not Seifert fibered. Hence we may assume that $W(\gamma)$ is toroidal. Then by Theorem 4.2 k is a $J_\varepsilon(\ell,m)$–satellite in $W(\mu) \cong S^1 \times D^2$. Then $W(\gamma) = N_1 \cup_T M_2$ as in conclusion (3). Let φ be the fiber of N_1 on $T_0 = \partial W(\gamma)$.

Consider the component $Z \neq W$ of the JSJ–decomposition of X with $T_0 \subset \partial Z$. Assume that Z is a Seifert fiber space, with fiber ψ on T_0. We will show that $\psi \neq \varphi$; hence $\mathbf{T}(X(\gamma)) = \mathbf{T}(X) \cup T$ and (3) holds.

Let μ_0 be the meridian of $W(\mu) \cong S^1 \times D^2$. Since $\Delta(\mu_0,\varphi) \geq 2$ by the Addendum to Theorem 4.3, it suffices to prove that $\Delta(\mu_0,\psi) \leq 1$.

By Lemma 5.1, Z is either (i) a torus knot exterior, or (ii) a cable space, or (iii) a composing space. Let $X_0 = \overline{X - W}$; so $\partial X_0 = T_0$. Since $X_0(\mu_0) \cong S^3$, we must have $\Delta(\mu_0, \psi) = 1$ in case (i), $\Delta(\mu_0, \psi) = 1$ in case (ii) (since $Z \cup W(\mu)$ is a solid torus), and, $\mu_0 = \psi$ in case (iii) (since $\partial(Z \cup W(\mu))$ must be compressible).

(B) W **Seifert fibered.**

Then W is a cable space. Thus k is a (p, q)–cable, $q \geq 2$, of a non-trivial knot k_0. Let φ be the slope on ∂X of the Seifert fiber of W; thus $\varphi = pq/1$ with respect to the usual meridian-longitude basis.

If $\Delta(\gamma, \varphi) \geq 2$ then $W(\gamma)$ is a Seifert fiber space over D^2 with two exceptional fibers, which is atoroidal. Therefore $\mathbf{T}(X(\gamma)) = \mathbf{T}(X)$.

Since $\Delta(\gamma, \mu) \geq 2$ ($\mu = 1/0 =$ meridian of k), $\gamma \neq \varphi$. So assume $\Delta(\gamma, \varphi) = 1$; ie, $\gamma = \frac{npq \pm 1}{n}$, $n \geq 2$. Then $W(\gamma)$ is a solid torus, with meridian γ_0, say, on T_0. Let μ_0 be the slope on T_0 of the meridian of the solid torus $W(\mu)$. Then $\Delta(\gamma_0, \mu_0) = q^2 \Delta(\gamma, \mu) \geq 8$. Let $X_0 = \overline{X - W}$, the exterior of k_0. By induction on the number of components of $\mathbf{T}(X)$, we may assume that the theorem holds for X_0. (The start of the induction is Case (1) above.) Since $\Delta(\gamma_0, \mu_0) > 2$, conclusions (2) and (3) of the theorem do not hold for X_0. If (1) holds for X_0, then we get $\mathbf{T}(X(\gamma)) = \mathbf{T}(X_0(\gamma_0)) = \mathbf{T}(X_0) = \mathbf{T}(X) - T_0$, which is conclusion (4) for X. Finally, assume that (4) holds for X_0. So k_0 is a (p_1, q_1)–cable, $q_1 \geq 2$, of a non-trivial knot k_1, and $\gamma_0 = \frac{n_1 p_1 q_1 \pm 1}{n_1}$, $n_1 \geq 2$, with respect to the usual basis for k_0. But we also have $\gamma_0 = \frac{npq \pm 1}{nq^2}$, $n \geq 2$, and this is a contradiction (see [8, page 704]).

Case III $|\partial W| \geq 3$

Let the components of $\partial W - \partial X$ be $T_1, \dots, T_n$, $n \geq 2$. Then the components of $\overline{X - W}$ are $Y_1, \dots, Y_n$, say, where $\partial Y_i = T_i$ and T_i is incompressible in Y_i, $1 \leq i \leq n$. Since T_i compresses in $S^3 = X(\mu)$, it compresses in $W(\mu)$. Hence $W(\mu)$ is reducible. (In fact it is easy to show that $W(\mu)$ is a connected sum of n solid tori.)

Again we distinguish two subcases.

(A) W **hyperbolic**

Since $\Delta(\gamma, \mu) \geq 2$, $W(\gamma)$ is irreducible by [10]. Also, since $|\partial W| \geq 3$, $W(\gamma)$ is atoroidal and anannular by [37]. Therefore $W(\gamma)$ is hyperbolic, and $\mathbf{T}(X(\gamma)) = \mathbf{T}(X)$. This is conclusion (1).

(B) W **Seifert fibered**

Since $|\partial W| = n + 1$, $n \geq 2$, W is a composing space by Lemma 5.1. Also, the meridian μ is the Seifert fiber of W, since $W(\mu)$ is reducible. Hence $W(\gamma)$ is a Seifert fiber space over an n–punctured sphere with one exceptional fiber, of multiplicity $\Delta(\gamma, \mu)$. Since the Seifert fibers of W and $W(\gamma)$ are the same on each T_i, we have $\mathbf{T}(X(\gamma)) = \mathbf{T}(X)$, and (1) holds. □

Proof of Addendum 5.3 This follows by examining the proof of Theorem 5.2. □

6 Main theorem

Recall from Section 2 the definition of the characteristic 2–sided toric 2–suborbifold $\mathbf{T}(K)$ of a prime knot K in S^3. Let $\mathrm{Seif}(K) = \mathrm{Seif}(O(K))$ be the disjoint union of the S^1–fibered components of $O(K)$ cut along $\mathbf{T}(K)$.

Let $(a, \partial a)$ be an unknotting arc for K. As described in Section 2, a relative regular neighborhood of $(a, \partial a)$ in (S^3, K) determines a marked tangle $\mathcal{T}_0$ which is replaced with a tangle $\mathcal{T}_0'$ under the crossing move. Let M be the double branched cover of S^3 along K, V_0 be the solid torus preimage of $\mathcal{T}_0$ under the branched covering, and $X = M - V_0$. Then $M = X(\gamma)$ and $S^3 = X(\mu)$ where $\Delta(\mu, \gamma) = 2$. Let $k = k_\mu$ be the knot in S^3 of which X is the exterior, and let k_γ be the core of V_0 in M.

Definition The unknotting arc $(a, \partial a)$ is said to be an (r, s)*–cable of an exceptional fiber of* $\mathrm{Seif}(K)$ iff k_γ is an (r, s)–cable of an exceptional fiber in $\mathrm{Seif}(M)$.

Remark If $(a, \partial a)$ is an (r, s)–cable of an exceptional fiber of $\mathrm{Seif}(K)$, then the corresponding $\mathcal{T}_0$ lies in a rational tangle $\mathcal{R}(p/q)$ in $\mathrm{Seif}(K)$ which is the quotient of a neighborhood of this exceptional fiber. The tangle $\mathcal{R}(p/q) - \mathcal{T}_0$ in $S^2 \times I$, has a double branched cover which is a cable space. By Lemma 3.8, $\mathcal{R}(p/q) - \mathcal{T}_0$ is homeomorphic to $\mathcal{M}(\frac{v}{w}, *)$ for some $v, w \in \mathbb{Z}$. The results of applying the crossing move associated to such an $(a, \partial a)$ are further discussed in Lemmas 8.1, 12.3, 12.4, and Theorem 8.2.

Lemma 6.1 *Let $(a, \partial a)$ be an unknotting arc for K. One of the following holds:*

(1) *$(a, \partial a)$ can be isotoped in (S^3, K) to be disjoint from $\mathbf{T}(K)$. Furthermore, if $\mathbf{T}(K) \neq \varnothing$ and $(a, \partial a)$ can be isotoped into $\mathrm{Seif}(K)$, then a is isotopic to an (r, s)–cable of an exceptional fiber in $\mathrm{Seif}(K)$, for some $s \geq 1$.*

(2) (a) *K is an EM–knot $K(\ell, m, n, p)$.*

(b) *$O(K)$ has a unique connected, incompressible, 2–sided, toric 2–suborbifold S, a Conway sphere, K has an unknotting arc $(b, \partial b)$ with $|b \cap S| = 1$ (the standard unknotting arc for $K(\ell, m, n, p)$), and no unknotting arc is disjoint from S.*

(3) *K is the union of essential tangles $\mathcal{P} \cup \mathcal{P}_0$, where $\mathcal{P}_0$ is the EM–tangle $\mathcal{A}_\varepsilon(\ell, m)$, $\partial \mathcal{P}_0 \subset \mathbf{T}(K)$, and $(a, \partial a)$ can be isotoped into $\mathcal{P}_0$. If $\mathcal{T}(*, *)$ is the exterior in $\mathcal{P}_0$ of the crossing ball corresponding to $(a, \partial a)$, then $\mathcal{T}(*, *)$ is as described in Theorem 4.3.*

Proof of Lemma 6.1 Let M, $k = k_\mu, k_\gamma$ be as described above. We are now in the context of Theorem 5.2. Possibilities (1), (2), and (3) in the conclusion of Theorem 5.2 will lead to conclusions (1), (2), and (3), respectively of Lemma 6.1, and possibility (4) will lead to conclusion (1).

Let $h\colon M \to M$ be the covering involution, with quotient orbifold $\widehat{M} = O(K)$. Write $V_0 = V_\gamma$, with quotient $\widehat{V}_\gamma$ the 3–suborbifold $\mathcal{T}_0$ of $O(K)$.

(1) Here $\mathbf{T}(M) = \mathbf{T}(X)$. The covering involution $h\colon M \to M$ restricts to $h\colon X \to X$, and we can isotop $\mathbf{T}(X)$ in X to be h–invariant [20]. Let T be a component of $\mathbf{T}(X)$. Then T separates X into two components, one of which contains ∂X. It follows that if $h(T) = T$ then h preserves the sides of T, and hence $\mathbf{T}^+(M) = \mathbf{T}^+(X) = \mathbf{T}(X) = \mathbf{T}(M)$.

By (1)(a) of Addendum 5.3, if $\mathbf{T}(X) = \varnothing$ then $M = X(\gamma)$ is atoroidal, and hence $\mathbf{T}(K) = \varnothing$. Thus conclusion (1) holds trivially.

If $\mathbf{T}(X) \neq \varnothing$, then $X = X_0 \cup W$, say, $X_0 \neq \varnothing$. Hence $M = X_0 \cup W(\gamma)$, and, taking quotients, $O(K) = \widehat{X}_0 \cup \widehat{W}(\gamma)$, where $\widehat{V}_\gamma = \mathcal{T}_0 \subset \widehat{W}(\gamma)$. By (1)(b) of Addendum 5.3, if W is hyperbolic then $W(\gamma)$ is hyperbolic, and so $\widehat{W}(\gamma)$ is atoroidal and is a component of $O(K)$ cut along $\mathbf{T}(K)$. Hence $\mathbf{T}(K)$ can be orbifold-isotoped off $\widehat{W}(\gamma)$, in particular, off $\mathcal{T}_0$. If W is Seifert fibered, then by (1)(c) of Addendum 5.3 the Seifert fibering of W extends to a Seifert fibering of $W(\gamma)$. Thus $\widehat{W}(\gamma)$ is S^1–fibered, and $\widehat{V}_\gamma = \mathcal{T}_0$ is a neighborhood of an exceptional (orbifold) fiber. Also, since $\mathbf{T}(M) = \mathbf{T}(X)$, $W(\gamma)$ is a component of $\mathrm{Seif}(M)$, and hence $\widehat{W}(\gamma)$ is a component of $\mathrm{Seif}(K)$.

(2) Here k^* is an Eudave-Muñoz knot $k(\ell, m, n, p)$. By Theorem 3.4(1), $K = K(\ell, m, n, p)$, $M = M_1 \cup_T M_2$, $\mathbf{T}(M) = T$, and we may isotop T so that $h(M_i) = M_i$, $i = 1, 2$. Hence $\mathbf{T}(K) = S = \widehat{T}$. The facts that K has an unknotting arc b with $|b \cap S| = 1$, and no unknotting arc disjoint from S, are proved in [6].

(3) Here k^* is a $J_\varepsilon(\ell, m)$–satellite of k_0. Thus $X = X_0 \cup_{T_0} W$, where X_0 is the exterior of k_0, $T_0 = \partial X_0$, and W is the exterior of $J_\varepsilon(\ell, m)$ in $S^1 \times D^2$. Also, $M = X_0 \cup_{T_0} W(\gamma)$, $W(\gamma) \cong N_1 \cup_T M_2$ as in Theorem 5.2, and $\mathbf{T}(M) = \mathbf{T}(X) \cup T$. Since W is hyperbolic, $T_0 \subset \mathbf{T}(X)$.

Now $h\colon M \to M$ leaves X invariant. Hence we can isotop $\mathbf{T}(X)$ in X to be h–invariant [20]. In particular, we must clearly have $h(T_0) = T_0$. Hence h leaves $W(\gamma)$ invariant. $\mathrm{Fix}(h)$ cannot be disjoint from T_0 or completely lie in T_0, otherwise h would give rise to an involution on S^3 whose fixed set was k_0 or a satellite of k_0—contradicting the $\mathbb{Z}_2$–Smith Conjecture [36]. In particular the quotient of W

under h, the exterior of the crossing ball corresponding to $(a,\partial a)$, is a tangle $\mathcal{T}(*,*)$ in $S^2\times I$ satisfying the hypotheses of Theorem 4.3; hence $\mathcal{T}(*,*)$ is as described there. This implies $\mathcal{P}_0 \cong \widehat{W}(\gamma)=\mathcal{T}(*,\beta)\cong \mathcal{A}_\varepsilon(\ell,m)$ for some ε,ℓ,m. Since $\mathbf{T}(M)=\mathbf{T}(X)\cup T$, N_1 and M_2 of $W(\gamma)$ are components of $\mathrm{Seif}(M)$. Furthermore, $\mathbf{T}^+(M)=\mathbf{T}(M)$ since h preserves the sides of T and since $\mathbf{T}^+(X)=\mathbf{T}(X)$ by the argument for (1). Thus $\partial\mathcal{P}_0\in\mathbf{T}(K)$ and $\widehat{N}_1$, $\widehat{M}_2$ are components of $\mathrm{Seif}(K)$.

(4) Here W is a cable space and $W(\gamma)$ is a solid torus. Let $X_0=\overline{X-W}$, and $\partial X_0=T_0$. Then $M=X_0\cup_{T_0}W(\gamma)$, and $\mathbf{T}(M)=\mathbf{T}(X)-T_0=\mathbf{T}(X_0)$. Since h leaves X invariant, we can isotop $\mathbf{T}(X)$ in X to be h–invariant. Then $h(T_0)=T_0$, $\mathbf{T}(X_0)$ is h–invariant, and $\mathbf{T}^+(M)=\mathbf{T}^+(X_0)=\mathbf{T}(X_0)=\mathbf{T}(M)$. Therefore $\mathbf{T}(K)$ is the quotient $\widehat{\mathbf{T}}(X_0)$.

Let μ_0 be the meridian of K_0 on T_0, and let γ_0 be the meridian of the solid torus $W(\gamma)$. Then $\Delta(\gamma_0,\mu_0)>2$, and $M=X(\gamma)=X_0(\gamma_0)$. Let W_0 be the component of X_0 cut along $\mathbf{T}(X_0)$ that contains ∂X_0. By case (4)(b) of Addendum 5.3, (1) holds for $X_0(\gamma_0)$; thus (1)(a), (1)(b) and (1)(c) of Addendum 5.3 hold for X_0, W_0. Conclusions 1(a) and 1(b) now follow from the argument in case (1) above applied to X_0, W_0, γ_0. □

Theorem 6.2 *Let K be a knot with unknotting number 1. Then one of the following three possibilities holds.*

(1) (a) *Any unknotting arc $(a,\partial a)$ for K can be isotoped in (S^3,K) so that $a\cap\mathbf{T}(K)=\varnothing$.*

(b) *If $\mathbf{T}(K)\neq\varnothing$ and K has an unknotting arc a in* Seif(K) *then a is isotopic to an (r,s)–cable of an exceptional fiber of* Seif(K)*, for some $s\geq 1$.*

(2) (a) *K is an EM–knot $K(\ell,m,n,p)$.*

(b) *$O(K)$ has a unique connected incompressible 2–sided toric 2–suborbifold S, a Conway sphere, K has an unknotting arc a with $|a\cap S|=1$ (the standard unknotting arc for $K(\ell,m,n,p)$), and K has no unknotting arc disjoint from S.*

(3) *K is the union of essential tangles $\mathcal{P}\cup\mathcal{P}_0$, where $\mathcal{P}_0$ is an EM–tangle $\mathcal{A}_\varepsilon(\ell,m)$ and $\partial\mathcal{P}_0$ is in $\mathbf{T}(K)$. Any unknotting arc for K can be isotoped into $\mathcal{P}_0$. The standard unknotting arc for $\mathcal{A}_\varepsilon(\ell,m)$ is an unknotting arc for K.*

Remarks (A) If $K=K(\ell,m,n,p)$, then (2) must hold. If $K=\mathcal{P}\cup\mathcal{P}_0$ where $\mathcal{P}_0$ is an EM–tangle then (1) or (3) may hold. If (1) holds then, still, by Lemma 9.6 (and Definition 3.3), any unknotting arc of K can be isotoped into $\mathcal{P}_0$ (hence into an exceptional fiber of $\mathcal{P}_0$).

(B) In conclusion (3), to say that K is unknotted by the standard unknotting arc for $\mathcal{A}_\varepsilon(\ell,m)$ (as described in Section 4) we mean that there is a tangle homeomorphism from $\mathcal{P}_0$ to $\mathcal{A}_\varepsilon(\ell,m)$ which makes this identification. Any two such will differ by an isotopy of the tangle ball fixed on the boundary, which will isotop the two unknotting arcs. Indeed, any homeomorphism of tangles, $h\colon \mathcal{A}_\varepsilon(\ell,m)\to\mathcal{A}_\varepsilon(\ell,m)$, preserves the markings, hence is isotopic to the identity.

Question *Is any unknotting arc for $K(\ell,m,n,p)$ or for $\mathcal{A}_\varepsilon(\ell,m)$ isotopic to its standard unknotting arc?*

One approach to the above question would be to prove an analog of Theorem 3.4 for the exteriors of $k(\ell,m,n,p)$, $J_\varepsilon(\ell,m)$ (resp.). That is, show that there is a unique tangle quotient arising from involutions on any such knot exterior.

Proof of Theorem 6.2 Theorem 6.2(2) is the same as Lemma 6.1(2). So we assume K is not an EM–knot. Furthermore we may assume that K can be written as the union of essential tangles $\mathcal{P}\cup\mathcal{P}_0$ with $\mathcal{P}_0$ homeomorphic to $\mathcal{A}_\varepsilon(\ell,m)$ and $\partial\mathcal{P}_0\subset\mathbf{T}(K)$ (otherwise (1) holds for K by Lemma 6.1).

If $(a,\partial a)$ is an unknotting arc for K which cannot be isotoped into $\mathcal{P}_0$, then Lemma 6.1(1) applies. But this says the unknot can be written as $\mathcal{T}\cup\mathcal{A}_\varepsilon(\ell,m)$ for some tangle $\mathcal{T}$ and some ε,ℓ,m. This contradicts Lemma 9.6 and Definition 3.3. Thus any unknotting arc for K can be isotoped into $\mathcal{P}_0$.

If conclusion (1) of Theorem 6.2 does not hold then there is an unknotting arc $(a,\partial a)$ satisfying Lemma 6.1(3). Let $\mathcal{T}(*,*)$ be the exterior of the crossing ball corresponding to $(a,\partial a)$ and $h_1\colon \mathcal{T}(*,\beta)\to\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{2})$, $h_2\colon \mathcal{T}(*,\alpha)\to\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$ be the tangle homeomorphisms provided by Theorem 4.3. The standard unknotting arc for $\mathcal{A}_\varepsilon(\ell,m)$ corresponds to the crossing move $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{2})\to\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$. Thus h_1 identifies the standard unknotting arc of $\mathcal{P}_0\cong\mathcal{A}_\varepsilon(\ell,m)$ for which we are looking. That is, performing the crossing move on K dictated by the standard unknotting arc gives the knot gotten by gluing $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$ to $\mathcal{P}$ via $h_1^{-1}|\partial$. Since $(h_2|\partial)(h_1|\partial)^{-1}$ is the identity, this is the same as gluing $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$ to $\mathcal{P}$ via $h_2^{-1}|\partial$, which is the unknot by assumption. □

The following is a generalization of the result of [30; 31] (see also [17]) that an unknotting arc for a satellite knot can always be taken to be disjoint from the companion 2–torus.

Corollary 6.3 *Let K be a knot with unknotting number 1, that is neither an EM–knot nor a knot containing an EM–tangle with essential boundary. Let F be an* *incompressible 2–sided toric 2–suborbifold of $O(K)$. Then any unknotting arc $(a, \partial a)$ for K can be isotoped in (S^3, K) so that $a \cap F = \varnothing$.*

This is an immediate consequence of Theorem 6.2 and the following lemma.

Lemma 6.4 *Let F be an incompressible 2–sided toric 2–suborbifold of $O(K)$. Then F is orbifold isotopic to a vertical suborbifold of* $\mathrm{Seif}(K)$.

Proof The fact that F is isotopic into $\mathrm{Seif}(K)$ follows from the discussion in [3] beginning at the paragraph immediately preceding Lemma 7 on page 456, and ending at the statement "and therefore that $F \cap F' = \varnothing$" near the bottom of page 457.

So assume $F \subset \mathrm{Seif}(K)$. Any component of F that is boundary parallel in $\mathrm{Seif}(K)$ can be isotoped to be vertical, so by [3, Verticalization Theorem 4] it is enough to show that F cannot be isotoped to be horizontal. Let $p\colon M \to O(K)$ be the double branched covering projection. If F were a horizontal 2–suborbifold of $O(K)$, then $p^{-1}(F)$ would be a horizontal surface in M. But $H_2(M; \mathbb{Q}) = 0$, so M contains no horizontal surface. □

7 Mutation

Let $\mathcal{T} = (B, A)$ be a knot in S^3 or a tangle. Let $\mathcal{T}_0 = (B_0, A_0)$ be a subtangle of $\mathcal{T}$ such that B_0 is a 3–ball. Let $S_0 = \partial B_0$. Let $h\colon B_0 \to B^3$ be a homeomorphism such that $h(S_0 \cap A) = Q \subset S^2$. Let $\Gamma_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be the group of automorphisms of (S^2, Q) consisting of rotations through π about any one of the three co-ordinate axes in $\mathbb{R}^3$ together with the identity. Let $\mathcal{T}_1 = \mathcal{T} - \operatorname{int} \mathcal{T}_0$ and regard $\mathcal{T}$ as $\mathcal{T}_0 \cup \mathcal{T}_1$, where $\mathcal{T}_0$ is glued to $\mathcal{T}_1$ by the identity map on S_0. Now for $g \in \Gamma_0 - \{1\}$, let $\mu = h^{-1} g h\colon S_0 \to S_0$ and define $\mathcal{T}' = \mathcal{T}_0 \cup_\mu \mathcal{T}_1$. We say $\mathcal{T}'$ is a *mutant* of $\mathcal{T}$. The operation of replacing $\mathcal{T}$ by $\mathcal{T}'$ is *mutation* of $\mathcal{T}$ along S_0, by the *mutation involution* μ.

Boileau has asked [16, Problem 1.69(c)] if the unknotting number of a link is a mutation invariant. We prove that this is at least true for knots with unknotting number 1.

Theorem 7.1 *Having unknotting number 1 is invariant under mutation.*

Proof Let K be a knot with unknotting number 1, and let K' be a mutant of K. Then there is a Conway sphere S which decomposes K into two tangles $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $K' = \mathcal{T}_1 \cup \rho(\mathcal{T}_2)$, where ρ is rotation of the ball B_2 containing $\mathcal{T}_2$ through π about one of the co-ordinate axes. Note that we also have $K' = \rho(\mathcal{T}_1) \cup \mathcal{T}_2$. If either

$\mathcal{T}_1$ or $\mathcal{T}_2$ is trivial then $K = K'$. Also, since $u(K) = 1$, K is prime [28]. Hence we may assume that the tangles $\mathcal{T}_i$ are prime, and therefore that S is essential.

First suppose that K is not an EM–knot nor the union of two essential tangles, one of which is an EM–tangle. Then, taking $F = S$ in Corollary 6.3 we get that there is an unknotting arc a for K disjoint from S, and therefore, without loss of generality, contained in B_2. As marked tangles, the crossing move determined by a transforms $\mathcal{T}_2$ to a rational tangle $\mathcal{R}$. Then the crossing move on K' determined by $\rho(a)$ transforms $\rho(\mathcal{T}_2)$ to the rational tangle $\rho(\mathcal{R}) = \mathcal{R}$, and hence transforms K' to $\mathcal{T}_1 \cup \mathcal{R} =$ unknot.

If K is an EM–knot, then $K' = K$ [6].

Finally, suppose K is the union of essential tangles $\mathcal{P} \cup \mathcal{P}_0$, where $\mathcal{P}_0$ is an EM–tangle. Then $\mathcal{P}_0$ is of the form $\mathcal{S}(\alpha, \beta; \gamma, *)$. If S is not isotopic in $O(K)$ to $S_0 = \partial\mathcal{M}(\alpha, \beta)$, then the argument above shows that K' has unknotting number 1. So assume that $S = S_0$. Since rotating $\mathcal{M}(\alpha, \beta)$ about the horizontal axis leaves it invariant, we may assume that ρ is rotation about the axis perpendicular to the plane of the paper. Thus $K' = \mathcal{P} \cup \mathcal{P}'_0$, where $\mathcal{P}'_0 = \mathcal{S}(\beta, \alpha; \gamma, *)$.

By part (3) of Theorem 6.2, K has an unknotting arc a that lies in the 3–ball B_0 containing the tangle $\mathcal{P}_0$. By Lemma 2.1, there is a homeomorphism $h\colon \mathcal{S}(\alpha, \beta; \gamma, *) \to \mathcal{S}(\beta, \alpha; \gamma, *)$ such that $h|\partial D$ is rotation through π about the horizontal axis. Note that h is isotopic to the corresponding rotation of the ball $\overline{S^3 - D}$. Hence there is a rotation g of the ball B_0 which takes $\mathcal{P}_0$ to $\mathcal{P}'_0$. The crossing change of K determined by a converts $\mathcal{P}_0$ to a rational tangle $\mathcal{R}$, where $\mathcal{P} \cup \mathcal{R}$ is the unknot. Therefore the crossing change of K' determined by $g(a)$ converts $K' = \mathcal{P} \cup \mathcal{P}'_0 = \mathcal{P} \cup g(\mathcal{P}_0)$ to $\mathcal{P} \cup g(\mathcal{R}) = \mathcal{P} \cup \mathcal{R} =$ unknot. □

8 Algebraic knots

For the definition of an *algebraic* knot or link (in the sense of Conway) see Section 2 of [33]. We briefly summarize this in a form suitable for our present purposes.

An *elementary (algebraic) tangle* is a tangle of the form $\mathcal{M}(\alpha, \beta)$, $\mathcal{M}(\gamma, *)$, or $\mathcal{M}(*, *)$, where $\alpha, \beta, \gamma \in \mathbb{Q} - \mathbb{Z}$. We shall refer to these as elementary tangles of type I, II, or III respectively. See Figure 8.1.

A *Montesinos tangle of length 3*, $\mathcal{M}(\alpha, \beta, \gamma)$, $\alpha, \beta, \gamma \in \mathbb{Q} - \mathbb{Z}$, is defined in the obvious way; see Figure 8.2.

Recall [4] that if $\mathcal{T}$ is a marked tangle in B^3 then $1^*\mathcal{T}$ is the knot or link obtained by capping off $\mathcal{T}$ with the rational tangle $\mathcal{R}(0)$ (ie, $1^*\mathcal{T}$ is the numerator closure of $\mathcal{T}$).

Then an *algebraic* knot is a knot of one of the following forms:

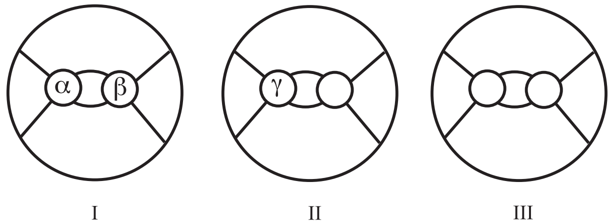


Figure 8.1

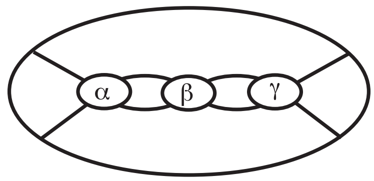


Figure 8.2

(a) $1^*\mathcal{R}(\alpha)$;

(b) $1^*\mathcal{M}(\alpha,\beta,\gamma)$;

(c) a union along boundary components of elementary tangles.

The knots of type (a) are the 2–bridge or rational knots. (Any knot $1^*\mathcal{M}(\alpha,\beta)$ can also be expressed as $1^*\mathcal{R}(\gamma)$.) Those of type (b) are the *Montesinos knots of length 3*. We will call a knot of type (c) a *large* algebraic knot. Thus an algebraic knot is large if and only if it has an essential Conway sphere. If K is an algebraic knot, of type (a), (b), or (c), then the double branched cover of K is a lens space, a Seifert fiber space over S^2 with three exceptional fibers, or a toroidal graph-manifold, respectively.

An elementary tangle comes equipped with a marking, given by Figure 8.1. In constructing a large algebraic knot K, the gluing homeomorphisms between the boundary components of the elementary tangles will not in general preserve the markings. To describe K as a marking-preserving union of marked tangles we need to interpolate marked tangles of 4–string braids in $S^2\times I$ between the boundary components.

This can also be described in terms of diagrams. Figure 8.1 III is a diagram in a pair of pants of an elementary tangle of type III. A diagram of an elementary tangle of type I or II, in a disk or annulus respectively, may be obtained by inserting diagrams of the appropriate rational tangles into the diagrams in Figure 8.1, I or II. Also, a 4–string braid in $S^2\times I$ has a diagram in an annulus. Then a knot is large algebraic if and only if it has a diagram in S^3 that is a union along boundary components of such elementary tangle diagrams and 4–string braid diagrams.

Lemma 8.1 *$\mathcal{R}(p/q)$ can be transformed to $\mathcal{R}(1/0)$ by a crossing move if and only if there exist coprime integers r, s such that $p/q = \frac{2rs\pm1}{2s^2}$.*

Proof If $p/q = \frac{2rs\pm1}{2s^2}$, $q \neq 0$, then p/q has a continued fraction expansion of the form $[a_1, a_2, \dots, a_k, \pm2, -a_k, \dots, , -a_2, -a_1, a]$, (see [15] or [18]), and hence can be transformed to $\mathcal{R}(1/0)$ by a crossing move.

Conversely, suppose $\mathcal{R}(p/q)$ can be transformed to $\mathcal{R}(1/0)$ by a crossing move. Let the double branched covers of $\mathcal{R}(1/0)$ and $\mathcal{R}(p/q)$ be V and V' respectively. Then there is a knot K in V such that (with respect to some framing of K), $m/2$–Dehn surgery on K gives V'. Note that, with respect to the basis of $H_1(\partial V)$ corresponding to the standard marking, the meridian of V' has slope p/q.

If K is unknotted in V, then $p/q = 1/0$, while if K is a core of V, then $p/q = m/2$; in both cases p/q is of the stated form.

Otherwise, it follows from [5, Theorem 2.4.4] that K is an (r, s)–cable of the core of V. With respect to the usual framing on K, the Seifert fiber of the cable space $V - \operatorname{int} N(K)$ has slope rs on $\partial N(K)$. Hence, $K(m/2)$ will be a solid torus V' if and only if $\Delta(m/2,\ rs/1) = 1$, ie $m = 2rs \pm 1$. The meridian of V' then has slope $m/2s^2 = \frac{2rs\pm1}{2s^2}$. □

Theorem 8.2 *Let K be a large algebraic knot with unknotting number 1. Then either*

(1) *any unknotting arc for K can be isotoped into either*

  (a) *one of the rational tangles $\mathcal{R}(p/q)$ in an elementary tangle of type I; or*

  (b) *the rational tangle $\mathcal{R}(p/q)$ in an elementary tangle of type II.*

In case (a), the crossing move transforms $\mathcal{R}(p/q)$ to $\mathcal{R}(k/1)$ for some integer k, and $p/q = \frac{2s^2}{2rs\pm1} + k$, where $s \geq 1$ and $(r, s) = 1$.

In case (b), the crossing move transforms $\mathcal{R}(p/q)$ to $\mathcal{R}(1/0)$, and $p/q = \frac{2rs\pm1}{2s^2}$, where $s \geq 1$ and $(r, s) = 1$.

(2) (a) *K is an EM–knot $K(\ell, m, n, p)$.*

  (b) *$O(K)$ has a unique connected incompressible 2–sided toric 2–suborbifold S, a Conway sphere, K has an unknotting arc a with $|a \cap S| = 1$ (the standard unknotting arc for $K(\ell, m, n, p)$), and K has no unknotting arc disjoint from S.*

(3) *K is the union of essential tangles $\mathcal{P} \cup \mathcal{P}_0$, where $\mathcal{P}_0$ is an EM–tangle $\mathcal{A}_\varepsilon(\ell, m)$ and $\partial\mathcal{P}_0$ is in $\mathbf{T}(K)$. Any unknotting arc for K can be isotoped into $\mathcal{P}_0$. The standard unknotting arc for $\mathcal{A}_\varepsilon(\ell, m)$ is an unknotting arc for K.*

Remark The remarks (A), (B) following Theorem 6.2 also apply here.

Proof Note that the characteristic orbifold decomposition of $O(K)$ is gotten by amalgamating subcollections of the constituent elementary tangles. Applying Theorem 6.2, we are left to check that Theorem 6.2(1) implies Theorem 8.2(1). Theorem 6.2(1) implies that the unknotting move replaces $\mathcal{R}(p/q)$ in some elementary tangle of type I or II with another tangle $\mathcal{T}$. $\mathcal{T}$ must be an integer tangle, $\mathcal{R}(k/1)$, if the elementary tangle is of type I, and $\mathcal{R}(1/0)$ if it is of type II. Lemma 8.1 gives the desired result (using $\mathcal{R}(1/(\frac{p}{q}-k))$ for $\mathcal{R}(p/q)$ in type I). □

9 Some algebraic tangle calculations

In this section we do some calculations concerning crossing moves on certain algebraic tangles. These will be used in Sections 10 and 11.

Lemma 9.1 *Suppose $q > 1$, and that $\mathcal{M}(p/q, \chi)$ is a rational tangle, where $\chi \in \mathbb{Q} \cup \{\infty\}$. Then*

(1) $\chi = k \in \mathbb{Z}$;

(2) $\mathcal{M}(p/q, k) = \mathcal{R}\left(\frac{kq+p}{q}\right)$

(3) *if $\mathcal{M}(p/q, k) = \mathcal{R}(1/x)$, $x \in \mathbb{Z}$, then there exists $\varepsilon = \pm 1$ such that $x = \varepsilon q$ and $kq + p = \varepsilon$.*

Proof (1) Since $\mathcal{M}(p/q, \chi)$ is a disk sum of $\mathcal{R}(p/q)$ and $\mathcal{R}(\chi)$, and $q > 1$, we must have $\chi = k \in \mathbb{Z}$.

(2) Incorporating the k horizontal twists into $\mathcal{R}(p/q)$, we see that $\mathcal{M}(p/q, \chi) = \mathcal{R}\left(\frac{kq+p}{q}\right)$.

(3) This follows immediately from (2). □

Lemma 9.2 *Suppose $q_1, q_2 > 1$, and that $\mathcal{M}(p_1/q_1,\ p_2/q_2)$ can be transformed to a rational tangle $\mathcal{R}$ by a crossing move. Then*

(1) *the crossing arc is isotopic to an (r, s)–cable, $s \geq 1$, of one of the two exceptional fibers of $\mathcal{M}(p_1/q_1,\ p_2/q_2)$;*

(2) *the crossing move transforms the corresponding rational tangle, $\mathcal{R}(\frac{p_1}{q_1})$, say, of $\mathcal{M}$, to an integral tangle $\mathcal{R}(k)$;*

(3) $\mathcal{R} = \mathcal{R}\left(\frac{kq_2+p_2}{q_2}\right)$;

(4) $p_1/q_1 = k + \frac{2s^2}{2rs\pm 1}$.

Proof (1) The argument is very similar to the proofs of Lemma 6.1 and Theorem 5.2. Let M be the double branched cover of B^3 along $\mathcal{M}(\frac{p_1}{q_1}, \frac{p_2}{q_2})$. M is a Seifert fiber space over the disk with two exceptional fibers. The crossing move corresponds to replacing a marked tangle $\mathcal{T}_0$ in $\mathcal{M}(\frac{p_1}{q_1}, \frac{p_2}{q_2})$ with a marked tangle $\mathcal{T}_0'$, resulting in a rational tangle. Let V_0 be the solid torus preimage of $\mathcal{T}_0$ in M and $X = M - \mathrm{int}(V_0)$ be its exterior. Let γ be the meridian of V_0. Then $M = X(\gamma)$ and $X(\mu)$ is a solid torus for some μ with $\Delta(\gamma, \mu) = 2$. By [23, Proposition 9], X must be either Seifert fibered or toroidal. In the first case, X is the exterior of an exceptional fiber in M and we are done. So X is toroidal and we let W be the component of X cut along $\mathbf{T}(X)$ (canonical torus decomposition) that contains ∂X. Then $\partial W - \partial X$ is compressible in $W(\gamma), W(\mu)$. Furthermore $W(\gamma), W(\mu)$ are irreducible. Then $W(\gamma), W(\mu)$ are solid tori. By [5, Theorem 2.4.4], W is a cable space. In particular, say that W is the exterior of the (p,q) curve in the solid torus $W(\mu)$, $q \geq 2$. If γ_0, μ_0 are the slopes of the meridian disks on $\partial W(\gamma)$, $\partial W(\mu)$ respectively, then $\Delta(\gamma_0, \mu) = q^2(\Delta\gamma, \mu) \geq 8$. Thus, if $X_0 = X - W$, we may argue as above to conclude that X_0 is the exterior of a (p,q)–cable on some knot k_0 in $X(\mu)$. But then a coordinate calculation, as in the proof of Theorem 5.2 (Case IIB), says that $\gamma_0 = \frac{n_1 p_1 q_1 \pm 1}{n_1}$, $n_1 \geq 2$ and $\gamma_0 = \frac{npq\pm 1}{nq^2}$ for $n \geq 2$. This contradiction finishes the proof of Lemma 9.2(1).

Lemma 9.2(2) and (3) now follow from Lemma 9.1, (1) and (2). Finally, (4) follows from Lemma 8.1 applied to $\mathcal{R}(1/(\frac{p_1}{q_1} - k))$. □

Corollary 9.3 *Suppose $q_1, q_2 > 1$ and that $\mathcal{M}(p_1/q_1,\ p_2/q_2)$ can be transformed to a vertical twist tangle $\mathcal{R}(1/x)$ by a crossing move. Then there exist $\varepsilon = \pm 1$ and $k \in \mathbb{Z}$ such that, after possibly interchanging p_1/q_1 and p_2/q_2,*

(1) $x = \varepsilon q_2$;

(2) $kq_2 + p_2 = \varepsilon$;

(3) $p_1/q_1 = k + \frac{2s^2}{2rs\pm 1}$, *for some* $s \geq 1$, $(r,s) = 1$.

Proof This follows from Lemmas 9.1 and 9.2. □

Lemma 9.4 $1^*\mathcal{M}(\alpha, \beta)$ *is the unknot if and only if* $\Delta(\alpha, -\beta) = 1$.

Proof $1^*\mathcal{M}(\alpha, \beta)$ is the unknot if and only if its double branched cover M is S^3. But M is the union of the two solid tori $\widetilde{\mathcal{R}}(\alpha)$ and $\widetilde{\mathcal{R}}(\beta)$, whose meridians have slopes α and $-\beta$ respectively on the torus $T = \partial\widetilde{\mathcal{R}}(\alpha)$ with respect to the basis of $H_1(T)$ corresponding to the lifts of the slopes $1/0$ and $0/1$ on $\partial\mathcal{R}(\alpha)$. Hence M is S^3 if and only if $\Delta(\alpha, -\beta) = 1$. □

Lemma 9.5 *Let K be the knot shown in Figure 9.1, where $q_1, q_2 > 1$ and $\mathcal{T}$ is some marked tangle. Then*

(1) *K is the unknot if and only if $\mathcal{T} = \mathcal{R}(x)$, where $x \in \mathbb{Z}$ satisfies*

$$xq_1q_2 + p_1q_2 + p_2q_1 = \pm 1\,;$$

(2) *if $|p_1/q_1|, |p_2/q_2| < 1$, and x is as in (1), then $|x| \le 1$.*

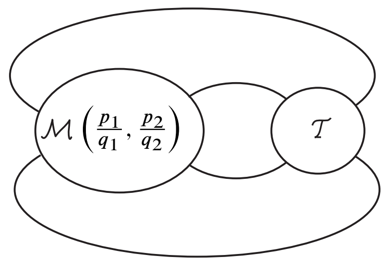


Figure 9.1

Proof Suppose K is the unknot. Passing to double branched covers, we see that $\widetilde{\mathcal{T}}$ must be a solid torus, implying that $\mathcal{T}$ is a rational tangle. Moreover, the meridian of $\widetilde{\mathcal{T}}$ must be distance 1 from the Seifert fiber of $\widetilde{\mathcal{M}}(p_1/q_1,\ p_2/q_2)$. Since the latter projects to the slope $1/0$ on $\partial\mathcal{M}(p_1/q_1,\ p_2/q_2)$, this implies that $\mathcal{T} = \mathcal{R}(x)$ for some $x \in \mathbb{Z}$. Incorporating this twist tangle $\mathcal{R}(x)$ with $\mathcal{R}(p_1/q_1)$, we see that the unknot is the union of the rational tangles $\mathcal{R}(x + \frac{p_1}{q_1}) = \mathcal{R}\left(\frac{xq_1+p_1}{q_1}\right)$ and $\mathcal{R}(p_2/q_2)$. Then $\Delta\left(\frac{xq_1+p_1}{q_1}, \frac{-p_2}{q_2}\right) = 1$ by Lemma 9.4, giving the equation in (1).

Conversely, if $\mathcal{T} = \mathcal{R}(x)$ where x satisfies the given equation then $\Delta(\frac{xq_1+p_1}{q_1}, -\frac{p_2}{q_2}) = 1$ and K is the unknot by Lemma 9.4.

To prove (2), suppose $|p_1/q_1|, |p_2/q_2| < 1$. Then from (1) we have

$$x + \frac{p_1}{q_1} + \frac{p_2}{q_2} = \pm\frac{1}{q_1q_2}\ ,$$

giving $|x| \le \frac{(q_1-1)}{q_1} + \frac{(q_2-1)}{q_2} + \frac{1}{q_1q_2} < 2$. □

Lemma 9.6 *Suppose $\alpha, \beta, \gamma \in \mathbb{Q} - \mathbb{Z}$, and $|\alpha|, |\beta| < 1$. Then $\mathcal{S}(\alpha, \beta; \gamma, \mathcal{T})$ is not the unknot, for any tangle $\mathcal{T}$.*

Proof Suppose $\mathcal{S}(\alpha,\beta;\gamma,\mathcal{T})$ is the unknot. Then passing to double branched coverings as usual we see that $\mathcal{T}$ must be a rational tangle $\mathcal{R}(\chi)$. Furthermore, by Lemma 9.5, $\mathcal{M}(\gamma,\chi)$ must be a vertical twist tangle $\mathcal{R}(-1/x)$, where $|x|\le 1$ since $|\alpha|,|\beta|<1$. But by Lemma 9.1(3), $|x|\ge 2$. $\square$

Corollary 9.7 *Let $K=\mathcal{S}(\alpha,\beta;\gamma,\delta)$, where $\alpha,\beta,\gamma,\delta\in\mathbb{Q}-\mathbb{Z}$, and $|\alpha|,|\beta|<1$. Then K cannot be unknotted by a crossing move in $\mathcal{M}(\gamma,\delta)$.*

Proof By Lemma 9.2(2), the crossing move has the effect of replacing one of the rational tangles $\mathcal{R}(\gamma)$ or $\mathcal{R}(\delta)$ in $\mathcal{M}(\gamma,\delta)$ with some other (rational) tangle. But this contradicts Lemmas 9.6 and 2.2(2). $\square$

Recall (Corollary 3.2) that the EM–knots are all of the form $\mathcal{S}(\alpha,\beta;\gamma,\delta)$ with $\alpha,\beta,\gamma,\delta \in\mathbb{Q}-\mathbb{Z}$ and $|\alpha|,|\beta|,|\gamma|,|\delta|<1$.

Theorem 9.8 *Let $K=\mathcal{S}(\alpha,\beta;\gamma,\delta)$ where $\alpha,\beta,\gamma,\delta\in\mathbb{Q}-\mathbb{Z}$ and $|\alpha|,|\beta|,|\gamma|,|\delta|<1$. Then K has unknotting number 1 if and only if K is an EM–knot $K(\ell,m,n,p)$.*

Proof K has a unique essential Conway sphere $S=\partial\mathcal{M}(\alpha,\beta)=\partial\mathcal{M}(\gamma,\delta)$. Therefore, by Corollary 6.3, if K has unknotting number 1 then either K is an EM–knot or K can be unknotted by a crossing move disjoint from S. But the latter is impossible by Corollary 9.7. $\square$

10 Examples

In this section we apply our results to certain families of knots defined in terms of the notation of Conway [4]; since that notation naturally encodes the characteristic toric orbifold decomposition of a knot it is eminently suited to our techniques. In particular, we consider all the knots up to 11 crossings in Conway's tables [4] with the property that their description in the tables makes it clear that they contain an essential Conway sphere. It turns our that these are all large algebraic knots. Specifically, they are the knots that are listed in [4] as either $.a.b$, $.a.b.c$, $.a.b.c.d$, $.a.(b,c)$, $.(a,b).c$, $(a,b)(c,d)$, $(a,b)1(c,d)$, or a,b,c,d. We determine exactly which of them have unknotting number 1. Note that the knots a,b,c,d are Montesinos knots of length 4, so they have unknotting number greater than 1 by [25].

Throughout this section, a,b,c and d will denote rational numbers.

We start with the knots $.a.b.c.d$. Recall [4, page 335] that in Conway's tables the form $.a.b.c.d$ is used only when a,b,c and d are >0. It turns out that all the EM–knots

$K(\ell, m, n, p)$ are of this form (up to mirror image), and that a knot $.a.b.c.d$ with $a, b, c, d > 0$ has unknotting number 1 if and only if it is an EM–knot. Recall also that in [4] $.a.b.c.1$ is abbreviated to $.a.b.c$, and $a.b.1$ to $.a.b$.

First we describe some symmetries of Conway's $.x.y.z.w$ notation. Recall that if x, y, z and w are arbitrary marked tangles, then $.x.y.z.w$ is the knot shown in Figure 10.1 (where the leftmost and rightmost horizontal arcs are understood to meet at the point at infinity in the projection S^2).

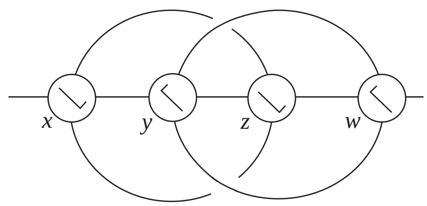


Figure 10.1

Let D_8 be the dihedral group of order 8, the group of symmetries of the square. By cyclically numbering the vertices of the square 1,2,3,4, we regard D_8 as a subgroup of S_4. Then D_8 acts on the set of expressions of the form $.x.y.z.w$ by permuting the substituent tangles; thus, by a slight abuse of notation, we write $\pi(.x_1.x_2.x_3.x_4) = .x_{\pi(1)}.x_{\pi(2)}.x_{\pi(3)}.x_{\pi(4)}$, for $\pi \in D_8$.

Recall also [4, pages 330–331] that $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the set of marked tangles as follows. If x is a marked tangle, then x_h, x_v and x_r are the marked tangles obtained by rotating x through $180°$ about the horizontal axis, the vertical axis, and the axis perpendicular to the plane of the paper, respectively. Let $\mathrm{Mut} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be the group $\{h, v, r, \mathrm{id}\}$. (Then mutation is the equivalence relation on knots generated by replacing a tangle x in (some diagram of) K by x_f for some $f \in \mathrm{Mut}$.) Again by a slight abuse of notation, we write $(.x.y.z.w)_f = .x_f.y_f.z_f.w_f$, for $f \in \mathrm{Mut}$.

Let μ: $D_8 \to \mathrm{Mut}$ be the epimorphism defined by $\mu((1\,2\,3\,4)) = h$, and $\mu((14)(23)) = v$. Finally, let ε: $\mathrm{Mut} \to \mathbb{Z}_2 = \{\pm 1\}$ be the homomorphism defined by $\varepsilon(h) = \varepsilon(v) = -1$, and recall that – denotes mirror-image.

Theorem 10.1 *If $\pi \in D_8$ then:*

$$\pi(.x.y.z.w) = \varepsilon(\mu(\pi))(.x.y.z.w)_{\mu(\pi)}$$

Corollary 10.2 *Up to mutation and mirror-image, $.x.y.z.w$ is invariant under the action of D_8.*

Since $t_f = t$ for rational tangles t, for all $f \in \mathrm{Mut}$, we have

Corollary 10.3 *If $\pi \in D_8$ then:*

$$\pi(.a.b.c.d) = \varepsilon(\mu(\pi)).a.b.c.d$$

Proof of Theorem 10.1 Consider $.x.y.z.w$, as shown in Figure 10.1. By sliding the tangle x around the point at infinity we get Figure 10.2. Changing all crossings, we see that $.x.y.z.w = -.y_h.z_h.w_h.x_h$. This shows that the theorem holds for $\pi = (1\,2\,3\,4)$.

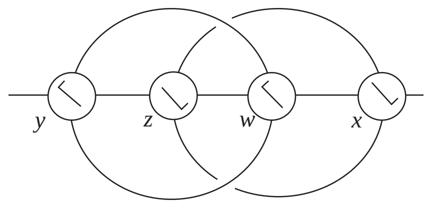


Figure 10.2

Rotating Figure 10.1 through $180°$ about the central axis perpendicular to the plane of the paper, we get Figure 10.3. Changing all crossings shows that $.x.y.z.w = -.w_v.z_v.y_v.x_v$, in other words, the theorem holds for $\pi = (14)(23)$.

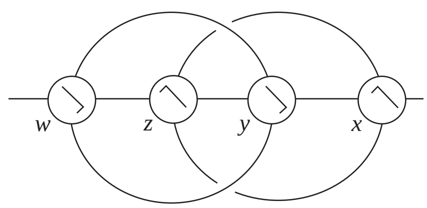


Figure 10.3

Since $(1\,2\,3\,4)$ and $(14)(23)$ generate D_8, the result follows. □

We consider a further symmetry. Recall [4, page 331] that $t0$ denotes the reflection of the tangle t in a plane perpendicular to the paper through the NW/SE–diagonal. Note that $(t0)_r$ is then the reflection of t in a plane through the NE/SW–diagonal.

Theorem 10.4 $.x.y.z.w = -.x0.(y0)_r.z0.(w0)_r$

In particular, for rational tangles we have

Corollary 10.5 $.a0.b0.c0.d0 = -.a.b.c.d$

Proof $t00 = t$ and $a_r = a$ when a is rational. □

Corollary 10.6

(1) $.a.b.c0 = -(.a0.b0.c)$

(2) $.a.b = -(.a0.b0)$

(3) $.a0.b = -(.a.b0)$

Proof These follow immediately from Corollary 10.5, along with the facts that $10 = 1$ and $t00 = t$. □

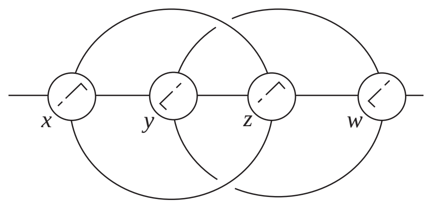


Figure 10.4

Proof of Theorem 10.4 Rotating Figure 10.1 through $180°$ about the horizontal axis gives Figure 10.4. Changing all crossings, we now get $.x0.(y0)_r.z0.\ (w0)_r$. □

The following lemma describes $.a.b.c.d$ in terms of the square tangle $\mathcal{S}$.

Lemma 10.7 $.a.b.c.d = \mathcal{S}(\frac{-1}{c+1}, \frac{a}{a+1}; \frac{1}{b+1}, \frac{-d}{d+1})$

Proof This follows from the second deformation shown in [4, Figure 10], possibly together with Lemma 2.2. □

Recall that if K is an EM–knot $K(\ell, m, n, p)$, then, by taking the mirror image of K if necessary, we may assume that $\ell > 1$.

Lemma 10.8 *Assume $\ell > 1$. Then $K(\ell, m, n, p) = .a.b.c$, where a, b, c are > 0 and are given by:*

$$p = 0 : \quad a = \frac{m}{(\ell-1)m-1}, \quad b = \frac{2mn-m+n}{2mn-m-n+1}, \quad c = \ell - 1$$

$$n = 0 : \quad a = \frac{2mp-m-p}{(\ell-1)(2mp-m-p)-2p+1}, \quad b = \frac{m}{m-1}, \quad c = \ell - 1$$

Proof This follows from Lemmas 3.1 and 10.7. □

Theorem 10.9 *Let $K = .a.b.c.d$, with $a, b, c, d > 0$. Then K has unknotting number 1 if and only if it is an EM–knot $K(\ell, m, n, p)$.*

Proof By Lemma 10.7, $.a.b.c.d = \mathcal{S}(\alpha, \beta; \gamma, \delta)$ where $\alpha, \beta, \gamma, \delta \in \mathbb{Q} - \mathbb{Z}$ and $|\alpha|, |\beta|, |\gamma|, |\delta| < 1$. The result is now an immediate consequence of Theorem 9.8. □

We now determine the EM–knots up to 11 crossings.

Theorem 10.10 *The EM–knots $K(\ell, m, n, p)$ with at most 11 crossings are listed in Table EM 11, up to mirror image.*

Rolfsen	Conway	$K(\ell, m, n, p)$
8_{17}	.2.2	$K(3, 2, 0, 1)$
9_{33}	.21.2	$-K(2, 3, 0, 1)$
10_{82}	.4.2	$K(2, 2, 0, 2)$
10_{84}	.22.2	$K(2, 2, 0, -1)$
10_{88}	.21.21	$K(2, 3, 0, 0)$
10_{91}	.3.2.20	$-K(4, 1, 1, 0)$
10_{95}	.210.2.2	$K(3, 2, 0, 0)$
	.311.2	$-K(2, 2, -1, 0)$
	.23.2	$K(2, 2, 0, 3)$
	.212.2	$K(2, 2, 0, -2)$
	.2111.2	$-K(2, 2, 2, 0)$
	.31.21	$-K(2, -3, 1, 0)$
	.22.2.20	$-K(3, -2, 1, 0)$
	.210.21.2	$K(3, 2, 1, 0)$

$K(\ell, m, n, p)$ with at most 11 crossings
Table EM 11

Remark The third column of Table EM 11 represents the knot in $K(\ell, m, n, p)$ form. These representations are not unique.

Proof of Theorem 10.10 By possibly taking mirror images, Proposition 1.4 of [6] allows us to assume that $\ell > 1$. By Lemma 10.8, $K(\ell, m, n, p) = .a.b.c$ with a, b, c positive rational numbers. Inserting alternating diagrams of the corresponding rational tangles $\mathcal{R}(a)$, $\mathcal{R}(b)$, $\mathcal{R}(c)$ into Conway's 6^{**} polyhedron gives an alternating diagram of $.a.b.c$. This alternating diagram will be a minimal crossing diagram from which we can compute the crossing number of $.a.b.c$. To get alternating diagrams of the rational tangles we can use the positive continued fraction expansions of a, b, and c. That is, if r is a positive rational number and $r = [a_1, a_2, \dots, a_n]$ (see Section 2) with $a_i > 0$ if $i \neq n$ and $a_n \geq 0$, then there is an alternating diagram of the rational tangle $\mathcal{R}(r)$ that has exactly $\sum_{i=1}^{n} a_i$ crossings. If $r \neq 1$, we will take $a_1 > 1$.

In Tables E1–E3 below, we list nonnegative continued fraction expansions of a, b, c.

m, p	a	b	c
$m > 0,\ p > 0$	$[p-1, 2, m-2, 1, \ell-2, 0]$	$[m-1, 1]$	$\ell - 1$
$m > 0,\ p < 0$	$[\lvert p\rvert, 1, 1, m-2, 1, \ell-2, 0]$	$[m-1, 1]$	$\ell - 1$
$m < 0,\ p > 0$	$[p-1, 1, 1, \lvert m\rvert, \ell-1, 0]$	$[\lvert m\rvert, 1, 0]$	$\ell - 1$
$m < 0,\ p < 0$	$[\lvert p\rvert, 2, \lvert m\rvert, \ell-1, 0]$	$[\lvert m\rvert, 1, 0]$	$\ell - 1$

$n = 0 \quad [\ell > 1, \quad m \neq 0, 1, \quad (\ell, m, p) \neq (2, 2, 1)]$

Table E1

m, n	a	b	c
$m > 0,\ n > 0$	$[m-1, 1, \ell-2, 0]$	$[n-1, 1, 1, m-1, 1]$	$\ell - 1$
$m > 0,\ n < 0$	$[m-1, 1, \ell-2, 0]$	$[\lvert n\rvert, 2, m-1, 1]$	$\ell - 1$
$m < 0,\ n > 0$	$[\lvert m\rvert, \ell-1, 0]$	$[n-1, 2, \lvert m\rvert-1, 1, 0]$	$\ell - 1$
$m < 0,\ n < 0$	$[\lvert m\rvert, \ell-1, 0]$	$[\lvert n\rvert, 1, 1, \lvert m\rvert-1, 1, 0]$	$\ell - 1$

$p = 0 \quad [\ell > 1, \quad m \neq 0, \quad (\ell, m) \neq (2, 1);\ (m, n) \neq (1, 0), (-1, 1)]$

Table E2

m	a	b	c
$m > 0$	$[m-1, 1, \ell-2, 0]$	$[m-1, 1]$	$\ell - 1$
$m < 0$	$[\lvert m\rvert, \ell-1, 0]$	$[\lvert m\rvert, 1, 0]$	$\ell - 1$

$n = 0 = p \qquad (\ell > 1;\ m \neq 0, 1)$

Table E3

If 0 appears in any but the last entry of one of these expansions (for certain ℓ, m, n, p), we may use one of the following rules to eliminate it:

(1) $[0, a, b, c, \cdots] \longrightarrow [b, c, \cdots]$

(2) $[\cdots a, b, 0, c, d, \cdots] \longrightarrow [\cdots a, b+c, d, \cdots]$

Note that the sum of the entries is changed by (1) only and that amounts to deleting the second entry from the sum. In this way we enumerate those $K(\ell, m, n, p)$ with crossing number at most 11.

As an example, consider the case when $n=0$, $m>0$, $p>0$. Assuming $p>1$, Table E1 gives the crossing number of $K(\ell,m,0,p)$ as $2m+2\ell+p$ (the crossings from $\mathcal{R}(a)$, $\mathcal{R}(b)$, $\mathcal{R}(c)$ plus 3 from the Conway polyhedron 6^{**}). Thus if $K(\ell,m,n,p)$ has at most 11 crossings, $(\ell,m,p)\in\{(2,2,2),(2,2,3)\}$ (noting that $m\neq 1$ when $n=0$). If $p=1$ and $m>2$, then the crossing number of $K(\ell,m,0,1)$ is $2m+2\ell-1$. Thus $(\ell,m)\in\{(2,3),(2,4),(3,3)\}$. Finally, if $p=1$ and $m=2$, then $\ell>2$ and the crossing number is $2\ell+2$. That is, $\ell\in\{3,4\}$.

Once having enumerated the $K(\ell,m,n,p)$ with at most 11 crossings and written them in the form $.a.b.c$, we can now locate them in the tables. To do this we use the symmetries given by Corollaries 10.3 and 10.6 (note that if $t=[a_1,\dots,a_n]$ is rational then $t0=[a_1,a_2,\dots,a_n,0]$).

This completes the proof of Theorem 10.10. □

Of the knots listed in Conway's tables in the form $.a.b.c.d$ (or $.a.b$, or $.a.b.c$) that are not EM–knots, there are: 1 with 8 crossings, 3 with 9 crossings, 13 with 10 crossings, and 45 with 11 crossings. By Theorem 10.9 these all have unknotting number greater than 1.

Next we consider the knots of the form $(a,b)(c,d)$ in Conway's notation. For these we first have the following result.

Theorem 10.11 *Let $K=(a,b)(c,d)$, where $|a|,|b|,|c|$ and $|d|$ are >1 and either $ab>0$ or $cd>0$. Then K does not have unknotting number 1.*

Proof It is easy to see (possibly using Lemma 2.2) that $(a,b)(c,d)=\mathcal{S}(-\frac{1}{a},-\frac{1}{b};\frac{1}{c},\frac{1}{d})$. It follows from Theorem 9.8 that K has unknotting number 1 if and only if K is an EM–knot. Now by Corollary 3.2, the EM–knots are all of the form $\mathcal{S}(\alpha,\beta;\gamma,\delta)$ with $|\alpha|,|\beta|,|\gamma|,|\delta|<1$ and $\alpha\beta<0$, $\gamma\delta<0$. Moreover, it is easy to verify, using Lemma 2.2, that if $\mathcal{S}(\alpha,\beta;\gamma,\delta)=\pm\mathcal{S}(\alpha',\beta';\gamma',\delta')$, where $|\alpha|,|\alpha'|$, etc. are all <1, and $\alpha\beta<0$ and $\gamma\delta<0$, then $\alpha'\beta'<0$ and $\gamma'\delta'<0$. It follows that a knot K of the form described in the theorem is never an EM–knot. □

In Conway's tables, there are 48 knots listed in the form $(a,b)(c,d)$: 3 10–crossing alternating knots, 7 10–crossing non-alternating knots, 10 11–crossing alternating knots, and 28 11–crossing non-alternating knots.

Theorem 10.12 *Up to 11 crossings, of the 48 knots listed in Conway's tables as $(a,b)(c,d)$, the only ones with unknotting number 1 are the four non-alternating 11–crossing knots $(3,2+)(21,2-)$, $(21,2+)(21,2-)$, $(3,2+)$–$(21,2)$, and $(21,2+)$–$(21,2)$.*

Proof All the knots in question satisfy the hypotheses of Theorem 10.11 except those with $(a,b) = (3, 2+)$ or $(21, 2+)$. Of these, the four listed in the theorem are easily seen to have unknotting number 1. The others have $(c,d) = (3,2)$, $(21,2)$, $(3, 2-)$ or $-(3,2)$. From now on, let K be a knot of the form $(a,b)(c,d)$ where (a,b) (resp. (c,d)) is one of the two (resp. four) possibilities listed. Recall (see proof of Theorem 10.11) that $K = \mathcal{S}(-\frac{1}{a}, -\frac{1}{b}; \frac{1}{c}, \frac{1}{d})$.

First note that K is not an EM–knot. For $\{\frac{1}{a}, \frac{1}{b}\} = \{\frac{1}{3}, \frac{3}{2}\}$ or $\{\frac{2}{3}, \frac{3}{2}\}$, and it follows easily from Lemma 2.2 that K is not of the form $\mathcal{S}(\alpha, \beta; \gamma, \delta)$ with $|\alpha|, |\beta|, |\gamma|, |\delta| < 1$, and hence not an EM–knot by Corollary 3.2.

By Theorem 6.2, if K has unknotting number 1 then it can be unknotted by a crossing move in $\mathcal{M}(-\frac{1}{a}, -\frac{1}{b})$ or $\mathcal{M}(\frac{1}{c}, \frac{1}{d})$. Since $|c|, |d| > 1$, the former is impossible, by Corollary 9.7. So the unknotting move is contained in $\mathcal{M}(\frac{1}{c}, \frac{1}{d})$, transforming $\mathcal{M}(\frac{1}{c}, \frac{1}{d})$ to a tangle $\mathcal{T}$ that unknots $\mathcal{M}(-\frac{1}{a}, -\frac{1}{b}) = \mathcal{M}(-\frac{1}{3}, -\frac{3}{2})$ or $\mathcal{M}(-\frac{2}{3}, -\frac{3}{2})$. By Lemma 9.5, we see that in both cases $\mathcal{T} = \mathcal{R}(-1/2)$.

Consider the case $(c,d) = (3,2)$. Then $\mathcal{M}(\gamma, \delta) = \mathcal{M}(\frac{1}{3}, \frac{1}{2})$. By Corollary 9.3, this can be transformed to $\mathcal{R}(-1/2)$ by a crossing move if and only if there is an integer k such that $k \cdot 2 + 1 = -1$, and $1/3 = k + \frac{2s^2}{2rs \pm 1}$ for some $s \geq 1$, $(r,s) = 1$. The first equation gives $k = -1$, and the second now gives $\pm 2s^2 = 1 - 3k = 4$, which is impossible.

The other three cases $(c,d) = (21,2)$, $(3, 2-)$ and $-(3,2)$ are similar. We omit the details. □

We now consider the knots of the form $.a.(b,c)$ or $.(b,c).a$. First we have the following, which is an immediate consequence of Theorem 10.1.

Lemma 10.13 *$.a.(b,c)$ and $-.(b,c).a$ are mutants.*

Lemma 10.13 and Theorem 7.1 imply that $.a.(b,c)$ has unknotting number 1 if and only if $.(b,c).a$ does, so we restrict attention to knots of the first type.

Theorem 10.14 *Suppose that $a > 0$ and $|b|, |c| > 1$. Then $.a.(b,c)$ has unknotting number 1 if and only if $\Delta(\frac{a}{a+1}, \frac{1}{2}) = \Delta(-\frac{1}{b}, \frac{c+1}{c}) = 1$.*

Proof The knot $.x.y$ has the form shown in Figure 10.5. Therefore $K = .a.(b,c)$ is of the form $\mathcal{M}(\alpha, \beta) \cup \mathcal{M}(\gamma, \delta) \cup \mathcal{M}(-1/2, *)$, where $\alpha = a/(a+1)$, $\beta = -1/2$, $\gamma = -1/b$, and $\delta = -(c+1)/c$; see Figure 10.6. Note that the conditions on a, b and

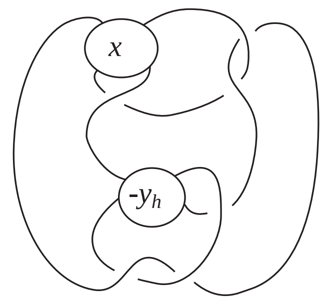


Figure 10.5

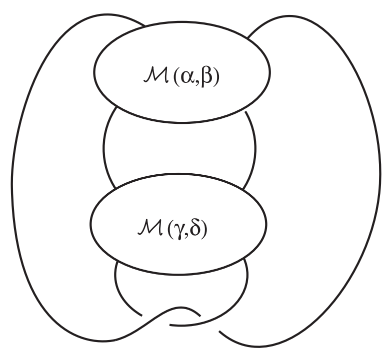


Figure 10.6

c guarantee that $\alpha, \beta, \gamma, \delta \in \mathbb{Q} - \mathbb{Z}$. This is a decomposition of K into elementary tangles.

Since K does not have a unique essential Conway sphere, K is not an EM–knot. Hence, if $u(K) = 1$, then either conclusion (1) or (3) of Theorem 8.2 holds.

First, suppose (1)(a) of Theorem 8.2 holds for $\mathcal{M}(\alpha, \beta)$. By Lemma 9.2(3), $\mathcal{M}(\alpha, \beta)$ is transformed to a rational tangle $\mathcal{R}(p'/q)$ with $q \geq 2$. If this unknots K, then $\mathcal{R}(p'/q) \cup \mathcal{M}(-1/2, *)$ must be an integral tangle; see Figure 10.7. Hence $p' = \pm 1$, and $\mathcal{R}(p'/q) \cup \mathcal{M}(-1/2, *) = \mathcal{R}(\frac{2}{1 \pm 2q})$. Therefore $q = 0$ or ± 1, a contradiction. Exactly the same argument shows that (1)(a) of Theorem 8.2 does not hold for $\mathcal{M}(\gamma, \delta)$.

Next suppose that we have conclusion (3) of Theorem 8.2 and that (1) of that theorem does not hold. By the remark after Theorem 8.2, for $\mathcal{M} = \mathcal{M}(\alpha, \beta)$ or $\mathcal{M}(\gamma, \delta)$, and $\mathcal{M}' = \mathcal{M}(\gamma, \delta)$ or $\mathcal{M}(\alpha, \beta)$, respectively, we have $\mathcal{M} \cup \mathcal{M}(-1/2, *) = \mathcal{A}$, an EM–tangle, and the rational tangle $\mathcal{R}$ resulting from the standard unknotting move in $\mathcal{A}$ must unknot $\mathcal{M}'$. Hence $\mathcal{R}$ must be an integral tangle, $\mathcal{R}(r/1)$. Here $r = \pm\Delta(\alpha, \beta)$ where α is the tangle slope corresponding to $\mathcal{R}$ and β is the tangle slope of the Conway

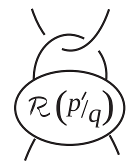


Figure 10.7

disk of $\mathcal{A}$. From Figure 4.3, we determine that $r = \pm 4m$ when $\mathcal{A} = \mathcal{A}_1(\ell, m)$ (the rational tangle is $\mathcal{R}(\frac{1-2m}{4m})$) and that $r = \pm(\ell(1-2m)+2)$ (the rational tangle is $\mathcal{R}(\frac{2m-1}{\ell(1-2m)+2})$) when $\mathcal{A} = \mathcal{A}_1(\ell, m)$.

Since $|\alpha|, |\beta| < 1$, it follows from Lemma 9.5(2) that $\mathcal{M} = \mathcal{M}(\alpha, \beta)$, $\mathcal{M}' = \mathcal{M}(\gamma, \delta)$. Let $-\frac{1}{b} = \frac{r_1}{t_1}$, $-\frac{1}{c} = \frac{r_2}{t_2}$ where $r_i, t_i \in \mathbb{Z}$ and $t_i > 1$. Applying Lemma 9.5(1) with $x = r$ we get that $r - \frac{1}{b} - \frac{1}{c} - 1 = \frac{\pm 1}{t_1 t_2}$. As $|b|$, $|c|$, t_1, $t_2 > 1$, $|r| < 4$. From the preceding paragraph, this implies that when $\mathcal{A} = \mathcal{A}_1(\ell, m)$, $|4m| < 4$. But $m \neq 0$, a contradiction. When $\mathcal{A} = \mathcal{A}_2(\ell, m)$, $|\ell(1-2m)+2| < 4$. Thus $|\ell|\,|1-2m| < 6$. Since $|\ell| \geq 2$ and $m \notin \{0, 1\}$, this is again a contradiction. Thus conclusion (3) of Theorem 8.2 cannot hold.

We conclude that (1)(b) of Theorem 8.2 must hold. The crossing move transforms $\mathcal{M}(-\frac{1}{2}, *)$ to $\mathcal{M}(\frac{1}{0}, *)$, which transforms K to the connected sum of $1^*\mathcal{M}(\alpha, \beta)$ and $1^*\mathcal{M}(\gamma, \delta)$. The result now follows from Lemma 9.4. □

In Conway's tables, there are 22 knots listed in the form $.a.(b, c)$ or $.(b, c).a$: 8 11–crossing alternating knots and 14 11–crossing non-alternating knots. (They come in mutant pairs $.a.(b, c)$ and $.(b, c).a$.) They all have $a = 2$ or $20 = 1/2$, the alternating knots have $(b, c) = (3, 2)$ or $(21, 2)$, while the non-alternating knots have $(b, c) = (3, 2-)$, $(21, 2-)$, $-(3, 2)$ or $-(21, 2)$.

Theorem 10.15 *Of the knots listed in Conway's tables of the form $.a.(b, c)$ or $.(b, c).a$, those with unknotting number 1 are precisely the 6 11–crossing non-alternating knots with $(b, c) = -(3, 2)$ or $-(21, 2)$.*

Proof This follows easily from Theorem 10.14. □

Finally, we consider the three knots in Conway's tables of the form $(a, b)1(c, d)$. First we have the following lemma.

Lemma 10.16 *If $a, b, c, d > 1$ then $(a, b)1(c, d)$ is not an EM–knot.*

Proof The knot $(a,b)1(c,d)$ has the form shown in Figure 10.8. Consider the arc that joins the SE–corners of the two Montesinos tangles. By swinging this arc over the right-hand tangle, one sees that $(a,b)1(c,d) = \mathcal{S}(1/a, 1/b+1; 1/c, 1/d+1)$. By Lemma 2.2, it is easy to see that this is never of the form $\mathcal{S}(\alpha,\beta;\gamma,\delta)$ with $|\alpha|, |\beta|, |\gamma|, |\delta| < 1$. But the EM–knots are all of this form, by Lemma 3.1. □

Theorem 10.17 *The 11–crossing alternating knots* $(3,2)1(3,2)$, $(3,2)1(21,2)$ *and* $(21,2)1(21,2)$ *do not have unknotting number 1.*

Proof Let $K = (a,b)1(c,d)$. Note that for the knots under consideration, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \in \mathbb{Q} - \mathbb{Z}$. Clearly K does not contain an EM–tangle. Also, it is not an EM–knot by Lemma 10.16.

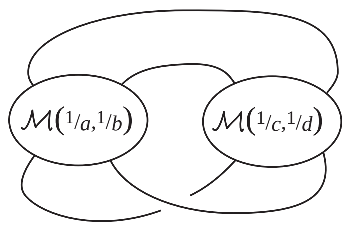


Figure 10.8

Hence, by Theorem 8.2, if $u(K) = 1$ then the unknotting move takes place in a rational substituent of $\mathcal{M}(\frac{1}{a}, \frac{1}{b})$ or $\mathcal{M}(\frac{1}{c}, \frac{1}{d})$, transforming it to a rational tangle $\mathcal{R}(r)$. By symmetry we may assume that the unknotting move takes place in $\mathcal{M}(\frac{1}{a}, \frac{1}{b})$. By Lemma 9.5(1), the tangle $\mathcal{R}(r)$ must be an integral tangle $\mathcal{R}(m)$. It follows that $r = [m, -1, 0] = \frac{m}{1-m}$. By Lemma 9.5(2), since $|\frac{1}{c}|, |\frac{1}{d}| < 1$, we must have $|m| \le 1$. On the other hand, by Lemma 9.2(3), there exists an integer k such that $\frac{kq+p}{q} = r = \frac{m}{1-m}$, where $p/q = 1/a$ or $1/b$. Since $q \ge 2$ and $(p,q) = 1$, $m = 0$ or 1 is impossible. Hence $m = -1$ and $q = 2$. Therefore $p/q = 1/2 = 1/a$ (since $(a,b) = (3,2)$ or $(21,2)$), and $2k+1 = m = -1$, giving $k = -1$. Also, by Lemma 9.2(4), the other rational substituent of $\mathcal{M}(\frac{1}{a}, \frac{1}{b})$, namely $1/b$, must satisfy $\frac{1}{b} = -1 + \frac{2s^2}{2rs \pm 1}$, $s \ge 1$, $(r,s) = 1$. But in both cases, $b = 3$ and $b = 21 = 3/2$, this is easily seen to be impossible. □

Remark For knots of up to 10 crossings, until the work described here and the work of Ozsváth and Szabó [27], there were 41 knots for which it was not known whether or not they had unknotting number 1. We ruled out 14 of them (see above): the two

9–crossing knots $9_{29} = .2.20.2$ and $9_{32} = .21.20$, and the twelve 10–crossing knots $10_{79} = (3,2)(3,2)$, $10_{81} = (21,2)(21,2)$, $10_{83} = .31.2$, $10_{86} = .31.20$, $10_{87} = .22.20$, $10_{90} = .3.2.2$, $10_{93} = .3.20.2$, $10_{94} = .30.2.2$, $10_{96} = .2.21.2$, $10_{148} = (3,2)(3,2-)$, $10_{151} = (21,2)(21,2-)$, and $10_{153} = (3,2)\!-\!(21,2)$. Meanwhile, Ozsváth and Szabó [27], using their remarkable Heegaard Floer homology theory, ruled out all 41 knots except 10_{153}. So the knots with 10 or fewer crossings and unknotting number 1 are completely determined.

11 Deciding if a large algebraic knot has unknotting number 1

It is unknown if the unknotting number of a knot is a computable invariant. Even the following special case is open:

Question 11.1 *Is there an algorithm to decide whether or not a given knot has unknotting number 1?*

Note that by Haken [12] there is an algorithm to decide if a knot has unknotting number 0.

Theorem 8.2 allows us to answer Question 11.1 affirmatively for large algebraic knots.

Theorem 11.2 *There is an algorithm to decide whether or not a given large algebraic knot K, described as a union of elementary marked tangles (Figure 8.1) and 4–braids in $S^2 \times [0,1]$, has unknotting number 1, and, if so, to identify an unknotting crossing move.*

Proof Note that none of the rational tangles in a constituent elementary tangle of K is integral or $\mathcal{R}(1/0)$ (that is, the distance between the slope of such a rational tangle and the orbifold S^1–fiber of the elementary tangle is at least two).

Definition Let $\mathcal{T}$ be a marked tangle in the 3–ball. The number $p/q \in \mathbb{Q} \cup \{\infty\}$ is called an *unknotting slope* for $\mathcal{T}$ if $\mathcal{T} \cup \mathcal{R}(-p/q)$ (ie $1^*(\mathcal{T} + \mathcal{R}(-p/q))$) is the unknot.

Remark As long as $\mathcal{T}$ is not rational, an unknotting slope for $\mathcal{T}$, if there is one, is unique. This follows, for example, from the fact that knots are determined by their complements and the $\mathbb{Z}_2$–Smith Conjecture (applied to the double branched covers).

Lemma 11.3 *There is an algorithm to decide whether or not a given algebraic tangle (ie a union of elementary tangles and 4–braids) has an unknotting slope, and, if so, to find it.*

Proof Again we assume the tangle is given as a union of elementary marked tangles and 4–braids in $S^2 \times [0,1]$. Begin with the innermost, constituent, elementary tangles, necessarily of type I, and compute their unknotting slopes (if they exist) via Lemma 9.5(1). Then work outward along elementary tangles of type II determining unknotting slopes at each step. This is equivalent to solving equations of the following form: given $\frac{r_1}{s_1}, \frac{r_2}{s_2}$ find a rational $\frac{x}{y}$ such that $\frac{x}{y} + \frac{r_1}{s_1} = \frac{r_2}{s_2}$. In working outward, if one comes to a constituent tangle of type III, the corresponding unknotting slope (if it exists) must be the slope of the orbifold S^1–fiber.

Working outward in this way, either one finds at some point that there is no unknotting slope, in which case there is none for the algebraic tangle, $\mathcal{T}$, or one determines the unknotting slope for $\mathcal{T}$. □

By Theorem 8.2, K has unknotting number 1 if and only if one of the following options holds: (A) K is an EM–knot; (B) K contains an EM–tangle for which the standard crossing change unknots K; or (C) K unknots by replacing a rational tangle in a constituent elementary tangle of type I or II with another rational tangle as described in Theorem 8.2(1).

We show that these options can be checked algorithmically.

(A) To see if K is an EM–knot, first check that there are two elementary tangles of type I whose union is K. If so, compute the orders of the exceptional orbifold S^1–fibers of each elementary tangle and list the finite number of EM–knots having exceptional fibers of the same order. Check if K is equivalent to one of these (eg, check that the orbifold S^1–fibers of the two elementary tangles intersect twice at the unique Conway sphere [ie, that the distance between the slopes of these fibers is 1 on the Conway sphere]. If so then K can be written in the form $\mathcal{S}(\alpha,\beta;\gamma,\delta)$ and Lemma 2.2 may be applied).

(B) To check the unknotting of K by a standard crossing change in an EM–tangle, we list all pairs $\{X_1, X_2\}$ of a type I and type II elementary tangle of K that share a common Conway sphere. The union $X_1 \cup X_2$ is a candidate for an EM–tangle. Let $-\frac{p}{q}$ be the unknotting slope of the complementary tangle $(S^3, K) - (X_1 \cup X_2)$. List the finitely many EM–tangles that have exceptional orbifold S^1–fibers of the same order as $X_1 \cup X_2$. Then check if there is a homeomorphism from the candidate $X_1 \cup X_2$ to one of these EM–tangles taking the slope $\frac{p}{q}$ to the slope of the rational tangle that results from the standard crossing change. (For example, check that the orbifold S^1–fibers of X_1 and X_2 intersect twice along the common Conway sphere. If so, $X_1 \cup X_2$ may be rewritten in the form $\mathcal{S}(*, \frac{p_1}{q_1}; \frac{p_2}{q_2}, \frac{p_3}{q_3})$. Then check that

$$\frac{p_1}{q_1} + n_1 = \frac{v_1}{w_1}, \quad \frac{p_2}{q_2} + n_2 = \frac{v_2}{w_2}, \quad \frac{p_3}{q_3} - n_2 = \frac{v_3}{w_3}$$

for some $n_1, n_2 \in \mathbb{Z}$, where $\mathcal{A}_\varepsilon(\ell, m) = \mathcal{S}(*, \frac{v_1}{w_1}; \frac{v_2}{w_2}, \frac{v_3}{w_3})$. If so, there is a unique homeomorphism up to isotopy taking $X_1 \cup X_2$ to $\mathcal{A}_\varepsilon(\ell, m)$. One then checks that $\frac{p}{q}$ is identified with the slope of the rational tangle gotten by the standard crossing move on $\mathcal{A}_\varepsilon(\ell, m)$.)

(C) To check the condition of Theorem 8.2(1), we check each constituent elementary tangle of type I or II as follows. For any elementary tangle of type II, we replace the rational tangle $\mathcal{R}(p/q)$ with $\mathcal{R}(1/0)$. If this yields the unknot (which can be checked algorithmically), then we check that $\frac{p}{q} = \frac{2rs \pm 1}{2s^2}$ for some $(r, s) = 1$. If so one identifies the unknotting crossing move of K as described, for example, in Lemma 8.1 (or Lemma 12.3).

Consider a constituent elementary tangle, X, of type I, and let $-\frac{v}{w}$ be the unknotting slope for the complementary tangle $(S^3 - K) - X$ (if there is none, K cannot be unknotted in this way). Decide if replacing some rational tangle, $\mathcal{R}(p/q)$, of X by an integer tangle, $\mathcal{R}(k)$, changes X to the rational tangle $\mathcal{R}(v/w)$. If so, determine if $\frac{p}{q} = \frac{2s^2}{2rs \pm 1} + k$ for some $(r, s) = 1$. If so, then a variation of Lemma 8.1 will determine an unknotting crossing move for K (see also Lemma 12.4). □

12 Unknotting in a minimal diagram

In [18], Kohn made the following conjecture, which he showed was true for 2–bridge knots and links.

Conjecture (Kohn) *Let K be a knot or link with $u(K) = 1$. There is a crossing in a minimal diagram of K which, when changed, unknots K.*

Note that the analog for knots with $u(K) > 1$ is false [1], [26].

We shall show that the conjecture is true for alternating large algebraic knots.

From [32], we take the following

Definition A tangle diagram D in a disk Δ is *prime* iff

(i) the underlying projection of D is a connected subset of the disk Δ

(ii) if C is a circle in Δ meeting D transversely in two points, then these points belong to the same edge of D (ie, diagrammatic connected sums are not allowed).

Proposition 12.1 (Flyping conjecture for alternating tangles in a 3–ball) *If D, D' are prime, alternating diagrams of the same marked tangle in a 3–ball, then D and D' differ by a sequence of flypes.*

Proof As outlined in [32, page 333] or [34, page 998], this follows from the rigid vertex version of the Flyping Conjecture proved in [22]. □

Definition A tangle (B, T) in a 3–ball B is *prime* iff

(i) (B, T) is not rational;

(ii) if S is a 2–sphere in $B - T$, then the 3–ball bounded by S does not meet T;

(iii) if a 2–sphere S in B meets T transversely in two points, then the 3–ball in B bounded by S meets T in an unknotted arc.

Corollary 12.2 *If D, D' are alternating diagrams of the same marked tangle in a 3–ball which is either prime or rational but not $\mathcal{R}(1/0)$, $\mathcal{R}(0/1)$, then any (marked) tangle obtained by changing a crossing in D can be gotten by changing a crossing in D'.*

Proof of Corollary 12.2 The hypotheses guarantee that if D is not a prime diagram, then it has a nugatory crossing. (A circle violating (ii) of primeness of the diagram must encircle an alternating diagram of the unknot. This must have a nugatory crossing by the minimality of crossing number for reduced, alternating diagrams of links.) Thus by reducing all nugatory crossings in D, we leave a prime, alternating diagram. Similarly for D'. Thus D, D' are related by a sequence of flypes and nugatory crossing reductions or creations. One checks that changing a crossing commutes with these operations. □

Recall (Lemma 8.1) that $\mathcal{R}(p/q)$ can be transformed to $\mathcal{R}(1/0)$ by a crossing move if and only if there are coprime integers r, s such that $p/q = \frac{2rs\pm1}{2s^2}$.

Lemma 12.3 *If $p/q = \frac{2rs\pm1}{2s^2}$, where $(r, s) = 1$ and $s \neq 0$, then $\mathcal{R}(p/q)$ can be transformed to $\mathcal{R}(1/0)$ by a crossing change in any alternating diagram for $\mathcal{R}(p/q)$.*

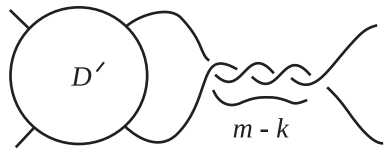


Figure 12.1

Proof By [18] the condition on p/q is equivalent to the condition that $\pm p/q =$

$$[c_1, \dots, c_{\ell-1}, c_\ell, 1, 1, c_\ell - 1, c_{\ell-1}, \dots, c_1, c_0] \text{ or}$$
$$[c_1, \dots, c_{\ell-1}, c_\ell - 1, 1, 1, c_\ell, c_{\ell-1}, \dots, c_1, c_0]$$

where $c_i \geq 1$, $0 \leq i \leq \ell - 1$, $c_\ell \geq 2$. The corresponding diagram of $\mathcal{R}(p/q)$ is alternating, and the crossing change is visible in that diagram. By Corollary 12.2, the transformation $\mathcal{R}(p/q) \mapsto \mathcal{R}(1/0)$ can be effected by a crossing change in any alternating diagram. □

Lemma 12.4 *If $p/q \in \mathbb{Q} - \mathbb{Z}$, and $\mathcal{R}(p/q)$ can be transformed to $\mathcal{R}(k)$ by a crossing move, then it can be transformed to $\mathcal{R}(k)$ by a crossing change in any alternating diagram of $\mathcal{R}(p/q)$, unless $p/q = \pm[\ell, 2, m]$, where $\ell > 0$, $m \geq 0$, and $k = \pm(\ell + m + 2)$.*

Proof By Corollary 12.2, we only need exhibit a crossing change in some alternating diagram of $\mathcal{R}(p/q)$. If $k = 0$ then the result holds by rotating and applying Lemms 8.1 and 12.3. So assume $k \neq 0$. We may also suppose, without loss of generality, that $p/q > 0$. Write $p/q = p'/q + m$, $0 < p'/q < 1$, $m \geq 0$. Then $\mathcal{R}(p/q - k) = \mathcal{R}(p'/q + (m-k))$ can be transformed to $\mathcal{R}(0)$ by a crossing move.

Let D' be a positive alternating diagram of $\mathcal{R}(p'/q)$ (see Figure 12.1). There is such a diagram since $\frac{p'}{q} > 0$.

Case (1) $\boldsymbol{m - k \geq 0}$

In this case $\mathcal{R}(p/q - k)$ has the alternating diagram shown in Figure 12.1

By the case $k = 0$ above, $\mathcal{R}(p/q - k)$ can be transformed to $\mathcal{R}(0)$ by changing a crossing c in this diagram. Clearly c must be a crossing of the diagram D' (eg, by computing the associated rational number after the crossing change). Let D be the alternating diagram of $\mathcal{R}(p/q) = \mathcal{R}(p'/q + m)$ obtained by putting m horizontal $\frac{1}{2}$–twists on the right of D'. Then changing the crossing c in D transforms $\mathcal{R}(p/q)$ to $\mathcal{R}(k)$.

Case (2) $\boldsymbol{m-k<0}$

Since $0<p'/q<1$, the diagram D' ends up with $r\geq 1$ vertical $\frac{1}{2}$–twists; see Figure 12.2. Hence $\mathcal{R}(p/q-k)=\mathcal{R}(p'/q+(m-k))$ has the diagram D_0 shown in Figure 12.3.

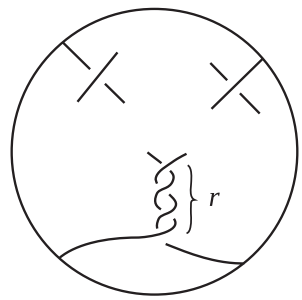


Figure 12.2

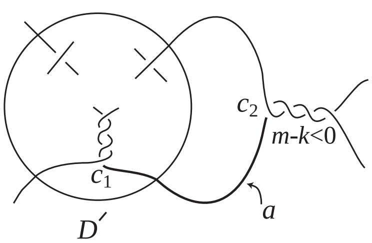


Figure 12.3

Let a be the arc indicated by the bold line in Figure 12.3. Swinging a underneath D' gives the diagram D_1 shown in Figure 12.4. Note that this is a (negative) alternating diagram. Therefore, by the case $k=0$ above, $\mathcal{R}(p/q-k)$ can be transformed to $\mathcal{R}(0)$ by changing some crossing c in D_1. Clearly c is either a crossing of D' (other than c_1) or the new crossing c_0.

In the first case, $\mathcal{R}(p'/q+m)=\mathcal{R}(p/q)$ is transformed to $\mathcal{R}(k)$ by changing the same crossing c in the alternating diagram D of $\mathcal{R}(p'/q+m)$ defined in Case (1).

In the second case, changing the crossing c_0 in D_1 and swinging the arc a over D' clearly gives the same tangle as changing the crossings c_1 and c_2 in the diagram D_0. By hypothesis, this is $\mathcal{R}(0)$. Hence changing only c_1 in D_0 gives $\mathcal{R}(-2)$. Therefore $r\leq 2$ (else the crossing change would not yield an integral tangle).

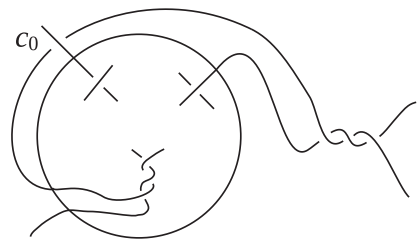


Figure 12.4

Claim *If $r = 1$, then $\frac{p}{q} = [\ell-1, 1, 1, m]$ where $\ell \geq 2$, $k = m+2-\ell$.*
If $r = 2$, then $\frac{p}{q} = [\ell, 2, m]$ where $\ell > 0$, $k = m+2+\ell$.

Proof of Claim Assume $r = 1$. The positive continued fraction expansion for the rational tangle of Figure 12.3 with crossing c_1 changed gives

$$-2 = (m-k) + \frac{1}{-1+\frac{s}{\ell}} = (m-k) + \frac{\ell}{s-\ell}$$

where $0 < \frac{s}{\ell} < 1$, $(s, \ell) = 1$. Integrality gives $s = \ell - 1$ and

$$\frac{p}{q} = m + \frac{1}{1+\frac{s}{\ell}} = m + \frac{1}{1+\frac{1}{1+\frac{1}{\ell-1}}}$$

where $\ell - 1 > 0$. Thus $\frac{p}{q} = [\ell-1, 1, 1, m]$. Furthermore

$$-2 = (m-k) + \frac{1}{-1+\frac{1}{1+\frac{1}{\ell-1}}} = m-k-\ell$$

as required.

Similarly, if $r = 2$ we have

$$-2 = (m-k) + \frac{1}{0+\frac{s}{\ell}}$$

This implies that $s = 1$, $\ell > 0$. Then $-2 = (m-k)+\ell$. Finally $\frac{p}{q} = [\ell, 2, m]$. □

By the Claim, if $r = 1$ then we are in Case (1). When $r = 2$, we obtain the list of tangles stated in the Lemma. □

Theorem 12.5 *Let K be an alternating large algebraic knot with unknotting number 1. Then K can be unknotted by a crossing change in any alternating diagram of K.*

Proof By [22], any two reduced alternating diagrams of K are related by flype moves. It follows easily that if K can be unknotted by a crossing change in some alternating diagram then it can be unknotted by a crossing change in any alternating diagram.

In what follows we use the notion of the "visibility" of a Conway sphere or disk in an alternating diagram as discussed in [33]. In particular, [21] shows that in an alternating diagram a Conway sphere is either visible, or *hidden* in a very specific way (see Figures 3(i), (ii) of [33]). In the latter case, there is a standard move on the diagram to make the sphere visible (see Figure 3(iii) of [33]). For a reduced alternating diagram of an elementary tangle, [33, page 326] shows that the arguments of [21] can also be used to say that the Conway disk must be visible.

First suppose that we are in Case (1) of Theorem 8.2.

Let D be a reduced alternating diagram of K. Suppose we are in subcase (a), so that the unknotting crossing move takes place in a rational subtangle of an elementary tangle $\mathcal{T}$ of type I. Let S be the boundary of $\mathcal{T}$, and suppose that S is visible in D. Then after flyping if necessary (see [33, page 326] for the visibility of the Conway disk), we may assume that D contains a subdiagram of the form shown in Figure 8.1(I). By Theorem 8.2, the crossing move transforms $\mathcal{R}(p/q)$ to $\mathcal{R}(k)$. Since $p/q \notin \mathbb{Z}$, it follows from Lemma 12.4 that this can be achieved by a crossing change in the diagram D_1, unless (without loss of generality) $p/q = [\ell, 2, m]$, $\ell > 0$, $m \geq 0$, and $k = \ell + m + 2$. Since $k > 0$ and D is alternating, we see that replacing D_1 with the standard diagram of $\mathcal{R}(k)$ gives an alternating diagram D'. Also, since D is reduced and S is essential, it is easy to see that D' is reduced. Hence D' is a diagram of a non-trivial knot, a contradiction.

Next suppose that we are in subcase (b) of Theorem 8.2, Case (1), and that the boundary components of the corresponding elementary tangle of type II are both visible in D (Figure 8.1(II)). The crossing move transforms $\mathcal{R}(p/q)$ to $\mathcal{R}(1/0)$, and, by Proposition 12.1 and (the proof of) Lemma 8.1, this can be achieved by a crossing change in the diagram D_3 ($q \neq 0$).

It remains to consider (a) and (b) when the relevant Conway spheres are hidden in D. So suppose we are in subcase (a), and the boundary S of the corresponding elementary tangle $\mathcal{T}$ of type I is hidden in D. Making S visible as described in [33], we get a diagram in which the tangle $\mathcal{T}$ appears as in Figure 12.5. Note that the diagrams D_1, D_2 of $\mathcal{R}(r/s)$, $\mathcal{R}(p/q)$ in Figure 12.5 will be alternating. Suppose, without loss of generality, that the unknotting crossing move takes place in the right-hand rational tangle $\mathcal{R}(p/q)$, transforming it to $\mathcal{R}(k)$ for some integer k.

We first argue that $k = 0$ or 1. Assume not. Let $\mathcal{R}(c/d)$ be the subtangle of $\mathcal{R}(r/s)$ encapsulated in the circle of D_1 of Figure 12.5. We then have the equation $\frac{d}{c} = 1 + \frac{s}{r}$.

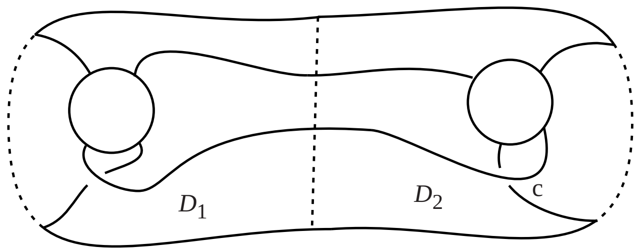


Figure 12.5: $\mathcal{T} = \mathcal{R}(r/s) + \mathcal{R}(p/q)$

Note that since the diagram of $\mathcal{R}(r/s)$ is alternating, $c/d < 0$. Now the crossing move we are considering turns $\mathcal{T}$ into $\mathcal{R}(\frac{r}{s}+k)$. This has to be of the form $\mathcal{R}(1/x)$, $x \in \mathbb{Z}$, or $\mathcal{R}(0/1)$. (Write $K = \mathcal{T} \cup \mathcal{T}'$, as in Figure 3(iii) of [33], where $\mathcal{T}'$ is also of the form of Figure 12.5. Since the corresponding subdiagrams, D_i', of $\mathcal{T}'$ are alternating, if either D_i' is a diagram of a rational tangle $\mathcal{R}(p/q)$, then $q > 0$. Arguing on the level of double branched covers, as in Lemma 9.5(1), we see that $\Delta(\frac{r}{s}+k, \frac{0}{1}) \leq 1$. Hence $\frac{r}{s}+k = \frac{1}{x}$ or $\frac{0}{1}$.) Since D_1 in Figure 12.5 for $\mathcal{T}$ is alternating, $|\frac{r}{s}| < 1$. Thus $\frac{r}{s}+k \neq \frac{0}{1}$. We assume $\frac{r}{s}+k = \frac{1}{x}$, $x \in \mathbb{Z}$. Then $\frac{r}{s} = \frac{1-kx}{x}$, hence $\frac{s}{r} = \frac{x}{1-kx}$. Therefore $0 > \frac{d}{c} = 1 + \frac{x}{1-kx} = (1-(k-1)x)/(1-kx)$, implying that $k = 0, 1$.

By Lemma 12.4, since $\frac{p}{q} \notin \mathbb{Z}$ and $k = 0, 1$, we see that $\mathcal{R}(p/q)$ can be transformed to $\mathcal{R}(k)$ by a crossing change in the minimal diagram D_2. Now D_2 can be obtained by adding a vertical right-handed twist, given by the crossing marked c, to the diagram of the tangle $\mathcal{R}(a/b)$ which was visible in the original alternating diagram of K. If the crossing changed in D_2 is not c, then this is a crossing change in the original diagram. So suppose the crossing changed is c. Since this gives $\mathcal{R}(k)$, we see that $\mathcal{R}(p/q)$ ($\mathcal{R}(a/b)$, resp.) is gotten by adding two (one, resp.) right-handed vertical twists to $\mathcal{R}(k)$. Since $\frac{p}{q} \neq \frac{0}{1}$, we see that $k \neq 0$. Thus $k = 1$ and $\frac{a}{b} = \frac{1}{2}$, $\frac{p}{q} = \frac{1}{3}$. But then one sees that the crossing change at c can be accomplished by a crossing change in the diagram of $\mathcal{R}(a/b)$ by Corollary 12.2. Thus K can be unknotted by a crossing change in D.

To finish the proof of Theorem 12.5 in Case I, we consider the case when the crossing move is in an elementary tangle of type II where one of the boundary components, S_1, is hidden in D. Making S_1 visible gives a diagram containing a subdiagram as shown in Figure 12.6.

Lemma 12.6 *Let F_1 be the disk pictured in Figure 12.6. Then F_1 is a Conway disk for S_1. Furthermore, any Conway disk for S_1 is parallel to F_1.*

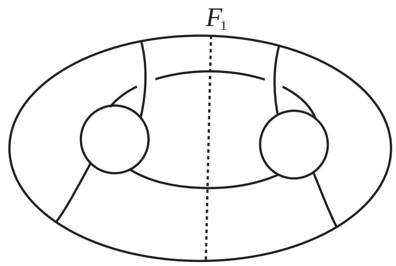


Figure 12.6

Proof By Corollary 3.3 of [33], F_1 is a Conway disk for the prime tangle bounded by S_1 in Figure 12.6. We assume for contradiction that there is a Conway disk, F_2, of S_1 that is not parallel to F_1. Then we may take F_2 to be disjoint from F_1. In particular, the slope of F_2 on S_1 is $\frac{1}{0}$ (in the diagram coordinates). Let $\mathcal{T}$ be the tangle containing F_2 after cutting Figure 12.6 along F_1. Without loss of generality assume this is the right-hand side of F_1. Then $\mathcal{T}$ is an alternating tangle for which F_2 is an essential Conway disk with slope $\frac{1}{0}$ (F_2 is not parallel to F_1). After possibly flyping, we can write $\mathcal{T}$ as the union of a positive braid in $S^2 \times I$, with at least one vertical twist, and a reduced alternating tangle $\mathcal{T}'$. See Figure 12.7.

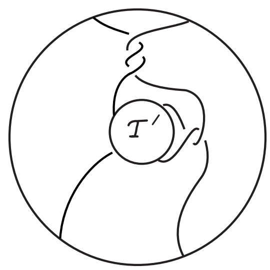


Figure 12.7

After an isotopy we may assume that F_2 intersects the boundary of $\mathcal{T}'$ in a single circle, thereby writing F_2 as the union of an annulus in $S^2 \times I$ and an essential Conway disk, F_2', in $\mathcal{T}'$. By [33, page 326], F_2' can be taken to be visible, or hidden in a very special way. If visible, then its slope on $\partial\mathcal{T}'$ (with coordinates from the diagram) is either $\frac{0}{1}$ or $\frac{1}{0}$. Since the braiding in $S^2 \times I$ is positive with at least one vertical twist, this means the slope of F_2 on $\mathcal{T}$ cannot be $\frac{1}{0}$, a contradiction (note that the flyping did not change the slope of F_2).

So we assume F_2' is hidden in $\mathcal{T}'$. But then [33] shows that $\mathcal{T}'$ has a subdiagram as in Figure 12.8 and shows that the slope of F_2' on $\partial\mathcal{T}'$ is either $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$ ($-\frac{1}{1}$ does not

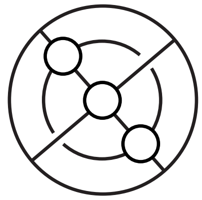


Figure 12.8

occur since the braiding is positive). Again, the fact that the braiding in $S^2 \times I$ is positive with at least one vertical twist guarantees that the slope of F_2 on $\partial\mathcal{T}$ is not $\frac{1}{0}$ as we have assumed. □

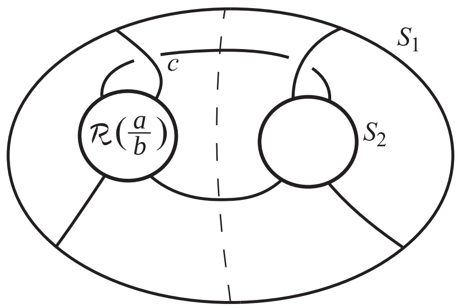


Figure 12.9

Lemma 12.6 allows us to say that, after isotoping to make S_1 visible, the elementary tangle of type II becomes visible as in Figure 12.9. In particular, we may assume the unknotting arc lies to the left of F_1 in this figure. By Theorem 8.2 and Lemma 12.3, this can be achieved by a crossing change in the (alternating) diagram on the left of Figure 12.9. If the crossing changed is not c, then this crossing change corresponds to a crossing change in the original diagram D. If the crossing changed is c, then $a/b = -1$. As argued above, changing the crossing c is equivalent to changing the single crossing in $\mathcal{R}(-1)$, which corresponds to a crossing in D.

This finishes the proof of Theorem 12.5 in Case (1). Cases (2) and (3) of the theorem are proved in Theorems 12.7 and 12.8. □

Theorem 12.7 *Let K be an EM–knot. Then K can be unknotted by a crossing change in any alternating diagram of K.*

Proof Again by the Flyping Conjecture [22], we need only show that K can be unknotted in some alternating diagram. Let $K = K(\ell, m, n, p)$. By [6, Proposition 1.4], we may assume, by taking the mirror-image of K if necessary, that $\ell > 1$. Lemmas 10.7 and 10.8 imply that $K = .a.b.c = \mathcal{S}(\frac{-1}{c+1}, \frac{a}{a+1}; \frac{1}{b+1}, \frac{-1}{2})$, where $a, b, c > 0$. Thus K has a diagram of the form shown in Figure 12.10. The unknotting arc a_0 as well as the

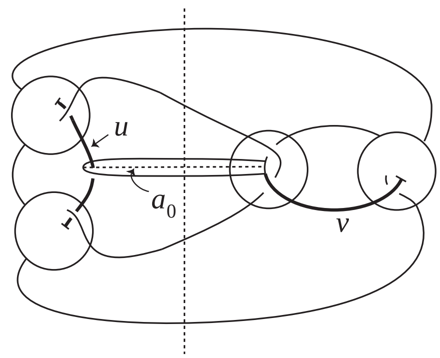


Figure 12.10

corresponding crossing move that unknots K are shown. Let u and v be the arcs of the diagram indicated by the bold lines. Swinging u "under" and v "over" gives the diagram shown in Figure 12.11. This diagram is alternating since $a, b, c > 0$. Also,

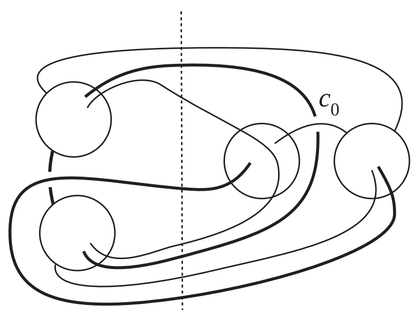


Figure 12.11

changing the crossing c_0 shown in Figure 12.11 has the same effect as performing the crossing move shown in Figure 12.10. □

Theorem 12.8 *Let K be an alternating algebraic knot containing an EM–tangle whose boundary is an essential Conway sphere. If $u(K) = 1$, then K can be unknotted by a crossing change in any alternating diagram of K.*

Proof Again, by [22] we need only verify this for some alternating diagram. Let K be the union of essential tangles $\mathcal{P} \cup \mathcal{P}_0$ where $\mathcal{P}_0$ is an EM–tangle $\mathcal{A}_\varepsilon(\ell, m)$. Applying

Theorem 6.2, we are either in Case (1) or (3). In Case (1), the proof of Theorem 12.5 guarantees the existence of an unknotting crossing change in an alternating diagram. Thus we assume we are in Case (3).

Lemma 12.9 *In an alternating diagram of K, $\partial\mathcal{P}_0$ must be visible.*

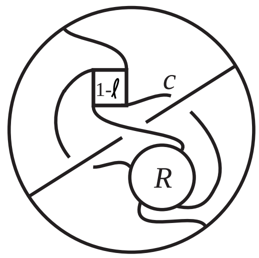
Proof Assume not. Then there is a diagram of $\mathcal{P}_0$ as in Figure 12.9. By [33], the disk F_1 in that figure is a Conway disk for $\mathcal{P}_0$. Thus it is the unique Conway disk for $\mathcal{P}_0$. Each side of F_1 is an alternating tangle. But this contradicts the fact that capping $\mathcal{A}_\varepsilon(\ell,m)$ along slope $\frac{1}{0}$ (ie, taking the denominator closure) gives either the unknot or Hopf link (by inspection of Figure 4.3). □

Thus $\partial\mathcal{P}_0$ is visible and let D_0 be the corresponding subdiagram. This allows us to regard $\mathcal{P}_0$ as a marked tangle. By Corollary 12.2 we need to find some alternating diagram of this marked tangle that exhibits a crossing change which unknots K.

Theorem 6.2(3) gives an (unmarked) tangle homeomorphism $h\colon \mathcal{P}_0 \to \mathcal{A}_\varepsilon(\ell,m) = \mathcal{T}_\varepsilon(\ell,m)(1/2)$ which identifies the standard crossing move on $\mathcal{A}_\varepsilon(\ell,m)$ as an unknotting, crossing move for K. After possibly rotating, reflecting, or applying a mutation involution to $\mathcal{A}_\varepsilon(\ell,m)$, there is an alternating braided tangle $\mathcal{C}$ in $S^2\times I$ such that extending $\mathcal{A}_\varepsilon(\ell,m)$ by adjoining $\mathcal{C}$ gives a marked tangle $\mathcal{A}_\varepsilon(\ell,m)\cup\mathcal{C}$ which is equivalent to $\mathcal{P}_0$ as a marked tangle — via an extension of h.

Definition Let E be the diagram in a disk Δ of a tangle in a 3–ball. A crossing c of E is said to be *inessential* iff there is a properly embedded arc in Δ that intersects E only in c, dividing the four arcs of E at c into pairs. A diagram is *reduced* iff it contains no inessential crossings.

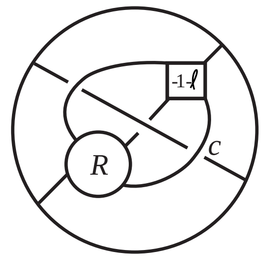
Figure 12.12 displays a reduced, prime, alternating diagram, E, for $\mathcal{A}_\varepsilon(\ell,m)$ which exhibits a crossing change sending $\mathcal{A}_\ell(\ell,m)$ to the rational tangle $\mathcal{T}_\varepsilon(\ell,m)(\frac{1}{0})$. Letting C denote an alternating braided diagram of the braided tangle $\mathcal{C}$, we have that $E\cup C$ is a prime diagram for the marked tangle $\mathcal{P}_0$, which also has the diagram D_0. If E and C have the same sign (defined in Section 2) as alternating tangles, then $E\cup C$ is the desired alternating representative of $\mathcal{P}_0$, exhibiting the appropriate crossing change. If C and D_0 have opposite sign, then adjoining to D_0 the diagram $\overline{C}$ of the reverse braiding of $\mathcal{C}$ gives an alternating diagram $D_0\cup\overline{C}$ for the marked tangle $\mathcal{A}_\varepsilon(\ell,m)$, which also has the diagram E. Since E is reduced, so is $D_0\cup\overline{C}$. (For, we may assume D_0 has no nugatory crossings, hence neither does $D_0\cup\overline{C}$. Then E and $D_0\cup\overline{C}$ have the same crossing number. But then a reducing arc for $D_0\cup\overline{C}$ suggests a capping of $D_0\cup\overline{C}$ and of E giving two alternating diagrams of the same link with the same number of crossings — one of which contains a nugatory crossing, the other does not). That is, C is empty and E, D_0 represent the same marked tangle. Thus E is the sought after diagram for $\mathcal{P}_0$.



$\ell > 0$

$R = \mathcal{R}\left(\frac{(1-\ell)m+1}{m}\right)$

crossing change at $c \to \mathcal{R}\left(\frac{1-2m}{1+2m}\right)$

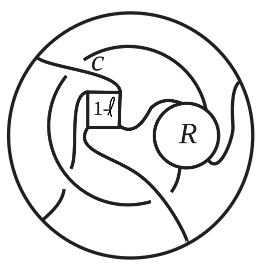


$\ell < 0$

$R = \mathcal{R}\left(\frac{(-1-\ell)m+1}{m}\right)$

crossing change at $c \to \mathcal{R}\left(\frac{1-2m}{1+2m}\right)$

$\mathcal{A}_1(\ell, m)$

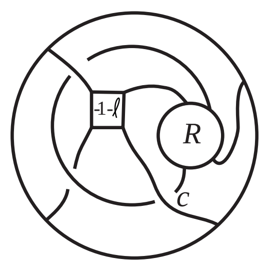


$\ell > 0$

$R = \mathcal{R}\left(\frac{m-1}{m}\right)$

crossing change at

$c \to \mathcal{R}\left(\frac{(\ell-1)(1-2m)+2}{1-2m}\right)$



$\ell < 0$

$R = \mathcal{R}\left(\frac{m-1}{m}\right)$

crossing change at

$c \to \mathcal{R}\left(\frac{2m-1}{(-1-\ell)(2m-1)+2}\right)$

$\mathcal{A}_2(\ell, m)$

Figure 12.12: Reduced, alternating diagrams of $\mathcal{A}_\varepsilon(\ell, m)$

Thus we assume E and C have opposite sign, and C and D_0 have the same sign. Let F be the diagram outside of D_0 in the alternating diagram of K. Figure 12.12 shows that after the crossing change in E the sign of the rational number corresponding to $\mathcal{T}_\varepsilon(\ell, m)(\frac{1}{0})$ is opposite to the sign of E. That is, the result of the crossing change has an alternating diagram R such that $R \cup C$ is an alternating diagram with the same sign as D_0. Then the diagram $R \cup C \cup F$ is an alternating diagram of the unknot. Hence it must have a nugatory crossing. Since we may assume neither F, C or R contain nugatory crossings, this means that either F or R is a split diagram (ie, there is a properly embedded arc that separates the arcs of the tangle). Since K cannot be a connected sum (by [28]), and since F is not rational, R must represent $\mathcal{R}(0/1)$ or $\mathcal{R}(1/0)$. But Figure 12.12 shows this is not true. □

Remarks about Figure 12.12

(1) The markings have been changed between Figure 4.3 and Figure 12.12. In particular, to get from the markings in Figure 12.12 to those in Figure 4.3 add the following braiding outside of Figure 12.12:

$$\begin{aligned}
\mathcal{A}_1(\ell,m),\ \ell>0 &: -1 \text{ vertical twist below}\\
\mathcal{A}_1(\ell,m),\ \ell<0 &: -1 \text{ vertical twist above}\\
\mathcal{A}_2(\ell,m),\ \ell>0 &: +1 \text{ vertical twist below, then}\\
& -1 \text{ horizontal twist to left}\\
\mathcal{A}_2(\ell,m),\ \ell<0 &: +1 \text{ vertical twist below}
\end{aligned}$$

(2) The diagrams for $\mathcal{A}_1(\ell,m)$ were obtained by the moves in Figures 12.10 and 12.11 as well as twistings that change the markings.

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