Dehn surgery, homology and hyperbolic volume

IAN AGOL
MARC CULLER
PETER B SHALEN

If a closed, orientable hyperbolic 3–manifold $M$ has volume at most 1.22 then $H_1(M;\mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M;\mathbb{Z}_2)$ and $H_1(M;\mathbb{Z}_7)$ have dimension at most 3. The proof combines several deep results about hyperbolic 3–manifolds. The strategy is to compare the volume of a tube about a shortest closed geodesic $C \subset M$ with the volumes of tubes about short closed geodesics in a sequence of hyperbolic manifolds obtained from $M$ by Dehn surgeries on $C$.

57M50; 57M27

1 Introduction

We shall prove:

**Theorem 1.1** Suppose that $M$ is a closed, orientable hyperbolic 3–manifold with volume at most 1.22. Then $H_1(M;\mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M;\mathbb{Z}_2)$ and $H_1(M;\mathbb{Z}_7)$ have dimension at most 3. Furthermore, if $M$ has volume at most 1.182, then $H_1(M;\mathbb{Z}_7)$ has dimension at most 2.

The bound of 2 for the dimension of $H_1(M;\mathbb{Z}_p)$ is sharp when $p$ is 3 or 5. Indeed, the manifolds $\mathfrak{m}003(-3,1)$, and $\mathfrak{m}007(3,1)$ from the list given in [10] have respective volumes 0.94... and 1.01..., and their integer homology groups are respectively isomorphic to $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

Apart from these two examples, the only example known to us of a closed, orientable hyperbolic 3–manifold with volume at most 1.22 is the manifold $\mathfrak{m}003(-2,3)$ from the list given in [10]. These three examples suggest that the bounds for the dimension of $H_1(M;\mathbb{Z}_p)$ given by Theorem 1.1 may not be sharp for $p \neq 3, 5$.

The proof of Theorem 1.1 depends on several deep results, including a strong form of the “log 3 Theorem” of Anderson, Canary, Culler and Shalen [4; 8]; the Embedded Tube Theorem of Gabai, Meyerhoff and N Thurston [9]; the Marden Tameness Conjecture,
recently proved by Agol [1] and by Calegari and Gabai [7]; and an even more recent result due to Agol, Dunfield, Storm and W Thurston [3]. The strategy of our proof is to compare the volume of a tube about a shortest closed geodesic $C$ in $M$ with the volumes of tubes about short closed geodesics in a sequence of hyperbolic manifolds obtained from $M$ by Dehn surgeries on $C$.

After establishing some basic conventions in Section 2, we carry out the strategy described above in Sections 3–6, for the case of manifolds which are “non-exceptional” in the sense that they contain shortest geodesics with tube radius greater than $(\log 3)/2$. In Section 5, for the case of non-exceptional manifolds with volume at most $1/22$, we establish a bound of $3$ for the dimension of $H_1(M;\mathbb{Z}_p)$ for any prime $p$. In Section 6, again for the case of non-exceptional manifolds with volume at most $1/22$, we establish a bound of $2$ for the dimension of $H_1(M;\mathbb{Z}_p)$ for any odd prime $p$. In Section 7 we use results from [9] to handle the case of exceptional manifolds, and complete the proof of Theorem 1.1.

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2 Definitions and conventions

2.1 If $g$ is a loxodromic isometry of hyperbolic 3-space $\mathbb{H}^3$ we shall let $A_g$ denote the hyperbolic geodesic which is the axis of $g$. The cylinder about $A_g$ of radius $r$ is the open set $Z_r(g) = \{ x \in \mathbb{H}^3 \mid \text{dist}(x, A_g) < r \}$.

2.2 Suppose that $M$ is a complete, orientable hyperbolic 3–manifold. Let us identify $M$ with $\mathbb{H}^3/\Gamma$, where $\Gamma \cong \pi_1(M)$ is a discrete, torsion-free subgroup of Isom$_+ \mathbb{H}^3$. If $C$ is a simple closed geodesic in $M$ then there is a loxodromic isometry $g \in \Gamma$ with $A_g\langle g \rangle = C$. For any $r > 0$ the image $Z_r(g)/\langle g \rangle$ of $Z_r(g)$ under the covering projection is a neighborhood of $C$ in $M$. For sufficiently small $r > 0$ we have

$$\{ h \in \Gamma \mid h(Z_r(g)) \cap Z_r(g) \neq \emptyset \} = \langle g \rangle.$$

Let $R$ denote the supremum of the set of $r$ for which this condition holds. We define $\text{tube}(C) = Z_R(g)/\langle g \rangle$ to be the maximal tube about $C$. We shall refer to $R$ as the tube radius of $C$, and denote it by $\text{tuberad}(C)$.

2.3 If $C$ is a simple closed geodesic in a closed hyperbolic 3–manifold $M$, it follows from [13], [2] that $M - C$ is homeomorphic to a hyperbolic manifold $N$ of finite volume having one cusp. The manifold $N$, which by Mostow rigidity is unique up to isometry, will be denoted $\text{drill}_C(M)$. 

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2.4 If $C$ is a shortest closed geodesic in a closed hyperbolic 3–manifold $M$, i.e., one such that length$(C) \leq$ length$(C')$ for every other closed geodesic $C'$, then in particular $C$ is simple, and the notions of 2.2 and 2.3 apply to $C$.

2.5 Suppose that $\mathbb{H}^3/\Gamma$ is a non-compact orientable complete hyperbolic manifold of finite volume. Let $\Pi \cong \mathbb{Z} \times \mathbb{Z}$ be a maximal parabolic subgroup of $\Gamma$ (so that $\Pi$ corresponds to a peripheral subgroup under the isomorphism of $\Gamma$ with $\pi_1(N)$). Let $\xi$ denote the fixed point of $\Pi$ on the sphere at infinity and let $B$ be an open horoball centered at $\xi$ such that $\{ g \in \Gamma \mid gB \cap B \neq \emptyset \} = \Pi$. Then $\mathcal{H} = B/\Pi$, which we identify with the image of $B$ in $N$, is called a cusp neighborhood in $N$.

If $\mathcal{H}$ is a cusp neighborhood in $N = \mathbb{H}^3/\Gamma$ then the inverse image of $\mathcal{H}$ under the covering projection $\mathbb{H}^3 \rightarrow N$ is a union of disjoint open horoballs. The cusp neighborhood $\mathcal{H}$ is maximal if and only there exist two of these disjoint horoballs whose closures have non-empty intersection.

2.6 If $N$ is a complete, orientable hyperbolic manifold of finite volume, $\hat{N}$ will denote a compact core of $N$. Thus $\hat{N}$ is a compact 3–manifold whose boundary components are all tori, and the number of these tori is equal to the number of cusps of $N$.

3 Drilling and packing

Lemma 3.1 Suppose that $M$ is a closed, orientable hyperbolic 3–manifold, and that $C$ is a shortest geodesic in $M$. Set $N = \text{drill}_C(M)$. If $\text{tuberad}(C) \geq (\log 3)/2$ then $\text{vol} \ N < 3.0177 \text{vol} \ M$.

Proof The proof is based on a result due to Agol, Dunfield, Storm and W Thurston [3]. We let $L$ denote the length of the geodesic $C$ in the closed hyperbolic 3–manifold $M$, and we set $R = \text{tuberad}(C)$ and $T = \text{tube}(C)$. Proposition 10.1 of [3] states that

$$\text{vol} \ N \leq (\coth^3 2R)(\text{vol} \ M + \frac{\pi}{2} L \tanh R \tanh 2R).$$

Note that

$$\text{vol} \ T = \pi L \sinh^2 R = \left(\frac{\pi}{2} L \tanh R \right) \left(2 \sinh R \cosh R \right) = \left(\frac{\pi}{2} L \tanh R \right) \left(\sinh 2R \right).$$

Thus

$$\text{vol} \ N \leq (\coth^3 2R) \left(\text{vol} \ M + \text{vol} \ T \frac{\tanh 2R}{\sinh 2R} \right) = (\coth^3 2R) \left(\text{vol} \ M + \frac{\text{vol} \ T}{\cosh 2R} \right).$$
In the language of [16], the quantity \((\text{vol } T)/(\text{vol } M)\) is the density of a tube packing in \(\mathbb{H}^3\). According to [16, Corollary 4.4], we have \((\text{vol } T)/(\text{vol } M) < 0.91\). Hence \(\text{vol } N < f(x) \text{vol}(M)\), where \(f(x)\) is defined for \(x \geq 0\) by

\[
f(x) = (\coth^3 2x) \left( 1 + \frac{0.91}{\cosh 2x} \right).
\]

Since \(f(x)\) is decreasing for \(x \geq 0\), and since a direct computation shows that \(f(0.5495) = 3.01762\ldots\), we have \(\text{vol } N < 3.0177 \text{vol } M\) whenever \(R \geq 0.5495\).

It remains to consider the case in which \(0.5495 > R \geq (\log 3)/2 = 0.5493\ldots\). In this case we use [16, Theorem 4.3], which asserts that the tube-packing density \((\text{vol } T)/(\text{vol } M)\) is bounded above by \((\sinh R)g(R)\), where \(g(x)\) is defined for \(x > 0\) by

\[
g(x) = \frac{\arcsin \frac{1}{2 \cosh x}}{\arcsinh \frac{\tanh x}{\sqrt{3}}}
\]

Since \(g(x)\) is clearly a decreasing function for \(x > 0\), and since \(\sinh R\) is increasing for \(x > 0\), we have

\[
(\text{vol } T)/(\text{vol } M) < (\sinh 0.5495)g((\log 3)/2) = 0.90817\ldots
\]

Hence \(\text{vol } N < f_1(x) \text{vol}(M)\), where \(f_1(x)\) is defined for \(x \geq 0\) by

\[
f_1(x) = (\coth^3 2x) \left( 1 + \frac{0.90817}{\cosh 2x} \right).
\]

Again, \(f_1(x)\) is decreasing for \(x \geq 0\), and we see by direct computation that \(f_1((\log 3)/2) = 3.017392\ldots\). Hence we have \(\text{vol } N < 3.0174 \text{vol } M\) in this case. ☐

**Lemma 3.2** Suppose that \(M\) is a closed, orientable hyperbolic 3–manifold such that \(\text{vol } M \leq 1.22\), and that \(C\) is a shortest geodesic in \(M\). Set \(N = \text{drill}_C(M)\). If \(\text{tuberad}(C) > (\log 3)/2\) then the maximal cusp neighborhood in \(N\) has volume less than \(\pi\).

**Proof** We let \(d(\infty) = .853276\ldots\) denote Böröczky’s lower bound [6] for the density of a horoball packing in hyperbolic space. It follows from the definition of the density of a horoball packing that the volume of a maximal cusp neighborhood in \(N\) is at most \(d(\infty) \text{vol } N\). **Lemma 3.1** gives \(\text{vol } N < 3.0177 \cdot 1.22 < \pi/d(\infty)\), and the conclusion follows. ☐
4 Filling

As in [4], we shall say that a group is semifree if it is a free product of free abelian groups; and we shall say that a group $\Gamma$ is $k$–semifree if every subgroup of $\Gamma$ whose rank is at most $k$ is semifree. Note that $\Gamma$ is 2–semifree if and only if every rank-2 subgroup of $\Gamma$ is either free or free abelian.

The following improved version of [4, Theorem 6.1] is made possible by more recent developments.

**Theorem 4.1** Let $k \geq 2$ be an integer and let $\Phi$ be a Kleinian group which is freely generated by elements $\xi_1, \ldots, \xi_k$. Let $z$ be any point of $\mathbb{H}^3$ and set $d_i = \text{dist}(z, \xi_i \cdot z)$ for $i = 1, \ldots, k$. Then we have

$$\sum_{i=1}^{k} \frac{1}{1 + e^{d_i}} \leq \frac{1}{2}.$$ 

In particular there is some $i \in \{1, \ldots, k\}$ such that $d_i \geq \log(2k - 1)$.

**Proof** If $\Gamma$ is geometrically finite this is included in [4, Theorem 6.1]. In the general case, $\Gamma$ is topologically tame according to [1] and [7], and it then follows from [15, Theorem 1.1], or from the corresponding result for the free case in [14], that $\Gamma$ is an algebraic limit of geometrically finite groups; more precisely, there is a sequence of geometrically finite Kleinian groups $(\Gamma_j)_{j \geq 1}$ such that each $\Gamma_j$ is freely generated by elements $\xi_{1j}, \ldots, \xi_{kj}$, and $\lim_{j \to \infty} \xi_{ij} = \xi_i$ for $i = 1, \ldots, k$. Given any $z \in \mathbb{H}^3$, we set $d_{ij} = \text{dist}(z, \xi_{ij} \cdot z)$ for each $j \geq 1$ and for $i = 1, \ldots, k$. According to [4, Theorem 6.1], we have

$$\sum_{i=1}^{k} \frac{1}{1 + e^{d_{ij}}} \leq \frac{1}{2}$$

for each $j \geq 1$. Taking limits as $j \to \infty$ we conclude that

$$\sum_{i=1}^{k} \frac{1}{1 + e^{d_i}} \leq \frac{1}{2}. \quad \square$$

Let us also recall the following definition from [4, Section 8]. Let $\Gamma$ be a discrete torsion-free subgroup of $\text{Isom}_+(\mathbb{H}^3)$. A positive number $\lambda$ is termed a strong Margulis number for $\Gamma$, or for the orientable hyperbolic 3–manifold $N = \mathbb{H}^3 / \Gamma$, if whenever $\xi$ and $\eta$ are non-commuting elements of $\Gamma$, we have

$$\frac{1}{1 + e^{\text{dist}(\xi \cdot z, z)}} + \frac{1}{1 + e^{\text{dist}(\eta \cdot z, z)}} \leq \frac{2}{1 + e^{\lambda}}.$$
The following improved version of [4, Proposition 8.4] is an immediate consequence of Theorem 4.1.

**Corollary 4.2** Let $\Gamma$ be a discrete subgroup of $\text{Isom}_+(\mathbb{H}^3)$. Suppose that $\Gamma$ is 2–semifree. Then $\log 3$ is a strong Margulis number for $\Gamma$.

**Lemma 4.3** Let $N$ be a non-compact finite-volume hyperbolic 3–manifold. Suppose that $S$ is a boundary component of the compact core $\hat{N}$, and $\mathcal{H}$ is the maximal cusp neighborhood in $N$ corresponding to $S$. If infinitely many of the manifolds obtained by Dehn filling $\hat{N}$ along $S$ have 2–semifree fundamental group then $\mathcal{H}$ has volume at least $\pi$.

**Proof** Suppose that $(N_i)$ is an infinite sequence of distinct hyperbolic manifolds obtained by Dehn filling $\hat{N}$ along $S$, and that $\pi_1(N_i)$ is 2–semifree for each $i$.

Thurston’s Dehn filling theorem [5, Appendix B], implies that for each sufficiently large $i$, the manifold $N_i$ admits a hyperbolic metric; that the core curve of the Dehn filling $N_i$ of $\hat{N}$ is isotopic to a geodesic $C_i$ in $N_i$; that the length $L_i$ of $C_i$ tends to 0 as $i \to \infty$; and that the sequence of maximal tubes $(\text{tube}(C_i))_{i \geq 1}$ converges geometrically to $\mathcal{H}$. In particular

$$\lim_{i \to \infty} \text{vol(\text{tube}(C_i))} = \text{vol} \mathcal{H}.$$ 

According to Corollary 4.2, $\log 3$ is a strong Margulis number for each of the hyperbolic manifolds $N_i$. It therefore follows from [4, Corollary 10.5] that $\text{vol \text{tube}(C_i)} > V(L_i)$, where $V$ is an explicitly defined function such that $\lim_{x \to 0} V(x) = \pi$. In particular, this shows that

$$\text{vol} \mathcal{H} \geq \lim_{i \to \infty} V(L_i) \geq \pi. \quad \Box$$

### 5 Non-exceptional manifolds, arbitrary primes

5.1 A closed hyperbolic 3–manifold $M$ will be termed *exceptional* if every shortest geodesic in $M$ has tube radius at most $(\log 3)/2$.

In this section we shall prove a result, Proposition 5.3, which gives a bound of 3 for the dimension of $H_1(M; \mathbb{Z}_p)$ for any prime $p$ when $M$ is a non-exceptional manifold with volume at most 1.22.
Lemma 5.2 Suppose that \( M \) is a compact, irreducible, orientable 3–manifold, such that every non-cyclic abelian subgroup of \( \pi_1(M) \) is carried by a torus component of \( \partial M \). Suppose that either

(i) \( \dim H_1(M; \mathbb{Q}) \geq 3 \), or

(ii) \( M \) is closed and \( \dim H_1(M; \mathbb{Z}_p) \geq 4 \) for some prime \( p \).

Then \( \pi_1(M) \) is 2–semifree.

Proof Let \( X \) be any subgroup of \( \pi_1(M) \) having rank at most 2. According to [11, Theorem VI.4.1], \( X \) is free, or free abelian, or of finite index in \( \pi_1(M) \). If \( \dim H_1(M; \mathbb{Q}) \geq 3 \), it is clear that \( X \) has infinite index in \( \pi_1(M) \). If \( M \) is closed and \( H_1(M; \mathbb{Z}_p) \geq 4 \) for some prime \( p \), then Proposition 1.1 of [17] implies that every 2–generator subgroup of \( \pi_1(M) \) has infinite index. Thus in either case \( X \) is either free or free abelian. This shows that \( \pi_1(M) \) is 2–semifree.

Proposition 5.3 Suppose that \( M \) is a closed, orientable, non-exceptional hyperbolic 3–manifold such that \( \text{vol} \ M \leq 1.22 \). Then \( H_1(M; \mathbb{Z}_p) \) has dimension at most 3 for every prime \( p \).

Proof Since \( M \) is non-exceptional, there is a shortest geodesic \( C \) in \( M \) with \( R = \text{tuberd}(C) > (\log 3)/2 \). We set \( \mathcal{N} = \text{drill}_C(M) \). Let \( \mathcal{H} \) denote the maximal cusp neighborhood in \( \mathcal{N} \). Since \( R > (\log 3)/2 \), Lemma 3.2 implies that \( \text{vol} \mathcal{H} < \pi \).

Now assume that \( \dim H_1(M; \mathbb{Z}_p) \geq 4 \) for some prime \( p \). There is an infinite sequence \( (M_i) \) of manifolds obtained by distinct Dehn fillings of \( \mathcal{N} \) such that \( H_1(M_i; \mathbb{Z}_p) \) has dimension at least 4 for each \( i \). (For example, if \( (\lambda, \mu) \) is a basis for \( H_1(\partial \mathcal{N}, \mathbb{Z}_p) \) such that \( \lambda \) belongs to the kernel of the inclusion homomorphism \( H_1(\partial \mathcal{N}, \mathbb{Z}_p) \rightarrow H_1(\mathcal{N}, \mathbb{Z}_p) \), we may take \( M_i \) to be obtained by the Dehn surgery corresponding to a simple closed curve in \( \partial \mathcal{N} \) representing the homology class \( \lambda + i p \mu \).) It follows from Thurston’s Dehn filling theorem [5, Appendix B] that for sufficiently large \( i \) the manifold \( M_i \) is hyperbolic. Hence by case (ii) of Lemma 5.2, the fundamental group of \( M_i \) is 2–semifree for sufficiently large \( i \). Thus Lemma 4.3 implies that \( \text{vol} \mathcal{H} \geq \pi \), a contradiction.

\section{Non-exceptional manifolds, odd primes}

Proposition 6.3, which is proved in this section, gives a bound of 2 for the dimension of \( H_1(M; \mathbb{Z}_p) \) for any odd prime \( p \) when \( M \) is a non-exceptional manifold with volume at most 1.22.
Definition 6.1  Let $N$ be a connected manifold, $\star \in N$ a base point, and $Q$ a subgroup of $\pi_1(N, \star)$. We shall say that a connected based covering space $r : (N', \star') \to (N, \star)$ carries the subgroup $Q$ if $Q \leq r_\#(\pi_1(N', \star')) \leq \pi_1(N, \star)$

Lemma 6.2  Suppose that $H$ is a maximal cusp neighborhood in a finite-volume hyperbolic 3–manifold $N$. Let $\star$ be a base point in $H$, and let $P \leq \pi_1(N, \star)$ denote the image of $\pi_1(H, \star)$ under inclusion. Then there is an element $\beta$ of $\pi_1(N, \star)$ with the following property:

\[ (\dagger) \quad \text{For every based covering space } r : (N', \star') \to (N, \star) \text{ which carries the subgroup } \langle P, \beta \rangle \text{ of } \pi_1(N, \star), \text{ there is a maximal cusp neighborhood } H' \text{ in } N' \text{ which is isometric to } H. \]

Proof  We write $N = \mathbb{H}^3 / \Gamma$, where $\Gamma$ is a discrete, torsion-free subgroup of $\text{Isom}(\mathbb{H}^3)$. Let $q : \mathbb{H}^3 \to N$ denote the quotient map and fix a base point $\star'$ which is mapped to $\star$ by $q$. The components of $q^{-1}(H)$ are horoballs. Let $B_0$ denote the component of $q^{-1}(H)$ containing $\star'$. The stabilizer $\Gamma_0$ of $B_0$ is mapped onto the subgroup $P$ of $\pi_1(N, \star)$ by the natural isomorphism $\iota : \Gamma \to \pi_1(N, \star)$.

Since $H$ is a maximal cusp, there is a component $B_1 \neq B_0$ of $q^{-1}(H)$ such that $\overline{B_1} \cap \overline{B_0} \neq \emptyset$. We fix an element $g$ of $\Gamma$ such that $g(B_0) = B_1$, and we set $\beta = \iota(g) \in \pi_1(N, \star)$.

To show that $\beta$ has property $(\dagger)$, we consider an arbitrary based covering space $r : (N', \star') \to (N, \star)$ which carries the subgroup $\langle P, \beta \rangle$ of $\pi_1(N, \star)$. We may identify $N'$ with $\mathbb{H}^3 / \Gamma'$, where $\Gamma'$ is some subgroup of $\Gamma$ containing $\langle \Gamma_0, g \rangle$.

Since $\Gamma_0 \subset \Gamma'$, the cusp neighborhood $H$ lifts to a cusp neighborhood $H'$ in $N'$. In particular $H'$ is isometric to $H$. The horoballs $B_0$ and $B_1 = g(B_0)$ are distinct components of $(q')^{-1}(H')$, where $q' : \mathbb{H}^3 \to N'$ denotes the quotient map. Since $g \in \Gamma'$ and $\overline{B_1} \cap \overline{B_0} \neq \emptyset$, the cusp neighborhood $H'$ is maximal.

Proposition 6.3  Suppose that $M$ is a closed, orientable, non-exceptional hyperbolic 3–manifold such that $\text{vol } M \leq 1.22$. Then $H_1(M; \mathbb{Z}_p)$ has dimension at most 2 for every odd prime $p$.

Proof  Since $M$ is non-exceptional, we may fix a shortest geodesic $C$ in $M$ with $R = \text{tuberad}(C) > (\log 3)/2$. We set $N = \text{drill}_C(M)$. Let $H$ denote the maximal cusp neighborhood in $N$. Since $R > (\log 3)/2$, Lemma 3.2 implies that $\text{vol } H < \pi$.

As in the statement of Lemma 6.2, we fix a base point $\star \in H$, and we denote by $P \leq \pi_1(N, \star)$ the image of $\pi_1(H, \star)$ under inclusion. We fix an element $\beta$ of $\pi_1(N, \star)$ having property $(\dagger)$ of Lemma 6.2. We set $Q = \langle P, \beta \rangle \leq \pi_1(N, \star)$.
Suppose that \( \dim H_1(M; \mathbb{Z}_p) \geq 3 \) for some prime \( p \). We shall prove the proposition by showing that this assumption leads to a contradiction if \( p \) is odd.

It follows from Poincaré duality that the image of the inclusion homomorphism \( \alpha : H_1(\partial \hat{N}; \mathbb{Z}_p) \to H_1(\hat{N}; \mathbb{Z}_p) \) has rank 1. Hence the image of \( P \) under the natural homomorphism \( \pi_1(N, \ast) \to H_1(N; \mathbb{Z}_p) \) has dimension 1. It follows that the image of \( Q \) of \( Q \) under this homomorphism has dimension either 1 or 2. In the case \( \dim \hat{Q} = 1 \) we shall obtain a contradiction for any prime \( p \). In the case \( \dim \hat{Q} = 2 \) we shall obtain a contradiction for any odd prime \( p \).

First consider the case \( \dim \hat{Q} = 1 \). We have assumed \( \dim H_1(M; \mathbb{Z}_p) \geq 3 \). Thus there is a \( \mathbb{Z}_p \times \mathbb{Z}_p \)-regular based covering space \((N', \ast')\) of \((N, \ast)\) which carries \( Q \). By property (†), there is a maximal cusp neighborhood \( \mathcal{H}' \) in \( N' \) which is isometric to \( \mathcal{H} \). In particular \( \text{vol} \mathcal{H}' < \pi \).

Since in particular \((N', \ast')\) carries \( P \), the boundary of the compact core \( \hat{N} \) lifts to \( \hat{N}' \). As \( N' \) is a \( p^2 \)-fold regular covering, it follows that \( \hat{N}' \) has \( p^2 \geq 4 \) boundary components.

It follows from Thurston’s Dehn filling theorem [5, Appendix B] that there are infinitely many hyperbolic manifolds obtained by Dehn filling one boundary component of \( \hat{N}' \). If \( Z \) is any hyperbolic manifold obtained by such a filling, then \( Z \) has at least three boundary components, and it follows from case (i) of Lemma 5.2 that \( \pi_1(Z) \) is 2-semifree. It therefore follows from Lemma 4.3 that each maximal cusp neighborhood in \( N' \) has volume at least \( \pi \). Since we have seen that \( \text{vol} \mathcal{H}' < \pi \), this gives the desired contradiction in the case \( \dim \hat{Q} = 1 \).

It remains to consider the case in which \( \dim \hat{Q} = 2 \) and the prime \( p \) is odd. Since we have assumed \( \dim H_1(M; \mathbb{Z}_p) \geq 3 \), there is a \( p \)-fold cyclic based covering space \((N', \ast')\) of \((N, \ast)\) which carries \( Q \). Since \( N' \) carries \( P \), the boundary of the compact core \( \hat{N} \) lifts to \( \hat{N}' \), and as \( N' \) is a \( p \)-fold regular covering, it follows that \( \hat{N}' \) has \( p \) boundary components.

We claim that the inclusion homomorphism \( \alpha' : H_1(\partial \hat{N}', \mathbb{Z}_p) \to H_1(\hat{N}', \mathbb{Z}_p) \) is not surjective. To establish this, we consider the commutative diagram

\[
\begin{array}{ccc}
H_1(\partial \hat{N}'; \mathbb{Z}_p) & \xrightarrow{\alpha'} & H_1(N'; \mathbb{Z}_p) \\
\downarrow & & \downarrow r_* \\
H_1(\partial \hat{N}; \mathbb{Z}_p) & \xrightarrow{\alpha} & H_1(N; \mathbb{Z}_p)
\end{array}
\]

where \( r : N' \to N \) is the covering projection. Since \((N', \ast')\) carries \( Q \) we have \( \hat{Q} \subset \text{Im} r_* \). Hence surjectivity of \( \alpha' \) would imply \( \hat{Q} \subset \text{Im} \alpha \). This is impossible.
observed above that \( \text{Im} \alpha \) has rank 1, and we are in the case \( \dim \widetilde{Q} = 2 \). Thus \( \alpha' \) cannot be surjective.

Since \( \tilde{N}' \) has \( p \) boundary components, it follows from Poincaré duality that \( \dim \text{Im} \alpha' = p \geq 3 \). Since \( \alpha' \) is not surjective and \( p \) is an odd prime, it follows that \( \dim H_1(N'; \mathbb{Z}_p) \geq p + 1 \geq 4 \).

Since \( (N', \star) \) carries \( Q \), some subgroup \( Q' \) of \( \pi_1(N', \star) \) is mapped isomorphically to \( Q \) by \( r_\# \). In particular \( Q' \) has rank at most 3. Since \( \dim H_1(N'; \mathbb{Z}_p) \geq 4 \), there is a \( p \)-fold cyclic based covering space \( (N'', \star'') \) of \( (N', \star') \) which carries \( Q' \). Hence \( (N'', \star'') \) is a \( p^2 \)-fold (possibly irregular) based covering space of \( (N, \star) \) which carries \( Q \). By property (\dagger), there is a maximal cusp neighborhood \( \mathcal{H}' \) in \( N'' \) which is isometric to \( \mathcal{H} \). In particular \( \text{vol} \mathcal{H}' < \pi \).

Since \( P \leq Q \), there is a component \( T \) of \( \partial \tilde{N}' \) such that \( Q' \) contains a conjugate of the image of \( \pi_1(T) \) under the inclusion homomorphism \( \pi_1(T) \to \pi_1(N') \). Hence \( T \) lifts to the \( p \)-fold cyclic covering space \( N'' \) of \( N' \). It follows that the covering projection \( r': N'' \to N' \) maps \( p \geq 3 \) components of \( (r')^{-1}(\partial \tilde{N}') \) to \( T \). As \( \tilde{N}' \) has at least three boundary components, \( \tilde{N}' \) must have at least five boundary components.

Hence if \( Z \) is any hyperbolic manifold obtained by Dehn filling one boundary component of \( \tilde{N}' \), we have \( \dim H_1(Z; \mathbb{Q}) \geq 4 > 3 \), and it follows from case (i) of Lemma 5.2 that \( \pi_1(Z) \) is \( 2 \)-semifree. It therefore follows from Lemma 4.3 and Thurston’s Dehn filling theorem that each maximal cusp neighborhood in \( N'' \) has volume at least \( \pi \). Since we have seen that \( \text{vol} \mathcal{H}' < \pi \), we have the desired contradiction in this case as well. \( \square \)

### 7 Exceptional manifolds

Our treatment of exceptional manifolds begins with Proposition 7.1 below, the proof of which will largely consist of citing material from [9]. In order to state it we must first introduce some notation.

For \( k = 0, \ldots, 6 \) we define constants \( \tau_k \) as follows:

\[
\begin{align*}
\tau_0 &= 0.4779 \\
\tau_1 &= 1.0756 \\
\tau_2 &= 1.0527 \\
\tau_3 &= 1.2599 \\
\tau_4 &= 1.2521 \\
\tau_5 &= 1.0239 \\
\tau_6 &= 1.0239
\end{align*}
\]
For $k = 0, \ldots, 6$ let $\mathcal{E}_k$ be the 2–generator group with presentation
\[
\mathcal{E}_k = \langle x, y : r_{1,k}, r_{2,k} \rangle,
\]
where the relators $r_{1,k} = r_{1,k}(x, y)$ and $r_{2,k} = r_{2,k}(x, y)$ are the words listed below (in which we have set $X = x^{-1}$ and $Y = y^{-1}$):
\[
\begin{align*}
    r_{1,0} &= xyXyyXyxyy, \\
    r_{2,0} &= Xyxyxyy, \\
    r_{1,1} &= XXYXYXYXYYXyXXy, \\
    r_{2,1} &= XXYxyxxXyxyXyy, \\
    r_{1,2} &= XyxyXyxyXyxyXyy, \\
    r_{2,2} &= XXYyXXyyXyxyXyy, \\
    r_{1,3} &= XXYyXyxyXXyyXXyXYXyXYXyy, \\
    r_{2,3} &= XXYyXXyyXXyXYXYXyXYXyy, \\
    r_{1,4} &= XXYyXXyXYxXyxyXXyXYXYXYyXYYy, \\
    r_{2,4} &= XXYyXXyXXyXYXyXXyXXyXYXYXyXYXYXYyXYYy, \\
    r_{1,5} &= XyXYyXyxyXXyyXyxyXyxy, \\
    r_{2,5} &= XyxyXXyXXyyXyxyXyxy, \\
    r_{1,6} &= XYYxXYXXyXYXYxyy, \\
    r_{2,6} &= XYYxyXXyXXyXYXyxyy.
\end{align*}
\]

The group $\mathcal{E}_0$ is the fundamental group of an arithmetic hyperbolic 3–manifold which is known as Vol3. This manifold, which was studied in [12], is described as $m007(3,1)$ in the list given in [10], and can also be described as the manifold obtained by a Dehn filling of the once-punctured torus bundle with monodromy $-R^2L$.

**Proposition 7.1** Suppose that $M$ is an exceptional closed, orientable hyperbolic 3–manifold which is not isometric to Vol3. Then there exists an integer $k$ with $1 \leq k \leq 6$ such that the following conditions hold:

1. $M$ has a finite-sheeted cover $\tilde{M}$ such that $\pi_1(\tilde{M})$ is isomorphic to a quotient of $\mathcal{E}_k$; and
2. there is a shortest closed geodesic $C$ in $M$ such that $\text{vol}(\text{tube}(C)) \geq \tau_k$.

**Proof** This is in large part an application of results from [9], and we begin by reviewing some material from that paper.

We begin by considering an arbitrary simple closed geodesic $C$ in a closed, orientable hyperbolic 3–manifold $M = \mathbb{H}^3 / \Gamma$. As we pointed out in 2.2, there is a loxodromic
isometry $f \in \Gamma$ with $A_f/\langle f \rangle = C$. If we set $R = \text{tuberad}(C)$ and $Z = Z_R(f)$, it follows from the definitions that $\text{tube}(C) = Z/\langle f \rangle$, that $h(Z) \cap Z = \emptyset$ for every $h \in \Gamma - \langle f \rangle$, and that there is an element $w \in \Gamma - \langle f \rangle$ such that $w(\bar{Z}) \cap \bar{Z} \neq \emptyset$.

Let us define an ordered pair $(f, w)$ of elements of $\Gamma$ to be a GMT pair for the simple geodesic $C$ if we have (i) $A_f/\langle f \rangle = C$, (ii) $w \notin \langle f \rangle$, and (iii) $w(\bar{Z}) \cap \bar{Z} \neq \emptyset$. Note that since $\langle f \rangle$ must be a maximal cyclic subgroup of $\Gamma$, condition (ii) implies that the group $\langle f, w \rangle$ is non-elementary.

Set $Q = \{ (L, D, R) \in \mathbb{C}^3 : \Re L, \Re D > 0 \}$. For any point $P = (L, D, R) \in Q$ we will denote by $(f_P, w_P)$ the pair $(f, w) \in \text{Isom}_+^+(\mathbb{H}^3) \times \text{Isom}_+^+(\mathbb{H}^3)$, where $f, w \in PGL_2(\mathbb{C}) = \text{Isom}_+^+(\mathbb{H}^3)$ are defined by

$$f = \begin{bmatrix} e^{L/2} & 0 \\ 0 & e^{-L/2} \end{bmatrix}$$

and

$$w = \begin{bmatrix} e^{R/2} & 0 \\ 0 & e^{-R/2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{D/2} & 0 \\ 0 & e^{-D/2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$ 

With this definition, $f_P$ has (real) translation length $\Re L$, and the (minimum) distance between $A_f$ and $w(A_f)$ is $(\Re D)/2$.

In [9, Section 1], it is shown that if $(f, w)$ is a GMT pair for a shortest geodesic $C$ in a closed, orientable hyperbolic 3–manifold and $\text{tuberad}(C) \leq (\log 3)/2$, then $(f, w)$ is conjugate by some element of $\text{Isom}_+^+(\mathbb{H}^3)$ to a pair of the form $(f_P, w_P)$ where $P \in Q$ is a point such that $\exp(P) = (e^L, e^D, e^R)$ lies in the union $X_0 \cup \cdots \cup X_6$ of seven disjoint open subsets of $\mathbb{C}^3$ that are explicitly defined in [9, Proposition 1.28].

For every $k$ with $0 \leq k \leq 6$ and every point $P = (L, D, R)$ such that $\exp(P) \in X_k$, it follows from [9, Definition 1.27 and Proposition 1.28] that

(I) the isometries $r_{1,k}(f_P, w_P)$ and $r_{2,k}(f_P, w_P)$ have translation length less than $\Re L$;

and it follows from [9, Table 1.1] that

(II) $\pi \Re(L) \sinh^2(\Re(D)/2) > \tau_k$.

According to [9, Proposition 3.1], if $C$ is a shortest geodesic in a closed, orientable hyperbolic 3–manifold, and if some GMT pair for $C$ has the form $(f_P, w_P)$ for some $P$ with $\exp(P) \in X_0$, then $M$ is isometric to Vol3.

Now suppose that $M$ is an exceptional closed, orientable hyperbolic 3–manifold. Let us choose a shortest closed geodesic $C$ in $M$. By the definition of an exceptional manifold, $C$ has tube radius $\leq (\log 3)/2$. Hence the facts recalled above imply that $C$ has a GMT pair of the form $(f_P, w_P)$ for some $P$ such that $\exp(P) \in X_k$ for some $k$.
with $0 \leq k \leq 6$; and furthermore, that if $M$ is not isometric to Vol3, then $1 \leq k \leq 6$. We shall show that conclusions (1) and (2) hold with this choice of $k$.

For $i = 1, 2$ it follows from property (I) above that the element $r_{i,k}(f, \omega)$ has real translation length less than the real translation length $\text{Re } L$ of $f$. Since $C$ is a shortest geodesic in $M$, it follows that the conjugacy class of $r_{i,k}(f, \omega)$ is not represented by a closed geodesic in $M$. As $M$ is closed it follows that $r_{i,k}(f, \omega)$ is the identity for $i = 1, 2$. Hence the subgroup of $\Gamma$ generated by $f$ and $\omega$ is isomorphic to a quotient of $\mathcal{E}_k$. Since we observed above that $(f, \omega)$ is non-elementary, there is a non-abelian subgroup $Y$ of $\pi_1(M)$ which is isomorphic to a quotient of $\mathcal{E}_k$. In particular $Y$ has rank 2, and it cannot be a free group of rank 2 since the relators $r_{1,k}$ and $r_{2,k}$ are non-trivial. Hence by [11, Theorem VI.4.1] we must have $|\pi_1(M):Y| < \infty$. This proves (1).

Finally, we recall that

$$\text{vol tube}(C) = \pi(\text{length}(C)) \sinh^2(\text{tuberad}(C)) = \pi(\text{Re } L) \sinh^2((\text{Re } D)/2).$$

Hence (2) follows from (II).  

We shall also need the following slight refinement of [17, Proposition 1.1].

**Proposition 7.2** Let $p$ be a prime and let $M$ be a closed 3–manifold. If $p$ is odd assume that $M$ is orientable. Let $X$ be a finitely generated subgroup of $\pi_1(M)$, and set $n = \dim H_1(X; \mathbb{Z}_p)$. If $\dim H_1(M; \mathbb{Z}_p) \geq \max(3, n+2)$, then $X$ has infinite index in $\pi_1(M)$. In fact, $X$ is contained in infinitely many distinct finite-index subgroups of $\pi_1(M)$.

**Proof** In this proof, as in [17, Section 1], for any group $G$ we shall denote by $G_1$ the subgroup of $G$ generated by all commutators and $p$-th powers, where $p$ is the prime given in the hypothesis. Since $\dim H_1(X; \mathbb{Z}_p) = n$ we may write $X = EX_1$ for some rank-$n$ subgroup $E$ of $X$.

We first assume that $n \geq 1$. Set $\Gamma = \pi_1(M)$. Let $S$ denote the set of all finite-index subgroups $\Delta$ of $\Gamma$ such that $\Delta \geq X$ and $\dim H_1(\Delta; \mathbb{Z}_p) \geq n + 2$. The hypothesis gives $\Gamma \in S$, so that $S \neq \emptyset$. Hence it suffices to show that every subgroup $\Delta \in S$ has a proper subgroup $D$ such that $D \in S$.

Any group $\Delta \in S$ may be identified with $\pi_1(\widetilde{M})$ for some finite-sheeted covering space $\widetilde{M}$ of $M$. In particular, $\widetilde{M}$ is a closed 3–manifold, and is orientable if $p$ is odd. Since $\Delta \in S$ we have $X \leq \Delta = \pi_1(\widetilde{M})$ and $\dim H_1(\widetilde{M}; \mathbb{Z}_p) = \dim H_1(\Delta; \mathbb{Z}_p) \geq n + 2$. Now set $D = E \Delta_1 \leq \Delta$. Applying [17, Lemma 1.5], with $\widetilde{M}$ in place of $M$, we deduce that
$D$ is a proper, finite-index subgroup of $\Delta$, and that $\dim H_1(D; \mathbb{Z}_p) \geq 2n + 1 \geq n + 2$. On the other hand, since $\Delta \in \mathcal{S}$, we have $X \leq \Delta$, and hence $X = EX_1 \leq E\Delta_1 = D$. It now follows that $D \in \mathcal{S}$, and the proof is complete in the case $n \geq 1$.

If $n = 0$ then, since $\dim H_1(M; \mathbb{Z}_p) \geq 3$, there exists a finitely generated subgroup $X' \geq X$ such that $H_1(X'; \mathbb{Z}_p)$ has dimension 1. The case of the Lemma which we have already proved shows that $X'$ has infinite index. Thus $X$ has infinite index as well.

**Corollary 7.3** Let $p$ be a prime and let $M$ be a closed, orientable 3–manifold. Let $X$ be a finite-index subgroup of $\pi_1(M)$, and set $n = \dim H_1(X; \mathbb{Z}_p)$. Then $\dim H_1(M; \mathbb{Z}_p) \leq \max(2, n + 1)$.

**Lemma 7.4** Suppose that $M$ is an exceptional hyperbolic 3–manifold with volume at most 1.22. Then $H_1(M; \mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M; \mathbb{Z}_2)$ and $H_1(M; \mathbb{Z}_7)$ have dimension at most 3. Furthermore, if $M$ has volume at most 1.182, then $H_1(M; \mathbb{Z}_7)$ has dimension at most 2.

**Proof** If $M$ is isometric to $\text{Vol}^3$ then $\pi_1(M)$ is generated by two elements, and the conclusions follow. For the rest of the proof we assume that $M$ is not isometric to $\text{Vol}^3$, and we fix an integer $k$ with $1 \leq k \leq 6$ such that conditions (1) and (2) of Proposition 7.1 hold.

By condition (2) of Proposition 7.1, we may fix a shortest closed geodesic $C$ in $M$ such that $\text{vol}(T) \geq \tau_k$, where $T = \text{tube}(C)$. It follows from a result of Przeworski’s [16, Corollary 4.4] on the density of cylinder packings that $\text{vol} T < 0.91 \text{vol} M$, and so $\text{vol} M > \tau_k/0.91$. If $k = 3$ we have $\tau_k/0.91 \geq 1.22$, and we get a contradiction to the hypothesis. Hence $k \in \{1, 2, 4, 5, 6\}$.

Furthermore, we have $\tau_1/0.91 > 1.182$. Hence if $\text{vol} M \leq 1.182$ then $k \in \{2, 4, 5, 6\}$.

By condition (1) of Proposition 7.1, $\pi_1(M)$ has a finite-index subgroup $X$ which is isomorphic to a quotient of $\mathcal{E}_k$. From the defining presentations of the groups $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$, $\mathcal{E}_5$ and $\mathcal{E}_6$, we find that $H_1(\mathcal{E}_1; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_7 \oplus \mathbb{Z}_7$, that $H_1(\mathcal{E}_2; \mathbb{Z})$ and $H_1(\mathcal{E}_4; \mathbb{Z})$ are isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$, while $H_1(\mathcal{E}_5; \mathbb{Z})$ and $H_1(\mathcal{E}_6; \mathbb{Z})$ are isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. (One can check that the two groups $\mathcal{E}_5$ and $\mathcal{E}_6$ are isomorphic to each other.) In particular, since $k \in \{1, 2, 4, 5, 6\}$ we have $\dim H_1(\mathcal{E}_k; \mathbb{Z}_p) \leq 1$ for any prime $p \neq 2, 7$, and $\dim H_1(\mathcal{E}_k; \mathbb{Z}_p) \leq 2$ for $p = 2$ or 7. As $X$ is isomorphic to a quotient of $\mathcal{E}_k$, it follows that $\dim H_1(X; \mathbb{Z}_p) \leq 1$ for any prime $p \neq 2, 7$, and $\dim H_1(X; \mathbb{Z}_p) \leq 2$ for $p = 2$ or 7. Hence by Corollary 7.3, we have $\dim H_1(M; \mathbb{Z}_p) \leq 2$ for $p \neq 2, 7$, and $\dim H_1(M; \mathbb{Z}_p) \leq 3$ for $p = 2, 7$. 

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It remains to prove that if $\text{vol } M \leq 1.182$ then $\dim H_1(M;\mathbb{Z}_7) \leq 2$. We have observed that in this case $k \in \{2, 4, 5, 6\}$. By the list of isomorphism types of the $H_1(\mathcal{E}_k;\mathbb{Z})$ given above, it follows that $\dim H_1(\mathcal{E}_k;\mathbb{Z}_7) = 0 < 1$. Hence in this case the argument given above for $p \neq 2, 7$ goes through in exactly the same way to show that $\dim H_1(M;\mathbb{Z}_7) \leq 2$.

**Proof of Theorem 1.1** For the case in which $M$ is non-exceptional, the theorem is an immediate consequence of Propositions 5.3 and 6.3. For the case in which $M$ is exceptional, the assertions of the theorem are equivalent to those of Lemma 7.4.

**References**


Department of Mathematics, Statistics, and Computer Science (M/C 249) University of Illinois at Chicago, 851 S Morgan St, Chicago, IL 60607-7045, USA

agol@math.uic.edu, culler@math.uic.edu, shalen@math.uic.edu

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