Z_2^k–actions fixing $\mathbb{R}P^2 \cup \mathbb{R}P^{\text{even}}$

ROGÉRIO DE OLIVEIRA
PEDRO L Q PERGHER
ADRIANA RAMOS

This paper determines, up to equivariant cobordism, all manifolds with Z_2^k–action whose fixed point set is $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^n$, where $n > 2$ is even.

57R85; 57R75

1 Introduction

Suppose M is a smooth, closed manifold and $T\colon M \to M$ is a smooth involution defined on M. It is well known that the fixed point set F of T is a finite and disjoint union of closed submanifolds of M. For a given F, a basic problem in this context is the classification, up to equivariant cobordism, of the pairs (M,T) for which the fixed point set is F. For related results, see for example Royster [16], Hou and Torrence [6; 7], Pergher [11], Stong [17; 18], Conner and Floyd [4, Theorem 27.6], Kosniowski and Stong [8, page 309] and Lü [9; 10].

For $F = \mathbb{R}\mathrm{P}^n$, the classification was established in [4] and [17]. D C Royster [16] then studied this problem with F the disjoint union of two real projective spaces, $F = \mathbb{R}\mathrm{P}^m \cup \mathbb{R}\mathrm{P}^n$. He established the results via a case-by-case method depending on the parity of m and n, with special arguments when one of the components is $\mathbb{R}\mathrm{P}^0 = \{\text{point}\}$, but his methods were not sufficient to handle the case when m and n are even and positive. If m and n are even and $m = n$, one knows from [8] that (M,T) is an equivariant boundary when $\dim(M) \geq 2n$; it was later shown in [7] that (M,T) also is a boundary when $n \leq \dim(M) < 2n$. To understand the case $(m,n) = (0,\text{even})$ and also the goal of this paper, consider the involution $(\mathbb{R}\mathrm{P}^{m+n+1}, T_{m,n})$, for any m and n, defined in homogeneous coordinates by

$$T_{m,n}[x_0, x_1, \ldots, x_{m+n+1}] = [-x_0, -x_1, \ldots, -x_m, x_{m+1}, \ldots, x_{m+n+1}].$$

The fixed set of $T_{m,n}$ is $\mathbb{R}\mathrm{P}^m \cup \mathbb{R}\mathrm{P}^n$. From $T_{m,n}$, it may be possible to obtain other involutions fixing $\mathbb{R}\mathrm{P}^m \cup \mathbb{R}\mathrm{P}^n$: in general, for a given involution (W,T) with fixed

Published: 23 February 2007 DOI: 10.2140/agt.2007.7.29

set F and W a boundary, the involution

$$\Gamma(W,T) = \Big(\frac{S^1 \times W}{-\operatorname{Id} \times T}, \tau\Big)$$

is equivariantly cobordant to an involution fixing F; here, S^1 is the 1–sphere, Id is the identity map and τ is the involution induced by $c \times \operatorname{Id}$, where c is complex conjugation (see Conner and Floyd [5]). If $(S^1 \times W)/(-\operatorname{Id} \times T)$ is a boundary, we can repeat the process taking $\Gamma^2(W,T)$, and so on. If F is nonbounding, this process finishes, that is, there exists a smallest natural number $r \geq 1$ for which the underlying manifold of $\Gamma^r(W,T)$ is nonbounding; this follows from the $(5/2)$–theorem of J Boardman in [1] and its strengthened version in [8]. In particular, if m and n are even and $m < n$, $\mathbb{R}\mathrm{P}^m \cup \mathbb{R}\mathrm{P}^n$ does not bound and $\mathbb{R}\mathrm{P}^{m+n+1}$ bounds, so this number r makes sense for $(\mathbb{R}\mathrm{P}^{m+n+1}, T_{m,n})$, and we denote r by $h_{m,n}$. In [16], Royster proved the following theorem:

Theorem *Let (M,T) be an involution fixing $\{\text{point}\} \cup \mathbb{R}\mathrm{P}^n$, where n is even. Then (M,T) is equivariantly cobordant to $\Gamma^j(\mathbb{R}\mathrm{P}^{n+1}, T_{0,n})$ for some $0 \leq j \leq h_{0,n}$.*

Later, in [15], R E Stong and P Pergher determined the value of $h_{0,n}$, thus answering the question posed by Royster in [16, page 271]: writing $n = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd, they showed that $h_{0,n} = 2$ if $p = 1$ and $h_{0,n} = 2^p - 1$ if $p > 1$.

In this paper, we contribute to this problem by solving the case $(m,n) = (2, \text{even})$. Specifically, we will prove the following:

Theorem 1 *Let (M,T) be an involution fixing $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^n$, where M is connected and $n \geq 4$ is even. If $n > 4$, then (M,T) is equivariantly cobordant to $\Gamma^j(\mathbb{R}\mathrm{P}^{n+3}, T_{2,n})$ for some $0 \leq j \leq h_{2,n}$. If $n = 4$, then (M,T) is either equivariantly cobordant to $\Gamma^j(\mathbb{R}\mathrm{P}^7, T_{2,4})$ for some $0 \leq j \leq h_{2,4}$, or equivariantly cobordant to $\Gamma^2(\mathbb{R}\mathrm{P}^3, T_{0,2}) \cup (\mathbb{R}\mathrm{P}^5, T_{0,4})$.*

In addition, we generalize the result of Stong and Pergher of [15], calculating the general value of $h_{m,n}$ (which, in particular, makes numerically precise the statement of Theorem 1).

Theorem 2 *For m,n even, $0 \leq m < n$, write $n - m = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd. Then $h_{m,n} = 2$ if $p = 1$, and $h_{m,n} = 2^p - 1$ if $p > 1$.*

Finally, we also extend the results for Z_2^k–actions. This extension is automatic from the combination of the above results and the case $F = \mathbb{R}\mathrm{P}^{\text{even}}$ with a recent paper of the first two authors [13]. The details concerning this extension will be given in Section 4. Section 2 and Section 3 will be devoted, respectively, to the proofs of Theorem 1 and Theorem 2.

Acknowledgements The authors would like to thank Professor Robert E Stong of the University of Virginia for suggestions and techniques. The second author was partially supported by CNPq and FAPESP, and the third was supported by CAPES.

2 Involutions fixing $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^{\mathrm{even}}$

We start with an involution (M, T) fixing $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^n$, where M is connected and $n \geq 4$ is even, and first establish some notations. We will always use $\lambda_r \to \mathbb{R}\mathrm{P}^r$ to denote the canonical line bundle over $\mathbb{R}\mathrm{P}^r$. Denote by $\alpha \in H^1(\mathbb{R}\mathrm{P}^2, Z_2)$ and $\beta \in H^1(\mathbb{R}\mathrm{P}^n, Z_2)$ the generators of the 1–dimensional Z_2–cohomology. The model involution $(\mathbb{R}\mathrm{P}^{n+3}, T_{2,n})$ fixes $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^n$ with normal bundles $(n+1)\lambda_2 \to \mathbb{R}\mathrm{P}^2$ and $3\lambda_n \to \mathbb{R}\mathrm{P}^n$. The total Stiefel–Whitney classes are $W((n+1)\lambda_2) = (1+\alpha)^{n+1}$, $W(3\lambda_n) = (1+\beta)^3$. Denote by $\eta \to \mathbb{R}\mathrm{P}^2$ and $\xi \to \mathbb{R}\mathrm{P}^n$ the normal bundles of $\mathbb{R}\mathrm{P}^2$ and $\mathbb{R}\mathrm{P}^n$ in M. To prove Theorem 1, it suffices to prove the following:

Lemma 3 *If $n > 4$, then $W(\eta) = (1+\alpha)^{n+1}$ and $W(\xi) = (1+\beta)^3$. If $n = 4$, then either $W(\eta) = (1+\alpha)^5$ and $W(\xi) = (1+\beta)^3$, or $W(\eta) = 1+\alpha$ and $W(\xi) = 1+\beta$.*

In fact, suppose Lemma 3 is true, and denote by R the trivial one-dimensional vector bundle over any base space. Set $k = \dim(\eta)$ and $l = \dim(\xi)$, that is, $k = \dim(M) - 2$ and $l = \dim(M) - n \geq 1$.

First consider $n > 4$. By [5], for $0 \leq j \leq h_{2,n}$, the involution $\Gamma^j(\mathbb{R}\mathrm{P}^{n+3}, T_{2,n})$ is equivariantly cobordant to an involution with fixed data

$$((n+1)\lambda_2 \oplus jR \to \mathbb{R}\mathrm{P}^2) \cup (3\lambda_n \oplus jR \to \mathbb{R}\mathrm{P}^n).$$

Using the notations $W = 1 + w_1 + w_2 + \dots$ for Stiefel–Whitney classes and $\binom{a}{b}$ for binomial coefficients mod 2, note that $w_3(\xi) = \binom{3}{3}\beta^3 = \beta^3 \neq 0$ and thus $l \geq 3$. Then

$$\eta \cup \xi \quad \text{and} \quad ((n+1)\lambda_2 \oplus (l-3)R) \cup (3\lambda_n \oplus (l-3)R)$$

are cobordant because they have the same characteristic numbers. If $l \leq 3 + h_{2,n}$, one then has from [4] that (M, T) and $\Gamma^{l-3}(\mathbb{R}\mathrm{P}^{n+3}, T_{2,n})$ are equivariantly cobordant, proving the result. By contradiction, suppose then $l > 3 + h_{2,n}$. Again from [4],

$$((n+1)\lambda_2 \oplus (l-3)R) \cup (3\lambda_n \oplus (l-3)R)$$

is the fixed data of an involution (W,S), and by removing sections if necessary we can suppose, with no loss, that $\dim(W)=n+h_{2,n}+4$ [4, Theorem 26.4]. Let (N,T') be an involution cobordant to $\Gamma^{h_{2,n}}(\mathbb{R}\mathrm{P}^{n+3},T_{2,n})$ and with fixed data

$$((n+1)\lambda_2\oplus h_{2,n}R)\cup(3\lambda_n\oplus h_{2,n}R).$$

One knows that N is not a boundary. Then $\Gamma(N,T')\cup(W,S)$ is cobordant to an involution with fixed data $R\to N$, and from [4] $R\to N$ then is a boundary, which is impossible.

Now suppose $n=4$. The case $W(\eta)=(1+\alpha)^5$ and $W(\xi)=(1+\beta)^3$ is included in the above approach, hence suppose $W(\eta)=1+\alpha$ and $W(\xi)=1+\beta$. Since $h_{0,2}=2$, the involution $\Gamma^2(\mathbb{R}\mathrm{P}^3,T_{0,2})$ is cobordant to an involution with fixed data

$$(5R\to\{\text{point}\})\cup(\lambda_2\oplus 2R\to\mathbb{R}\mathrm{P}^2).$$

Then the involution $\Gamma^2(\mathbb{R}\mathrm{P}^3,T_{0,2})\cup(\mathbb{R}\mathrm{P}^5,T_{0,4})$ is cobordant to an involution (W^5,T) with fixed data $(\lambda_2\oplus 2R\to\mathbb{R}\mathrm{P}^2)\cup(\lambda_4\to\mathbb{R}\mathrm{P}^4)$, and the total Stiefel–Whitney classes are $W(\lambda_2\oplus 2R)=1+\alpha$ and $W(\lambda_4)=1+\beta$. Because $h_{0,2}=2$, the underlying manifold of $\Gamma^2(\mathbb{R}\mathrm{P}^3,T_{0,2})$ does not bound; since $\mathbb{R}\mathrm{P}^5$ bounds, W^5 does not bound. By contradiction, suppose $l\geq 2$. Using the hypothesis, [4] and removing sections if necessary, we can suppose with no loss that (M,T) has fixed data

$$(\lambda_2\oplus 3R\to\mathbb{R}\mathrm{P}^2)\cup(\lambda_4\oplus R\to\mathbb{R}\mathrm{P}^4).$$

Using the same above argument for $\Gamma(W^5,T)\cup(M,T)$, we conclude $R\to W$ is a boundary, which is false. Then $l=1$ and (M,T) and (W^5,T) (and hence the union $\Gamma^2(\mathbb{R}\mathrm{P}^3,T_{0,2})\cup(\mathbb{R}\mathrm{P}^5,T_{0,4})$) have fixed data with same characteristic numbers.

In order to prove Lemma 3, we will intensively use the following basic fact from [4]: the projective space bundles $\mathbb{R}\mathrm{P}(\eta)$ and $\mathbb{R}\mathrm{P}(\xi)$ with the standard line bundles $\lambda\to\mathbb{R}\mathrm{P}(\eta)$ and $\nu\to\mathbb{R}\mathrm{P}(\xi)$ are cobordant as elements of the bordism group $\mathcal{N}_{k+1}(BO(1))$. Then any class of dimension $k+1$, given by a product of the classes $w_i(\mathbb{R}\mathrm{P}(\eta))$ and $w_1(\lambda)$, evaluated on the fundamental homology class $[\mathbb{R}\mathrm{P}(\eta)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(\mathbb{R}\mathrm{P}(\xi))$ and $w_1(\nu)$, evaluated on $[\mathbb{R}\mathrm{P}(\xi)]$. To evaluate characteristic numbers, the following formula of Conner will be useful [2, Lemma 3.1]: if $\pi\colon\mu\to N$ is any r–dimensional vector bundle, c is the first Stiefel–Whitney class of the standard line bundle over $\mathbb{R}\mathrm{P}(\mu)$, $\overline{W}(\mu)=1+\bar{w}_1(\mu)+\bar{w}_2(\mu)+\dots$ is the dual Stiefel–Whitney class defined by $W(\mu)\overline{W}(\mu)=1$ and $\alpha\in H^*(N,Z_2)$, then

(1) $$c^j\pi^*(\alpha)[\mathbb{R}\mathrm{P}(\mu)]=\bar{w}_{j-r+1}(\mu)\alpha[N]\quad\text{when } j\geq r-1.$$

In this context, our numerical arguments will always be considered modulo 2. Write $W(\lambda) = 1 + c$ and $W(\nu) = 1 + d$ for the Stiefel–Whitney classes of λ and ν. The structure of the Grothendieck ring of orthogonal bundles over real projective spaces says that $W(\eta) = (1+\alpha)^p$ and $W(\xi) = (1+\beta)^q$ for some $p, q \geq 0$. From [4, 23.3], one then has

$$W(\mathbb{R}\mathrm{P}(\eta)) = (1+\alpha)^3 \Big(\sum_{i=0}^{2} (1+c)^{k-i} \tbinom{p}{i} \alpha^i \Big)$$

and
$$W(\mathbb{R}\mathrm{P}(\xi)) = (1+\beta)^{n+1} \Big(\sum_{i=0}^{l} (1+d)^{l-i} \tbinom{q}{i} \beta^i \Big),$$

where here we are suppressing bundle maps.

Fact 1 The numbers p and q are odd; in particular, $w_1(\eta) = \alpha$ and $w_1(\xi) = \beta$.

Proof One has

$$w_1(\mathbb{R}\mathrm{P}(\eta)) = \tbinom{k}{1} c + \alpha + \tbinom{p}{1}\alpha \quad \text{and} \quad w_1(\mathbb{R}\mathrm{P}(\xi)) = \tbinom{l}{1} d + \beta + \tbinom{q}{1}\beta.$$

Since $k + 2 = l + n$ and n is even, $\binom{k}{1} = \binom{l}{1}$, and thus

$$w_1(\mathbb{R}\mathrm{P}(\eta)) + \tbinom{k}{1} c = (\tbinom{p}{1} + 1)\alpha \quad \text{and} \quad w_1(\mathbb{R}\mathrm{P}(\xi)) + \tbinom{l}{1} d = (\tbinom{q}{1} + 1)\beta$$

are corresponding characteristic classes. Because $n > 2$, it follows that

$$\begin{aligned} 0 = \big(\tbinom{p}{1} + 1\big)\alpha^n c^{l-1}[\mathbb{R}\mathrm{P}(\eta)] &= \big(\tbinom{q}{1} + 1\big)\beta^n d^{l-1}[\mathbb{R}\mathrm{P}(\xi)] \\ &= \big(\tbinom{q}{1} + 1\big)\beta^n[\mathbb{R}\mathrm{P}^n] = \tbinom{q}{1} + 1, \end{aligned}$$

which gives that q is odd. Also

$$\tbinom{p}{1} + 1 = \big(\tbinom{p}{1} + 1\big)\alpha^2 c^{k-1}[\mathbb{R}\mathrm{P}(\eta)] = (\tbinom{q}{1} + 1)\beta^2 d^{k-1}[\mathbb{R}\mathrm{P}(\xi)] = 0,$$

and p is odd. □

Fact 2 If $l = 1$, then $n = 4$, $W(\eta) = 1 + \alpha$ and $W(\xi) = 1 + \beta$.

Proof Since $l = 1$ and $w_1(\xi) = \beta$, we have $W(\xi) = 1 + \beta$. Then the involution $(M, T) \cup (\mathbb{R}\mathrm{P}^{n+1}, T_{0,n})$ is cobordant to an involution with fixed data

$$(\eta \to \mathbb{R}\mathrm{P}^2) \cup ((n+1)R \to \{\text{point}\}).$$

From [16] and the fact that $h_{0,2} = 2$, we have $W(\eta) = 1 + \alpha$ and $n = 4$. □

Fact 2 reduces Lemma 3 to the following assertion: if $l > 1$, then $W(\eta) = (1+\alpha)^{n+1}$ and $W(\xi) = (1+\beta)^3$; so we assume throughout the remainder of this section that $l > 1$. Note that $(1+\alpha)^{n+1} = (1+\alpha)^3$ if $\binom{n}{2} = 1$ and $(1+\alpha)^{n+1} = 1+\alpha$ if $\binom{n}{2} = 0$. Denote by r the greatest power of 2 that appears in the 2–adic expansion of n, that is, $4 \le 2^r \le n < 2^{r+1}$. We can assume $q < 2^{r+1}$ and $p < 4$. Then Fact 3 and Fact 4 show that $W(\eta) = (1+\alpha)^{n+1}$:

Fact 3 If $\binom{n}{2} = 1$, then $p = 3$.

Fact 4 If $\binom{n}{2} = 0$, then $p = 1$.

Set $p' = 4 - p$, $q' = 2^{r+1} - q$. Then the dual Stiefel–Whitney classes of η and ξ are given by $\overline{W}(\eta) = (1+\alpha)^{p'}$, $\overline{W}(\xi) = (1+\beta)^{q'}$. Since p and q are odd, p' and q' are odd; further, $\binom{p}{2} + \binom{p'}{2} = 1$ and $\binom{q}{2^u} + \binom{q'}{2^u} = 1$ for each $1 \le u \le r$.

Proof of Fact 3 We will use several times the fact that a binomial coefficient $\binom{a}{b}$ is nonzero modulo 2 if and only if the 2–adic expansion of b is a subset of the 2–adic expansion of a. We have $n = 4j+2$, with $j \ge 1$, and want to show that $p = 3$; since $p < 4$ is odd, it suffices to show that $\binom{p}{2} = 1$, or equivalently that $\binom{p'}{2} = 0$. Suppose by contradiction that $\binom{p'}{2} = 1$. By Conner's formula (1),

$$c^{k+1}[\mathbb{R}\mathrm{P}(\eta)] = \binom{p'}{2}\alpha^2[\mathbb{R}\mathrm{P}^2] = \binom{p'}{2} = d^{k+1}[\mathbb{R}\mathrm{P}(\xi)] = \binom{q'}{4j+2}.$$

Then $\binom{q'}{4j+2} = 1$ and consequently $\binom{q'}{2} = 1$. We formally introduce the class (with $l - 1 \ge 1$)

$$\widetilde{W}(\mathbb{R}\mathrm{P}(\ )) = \frac{W(\mathbb{R}\mathrm{P}(\ ))}{(1+c)^{l-1}}.$$

Since $k = l + 4j$ and p and q are odd, on $\mathbb{R}\mathrm{P}^2$ this class is

$$\widetilde{W}(\mathbb{R}\mathrm{P}(\eta)) = (1+\alpha)^3(1+c^4)^j(1+c+\alpha+(1+c)^{-1}\tbinom{p}{2}\alpha^2),$$

and on $\mathbb{R}\mathrm{P}^n$ it is

$$\widetilde{W}(\mathbb{R}\mathrm{P}(\xi)) = (1+\beta)^{4j+3}(1+d+\beta+(1+d)^{-1}\tbinom{q}{2}\beta^2+(1+d)^{-2}\tbinom{q}{3}\beta^3+\ldots).$$

Then $$\widetilde{w}_3(\mathbb{R}\mathrm{P}(\eta)) = \alpha^2 c + \tbinom{p}{2}\alpha^2 c = \tbinom{p'}{2}\alpha^2 c = \alpha^2 c,$$

and since $\binom{q}{2} + \binom{q}{3} = 0$ because q is odd, $\widetilde{w}_3(\mathbb{R}\mathrm{P}(\xi)) = \binom{q'}{2}\beta^2 d = \beta^2 d$. Now we observe that, if a and b are one-dimensional cohomology classes, then by the Cartan formula one has $\mathrm{Sq}^{2^u}(a^{2^u}b) = a^{2^{u+1}}b$, where Sq is the Steenrod operation and $u \ge 1$.

Also one has, by the Wu and Cartan formulae, that Sq^i evaluated on a product of characteristic classes gives a polynomial in the characteristic classes. Then

$$\mathrm{Sq}^{2^{r-1}}(\dots(\mathrm{Sq}^4(\mathrm{Sq}^2(\alpha^2 c)))\dots) = \alpha^{2^r} c$$

and
$$\mathrm{Sq}^{2^{r-1}}(\dots(\mathrm{Sq}^4(\mathrm{Sq}^2(\beta^2 d)))\dots) = \beta^{2^r} d$$

are corresponding classes on $\mathbb{R}\mathrm{P}^2$ and $\mathbb{R}\mathrm{P}^n$. Using Conner's formula (1) and the fact that $2^r \geq 4$, one then has

$$0 = (\alpha^{2^r} c)c^{4j+1-2^r+l-1}[\mathbb{R}\mathrm{P}(\eta)] = (\beta^{2^r} d)d^{4j+1-2^r+l-1}[\mathbb{R}\mathrm{P}(\xi)] = \binom{q'}{4j+2-2^r}.$$

Since $\binom{q'}{4j+2} = 1$ and 2^r belongs to the 2–adic expansion of $4j+2$, also $\binom{q'}{4j+2-2^r} = 1$, which is impossible. Hence Fact 3 is proved. □

Proof of Fact 4 We consider $n = 4j$ with $j \geq 1$; in this case, to show that $p = 1$, it suffices to show that $\binom{p'}{2} = 1$, and again by contradiction we suppose $\binom{p'}{2} = 0$. Then $\binom{p}{2} = 1$ and $k = l + 4j - 2$ gives

$$\widetilde{W}(\mathbb{R}\mathrm{P}(\eta)) = (1+\alpha)^3((1+c)^{4j-1} + (1+c)^{4j-2}\alpha + (1+c)^{4j-3}\alpha^2)$$

and $\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta)) = c^2 + \alpha^2 + c\alpha$. Also

$$\widetilde{W}(\mathbb{R}\mathrm{P}(\xi)) = (1+\beta)^{4j+1}(1 + d + \beta + (1+d)^{-1}\tbinom{q}{2}\beta^2 + (1+d)^{-2}\tbinom{q}{3}\beta^3 + \dots)$$

and $\widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi)) = \binom{q}{2}\beta^2 + \beta d + \beta^2$. Let 2^t be the lesser power of 2 of the 2–adic expansion of $n = 4j$ $(2^t \geq 4)$. For $t \leq x \leq r$ and with the same preceding tools, we then get

$$\begin{aligned}
&\mathrm{Sq}^{2^{x-1}}(\dots(\mathrm{Sq}^4(\mathrm{Sq}^2(\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta))c)))\dots)c^{4j+l-2^x-2}[\mathbb{R}\mathrm{P}(\eta)] \\
&\qquad = (c^{2^x}c + \alpha^{2^x}c + c^{2^x}\alpha)c^{4j+l-2^x-2}[\mathbb{R}\mathrm{P}(\eta)] \\
&\qquad = \tbinom{p'}{2} + 0 + \tbinom{p'}{1} \\
&\qquad = 1 \\
&\qquad = \mathrm{Sq}^{2^{x-1}}(\dots(\mathrm{Sq}^4(\mathrm{Sq}^2(\widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi))d)))\dots)d^{4j+l-2^x-2}[\mathbb{R}\mathrm{P}(\xi)] \\
&\qquad = (\tbinom{q}{2}\beta^{2^x}d + \beta d^{2^x} + \beta^{2^x}d)d^{4j+l-2^x-2}[\mathbb{R}\mathrm{P}(\xi)] \\
&\qquad = \tbinom{q}{2}\tbinom{q'}{4j-2^x} + \tbinom{q'}{4j-1} + \tbinom{q'}{4j-2^x} = \tbinom{q'}{2}\tbinom{q'}{4j-2^x} + \tbinom{q'}{4j-1},
\end{aligned}$$

$$0 = \tbinom{p'}{2} = c^{k+1}[\mathbb{R}\mathrm{P}(\eta)] = d^{k+1}[\mathbb{R}\mathrm{P}(\xi)] = \tbinom{q'}{4j}$$

and $\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta))c^{4j+l-3}[\mathbb{R}\mathrm{P}(\eta)] = \binom{p'}{2} + 1 + \binom{p'}{1} = 0$

$$= \widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi)d^{4j+l-3}[\mathbb{R}\mathrm{P}(\xi)]$$
$$= \binom{q}{2}\binom{q'}{4j-2} + \binom{q'}{4j-1} + \binom{q'}{4j-2}$$
$$= \binom{q'}{2}\binom{q'}{4j-2} + \binom{q'}{4j-1}.$$

That is, we get the equations:

$$0 = \binom{q'}{4j} \tag{2}$$

$$0 = \binom{q'}{2}\binom{q'}{4j-2} + \binom{q'}{4j-1} \tag{3}$$

$$1 = \binom{q'}{2}\binom{q'}{4j-2^x} + \binom{q'}{4j-1} \tag{4}$$

By using equations (3) and (4), we conclude that $\binom{q'}{2} = 1$ and $\binom{q'}{4j-2^x} \neq \binom{q'}{4j-2}$. Suppose $t < r$. If $\binom{q'}{4j-2^r} = 1$, equation (2) and the fact that 2^r belongs to the 2–adic expansion of $4j$ imply that 2^r is the only power of 2 of the 2–adic expansion of $4j$ that does not belong to the 2–adic expansion of q'. Hence $\binom{q'}{4j-2^t} = 0$, which is a contradiction. Then $\binom{q'}{4j-2^r} = \binom{q'}{4j-2^t} = 0$. In this case, equation (2) and $\binom{q'}{4j-2} = 1$ give that 2^t is the only power of 2 of the 2–adic expansion of $4j$ that does not belong to the 2–adic expansion of q', giving the contradiction $\binom{q'}{4j-2^t} = 1$. Now suppose $t = r$, that is, $n = 4j = 2^r$. One has

$$(\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta))^2 c^{2^r+l-5}[\mathbb{R}\mathrm{P}(\eta)] = \binom{p'}{2} + 0 + 1$$
$$= 1 = (\widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi))^2 d^{2^r+l-5}[\mathbb{R}\mathrm{P}(\xi)]$$
$$= \binom{q}{2}\binom{q'}{2^r-4} + \binom{q'}{2^r-2} + \binom{q'}{2^r-4}$$
$$= \binom{q'}{2}\binom{q'}{2^r-4} + \binom{q'}{2^r-2} = \binom{q'}{2^r-4} + \binom{q'}{2^r-2}.$$

Since $\binom{q'}{2} = 1$, we have $\binom{q'}{2^r-4} = \binom{q'}{2^r-2}$, which gives a contradiction. Thus Fact 4 is proved. □

Now we prove that $q = 3$. To do this, first we prove:

Fact 5 $\binom{q}{2} = 1$; in particular, $q \geq 3$.

Proof As before, first consider $n = 4j + 2$, with $j \geq 1$. In this case, we know that $0 = \binom{p'}{2} = \binom{q'}{4j+2}$, $\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta)) = \binom{p}{2}\alpha^2 + \alpha c = \alpha^2 + \alpha c$ and $\widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi)) = \binom{q}{2}\beta^2 + \beta d$. Then

$$(\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta)))^2 c^{4j+l-3}[\mathbb{R}\mathrm{P}(\eta)] = 1$$
$$= (\widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi)))^2 d^{4j+l-3}[\mathbb{R}\mathrm{P}(\xi)] = \binom{q}{2}\binom{q'}{4j-2} + \binom{q'}{4j}.$$

Since the sum $\binom{q}{2}+\binom{q'}{2}$ equals 1 and 2 belongs to the 2–adic expansion of $4j-2$, one has that $\binom{q}{2}\binom{q'}{4j-2}=0$, and thus $\binom{q'}{4j}=1$. Now $\binom{q'}{4j+2}=0$ and $\binom{q'}{4j}=1$ imply that $\binom{q'}{2}=0$, and thus $\binom{q}{2}=1$. Since q is odd, this means that $q \geq 3$.

Now suppose $n=4j$, with $j \geq 1$. One then has $\binom{p'}{2}=1$, $\widetilde{w}_3(\mathbb{R}\mathrm{P}(\eta))=c^3+\binom{p'}{2}\alpha^2 c = c^3+\alpha^2 c$ and $\widetilde{w}_3(\mathbb{R}\mathrm{P}(\xi))=\binom{q}{2}\beta^2 d$. Then

$$\begin{aligned}
\mathrm{Sq}^{2^{r-1}}(\ldots(\mathrm{Sq}^4(\mathrm{Sq}^2(\widetilde{w}_3(\mathbb{R}\mathrm{P}(\eta))))\ldots)c^{4j+l-2^r-2}[\mathbb{R}\mathrm{P}(\eta)]&\\
&= (c^{2^r}c+\alpha^{2^r}c)c^{4j+l-2^r-2}[\mathbb{R}\mathrm{P}(\eta)]\\
&= \tbinom{p'}{2}=1\\
&= \mathrm{Sq}^{2^{r-1}}(\ldots(\mathrm{Sq}^4(\mathrm{Sq}^2(\tbinom{q}{2}\beta^2 d))))\ldots)d^{4j+l-2^r-2}[\mathbb{R}\mathrm{P}(\xi)]\\
&= (\tbinom{q}{2}\beta^{2^r}d)d^{4j+l-2^r-2}[\mathbb{R}\mathrm{P}(\xi)]=\tbinom{q}{2}\tbinom{q'}{4j-2^r}.
\end{aligned}$$

Thus $\binom{q}{2}=1$, and Fact 5 is proved. □

To end our task, we will show that $q \leq 3$. The strategy will consist in finding nonzero characteristic numbers coming from characteristic classes involving α^{q-1}. To do this, we need the following:

Fact 6 $n+l-1>2(q-1)$.

Proof First suppose $n=4j+2$, $j \geq 1$. From the proof of Fact 5, $\binom{q'}{4j}=1$, and thus $\binom{q'}{2^r}=1$ and $\binom{q}{2^r}=0$. Since $q<2^{r+1}$, $q<2^r<4j+2$. In particular, $w_q(\xi)=\alpha^q \neq 0$ and $q \leq l$. Then $n+l-1=4j+2+l-1>2q-1>2(q-1)$. Now suppose $n=4j$, $j \geq 1$. In this case, $\binom{p'}{2}=1=\binom{q'}{4j}$, so the argument is the same. □

Fact 6 says that we can consider characteristic numbers coming from classes involving $\widetilde{w}_2^{q-1}$; in this direction, first consider $n=4j+2$, $j \geq 1$. In this case,

$$\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta))=\tbinom{p}{2}\alpha^2+\alpha c=\alpha(\alpha+c) \quad \text{and} \quad \widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi))=\tbinom{q}{2}\beta^2+\beta d=\beta(\beta+d).$$

Thus

$$(\alpha^{q-1}(\alpha+c)^{q-1}c^{4j+l-2q+3})[\mathbb{R}\mathrm{P}(\eta)]=(\beta^{q-1}(\beta+d)^{q-1}d^{4j+l-2q+3})[\mathbb{R}\mathrm{P}(\xi)].$$

The last term is the coefficient of β^{4j+2} in $\beta^{q-1}(1+\beta)^{q-1}(1+\beta)^{q'}$, by Conner's formula (1). If $n=4j$, $j \geq 1$, similarly one has

$$\begin{aligned}
\widetilde{w}_2(\mathbb{R}\mathrm{P}(\eta))+c^2&=(c^2+\tbinom{p}{2}\alpha^2+\alpha c)+c^2=\alpha c,\\
\widetilde{w}_2(\mathbb{R}\mathrm{P}(\xi))+d^2&=\tbinom{q'}{2}\beta^2+\beta d+d^2=(\beta+d)d,\\
((\alpha^{q-1}c^{q-1})c^{4j+l-2q+1})[\mathbb{R}\mathrm{P}(\eta)]&=(((\beta+d)d)^{q-1}d^{4j+l-2q+1})[\mathbb{R}\mathrm{P}(\xi)],
\end{aligned}$$

and the last term is the coefficient of β^{4j} in $(1+\beta)^{q-1}(1+\beta)^{q'}$. These numbers have value 1, since $(1+\beta)^{q-1}(1+\beta)^{q'} = (1+\beta)^{-1}$, which means that $\alpha^{q-1} \neq 0$ and $q-1 \leq 2$, thus ending the proof of Lemma 3.

3 Calculation of $h_{m,n}$

Denote by $\mathcal{W}_r$ the underlying manifold of $\Gamma^r(\mathbb{R}\mathrm{P}^{m+n+1}, T_{m,n})$ and by $\mathcal{P}_r$ the total space of the iterated fibration

$$\mathbb{R}\mathrm{P}((m+1)\mu_r \oplus (n+1)R) \to \mathbb{R}\mathrm{P}(\lambda_1 \oplus (r-1)R) \to \mathbb{R}\mathrm{P}^1,$$

where μ_r is the standard line bundle over $\mathbb{R}\mathrm{P}(\lambda_1 \oplus (r-1)R)$.

Lemma 4 *$\mathcal{W}_r$ is cobordant to $\mathcal{P}_r$.*

Proof If (W,T) is a free involution and $\lambda \to W/T$ is the usual line bundle, the sphere bundle $S(\lambda \oplus R)$ with the antipodal involution in the fibers can be identified to the free involution

$$\Big(\frac{W \times S^1}{T \times c}, \tau\Big),$$

where c is complex conjugation and τ is induced by $\mathrm{Id} \times -\mathrm{Id}$. Starting with $(S^1, -\mathrm{Id})$ and by iteratively applying this fact, we can see that $\mathcal{W}_r$ is diffeomorphic to the total space of the iterated fibration

$$\mathbb{R}\mathrm{P}((m+1)\xi_r \oplus (n+1)R) \to \mathbb{R}\mathrm{P}(\xi_{r-1} \oplus R) \to \ldots \to \mathbb{R}\mathrm{P}(\xi_2 \oplus R) \to \mathbb{R}\mathrm{P}(\xi_1 \oplus R) \to \mathbb{R}\mathrm{P}^1,$$

where $\xi_1 = \lambda_1$ and ξ_i is the standard line bundle over $\mathbb{R}\mathrm{P}(\xi_{i-1} \oplus R)$, for each $i > 1$. From [4], one knows that $\mathcal{N}_*(BO(1))$ is a free $\mathcal{N}_*$–module, where $\mathcal{N}_*$ is the unoriented cobordism ring, with one generator X_j in each dimension $j \geq 0$; these generators are characterized by the fact that $c^j[V^j] = 1$, where $\lambda \to V^j$ is a representative of X_j and c is the first Whitney class of λ. Further, it was shown by Conner in [3, Theorem 24.5] that there is a unique basis $\{X_j\}_{j=0}^{\infty}$ for $\mathcal{N}_*(BO(1))$ which satisfies two conditions:

(i) $\triangle(X_j) = X_{j-1}$, $j \geq 1$, where $\triangle\colon \mathcal{N}_j(BO(1)) \to \mathcal{N}_{j-1}(BO(1))$ is the Smith homomorphism.

(ii) If $\lambda \to V^j$ is a representative of X_j for $j \geq 1$, then V^j bounds.

Theorem 24.5 of [3] also showed that $X_1 = [\xi_1 \to \mathbb{R}\mathrm{P}^1]$ and $X_j = [\xi_j \to \mathbb{R}\mathrm{P}(\xi_{j-1} \oplus R)]$ for $j \geq 2$. For $j \geq 1$, set $Y_j = [\mu_j \to \mathbb{R}\mathrm{P}(\lambda_1 \oplus (j-1)R)]$. One has

$$c^j[\mathbb{R}\mathrm{P}(\lambda_1 \oplus (j-1)R)] = \overline{w}_1(\lambda_1)[S^1] = 1,$$
$$Y_1 = X_1$$

and $\quad \triangle([\mu_j \to \mathbb{R}\mathrm{P}(\lambda_1 \oplus (j-1)R)]) = [\mu_{j-1} \to \mathbb{R}\mathrm{P}(\lambda_1 \oplus (j-2)R)] \quad$ for $j \geq 2$.

Further, every projective space bundle over S^1 bounds [5, Lemma 2.2]. By the uniqueness, $Y_j = X_j$ for $j \geq 1$, and the result follows. □

With the Lemma 4 in hand, Theorem 2 can now be rephrased:

Theorem 2′ *For m, n even, $0 \leq m < n$, write $n - m = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd.*

(a) *If $p = 1$, $\mathcal{P}_1$ bounds and $\mathcal{P}_2$ does not bound.*

(b) *If $p > 1$, $\mathcal{P}_r$ bounds for each $1 \leq r \leq 2^p - 2$ and $\mathcal{P}_{2^p-1}$ does not bound.*

Denote by $\alpha \in H^1(\mathbb{R}\mathrm{P}^1, Z_2)$ the generator and by $\theta_r \to \mathcal{P}_r$ the standard line bundle; set $W(\mu_r) = 1 + c$ and $W(\theta_r) = 1 + d$. The following lemma, which follows from Conner's formula (1), will be useful in our computations:

Lemma 5 (i) *For $f + g + h = m + n + 1 + r$, $c^f (c+d)^g d^h[\mathcal{P}_r]$ is the coefficient of c^r in $(c^f(1+c)^g)/((1+c)^{m+1})$.*

(ii) *For $f + g + h = m + n + r$, $\alpha c^f (c+d)^g d^h[\mathcal{P}_r]$ is the coefficient of c^r in $(c^{f+1}(1+c)^g)/((1+c)^{m+1})$.*

If M is a closed manifold and $(1+t_1)(1+t_2)\ldots(1+t_l)$ is the factored form of $W(M)$, one has the s–class s_j given by the polynomial in the classes of M corresponding to the symmetric function $t_1^j + t_2^j + \ldots + t_l^j$. Since

$$W(\mathcal{P}_r) = (1+c+\alpha)(1+c)^{r-1}(1+c+d)^{m+1}(1+d)^{n+1},$$

$c^i = 0$ if $i > r$ and $\alpha^i = 0$ if $i > 1$, the s–class $s_{m+n+1+r}$ of $\mathcal{P}_r$ then is

$$\begin{aligned} s_{m+n+1+r} = \; & (c+\alpha)^{m+n+1+r} + (r-1)c^{m+n+1+r} \\ & + (m+1)(c+d)^{m+n+1+r} + (n+1)d^{m+n+1+r} \\ = \; & (c+d)^{m+n+1+r} + d^{m+n+1+r}. \end{aligned}$$

Using part (i) of Lemma 5 and the fact that

$$\frac{1}{(1+c)^{m+1}} = 1 + \sum_{i=1}^{r} \binom{m+i}{i} c^i$$

in $H^*(\mathcal{P}_r, Z_2)$, one then has

$$\begin{aligned} s_{m+n+1+r}[\mathcal{P}_r] &= \text{coefficient of } c^r \text{ in } (1+c)^{n+r} + \text{coefficient of } c^r \text{ in} \frac{1}{(1+c)^{m+1}} \\ &= \binom{n+r}{r} + \binom{m+r}{r}. \end{aligned}$$

Because $n = 2^p q + m$ and q is odd, one then gets

$$s_{m+n+1+2^p}[\mathcal{P}_{2^p}] = \binom{n+2^p}{2^p} + \binom{m+2^p}{2^p} = 1.$$

It follows that $\mathcal{P}_{2^p}$ does not bound. Because $\mathcal{P}_1$ is a projective space bundle over S^1 and hence a boundary, this in particular proves part (a) of Theorem $2'$. So we can assume from now that $p > 1$ and $r < 2^p$. Using again $n = 2^p q + m$, we rewrite $W(\mathcal{P}_r)$ as

$$W(\mathcal{P}_r) = (1+c+\alpha)(1+c)^{r-1}(1+c+d(c+d))^{m+1}(1+d^{2^p})^q.$$

Then a general characteristic number of $\mathcal{P}_r$ is a sum of terms of the form

$$\alpha^e c^f (d(c+d))^g d^{2^p h}[\mathcal{P}_r],$$

where $e + f + 2g + 2^p h = m + n + 1 + r$ and either $e = 0$ or $e = 1$. Since by Lemma 5,

$$\alpha^e c^f (d(c+d))^g d^{2^p h}[\mathcal{P}_r] = c^{f+1}(d(c+d))^g d^{2^p h}[\mathcal{P}_r],$$

we can assume $e = 0$. Thus, to prove the first statement of part (b) of Theorem $2'$, it suffices to show that $c^f(d(c+d))^g d^{2^p h}[\mathcal{P}_r] = 0$ when $f + 2g + 2^p h = m + n + 1 + r$ and $r < 2^p - 1$. Since $c^f = 0$ if $f > r$, we assume $f \le r$ and thus $0 \le r - f < 2^p - 1$. Take $s > p$ with $2^s > m + 1$; in particular, $2^s > 2^p > r$ and $1/((1+c)^{m+1}) = (1+c)^{2^s - m - 1}$. Then

$$\begin{aligned} c^f (d(c+d))^g d^{2^p h}[\mathcal{P}_r] &= \text{coefficient of } c^r \text{ in } c^f (1+c)^g/(1+c)^{m+1} \\ &= \text{coefficient of } c^r \text{ in } c^f (1+c)^g (1+c)^{2^s - m - 1} \\ &= \binom{2^s + g - m - 1}{r - f} \\ &= \binom{2^{p-1}(2^{s-p+1} + q - h) + (r - f + 1)/2 - 1}{r - f}. \end{aligned}$$

Write $r - f + 1 = 2^t a$, where a is odd. Since $r - f + 1 = 2g + 2^p h - m - n$ is even and $r - f + 1 < 2^p$, one has $1 \le t \le p - 1$. Then 2^{t-1} belongs to the 2–adic expansion of $r - f$ and does not belong to the 2–adic expansion of

$$2^{p-1}(2^{s-p+1} + q - h) + (r - f + 1)/2 - 1,$$

which means, as required, that the above number is zero.

Finally, we must to show that $\mathcal{P}_{2^p-1}$ does not bound. One has

$$w_2(\mathcal{P}_{2^p-1}) = \alpha c + \binom{m+1}{2} c^2 + d(c+d).$$

We have seen above that $c^f(d(c+d))^g d^{2^p h}[\mathcal{P}_r] = 0$ for $f+2g+2^p h = m+n+1+r$ and $0 \le r - f < 2^p - 1$; in particular, this is true for $r = 2^p - 1$ and $f > 0$. In this way,

$$\begin{aligned} w_2(\mathcal{P}_{2^p-1})^{\frac{m+n+2^p}{2}}[\mathcal{P}_{2^p-1}] &= (d(c+d))^{\frac{m+n+2^p}{2}}[\mathcal{P}_{2^p-1}] \\ &= \text{coefficient of } c^{2^p-1} \text{ in } ((1+c)^{\frac{m+n+2^p}{2}})/(1+c)^{m+1} \\ &= \text{coefficient of } c^{2^p-1} \text{ in } (1+c)^{\frac{n-m}{2}+2^{p-1}-1} \\ &= \binom{2^{p-1}q+2^{p-1}-1}{2^p-1} = 1, \end{aligned}$$

and $\mathcal{P}_{2^p-1}$ does not bound.

4 Z_2^k–actions fixing $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^{\text{even}}$

Let F^n be a connected, smooth and closed n–dimensional manifold satisfying the following property, which we call *property* $\mathcal{H}$: if N^m is any smooth and closed m–dimensional manifold with $m > n$ and $T\colon N^m \to N^m$ is a smooth involution whose fixed point set is F^n, then $m = 2n$. From [8], this implies that (N^m, T) is cobordant to the *twist involution* $(F^n \times F^n, t)$, given by $t(x,y) = (y,x)$. This concept was introduced and studied in Pergher and Oliveira [14], inspired by Conner and Floyd [4, 27.6] (or Conner [3, 29.2]), where it was shown that $\mathbb{R}\mathrm{P}^{\text{even}}$ has this property.

In [13], we studied the equivariant cobordism classification of smooth actions $(M;\Phi)$ of the group Z_2^k on closed and smooth manifolds M for which the fixed point set F of the action is the union $F = K \cup L$, where K and L are submanifolds of M with property $\mathcal{H}$ and with $\dim(K) < \dim(L)$. We showed that, for this F, the Z_2^k–classification is completely determined by the corresponding Z_2–classification. Specifically, the equivariant cobordism classes of Z_2^k–actions fixing $K \cup L$ can be represented by a special set of Z_2^k–actions which are explicitly obtained from involutions fixing $K \cup L$, K and L. Together with the results of Section 2 and Section 3 and the case $F = \mathbb{R}\mathrm{P}^{\text{even}}$, this gives a precise cobordism description of the Z_2^k–actions fixing $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^n$, where $n > 2$ is even; next we give this description.

Here, Z_2^k is the group generated by k commuting involutions $T_1, T_2, \ldots, T_k$. The *fixed data* of a Z_2^k–action $(M;\Phi)$, $\Phi = (T_1, T_2, \ldots, T_k)$, is $\eta = \bigoplus_\rho \varepsilon_\rho \to F$, where $F = \{x \in M \ / \ T_i(x) = x \text{ for all } 1 \le i \le k\}$ is the fixed point set of Φ and $\eta = \bigoplus_\rho \varepsilon_\rho$ is the normal bundle of F in M, decomposed into eigenbundles ε_ρ with ρ running through the 2^k-1 nontrivial irreducible representations of Z_2^k. A collection of Z_2^k–actions fixing F can be obtained from an involution fixing F through the following procedure: let (W,T) be any involution. For each r with $1\le r\le k$, consider the Z_2^k–action $\Gamma_r^k(W,T)$, defined on the cartesian product $W^{2^{r-1}}=W\times\ldots\times W$ (2^{r-1} factors) and described in the following inductive way: first set $\Gamma_1^1(W,T)=(W,T)$. Taking $k\ge 2$ and supposing by inductive hypothesis one has constructed $\Gamma_{k-1}^{k-1}(W,T)$, define

$$\Gamma_k^k(W,T)=(W^{2^{k-1}};T_1,T_2,\ldots,T_k),$$

where
$$\begin{aligned}(W^{2^{k-1}};T_1,T_2,\ldots,T_{k-1})&=(W^{2^{k-2}}\times W^{2^{k-2}};T_1,T_2,\ldots,T_{k-1})\\&=\Gamma_{k-1}^{k-1}(W,T)\times\Gamma_{k-1}^{k-1}(W,T),\end{aligned}$$

and T_k acts switching $W^{2^{k-2}}\times W^{2^{k-2}}$. This defines $\Gamma_k^k(W,T)$ for any $k\ge 1$. Next, define

$$\Gamma_r^k(W,T)=(W^{2^{r-1}};T_1,T_2,\ldots,T_k)$$

setting
$$(W^{2^{r-1}};T_1,T_2,\ldots,T_r)=\Gamma_r^r(W,T)$$

and letting $T_{r+1},\ldots,T_k$ act trivially.

If (W,T) fixes F and if $\eta\to F$ is the normal bundle of F in W, then $\Gamma_r^k(W,T)$ fixes F and its fixed data consists of 2^{r-1} copies of η, $2^{r-1}-1$ copies of the tangent bundle of F and 2^k-2^r copies of the zero-dimensional bundle over F. In particular, for the twist involution $(F\times F,t)$, we have $\Gamma_r^k(F\times F,t)=(F^{2^r};T_1,T_2,\ldots,T_k)$, where $(T_1,T_2,\ldots,T_r)$ is the usual twist Z_2^r–action on F^{2^r} which interchanges factors and $T_{r+1},\ldots,T_k$ act trivially, with the fixed data having in this case 2^r-1 copies of the tangent bundle of F and 2^k-2^r zero bundles. In this special case, we allow r to be zero, setting $\Gamma_0^k(F\times F,t)=(F;T_1,T_2,\ldots,T_k)$, where each T_i is the identity involution.

Now, from a given Z_2^k–action $(M;\Phi)$, $\Phi=(T_1,\ldots,T_k)$, we can obtain a collection of new Z_2^k–actions, described as follows: first, each automorphism $\sigma\colon Z_2^k\to Z_2^k$ yields a new action given by $(M;\sigma(T_1),\ldots,\sigma(T_k))$; we denote this action by $\sigma(M;\Phi)$. The fixed data of $\sigma(M;\Phi)$ is obtained from the fixed data of $(M;\Phi)$ by a permutation of eigenbundles, obviously depending on σ. Next, it was shown in [12] that if $(M;\Phi)$ has fixed data $\bigoplus_\rho\varepsilon_\rho\to F$ and one of the eigenbundles ε_θ is isomorphic to $\varepsilon'_\theta\oplus R$, then there is an action $(N;\Psi)$ with fixed data $\bigoplus_\rho\mu_\rho\to F$, where $\mu_\rho=\varepsilon_\rho$ if $\rho\ne\theta$ and $\mu_\theta=\varepsilon'_\theta$. We say in this case that $(N;\Psi)$ is obtained from $(M;\Phi)$ *by removing one section*. Thus, the iterative process of removing sections may possibly enlarge the set $\{\sigma(M;\Phi),\ \sigma\in\operatorname{Aut}(Z_2^k)\}$. Summarizing, from a given involution (W,T) that fixes

F, we obtain a collection of Z_2^k–actions fixing F by applying the operations $\sigma\Gamma_r^k$ on (W,T) and next by removing the (possible) sections from the resultant eigenbundles. The results of [13] say that when $F = K \cup L$, where K and L have property $\mathcal{H}$ and $\dim(K) < \dim(L)$, then up to equivariant cobordism, all Z_2^k–actions fixing F are obtained, with the above procedure, from involutions fixing $K \cup L$, K and L. Together with the Z_2–classification obtained in Section 2 and Section 3 and the case $F = \mathbb{R}\mathrm{P}^{\mathrm{even}}$, this gives the following Z_2^k–classification for $F = \mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^n$, where $n > 2$ is even (in our terminology, we agree that *the set obtained from* $(M;\Phi)$ *by removing sections* includes $(M;\Phi)$):

Theorem 6 *Let $(M;\Phi)$ be a Z_2^k–action fixing $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^n$, where $n > 2$ is even. Then $(M;\Phi)$ is equivariantly cobordant to an action belonging to the set $A \cup B$, where the sets A and B are described below in terms of n.*

(i) *$n - 2 = 2^p q$, with q odd and $p > 1$:*

$A = \varnothing$ = the empty set;

$B =$ the set obtained from $\{\sigma\Gamma_r^k\Gamma^{2^p-1}(\mathbb{R}\mathrm{P}^{n+3}, T_{2,n}),\ \sigma \in \mathrm{Aut}(Z_2^k),\ 1 \le r \le k\}$ by removing sections.

(ii) *$n - 2 = 2q$, with q odd, and n is not a power of 2:*

$A = \varnothing$;

$B =$ the set obtained from $\{\sigma\Gamma_r^k\Gamma^2(\mathbb{R}\mathrm{P}^{n+3}, T_{2,n}),\ \sigma \in \mathrm{Aut}(Z_2^k),\ 1 \le r \le k\}$ by removing sections;

(iii) *$n = 2^t$ is a power of 2 with $t \ge 3$:*

$$A = \big\{\sigma\Gamma_r^k(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, \mathrm{twist}) \cup \sigma'\Gamma_{r-t+1}^k(\mathbb{R}\mathrm{P}^{2^t} \times \mathbb{R}\mathrm{P}^{2^t}, \mathrm{twist}),\quad \sigma,\sigma' \in \mathrm{Aut}(Z_2^k),\ t-1 \le r \le k\big\};$$

$B =$ the set obtained from $\{\sigma\Gamma_r^k\Gamma^2(\mathbb{R}\mathrm{P}^{2^t+3}, T_{2,2^t}),\ \sigma \in \mathrm{Aut}(Z_2^k),\ 1 \le r \le k\}$ by removing sections (by dimensional reasons, in this case $A = \varnothing$ if $t - 1 > k$);

(iv) $n = 4$: *for $(W^5, T) = \Gamma^2(\mathbb{R}\mathrm{P}^3, T_{0,2}) \cup (\mathbb{R}\mathrm{P}^5, T_{0,4})$,*

$$A = \big\{\sigma\Gamma_{r+1}^k(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, \mathrm{twist}) \cup \sigma'\Gamma_r^k(\mathbb{R}\mathrm{P}^4 \times \mathbb{R}\mathrm{P}^4, \mathrm{twist}),\quad \sigma,\sigma' \in \mathrm{Aut}(Z_2^k),\ 0 \le r \le k-1\big\} \cup \{\sigma\Gamma_r^k(W^5, T),\ \sigma \in \mathrm{Aut}(Z_2^k),\ 1 \le r \le k\};$$

$B =$ the set obtained from $\{\sigma\Gamma_r^k\Gamma^2(\mathbb{R}\mathrm{P}^7, T_{2,4}),\ \sigma \in \mathrm{Aut}(Z_2^k),\ 1 \le r \le k\}$ by removing sections.

References

[1] **J M Boardman**, *On manifolds with involution*, Bull. Amer. Math. Soc. 73 (1967) 136–138 MR0205260

[2] **P E Conner**, *The bordism class of a bundle space*, Michigan Math. J. 14 (1967) 289–303 MR0227995

[3] **P E Conner**, *Differentiable periodic maps*, second edition, Lecture Notes in Mathematics 738, Springer, Berlin (1979) MR548463

[4] **P E Conner**, **E E Floyd**, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete 33, Academic Press, New York (1964) MR0176478

[5] **P E Conner**, **E E Floyd**, *Fibring within a cobordism class*, Michigan Math. J. 12 (1965) 33–47 MR0179796

[6] **D Hou**, **B Torrence**, *Involutions fixing the disjoint union of odd-dimensional projective spaces*, Canad. Math. Bull. 37 (1994) 66–74 MR1261559

[7] **D Hou**, **B Torrence**, *Involutions fixing the disjoint union of copies of even projective space*, Acta Math. Sinica (N.S.) 12 (1996) 162–166 MR1458687

[8] **C Kosniowski**, **R E Stong**, *Involutions and characteristic numbers*, Topology 17 (1978) 309–330 MR516213

[9] **Z Lü**, *Involutions fixing* $\mathbb{RP}^{\mathrm{odd}} \sqcup P(h,i)$ *I*, Trans. Amer. Math. Soc. 354 (2002) 4539–4570 MR1926888

[10] **Z Lü**, *Involutions fixing* $\mathbb{RP}^{\mathrm{odd}} \sqcup P(h,i)$ *II*, Trans. Amer. Math. Soc. 356 (2004) 1291–1314 MR2034310

[11] **P L Q Pergher**, *Involutions fixing an arbitrary product of spheres and a point*, Manuscripta Math. 89 (1996) 471–474 MR1383526

[12] **P L Q Pergher**, $(Z_2)^k$*–actions whose fixed data has a section*, Trans. Amer. Math. Soc. 353 (2001) 175–189 MR1783791

[13] **P L Q Pergher**, **R de Oliveira**, *Commuting involutions whose fixed point set consists of two special components*

[14] **P L Q Pergher**, **R de Oliveira**, Z_2^k*–actions with a special fixed point set*, Fund. Math. 186 (2005) 97–109 MR2162380

[15] **P L Q Pergher**, **R E Stong**, *Involutions fixing* (point)$\cup F^n$, Transform. Groups 6 (2001) 79–86 MR1825169

[16] **D C Royster**, *Involutions fixing the disjoint union of two projective spaces*, Indiana Univ. Math. J. 29 (1980) 267–276 MR563211

[17] **R E Stong**, *Involutions fixing projective spaces*, Michigan Math. J. 13 (1966) 445–447 MR0206979

[18] **R E Stong**, *Involutions fixing products of circles*, Proc. Amer. Math. Soc. 119 (1993) 1005–1008 MR1169050

Departamento de Ciências Exatas, Universidade Federal de Mato Grosso do Sul
Caixa Postal 210, Três Lagoas, MS 79603-011, Brazil

Departamento de Matemática, Universidade Federal de São Carlos
Caixa Postal 676, São Carlos, SP 13565-905, Brazil

Departamento de Matemática, Universidade Federal de São Carlos
Caixa Postal 676, São Carlos, SP 13565-905, Brazil

rogerio@ceul.ufms.br, pergher@dm.ufscar.br, aramos@dm.ufscar.br

Received: 30 October 2006