

Configuration space integral for long n -knots and the Alexander polynomial

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There is a higher dimensional analogue of the perturbative Chern–Simons theory in the sense that a similar perturbative series as in 3 dimensions, which is computed via configuration space integral, yields an invariant of higher dimensional knots (Bott–Cattaneo–Rossi invariant). This invariant was constructed by Bott for degree 2 and by Cattaneo–Rossi for higher degrees. However, its feature is yet unknown. In this paper we restrict the study to long ribbon n -knots and characterize the Bott–Cattaneo–Rossi invariant as a finite type invariant of long ribbon n -knots introduced by Habiro–Kanenobu–Shima [10]. As a consequence, we obtain a nontrivial description of the Bott–Cattaneo–Rossi invariant in terms of the Alexander polynomial.

[57Q45](#); [57M25](#)

1 Introduction

Witten [18] gave a path integral formulation of the Chern–Simons field theory which gives a framework for understanding many invariants of knots in a 3-manifold. But the path integral is not yet well defined mathematically. Though, the perturbative expansion method for the path integral gives a mathematical definition of knot invariants. It is defined by configuration space integral with certain uni-trivalent graphs called Jacobi diagrams. This construction of invariants has been studied by Axelrod–Singer [2], Bar-Natan [3], Guadagnini–Martellini–Mintchev [9], Kohno [12], Kontsevich [13], Bott–Taubes [5], Dylan Thurston [16] and others. It is known that the configuration space integral invariant of knots in S^3 is a universal finite type (or Vassiliev) invariant (see Altschuler–Freidel [1] and Thurston [16]).

There is a higher dimensional analogue of these theories with a certain variant of Jacobi diagrams, that is, isotopy invariants of long higher dimensional embeddings $\mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$ (long n -knots). The suitable diagrams for higher dimensions are graphs with two kinds of edges corresponding to the “angular forms” in \mathbb{R}^{n+2} and in \mathbb{R}^n respectively. The diagrams have degrees given by half the number of vertices. In his seminal paper, Bott [4] constructed an invariant for higher dimensional knots associated to degree 2 diagrams. After that Cattaneo and Rossi [7; 15] gave a path integral formula

for invariants of higher dimensional embedded manifolds into a manifold and, as a perturbative expansion, they obtained invariants for higher dimensional knots, which will be denoted by z_k in this paper, associated to degree- k diagrams. It may be an analogue of the 3-dimensional perturbative Chern–Simons theory and Bott’s invariant is the degree 2 part of Cattaneo–Rossi’s invariant. So we call z_k the Bott–Cattaneo–Rossi invariant, or the BCR invariant for short. More precisely, Cattaneo and Rossi proved that z_k (or its framing correction) is an isotopy invariant if n is odd (or $(n, k) = (2, 3)$). The BCR invariant currently seems to be a new invariant of long n -knots. However, it has not been known whether z_k is nontrivial or not. We try to understand its features, in particular, whether there is some connection with known invariants.

In this paper we study the invariant z_k restricting to a certain class of long n -knots called long ribbon n -knots. The class of long ribbon n -knots is a rather familiar class known to have similarities to the classical knot theory in 3-dimension. In particular, Habiro, Kanenobu and Shima [10] introduced the notion of finite type invariant of ribbon 2-knots, which is straightforwardly generalizable to long ribbon n -knots, and showed that the coefficients of the Alexander polynomial expanded as a power series in $t - 1$, are finite type invariants. Finite type invariant of type k is defined by the condition that a given invariant vanishes at any ‘ $(k+1)$ th order differential’ of a ribbon 2-knot. Moreover, Habiro and Shima [11] obtained a remarkable result saying that the set of all finite type invariants is isomorphic as a graded algebra to the polynomial algebra in the coefficients of the Alexander polynomial.

Restricting to long ribbon n -knots, we characterize the BCR invariant z_k as a finite type invariant (Theorem 4.1). In particular, we determine its ‘highest order term’ explicitly. As a consequence, we obtain a description in terms of the coefficients of the logarithm of the Alexander polynomial at $t = e^h$ with an explicit highest order term. In particular, it shows that the BCR invariant is nontrivial. The proof is done by choosing an embedding for a long ribbon n -knot in some extreme situation and then computing z_k explicitly at the k th order differentials of long ribbon n -knots. In the limiting situation, the computation of the integral is highly reduced and we can compute explicitly. The presence of two kinds of edges in the diagrams appearing in the definition of the BCR invariant makes the computation slightly complicated. But the spirit in the computation is similar to that of Altschuler and Friedel [1], and of Thurston [16].

This paper is organized as follows. In Section 2, the definition of the BCR invariant z_k is given and the Cattaneo–Rossi’s result concerning the invariance of their invariant is stated. The BCR invariant is defined as a linear combination of the integrals over the configuration spaces associated with certain graphs. In Section 3, we will explain the generality on the space of graphs from which each coefficient in z_k at a configuration

space integral of a graph are determined. This section is somewhat digressive but needed in the invariance proof of z_k in [Appendix A](#). In [Section 4](#), we restrict the study to long ribbon n -knots. The notion of Habiro, Kanenobu and Shima's finite type invariant is recalled and the main theorem ([Theorem 4.1](#)) about the characterization of z_k as a finite type invariant is stated and proved. The relation between z_k and the Alexander polynomial is explained at the end of [Section 4](#). [Appendix A](#) is devoted to the self-contained proof of invariance of the BCR invariant z_k , filling in the details not explicitly described by Rossi [[15](#)].

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2 Invariants of long n -knots

A *long n -knot* is the image of a smooth long embedding ψ of \mathbb{R}^n into \mathbb{R}^{n+2} that is standard near ∞ , that is, we assume that there exists an n -ball $D \subset \mathbb{R}^n$ such that $\psi|_{(\mathbb{R}^n \setminus D)} = i|_{(\mathbb{R}^n \setminus D)}$ where $i: \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$ is the natural inclusion.

In this section, we shall review the definition of the BCR invariant, which is an isotopy invariant of long n -knots. We review the definitions of

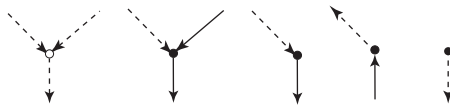
- Jacobi diagrams and weights w_k of them, and
- configuration space integral associated to a Jacobi diagram,

and define the BCR invariant as a linear combination of configuration space integrals for diagrams whose coefficients are their weights.

2.1 Jacobi diagrams

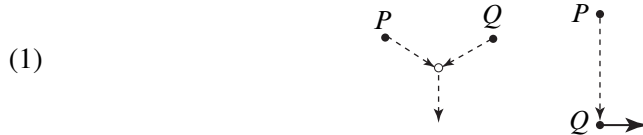
A *Jacobi diagram* is an oriented graph Γ valence at most 3 with two kinds of edges, θ -edges (depicted by directed dashed lines) and η -edges (depicted by directed solid lines) such that:

- The admissible combinations of incident oriented edges to a vertex are:



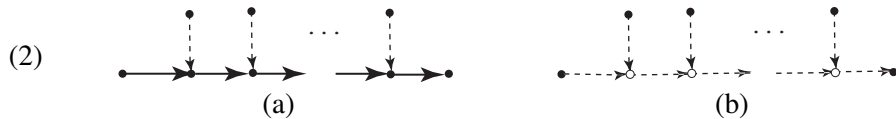
We will call a trivalent vertex with three incident θ -edges an *external vertex* and in pictures we depict such a vertex by a white dot. We will call a non-external vertex an *internal vertex* and in pictures we denote it by a black dot.

- Γ does not have a subgraph that looks like one of the following forms:

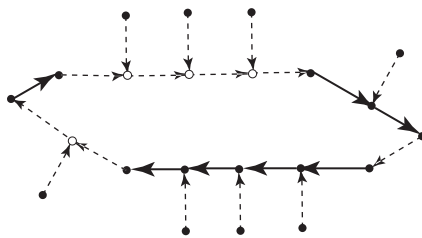


where no other edges are ingoing to both P and Q .

If Γ does not have subgraphs as in (1) and if Γ is connected, then the form of Γ is rather restricted: it must be a cyclic alternating sequence of the following two paths with θ -edges stuck into:



Here we allow a path to be cyclic. We will call a θ edge stuck into the path a *hair*. In particular, any such diagram includes just one cycle. The following picture is a typical example of this observation:



A *vertex orientation* of a Jacobi diagram Γ is a choice of orderings of two ingoing θ -edges incident to each external vertex modulo even number of swappings. In order to represent vertex orientations, we assume that diagrams may be depicted so that the three incident θ -edges

(the outgoing edge of v , first incident θ -edge, second incident θ -edge)

are arranged in the anti-clockwise order in a plane diagram.

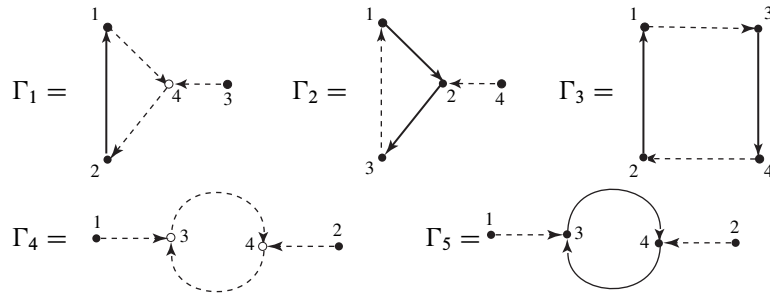


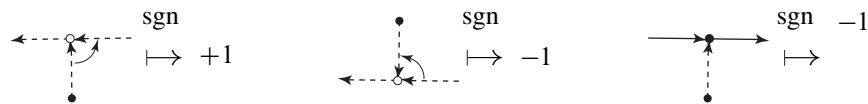
Figure 1: All degree-2 connected Jacobi diagrams up to automorphism

The *degree* of a Jacobi diagram is defined to be half the number of vertices. The complete list of connected Jacobi diagrams of degree 2 up to automorphism is shown in Figure 1.

An automorphism of a Jacobi diagram is an automorphism φ on the set of vertices sending internal vertices to internal vertices (and thus external vertices to external vertices) and inducing a bijection on the set of edges which sends each oriented edge $e = (i, j)$ to an oriented edge $(\varphi(i), \varphi(j))$. We call a θ -edge connecting two internal vertices a *chord*. We denote the group of automorphisms of Γ by $\text{Aut } \Gamma$.

2.1.1 The weight function w_k on Jacobi diagrams Let \mathcal{G}_k^0 denote the set of all degree- k connected Jacobi diagrams up to automorphism. We will define the invariants of higher dimensional knots in Section 2.3 whose values are \mathbb{R} -linear combinations of the weight function w_k on Jacobi diagrams, namely, a certain map $w_k: \mathcal{G}_k^0 \rightarrow \mathbb{R}$, so that their values are in \mathbb{R} . This map w_k is defined as follows.

We associate a sign with each trivalent vertex (not only external vertices) by the following rule:



and set

$$w_k(\Gamma) = \prod_{v \text{ trivalent}} \text{sgn}(v)$$

for any Γ .

2.2 Configuration space

Let $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$ be a smooth embedding for a long n -knot which is standard near ∞ . We consider that \mathbb{R}^{n+2} is the complement of a fixed point ∞ in S^{n+2} .

Let Γ be a Jacobi diagram with q internal and s external vertices. The *configuration space associated with* (Γ, ψ) denoted by $C_\Gamma^0(\psi)$ is defined by

$$\begin{aligned} C_\Gamma^0(\psi) := \{ & (x_1, \dots, x_q; x_{q+1}, \dots, x_{q+s}) \mid \\ & x_1, \dots, x_q \in \mathbb{R}^n, x_{q+1}, \dots, x_{q+s} \in \mathbb{R}^{n+2}; \\ & x_i \neq x_j \quad \text{if } i, j \geq q+1, i \neq j; \\ & \psi(x_i) \neq \psi(x_j) \quad \text{if } i, j \leq q, i \neq j; \\ & \psi(x_i) \neq x_j \quad \text{if } i \leq q, j \geq q+1 \}. \end{aligned}$$

To see the convergence of the integral over C_Γ^0 , we use the compactification $C_\Gamma(\psi)$ of $C_\Gamma^0(\psi)$ used by Axelrod–Singer [2] and Bott–Taubes [5], in analogy of Fulton–MacPherson [8], which is a smooth manifold with corners. Namely, C_Γ is obtained by a sequence of blow-ups along all the diagonals (see Section A.1) and along the infinity, in the way described in [5]. Detailed descriptions of the compactification used in this paper are found in Rossi [15]. We remark that there is a natural diffeomorphism between $\text{int}(C_\Gamma)$ and C_Γ^0 .

2.3 Integral over configuration space

Let $E_\theta(\Gamma)$ be the set of θ -edges that

- (1) are included in the cycle of Γ , or
- (2) end at an internal vertex.

The embedding $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$ determines an embedding of a configuration of points on \mathbb{R}^n into that of points on \mathbb{R}^{n+2} in C_Γ by setting $\psi(x_{q+i}) = x_{q+i}$ ($i \geq 1$). For each θ -edge $e = (i, j)$ of Γ (and each η -edge $f = (k, l)$ of Γ), let $\theta_e := u(\psi(x_j) - \psi(x_i))^* \omega_{n+1}$ (and $\eta_f := u'(x_k - x_l)^* \omega_{n-1}$) be the $(n+1)$ -form on the 2-point configuration space C_e (resp. the $(n-1)$ -form on C_f), where ω_p is the $SO(p+1)$ -invariant unit volume form on S^p , and the Gauss map $u: \mathbb{R}^{n+2} \setminus \{0\} \rightarrow S^{n+1}$ (and $u': \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$) is defined by

$$u(x) = \frac{x}{\|x\|}, \quad u'(x) = \frac{x}{\|x\|}.$$

We define a differential form on $C_\Gamma(\psi)$ by

$$\omega(\Gamma) := \bigwedge_{e \in E_\theta(\Gamma)} \omega(e),$$

$$\omega(e) := \begin{cases} \theta_e \wedge \eta_f & \text{if the } \eta\text{-edge } f \text{ goes out from the target of } e \\ \theta_e \wedge \theta_f & \text{if the } \theta\text{-edge } f \text{ (a hair) ends at the target of } e \\ & \text{with the vertex orientation determined by } (e, f) \\ \theta_f \wedge \theta_e & \text{if the } \theta\text{-edge } f \text{ (a hair) ends at the target of } e \\ & \text{with the vertex orientation determined by } (f, e) \end{cases}$$

Note that $\omega(\Gamma)$ is of the form $\pm \prod_e \theta\text{-edges of } \Gamma \theta_e \prod_f \eta\text{-edges of } \Gamma \eta_f$. When n is odd, the sign of the form is always $+$ because all θ - and η -forms have even degree. When n is even, both signs may occur.

Let Ω be an orientation of $C_\Gamma(\psi)$. We define

$$I(\Gamma, \Omega) := \int_{C_\Gamma(\psi)} \omega(\Gamma),$$

in which the orientation of C_Γ is given by Ω . Note that if n is even, the configuration space can be canonically oriented as a local product of even-dimensional oriented manifolds. Let

$$I(\Gamma) := I(\Gamma, \Omega(\Gamma)),$$

where $\Omega(\Gamma)$ is defined in [Lemma 2.1](#) below if n is odd.

Lemma 2.1 *Let n be an odd integer ≥ 3 . Let Γ be a vertex oriented Jacobi diagram and let $\bar{\Gamma}$ be Γ with its vertex orientation at one vertex reversed. Then there exists a choice of orientation $\Omega = \Omega(\Gamma)$ on $C_\Gamma(\psi)$ such that*

$$(3) \quad \Omega(\bar{\Gamma}) = -\Omega(\Gamma).$$

Proof Let $(X^1, \dots, X^{n+2}) \in \mathbb{R}^{n+2}$ be the coordinate of an external vertex of Γ and let $(X^1, \dots, X^n) \in \mathbb{R}^n$ be the coordinate of an internal vertex of Γ . Decompose each θ -edge into two half edges. To each half edge \bar{e} , we associate an odd form $\Omega_{\bar{e}}$ which are combined into an orientation of C_Γ as follows.

If \bar{e} is one of three half edges meeting at an external vertex v with the vertex orientation given by a bijection $o_v: \{\text{the two ingoing half edges}\} \rightarrow \{1, 2\}$, set

$$\Omega_{\bar{e}} := \begin{cases} dX_v^{o_v} & \text{if } \bar{e} \text{ is ingoing and } o_v(\bar{e}) = 1, 2 \\ dX_v^3 \wedge \dots \wedge dX_v^{n+2} & \text{if } \bar{e} \text{ is outgoing} \end{cases}$$

If \bar{e} is a half edge meeting an internal vertex, then set

$$\Omega_{\bar{e}} := dX_v^1 \wedge \dots \wedge dX_v^n.$$

In terms of this, we define $\Omega_e := \Omega_{\bar{e}_+} \wedge \Omega_{\bar{e}_-}$ for each θ -edge e with $e = (\bar{e}_-, \bar{e}_+)$ and define

$$\Omega(\Gamma) := \bigwedge_{e \in E_\theta(\Gamma)} \Omega_e.$$

Then the property (3) is because both $\Omega_{\bar{e}}$ and $\Omega_{\bar{e}'}$ are of odd degrees, and

$$(dX_v^1 \wedge \Omega_{\bar{e}}) \wedge (dX_v^2 \wedge \Omega_{\bar{e}'}) = -(dX_v^1 \wedge \Omega_{\bar{e}'}) \wedge (dX_v^2 \wedge \Omega_{\bar{e}})$$

for some half-edges \bar{e} and \bar{e}' . □

Further, $\omega(\bar{\Gamma}) = (-1)^{n+1} \omega(\Gamma)$ because

$$\theta_e \wedge \theta_f = (-1)^{n+1} \theta_f \wedge \theta_e.$$

This together with Lemma 2.1 implies

$$I(\bar{\Gamma}, \Omega(\bar{\Gamma})) = -I(\Gamma, \Omega(\Gamma))$$

for any n .

It follows from the definition of w_k that $w_k(\bar{\Gamma}) = -w_k(\Gamma)$. Therefore the product $I(\Gamma)w_k(\Gamma)$ does not depend on the vertex orientation of Γ . But it is still sensitive under a reversion of the orientation of an edge. This is the reason why we fix the edge-orientation.

2.4 Invariants of long n -knots

For $k \geq 2$ and an embedding $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$, let

$$(4) \quad z_k(\psi) := \frac{1}{2} \sum_{\Gamma \in \mathcal{G}_k^0} \frac{I(\Gamma)(\psi)w_k(\Gamma)}{|\text{Aut } \Gamma|} \in \mathbb{R}$$

where $|\text{Aut } \Gamma|$ denotes the order of the group $\text{Aut } \Gamma$. Recall that \mathcal{G}_k^0 is the set of all connected Jacobi diagrams of degree k up to automorphism. Here we assume that each term $I(\Gamma)(\psi)w_k(\Gamma)$ is given with any choice of vertex orientation of Γ .

The tangent of the embedding ψ defines the map

$$d\psi: \mathbb{R}^n \rightarrow I_n(\mathbb{R}^{n+2}),$$

where $I_n(\mathbb{R}^{n+2})$ is the space of all linear injective maps $\mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$. Namely, $d\psi(x)$ for $x \in \mathbb{R}^n$ is defined by the linear part of the local embedding near x .

Cattaneo and Rossi [7; 15] remark that they proved the following theorem.

Theorem 2.2 *If n is an odd integer ≥ 3 and $k \geq 2$, then z_k is an isotopy invariant of long n -knots.*

If $n = 2$, then there exists a certain 2-form ρ on $I_2(\mathbb{R}^4)$ (defined in Proposition A.12) such that

$$\widehat{z}_3(\psi) := z_3(\psi) + \int_{C_1} (d\psi)^* \rho$$

is an isotopy invariant of long 2-knots, where $C_1 := Bl(S^2, \{\infty\})$ denotes the blow-up of S^2 along $\infty \in S^2$.

We give a self-contained proof of Theorem 2.2 in Appendix A. For $n = 2$, the vanishing of z_2 is proved by Rossi [15] (see also Proposition 2.5). So \widehat{z}_3 is the first possibly nontrivial invariant.

By Theorem 2.2, we may write $z_k(K)$ (or $\widehat{z}_3(K)$) instead of $z_k(\psi)$ (resp. $\widehat{z}_3(\psi)$) if $K = \text{Im } \psi$. We will call z_k the k th BCR invariant.

Example 2.3 From Figure 1 and the weights,

$$w_2(\Gamma_1) = -w_2(\Gamma_2) = w_2(\Gamma_3) = w_2(\Gamma_4) = w_2(\Gamma_5) = 1,$$

the degree 2 term z_2 may be given as follows:

$$(5) \quad z_2 = \frac{1}{2}I(\Gamma_1) - \frac{1}{2}I(\Gamma_2) + \frac{1}{4}I(\Gamma_3).$$

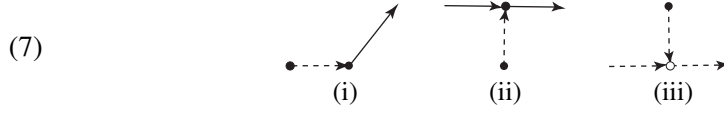
Here, $I(\Gamma_4) = I(\Gamma_5) = 0$ because $I(\Gamma_4)$ is the integral of $4(n+1)$ -form over $3(n+1)$ -dimensional submanifold of the target manifold of the Gauss maps (since the targets of two edges by the map u coincide) and similarly for $I(\Gamma_5)$. We will see here that (5) is equal to ($\frac{1}{2}$ times) the second invariant in Cattaneo–Rossi [7], which, in this degree, was first obtained by Bott [4]:

$$(6) \quad z_2 = \frac{1}{2} \int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{12} - \frac{1}{2} \int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34} + \frac{1}{4} \int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23}$$

2.5 Vanishing of some terms of z_k

By the following proposition, most terms of z_k in lower odd degrees vanish in the case n is odd. More precisely, for degrees up to three, the direct proof of Proposition 2.4 is given by Rossi [15] for all individual diagrams. Proposition 2.4 is in some sense a generalization of the arguments in [15]. Let k be the degree of a Jacobi diagram Γ and

k_1, k_2, k_3 ($k = k_1 + k_2 + k_3$) be the numbers of the following three parts respectively:



Proposition 2.4 Suppose that Γ is a vertex oriented Jacobi diagram of degree k . Then

$$(8) \quad I(\Gamma^*)w_k(\Gamma^*) = \begin{cases} (-1)^k I(\Gamma)w_k(\Gamma) & \text{if } n \text{ is odd} \\ (-1)^{k-1} I(\Gamma)w_k(\Gamma) & \text{if } n \text{ is even} \end{cases}$$

where Γ^* denotes Γ with the orientations of all edges involved in its cycle reversed.

Proof If n is odd, then one can show that

- (1) $w_k(\Gamma^*) = (-1)^{k_3} w_k(\Gamma)$.
- (2) $\Omega(\Gamma^*) = (-1)^{k_1+k_3} \Omega(\Gamma)$.
- (3) $\omega(\Gamma^*) = (-1)^{k_2+k_3} \omega(\Gamma)$.

(1) follows from the definition. (2) follows because the number of θ -edges involved in the cycle of Γ is $k_1 + k_3$. (3) follows because the number of edges involved in the cycle of Γ is $2k_1 + k_2 + k_3$. Hence (8) holds.

If n is even, then (8) follows from

- (1) $w_k(\Gamma^*) = (-1)^{k_3} w_k(\Gamma)$.
- (2) $\Omega(\Gamma^*) = \Omega(\Gamma)$.
- (3) $\omega(\Gamma^*) = (-1)^{k_1+k_2-1} \omega(\Gamma)$.

Here, the changing of the orientations in the cycle permutes the integrand form, which is a product of symmetric odd-degree forms. In the case (iii) in (7), let a, b, c denote the three edges in the middle, left, right respectively in the picture. In Γ , the integrand form locally defined as $\theta_a \theta_b \theta_c = (\theta_a \theta_b) \theta_c$, and in Γ^* , the integrand form locally defined as $\theta_c \theta_a \theta_b = \theta_c (\theta_a \theta_b)$. Hence in a path as in (2)(b), the change of the sign of the integrand form, under the change of the orientations in the cycle, is equal to the change of the sign when the intermediate hairs are removed. Since the part as in (ii) may be read equally as (i) in the integrand form, it is enough to show the identity (3) for the diagram with only the case (i) $k_1 + k_2$ times. In this restricted case, the integrand form is of the form $\theta_1 \eta_1 \dots \theta_{k_1+k_2} \eta_{k_1+k_2}$. Then the reversed form is $\theta_1 \eta_{k_1+k_2} \theta_{k_1+k_2} \dots \eta_1 = (-1)^{k_1+k_2-1} \theta_1 \eta_1 \dots \theta_{k_1+k_2} \eta_{k_1+k_2}$. \square

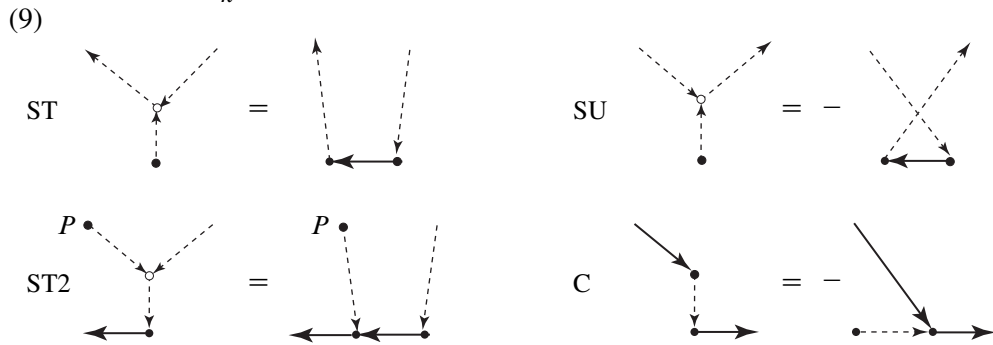
Proposition 2.4 implies the following proposition.

Proposition 2.5 *If n is odd, then $z_k = 0$ for k odd. If n is even, then $z_k = 0$ for k even.**

3 The space of Jacobi diagrams

We shall see that the weight function $w_k: \mathcal{G}_k^0 \rightarrow \mathbb{R}$ satisfies some axioms, which are needed in the proof of Theorem 2.2 and see that w_k may arise naturally from the generality on some space of Jacobi diagrams in which the axioms are satisfied. In three dimensions, the graph valued perturbative invariant is very strong in the sense that it yields a lot of scalar valued invariants by composing some weight functions on graphs. But in higher dimensions, the graph valued one is not so strong than a scalar valued one, as observed in this section.

Let $\mathbb{R}\mathcal{G}_k^0$ denote the vector space spanned by elements of \mathcal{G}_k^0 . We define *ST, SU, ST2, C relations* on $\mathbb{R}\mathcal{G}_k^0$ as follows[†]:



Here the graphs in (9) are subgraphs of Jacobi diagrams and P be a univalent vertex. Other edges may incident to vertices in the subgraphs. Let

$$\mathcal{A}_k := \mathbb{R}\mathcal{G}_k^0 / \text{relations in (9)}.$$

This quotient is the minimal in the sense that the span of the relations in (9) is the minimal one required to prove Theorem 2.2. Then as in the 3-dimensional perturbative

*Proposition 2.4, which is a generalization of that in the previous version of this paper, and Proposition 2.5 were pointed out by C Lescop.

[†]If the internal paths are replaced by 1-handles and the θ -part is replaced by “oriented tree clasps” [17], then these relations are essentially those appeared in Habiro–Shima [11].

Chern–Simons theory, one could define a graph valued series

$$z_k^*(\psi) := \frac{1}{2} \sum_{\Gamma \in \mathcal{G}_k^0} \frac{I(\Gamma)(\psi)[\Gamma]}{|\text{Aut } \Gamma|} \in \mathcal{A}_k.$$

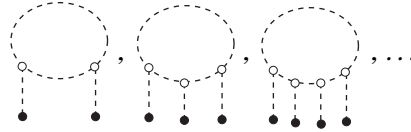
Since the dimension of \mathcal{A}_k is one as proved in Proposition 3.1 below, any linear map $\mathcal{A}_k \rightarrow \mathbb{R}$ is a scalar multiple of w_k , linearly extended to \mathcal{A}_k (well-definedness of this extension is proved also in Proposition 3.1). Thus, we obtain the identity

$$z_k^*(\psi) = z_k(\psi)[\Gamma], \quad z_k(\psi) = w_k \circ z_k^*(\psi)$$

for some $[\Gamma] \in \mathcal{A}_k$. Therefore, the BCR invariant is equivalent to the graph valued one.

The vector space structure on \mathcal{A}_k is determined as in the following propositions.

Proposition 3.1 (1) *The θ part of a Jacobi diagram forms a disjoint union of chords, trees and wheels. Here wheels are Jacobi diagrams of the following forms:*



- (2) *Extend w_k linearly to a map $w_k: \mathbb{R}\mathcal{G}_k^0 \rightarrow \mathbb{R}$. Then the map $w_k: \mathbb{R}\mathcal{G}_k^0 \rightarrow \mathbb{R}$ descends to a well defined \mathbb{R} –linear map $w_k: \mathcal{A}_k \rightarrow \mathbb{R}$ which spans \mathcal{A}_k^* .*
- (3) *The dimension of \mathcal{A}_k is 1 for each $k \geq 2$.*

Proof (1) immediately follows from the edge-orientation condition of external vertices.

(2) follows from a direct check of the fact that w_k satisfies the relations in (9).

For (3), observe first that any Jacobi diagram can be transformed into a Jacobi diagram without η –edges by ST, SU, C relations. The resulting graph must be a wheel-like graph. So the dimension of \mathcal{A}_k is at most one. By (2), w_k descends to a nontrivial linear map $w_k: \mathcal{A}_k \rightarrow \mathbb{R}$, so the dimension of \mathcal{A}_k is at least one. Therefore (3) follows. \square

4 BCR invariant for long ribbon n –knots

We restrict our study to a certain class of long n –knots called long ribbon n –knots and characterize the BCR invariant as a finite type invariant in the sense of Habiro–Kanenobu–Shima [10]. By this characterization, we obtain a description of the BCR invariant in terms of the Alexander polynomial.

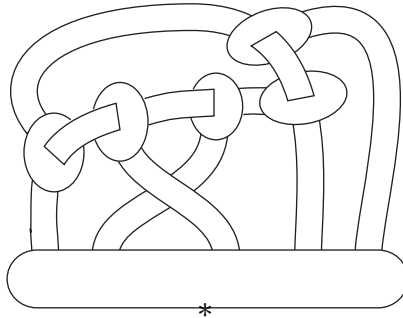


Figure 2: A ribbon presentation

4.1 Long ribbon n -knots and ribbon presentations

4.1.1 Long ribbon n -knots A long ribbon $(n+1)$ -disk is the image of an immersion f of a lower half $(n+1)$ -plane $D_-^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \leq 0\}$ into \mathbb{R}^{n+2} such that the singularity of f consists of finitely many ribbon singularities and such that f is standard outside a sufficiently large $(n+2)$ -ball in \mathbb{R}^{n+1} whose center is the origin. Here a *ribbon singularity* is an n -disk consisting of transverse double points and whose pre-image consists of a proper n -disk in D_-^{n+1} and an n -disk in the interior of D_-^{n+1} . A *long ribbon n -knot* is a long n -knot bounding a long ribbon $(n+1)$ -disk.

4.1.2 Ribbon presentations We use ribbon presentations to present long ribbon n -knots. A *ribbon presentation* $P = \mathcal{D} \cup \mathcal{B}$ in \mathbb{R}^3 is an immersed oriented 2-disk into \mathbb{R}^3 with a base point on its boundary having a decomposition consists of $p+1$ disjoint embedded 2-disks $\mathcal{D} = D_0 \sqcup D_1 \sqcup D_2 \sqcup \dots \sqcup D_p$ and p disjoint embedded bands $\mathcal{B} = B_1 \sqcup B_2 \sqcup \dots \sqcup B_p$ satisfying the following conditions.

- (1) The base point is on the boundary of D_0 .
- (2) Each band B_i transversely intersects the interiors of the disks in \mathcal{D} as in Figure 3(i).
- (3) Each end of a band is attached to the boundary of some disk in \mathcal{D} .

An example of a ribbon presentation is depicted in Figure 2.

4.1.3 Associating a long ribbon n -knot to a ribbon presentation We can construct a ribbon $(n+1)$ -disk V_P associated to each ribbon presentation P by

$$V_P = N(\mathcal{D} \times [-2, 2]^{n-1} \cup \mathcal{B} \times [-1, 1]^{n-1}) \subset \mathbb{R}^3 \times \mathbb{R}^{n-1},$$

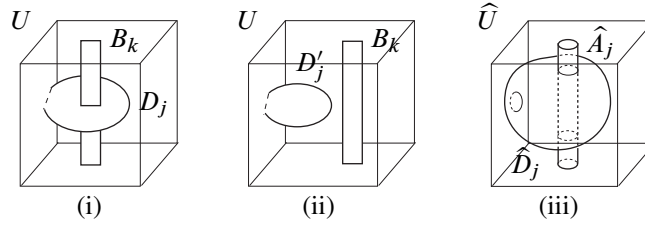


Figure 3: Crossing, unclasp

where $N(\cdot)$ denotes a smoothing of corners. Then we obtain a long ribbon n -knot associated to P by taking the boundary of V_P followed by connect summing the standardly embedded plane \mathbb{R}^n at the point associated with the base point (the base point may be thickened to form an $(n-1)$ -disk. Then define the base point of the n -knot to be its center). Note that the connect summing of \mathbb{R}^n is not unique even up to isotopy, depending on the position of the base point. We denote by K_P the long ribbon n -knot associated to a (based) ribbon presentation P and by $[K_P]$ its isotopy class. It is known that any (long) ribbon n -knot is isotopic to the one associated to some ribbon presentation.

4.1.4 Crossings Now we give a definition of crossings of both ribbon presentations and long ribbon n -knots. For crossings of ribbon presentations, consider a regular neighborhood U of a disk $D_j \subset \mathcal{D}$ (of $P = \mathcal{D} \cup \mathcal{B}$) including a ribbon singularity inside so that $U \cap P$ consists of

- the disk D_j ,
- a part of some band B_k intersecting D_j .

We call the triple $[U, U \cap D_j, U \cap B_k]$ a *crossing of P* (see Figure 3(i)).

In the associating process of $P \rightsquigarrow K_P$ above, U may also be thickened by taking a direct product with $[-3, 3]^{n-1}$ to include the associated pieces thickened from $U \cap D_j$ and $U \cap B_k$. We call the triple $(\widehat{U}, \widehat{D}_j, \widehat{A}_j)$ a *crossing of K_P* associated to $[U, U \cap D_j, U \cap B_k]$ where

- $\widehat{U} = U \times [-3, 3]^{n-1} \subset \mathbb{R}^{n+1}$,
- \widehat{D}_j is one of the components of $\widehat{U} \cap K_P$ associated to $U \cap D_j$,
- \widehat{A}_j is one of the components of $\widehat{U} \cap K_P$ associated to $U \cap B_k$.

Observe that $\widehat{D}_j \cong D^n$ and $\widehat{A}_j \cong S^{n-1} \times I$. $(\widehat{U}, \widehat{D}_j, \widehat{A}_j)$ looks like Figure 3(iii).

A crossing of a long ribbon n -knot isotopic to K_P is a triple $(\widetilde{U}, \widetilde{D}_j, \widetilde{A}_j)$, $(\widetilde{D}_j \cup \widetilde{A}_j \subset \widetilde{U})$ which is obtained from an associated crossing $(\widehat{U}, \widehat{D}_j, \widehat{A}_j)$ on K_P as above by an isotopy deforming $K_P \cup \widehat{U}$ such that

- it induces an isotopy deforming the trivial link $K_P \cap \partial \widehat{U}$ in $\partial \widehat{U}$,
- the number of the components in $K_P \cap \partial \widehat{U}$ is preserved during the isotopy,
- it sends \widehat{D}_j and \widehat{A}_j into \widetilde{D}_j and \widetilde{A}_j respectively.

4.1.5 Unclasp of a crossing An *unclasp of a crossing* on a ribbon presentation is defined as a replacement of a crossing $[U, U \cap D_j, U \cap B_k]$ in Figure 3(i) with another triple $[U, U \cap D'_j, U \cap B_k]$ in Figure 3(ii) keeping near ∂U unchanged. It is easy to see that any ribbon presentation can be made into the one without ribbon singularities by a sequence of unclasps. We can also define an unclasp of a crossing on a long ribbon n -knot by applying the associating procedure to both (i) and (ii) in Figure 3 keeping near the boundary unchanged.

4.1.6 Connected sum A connected sum of two long n -knots $K \# L$ is defined up to isotopy by arranging K and L along the standard plane $\mathbb{R}^n \subset \mathbb{R}^{n+2}$.

4.2 Finite type invariants of long ribbon n -knots

First we recall the notion of k -schemes defined by Habiro–Kanenobu–Shima [10] which is in some sense a higher dimensional analogue of singular knots in Vassiliev’s theory of knot invariants.

Let $P = \mathcal{B} \cup \mathcal{D}$ be a based ribbon presentation and let $\{c_1, \dots, c_k\}$ be a set of disjoint crossings on P . A k -scheme $[P; c_1, \dots, c_k]$ is defined by

$$[P; c_1, \dots, c_k] = \sum_{S \subset \{c_1, \dots, c_k\}} (-1)^{|S|} [K_{P^S}]$$

where $|S|$ is the size of S and P^S is the ribbon presentation obtained from P by unclasp at all the crossings whose labels are in S .

Let \mathcal{J}_k be the subspace of

$$\mathcal{L}^n := \text{span}_{\mathbb{R}} \{\text{isotopy classes of long ribbon } n\text{-knots}\}$$

spanned by all k -schemes. It is easy to check that this constitutes a descending filtration on \mathcal{L}^n :

$$(10) \quad \mathcal{L}^n = \mathcal{J}_0 \supset \mathcal{J}_1 \supset \mathcal{J}_2 \supset \dots \supset \mathcal{J}_k \supset \dots$$

with the product given by the connected sum. Let g be an \mathbb{R} -valued invariant of long ribbon n -knots. Then we can extend g naturally to \mathcal{L}^n by linearity. We say that g is of type k if it vanishes on \mathcal{J}_{k+1} , or equivalently, if g is an element of $(\mathcal{L}^n/\mathcal{J}_{k+1})^*$.

There is a filtration

$$(11) \quad \mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_k \subset \cdots = \mathcal{I}$$

on the set \mathcal{I} of all \mathbb{R} -valued finite type invariants, with \mathcal{I}_k being the set of all type k invariants. The two filtrations (10) and (11) are dual to each other in the sense of the following isomorphism, proved by Habiro–Shima [11]:

$$(12) \quad \mathcal{I}_k/\mathcal{I}_{k-1} \cong (\mathcal{J}_k/\mathcal{J}_{k+1})^*$$

In [11], it is also proved that the elements of the following form constitute a basis

$$B_{(k_1, \dots, k_r)} := [W_{k_1}; c_1, \dots, c_{k_1}] \# [W_{k_2}; c_1, \dots, c_{k_2}] \# \cdots \# [W_{k_r}; c_1, \dots, c_{k_r}],$$

of $\mathcal{J}_k/\mathcal{J}_{k+1}$, where $[W_i; c_1, \dots, c_i]$ is defined in Figure 4(a), for possible sequences (k_1, \dots, k_r) of positive integers with $k_1 + \cdots + k_r = k$, $2 \leq k_1 \leq \cdots \leq k_r$. We will call the scheme of this form a *wheel-like* scheme. For $g \in \mathcal{I}_k$, we call its projected image in $\mathcal{I}_k/\mathcal{I}_{k-1}$ a *principal term* of g , which is in some sense an analogue of highest order term of a polynomial. Then (12) says that the principal term is determined by a linear functional on $\mathcal{J}_k/\mathcal{J}_{k+1}$.

We set $\hat{z}_k = z_k$ when n is odd and let \hat{z}_3 be as in Theorem 2.2 when $n = 2$. The main theorem of this paper is stated as follows.

Theorem 4.1 *Let n is an odd integer ≥ 3 and $k \geq 2$, or $(n, k) = (2, 3)$. Then*

- (1) \hat{z}_k is a finite type invariant of type k of long ribbon n -knots,
- (2) If $n \not\equiv k \pmod{2}$, the principal term of \hat{z}_k corresponds to the linear functional f on $\mathcal{J}_k/\mathcal{J}_{k+1}$ such that $f(B_{(k)}) = 1$ and $f(\text{other basis element}) = 0$,
- (3) If $n \equiv k \pmod{2}$, the principal term of \hat{z}_k is zero.

Remark 4.2 Here, although the set of ribbon n -knots and the set of long ribbon n -knots are different, it can be shown that the filtrations of finite type invariants of both are the same[‡].

[‡]For instance, any “long” k -scheme may be reduced modulo long $(k+1)$ -schemes to “wheel-like” ones (see Figure 4). So the graded piece $\text{span}_{\mathbb{R}}\{\text{long } k\text{-schemes}\}/\text{span}_{\mathbb{R}}\{\text{long } (k+1)\text{-schemes}\}$ may be isomorphic to that of usual (non-long) k -schemes.

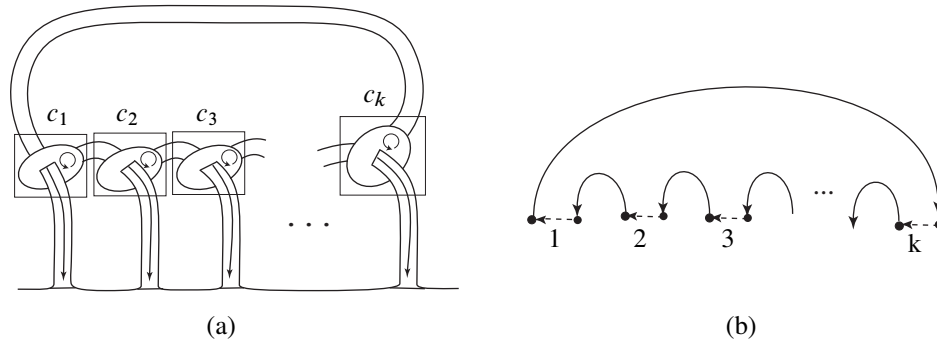


Figure 4: $\{W_k; c_1, \dots, c_k\}$ and Γ_k

4.3 Proof of Theorem 4.1

Let $\{P; c_1, \dots, c_k\}$ denote a combination of a ribbon presentation $P = \mathcal{D} \cup \mathcal{B}$ and a set of crossings $\{c_1, \dots, c_k\}$ on it. We will call such a combination a k -marked ribbon presentation.

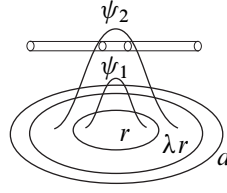
4.3.1 Convenient choices of embeddings Assume that n is an odd integer ≥ 3 and $k \geq 2$ or $(n, k) = (2, 3)$. Theorem 4.1 will be proved by computing the values of \hat{z}_k for m -schemes $[P; c_1, \dots, c_m]$ with $m \geq k$. Let

$$\psi^S(P; c_1, \dots, c_m): \mathbb{R}^n \rightarrow \mathbb{R}^{n+2} \quad (S \subset \{1, \dots, m\})$$

be a choice of an embedding for each $[K_{PS}]$ which is standard near ∞ . We will often write ψ^S for $\psi^S(P; c_1, \dots, c_m)$ when the marked ribbon presentation $\{P; c_1, \dots, c_m\}$ considered is understood from the context.

Now we fix some choices of embeddings in the proof, which do not lose the generality because \hat{z}_k is an isotopy invariant, to make the computation simpler. Let $[\check{U}_i, \check{D}_i, \check{A}_i]$ ($i = 1, \dots, m$) denote the crossings on P corresponding to c_1, \dots, c_m and (U_i, D_i, A_i) ($i = 1, \dots, m$) denote the corresponding crossings on $\text{Im } \psi^\emptyset$. We choose the embeddings satisfying the following condition, which is made depending on small parameters $\varepsilon > 0$, $\varepsilon_i > 0$ ($i = 1, \dots, m$) with $\varepsilon_i < \varepsilon$.

(Emb-0) For any S and $i \in \{1, \dots, m\} \setminus S$, ψ^S and $\psi^{S \cup \{i\}}$ is chosen so that they coincide outside an $(n+2)$ -ball in U_i with radius $\varepsilon_i < \varepsilon$. Indeed, such ψ^S and $\psi^{S \cup \{i\}}$ may be obtained by contracting around the center $\{\frac{1}{2}\} \times S^{n-1} \subset I \times S^{n-1}$ of A_i into a very thin cylinder with small S^{n-1} component and let them approach near D_i .

Figure 5: Change of a crossing preserving $d\psi$

This assumption is explained in the lower right of Figure 6. In addition to this assumption, we make another assumption in the following lemma in the case $n = 2$ to make the correction term uniformly over the terms of a k -scheme. From the procedure of framing correction in this case (see Section A.6.1), the correction term depends only on the map

$$f = d\psi: \mathbb{R}^2 \rightarrow I_2(\mathbb{R}^4)$$

giving the tangent 2-framing in \mathbb{R}^4 of $\psi(x) \in \mathbb{R}^4$ ($x \in \mathbb{R}^2$) where $I_2(\mathbb{R}^4)$ is the space of all linear injective maps $\mathbb{R}^2 \rightarrow \mathbb{R}^4$, or the set of 2-framings in \mathbb{R}^4 . The choice of the embeddings in the following lemma can be made compatible with (Emb-0).

Lemma 4.3 *Let P_1 and P_2 be two ribbon presentations which differ only by one unclasping of a crossing. For long ribbon 2-knots, we can choose certain embeddings $\psi_i: \mathbb{R}^2 \rightarrow \mathbb{R}^4$, $i = 1, 2$ for the isotopy classes $[K_{P_1}]$ and $[K_{P_2}]$ respectively, so that their tangent 2-frames f_i coincide.*

Proof We can assume without losing generality that there exist some positive real numbers r and a with $0 < r < a$ such that $f_1(x)$ is equal to a constant $g \in I_2(\mathbb{R}^4)$ in $r \leq \|x\| \leq a$ and such that $f_1(x)$ is not a constant function in $\|x\| < r$. Then we can take ψ_2 as

$$\psi_2(x) = \begin{cases} \lambda \psi_1(\lambda^{-1}x) & \|x\| \leq \lambda r \\ \psi_1(x) & \|x\| > \lambda r \end{cases}$$

for some constant λ with $0 < \lambda < a/r$. See Figure 5(a) for an explanation of this condition. The naturally induced framing f_2 for ψ_2 from f_1 is

$$f_2(x) = \begin{cases} f_1(\lambda^{-1}x) & \|x\| \leq \lambda r \\ f_1(x) & \|x\| > \lambda r \end{cases}$$

After some change of parametrization in the disk $\|x\| \leq a$, f_2 becomes equal to f_1 without changing the image of ψ_2 . \square

We define the formal alternating sum

$$\psi(P; c_1, \dots, c_m) := \sum_{S \subset \{1, \dots, m\}} (-1)^{|S|} \psi^S.$$

Then its configuration space integral $I(\Gamma)$ is given by

$$I(\Gamma)(\psi(P; c_1, \dots, c_m)) = \sum_{S \subset \{1, \dots, m\}} (-1)^{|S|} I(\Gamma)(\psi^S).$$

Proof of Theorem 4.1(1), finiteness of \hat{z}_k Assume that n is an odd integer ≥ 3 and $k \geq 2$ or $(n, k) = (2, 3)$. We will show that $\hat{z}_k(\psi(P; c_1, \dots, c_m)) = 0$ for any m -scheme $[P; c_1, \dots, c_m]$ with $m > k$, with the choices of embeddings ψ^S ($S \subset \{1, \dots, m\}$) satisfying (Emb-0) and the condition of Lemma 4.3 if $n = 2$.

Let $\pi^S: C_\Gamma(\psi^S) \rightarrow C_q(\mathbb{R}^n)$ be the natural projection where $C_q(\mathbb{R}^n)$ denotes the configuration space for internal vertices in Γ . Then π^S can be considered as a bundle and we can rewrite the integral by the pushforward:

$$I(\Gamma)(\psi^S) = \int_{C_\Gamma(\psi^S)} \omega(\Gamma)(\psi^S) = \int_{C_q(\mathbb{R}^n)} \pi_*^S \omega(\Gamma)(\psi^S)$$

for some qn -form $\pi_*^S \omega(\Gamma)(\psi^S)$ on $C_q(\mathbb{R}^n)$. By using this expression, we can rewrite the alternating sum $I(\Gamma)(\psi(P; c_1, \dots, c_m))$ as an integral of an alternating sum of forms over the common space $C_q(\mathbb{R}^n)$:

$$I(\Gamma)(\psi(P; c_1, \dots, c_m)) = \int_{C_q(\mathbb{R}^n)} \sum_{S \subset \{1, \dots, m\}} (-1)^{|S|} \pi_*^S \omega(\Gamma)(\psi^S)$$

Let $V_1(D_i)$ be the subset of $C_q(\mathbb{R}^n)$ consisting of configurations such that no points are mapped into $D_i \subset U_i$ by ψ^\emptyset (and thus by ψ^S) and $V_1(A_i)$ is similarly defined for $A_i \subset U_i$. Then we have

$$(13) \quad \int_{V_1(D_i) \cup V_1(A_i)} \sum_{S \subset \{1, \dots, m\}} (-1)^{|S|} \pi_*^S \omega(\Gamma)(\psi^S) = O(\varepsilon_i).$$

Here $\varepsilon_i < \varepsilon$ is the number taken in (Emb-0) and $O(\varepsilon_i)$ denotes a term which vanishes at the limit $\varepsilon_i \rightarrow 0$. (13) is because the integral of the LHS is equal to

$$\sum_{S \subset \{1, \dots, \hat{i}, \dots, m\}} (-1)^{|S|} \int_{V_1(D_i) \cup V_1(A_i)} (\pi_*^S \omega(\Gamma)(\psi^S) - \pi_*^{S \cup \{i\}} \omega(\Gamma)(\psi^{S \cup \{i\}}))$$

and this contributes as $O(\varepsilon_i)$ by (Emb-0).

By (13), the computation reduces to the one for configurations such that both D_i and A_i includes at least one internal vertices. If $m > k$, this is impossible. So we have $\hat{z}_k(\psi(P; c_1, \dots, c_m)) = O(\varepsilon)$ and this must be zero because \hat{z}_k is an invariant by Theorem 2.2. Hence Theorem 4.1(1) is proved. \square

Proof of Theorem 4.1(2), (3), the principal term of \hat{z}_k Assume that n is an odd integer ≥ 3 and $k \geq 2$ or $(n, k) = (2, 3)$. By Theorem 4.1(1) and by the fact that $\mathcal{J}_k/\mathcal{J}_{k+1}$ is generated by wheel-like schemes, we may restrict the computation to wheel-like ones and we assume that $\{P; c_1, \dots, c_m\}$ is such one. We will call the disk $\check{D}_j \cup B_j$, where \check{D}_j is the disk of P in the crossing $c_j = [\check{U}_j, \check{D}_j, \check{A}_j]$ and B_j is the band connecting the based disk and \check{D}_j , a *branch* of a marked wheel-like ribbon presentation. Then label the branches of $\{P; c_1, \dots, c_m\}$ by integers in $\{1, \dots, m\}$. We choose embeddings $\psi^S(P; c_1, \dots, c_m)$ satisfying the following conditions, which are compatible with (Emb–0) and Lemma 4.3. $\varepsilon > 0$ is the same as in (Emb–0).

There are disjoint subsets D_j and A_j ($j = 1, \dots, m$) of \mathbb{R}^n each diffeomorphic to D^n and $S^{n-1} \times I$ respectively such that:

(Emb–1) $\psi^S = \psi^\emptyset$ on $\mathbb{R}^n \setminus (\bigcup_j D_j \cup \bigcup_j A_j)$.

(Emb–2) $U_i \cap \text{Im } \psi^S = \psi^S(D_i \cup A_i)$.

(Emb–3) If the intersection of the j th branch of P and $\check{U}_1 \cup \dots \cup \check{U}_m$ is $\check{A}_{j_1} \cup \check{D}_j$, then

$$A_{j_1} \cup D_j \subset S_j$$

where

$$S_j := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \|x_1 - j\|^2 + \|x_2\|^2 + \dots + \|x_n\|^2 \leq \varepsilon^2\} \subset \mathbb{R}^n.$$

(Emb–4) The distance between the crossings U_i and U_j for $i \neq j$ is very large relative to the diameters of both U_i and U_j . More precisely, the distance is assumed larger than $\frac{1}{\varepsilon} \max\{\text{diam } U_i, \text{diam } U_j\}$.

Figure 6 is a picture explaining these assumptions. One may check that these assumptions are compatible. Note that (Emb–1) and (Emb–4) are the assumptions concerning the image of embeddings and (Emb–2) and (Emb–3) are those concerning the choice of parameterization of \mathbb{R}^n . Note also that we can not define a limit of the embedding at $\varepsilon = 0$ while we can define for the limit $\varepsilon_i = 0$ (in (Emb–0)).

We prove the following claims with the choices of embeddings ψ^S ($S \subset \{1, \dots, k\}$) satisfying (Emb–0)–(Emb–4) and the condition of Lemma 4.3 if $n = 2$.

Claim 1 $\hat{z}_k(\psi(W_i; c_1, \dots, c_i) \# \psi(W_j; c_1, \dots, c_j)) = 0$ with $i + j = k$, $i, j > 0$.

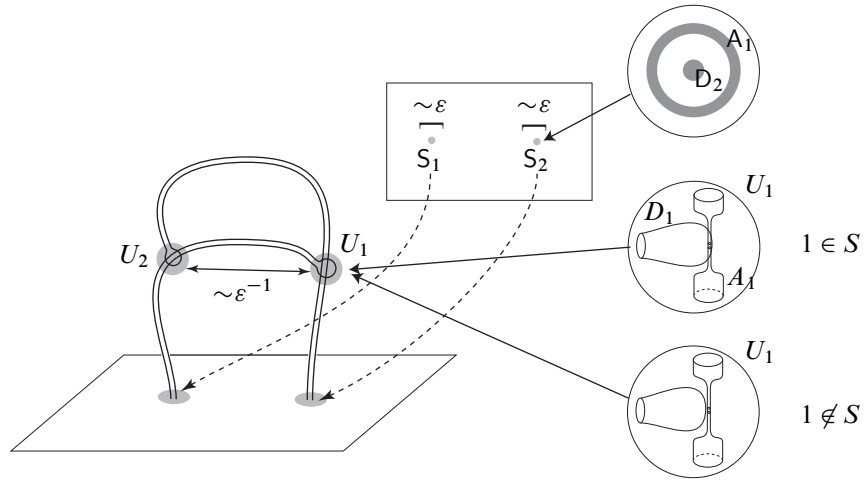


Figure 6: (Emb-0)-(Emb-4)

Claim 2 If $n \not\equiv k \pmod 2$, $\widehat{z}_k(\psi(W_k; c_1, \dots, c_k)) = w_k(\Gamma_k)$ where Γ_k is defined in Figure 4(b).

Claim 3 If $n \equiv k \pmod 2$, $\widehat{z}_k(\psi(W_k; c_1, \dots, c_k)) = 0$.

The reason for Claim 1-3 to be sufficient to determine the principal term of z_k is as follows. Claim 1 implies that \widehat{z}_k can be non trivial only on the subspace $\mathcal{P}(\mathcal{J}_k/\mathcal{J}_{k+1}) := \text{span}_{\mathbb{R}}\{B_{(k)}\}$. So the principal term of \widehat{z}_k reduces to $\mathcal{P}(\mathcal{J}_k/\mathcal{J}_{k+1})^*$. Thus Claim 2 and 3 characterize the element of $\mathcal{P}(\mathcal{J}_k/\mathcal{J}_{k+1})^*$ for \widehat{z}_k .

By the proof of Theorem 4.1(1), we may assume $m = k$ to prove Claim 1-3. Proofs of the claims are done by explicit computations of $I(\Gamma)(\psi(P; c_1, \dots, c_k))$ for $\deg \Gamma = k$ and a wheel-like k -marked ribbon presentation $\{P; c_1, \dots, c_k\}$.

Proof of Claim 1 According to the observations in the proof of Theorem 4.1(1), the Jacobi diagrams giving non-zero contribution of $I(\Gamma)(\psi(P; c_1, \dots, c_k))$ are those without external vertices and so we restrict to such Jacobi diagrams in the following. Note that $C_\Gamma = C_q(\mathbb{R}^n)$ for such Γ . Let $C'_q(\mathbb{R}^n) = C_q(\mathbb{R}^n) \setminus \bigcup_i (V_1(D_i) \cup V_1(A_i))$.

Lemma 4.4 Let $V_2(e)$, for e a θ -edge of Γ , be the set of configurations for Γ such that the two ends of e are mapped by the embeddings into some pair of two different balls U_i and U_j respectively. Then

$$\int_{V_2(e)} \sum_{S \subset \{1, \dots, m\}} (-1)^{|S|} \omega(\Gamma)(\psi^S) = O(\varepsilon).$$

Proof of Lemma 4.4 By the assumption (Emb–4), the locus of the image of the relative vector connecting a point in U_i and another point in U_j ($i \neq j$) via the Gauss map $u: \mathbb{R}^{n+2} \setminus \{0\} \rightarrow S^{n+1}$ (defined in Section 2.3) for the θ -edge e , is included in an arbitrarily small ball embedded into S^{n+1} , as ε tends to 0. So the integral is $O(\varepsilon)$. \square

According to Lemma 4.4, only the Jacobi diagrams, each θ -edge of which is entirely mapped into some U_i , contribute. Since Γ is of degree k , it follows that such diagrams can not have external vertices.

Lemma 4.5 Let $V_3(e)$, for e an η -edge of Γ , be the subset of $C'_q(\mathbb{R}^n)$ consisting of configurations for Γ such that two successive internal vertices connected by e split into S_i and S_j ($i \neq j$). Then

$$\int_{V_3(e)} \sum_{S \subset \{1, \dots, m\}} (-1)^{|S|} \omega(\Gamma)(\psi^S) = O(\varepsilon).$$

Proof of Lemma 4.5 By (Emb–3), the locus of the image of the relative vector connecting the point in S_i and another point in some S_j ($i \neq j$) via the Gauss map $u': \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ (defined in Section 2.3) for the η -edge e , is included in an arbitrarily small ball in S^{n-1} as ε tends to 0, by the definition of S_i 's. So the integral is $O(\varepsilon)$. \square

For k -schemes of the form of connected sum $K_i \# K_j$ of i - and j -schemes with $i + j = k$, $i, j > 1$, any configurations not in $V_2(e)$'s associated to a connected Jacobi diagram, that is, each θ -edge is mapped entirely into some U_i , must be in some $V_3(e)$ because the image of any θ -edge cannot connect K_i and K_j and then some η -edge must connect them. Hence Claim 1 is proved. \square

Proof of Claim 2 Now we compute the precise value of $\widehat{z}_k(\psi(W_k; c_1, \dots, c_k))$. By Lemma 4.4 and 4.5, the only contributing term in the sum (4) is the term

$$\frac{1}{2} \frac{I(\Gamma_k)(\psi(W_k; c_1, \dots, c_k)) w_k(\Gamma_k)}{|\text{Aut } \Gamma_k|}.$$

By Lemma 4.6 below, this equals

$$\frac{|\text{Aut}' \Gamma_k| w_k(\Gamma_k)}{2|\text{Aut } \Gamma_k|} + O(\varepsilon) = \frac{2|\text{Aut } \Gamma_k| w_k(\Gamma_k)}{2|\text{Aut } \Gamma_k|} + O(\varepsilon) = w_k(\Gamma_k) + O(\varepsilon),$$

where $\text{Aut}' \Gamma$ denotes the group of automorphisms of Γ considered as an unoriented graph. Roughly, by the above observations, the computation reduces in Lemma 4.6 to the integral over a direct product of some simple spaces and then it is computed as a product of ‘linking numbers’. This completes the proof of Claim 2. \square

Proof of Claim 3 In the case $n \equiv k \pmod 2$, Claim 3 follows directly from Proposition 2.5. □

This completes the proof of Theorem 4.1(2), (3). □

Lemma 4.6 Suppose that $n \not\equiv k \pmod 2$. Under the assumptions (Emb-0)–(Emb-4), we have $I(\Gamma_k)(\psi(W_k; c_1, \dots, c_k)) = |\text{Aut}' \Gamma_k| + O(\varepsilon)$ where $\text{Aut}' \Gamma$ denotes the group of automorphisms of Γ considered as an unoriented graph.

Proof There are exactly $|\text{Aut}' \Gamma_k|$ connected components in the reduced configuration space $C'_q(\mathbb{R}^n) \setminus (\bigcup_e V_2(e) \cup \bigcup_f V_3(f))$ for Γ_k each of which is equal up to permutations to

$$M_k := A_1 \times D_1 \times A_2 \times D_2 \times \dots \times A_k \times D_k$$

By symmetry, all $|\text{Aut}' \Gamma_k|$ components contribute as a common value up to sign. Note that in the computation of $I(\Gamma_k)(\psi(W_k; c_1, \dots, c_k))$, the integration domain M_k may be assumed to be common for all S because of (Emb-1) and (Emb-2) while the integrand form varies depending on ψ^S 's.

After a suitable $\text{Diff}_+(\mathbb{R}^n)$ action on \mathbb{R}^n fixing outside $S_1 \cup \dots \cup S_k \subset \mathbb{R}^n$, we may assume in addition to (Emb-0)–(Emb-4) that

(Emb-5)

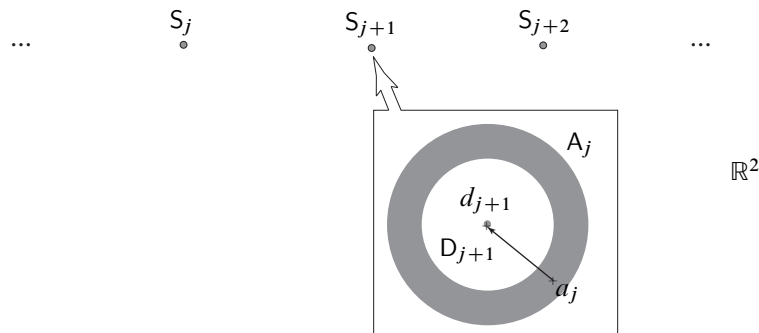
$$A_i = B^n((i + 1, 0, \dots, 0); 2\varepsilon/3) \setminus \text{int } B^n((i + 1, 0, \dots, 0); \varepsilon/2) \text{ for } 1 \leq i < k,$$

$$A_k = B^n((1, 0, \dots, 0); 2\varepsilon/3) \setminus \text{int } B^n((1, 0, \dots, 0); \varepsilon/2),$$

$$D_i = B^n((i, 0, \dots, 0); \varepsilon^2),$$

where $B^n(a; r) \subset \mathbb{R}^n$ denotes the n -ball with radius r whose center is $a \in \mathbb{R}^n$.

This additional assumption can be made so as not to affect the assumptions (Emb-0)–(Emb-4) by deforming all ψ^S 's for $S \subset \{1, \dots, k\}$ simultaneously. In the case of $n = 2$, S_j, D_j, A_j look like the following picture.



Let $(a_1, d_1, \dots, a_k, d_k)$ ($a_i \in A_i, d_i \in D_i$) be the coordinate of M_k . The explicit expression for the LHS integral restricted to M_k is

$$(14) \quad \sum_{S \subset \{1, \dots, k\}} (-1)^{|S|} \int_{M_k} \bigwedge_i (u(\psi^S a_i - \psi^S d_i)^* \omega_{n+1} \wedge u'(d_{i+1} - a_i)^* \omega_{n-1}).$$

It suffices to show that this quantity equals $1 + O(\varepsilon)$ because in the case $n \not\equiv k \pmod{2}$, any automorphism on M_k induced from an automorphism of Γ_k does not change the orientation of the configuration space and the integrand form. So all $|\text{Aut}' \Gamma_k|$ connected components contribute as a common value modulo $O(\varepsilon)$.

From the fact that

varying d_i fixing all other points only affects (modulo $O(\varepsilon)$) to $u(\psi^S a_i - \psi^S d_i)^* \omega_{n+1}$ in $\omega(\Gamma_k)$ because the other form $u'(d_i - a_{i-1})^* \omega_{n-1}$, which may also depend on d_i , contributes as $O(\varepsilon)$ by (Emb-5),

we can write (14) as

$$\sum_{S \subset \{1, \dots, k\}} (-1)^{|S|} \int_{M_k} \bigwedge_i (u(\psi^S a_i - \psi^S d_i)^* \omega_{n+1} \wedge u'(d_{i+1}^0 - a_i)^* \omega_{n-1}) + O(\varepsilon)$$

where $d_{i+1}^0 = (i+1, 0, \dots, 0) \in \mathbb{R}^n$ for $1 \leq i < k$, $d_{k+1}^0 = (1, 0, \dots, 0)$. We will write $\Phi_i(\psi) := u(\psi a_i - \psi d_i)^* \omega_{n+1} \wedge u'(d_{i+1}^0 - a_i)^* \omega_{n-1}$ for simplicity bearing in mind that it depends on a_i and d_i . We have

$$\begin{aligned} \int_{M_k} \bigwedge_i \Phi_i(\psi^S) - \int_{M_k} \bigwedge_i \Phi_i(\psi^{S \cup \{j\}}) \\ = \int_{M_k(j)} \bigwedge_{i \neq j} \Phi_i(\psi^S) \int_{A_j \times D_j} (\Phi_j(\psi^S) - \Phi_j(\psi^{S \cup \{j\}})) \end{aligned}$$

($S \subset \{1, \dots, \widehat{j}, \dots, k\}$) where $M_k(j) := A_1 \times D_1 \times \dots \times \widehat{A_j \times D_j} \times \dots \times A_k \times D_k$, because $\Phi_i(\psi^S) = \Phi_i(\psi^{S \cup \{j\}})$ on $M_k(j)$ for $i \neq j$. Thus (14) equals

$$\begin{aligned} \sum_{S \subset \{1, \dots, \widehat{j}, \dots, k\}} (-1)^{|S|} \int_{M_k(j)} \bigwedge_{i \neq j} \Phi_i(\psi^S) \int_{A_j \times D_j} (\Phi_j(\psi^S) - \Phi_j(\psi^{S \cup \{j\}})) + O(\varepsilon) \\ = \left[\int_{A_j \times D_j} (\Phi_j(\psi^\emptyset) - \Phi_j(\psi^{\{j\}})) \right] \sum_{S \subset \{1, \dots, \widehat{j}, \dots, k\}} (-1)^{|S|} \int_{M_k(j)} \bigwedge_{i \neq j} \Phi_i(\psi^S) + O(\varepsilon) \\ = \prod_{j \in \{1, \dots, k\}} \int_{A_j \times D_j} (\Phi_j(\psi^\emptyset) - \Phi_j(\psi^{\{j\}})) + O(\varepsilon). \end{aligned}$$

Here the fact that $\Phi_j(\psi^S) - \Phi_j(\psi^{S \cup \{j\}}) = \Phi_j(\psi^\emptyset) - \Phi_j(\psi^{\{j\}})$ on $A_j \times D_j$ is used. Then by using [Lemma 4.7](#) below k times, we conclude that (14) is $1 + O(\varepsilon)$. \square

Lemma 4.7

$$\int_{A_j \times D_j} (\Phi_j(\psi^\emptyset) - \Phi_j(\psi^{\{j\}})) = 1.$$

Roughly, [Lemma 4.7](#) says that the ‘linking number’ of D_j and the axis of A_j is 1.

Proof For each crossing (U_i, D_i, A_i) , consider the three unclasplings of it defined as follows.

- (U_i, D'_i, A_i) The result of an unclaspling of (U_i, D_i, A_i) keeping A_i fixed.
- (U_i, D_i, A'_i) The result of an unclaspling of (U_i, D_i, A_i) keeping D_i fixed.
- (U_i, D'_i, A'_i) Obtained by mixing the D'_i and A'_i in the previous two.

Note that these three are isotopic each other and that (U_i, D_i, A'_i) and (U_i, D'_i, A'_i) fail to satisfy the condition of [Lemma 4.3](#). However, such replacements do not change the integral, as observed below.

Denote by $\psi(D'_j), \psi(A'_j), \psi(D'_j, A'_j)$ the corresponding embeddings for the three which are all isotopic to $\psi^{\{j\}}$. Since the three are all isotopic relative to $\partial(D_j \times A_j)$ and the integrand form is closed,

$$\int_{A_j \times D_j} \Phi_j(\psi(D'_j)) = \int_{A_j \times D_j} \Phi_j(\psi(A'_j)) = \int_{A_j \times D_j} \Phi_j(\psi(D'_j, A'_j)) = \int_{A_j \times D_j} \Phi_j(\psi^{\{j\}}).$$

Thus the LHS of the lemma is rewritten as

$$\int_{A_j \times D_j} (\Phi_j(\psi^\emptyset) - \Phi_j(\psi(D'_j)) - \Phi_j(\psi(A'_j)) + \Phi_j(\psi(D'_j, A'_j))) = \int_{A_j^\circ \times D_j^\circ} \Phi_j(\psi_j^{\circ\circ}).$$

Here X° denotes $X \cup_\partial (-X)$ and $\psi_j^{\circ\circ}$ is the embedding of $A_j^\circ \cup D_j^\circ$ into $U_j \cup_\partial (-U_j) \cong S^{n+2}$ obtained by gluing $\psi(*)$'s along boundaries.

The computation of the last expression decomposes with respect to the splitting $A_j^\circ \cong S^{n-1} \times S^1$ as follows:

$$\begin{aligned} \int_{(v,l,d_j) \in S^{n-1} \times S^1 \times D_j^\circ} u(\psi_j^{\circ\circ}(v,l) - \psi_j^{\circ\circ}(d_j))^* \omega_{n+1} \wedge u'(d_{j+1}^0 - (v,l))^* \omega_{n-1} \\ = \int_{v \in S^{n-1}} \left(\int_{L_j^\circ(v) \times D_j^\circ} u(\psi_j^{\circ\circ}(v,l) - \psi_j^{\circ\circ}(d_j))^* \omega_{n+1} \right) \wedge \omega_{n-1}(v). \end{aligned}$$

where we use the fact that

varying a_i along the line

$$(15) \quad L_i(v) := \{(i+1, 0, \dots, 0) + tv \mid t > 0\} \cap A_i \subset \mathbb{R}^n, \quad v \in S^{n-1}$$

fixing all other points only affects (modulo $O(\varepsilon)$) to the form $u(\psi^\circledast a_i - \psi^\circledast d_i)^* \omega_{n+1}$ by (Emb-5).

To compute the last integral, we choose an orientation of $L_j(v)$ given by the direction of t (in the definition of $L_j(v)$) increases. Then the induced orientation of S^{n-1} from that of $S^{n-1} \times L_j(\cdot) = A_j$ and from that of $L_j(\cdot)$ differs by $(-1)^n$ from the one naturally induced from the orientation of \mathbb{R}^n by the outward-normal first convention.

We consider first the integral along $L_j^\circ(v) \times D_j^\circ$. Here the induced orientation of S^{n+1} by mapping (orientation of L_j) \wedge (orientation of D_i) via u coincides with the one naturally induced from \mathbb{R}^{n+2} . To see this, consider the orientation at a point $(l_j(v), d_j) \in L_j(v) \times D_j$, identified with a point on the image of the embedding

$$\begin{aligned} l_j(v) &= (p, y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^3 \times [-1, 1]^{n-1} \subset \mathbb{R}^{n+2}, \\ d_j &= (q, y'_1, y'_2, \dots, y'_{n-1}) \in \mathbb{R}^3 \times [-2, 2]^{n-1} \subset \mathbb{R}^{n+2} \end{aligned}$$

where (p, q) are points on the boundary of the two components in a crossing of a ribbon presentation such that the orientation for the factor of p corresponds to the orientation on the band, represented by an oriented arc going forward to the based disk (depicted in Figure 4(a)), and such that the orientation for the factor of q corresponds to the induced orientation on the boundary from the orientation of the ribbon disk. By the definition of the ribbon presentation W_k (Figure 4(a)) and the orientation of $L_j(v)$, $dp \wedge dq$ is mapped by the Gauss map into the natural orientation of $S^2 \subset \mathbb{R}^3$ because the linking number of the two arcs is 1. The orientation of $L_j(v) \times D_j$ corresponding to varying p fixing y_1, \dots, y_{n-1} (varying along the orientation of $L_j(v)$) and varying d_j , is $dp \wedge dq \wedge dy'_1 \wedge \dots \wedge dy'_{n-1}$. This is mapped by the map u into the natural orientation of $S^{n+1} \subset \mathbb{R}^{n+2}$. Further, the linking number of $\text{Im } \psi_j^\circledast | L_j^\circ(v)$ and D_j° is ± 1 because it is equal to the intersection number of $\text{Im } \psi_j^\circledast | L_j^\circ(v)$ and an $(n+1)$ -disk bounded by D_j° . Thus the integral along $L_j^\circ(v) \times D_j^\circ$ contributes by 1.

For the integral along S^{n-1} , since the direction of vector from $l_j(v)$ to d_{j+1}^0 coincides with the ingoing normal vector on S^{n-1} , the integral contributes by $(-1)^n \times (-1)^n = 1$ and the result follows. \square

4.4 Relation with the Alexander polynomial

The Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ for a (long) n -knot K is defined by using the Fox calculus for the knot group. See Habiro–Kanenobu–Shima [10] for

detailed definition. The Alexander polynomial is uniquely determined by the conditions: $\Delta_K(1) = 1$ and $(d\Delta_K/dt)(1) = 0$, and we use such a normalized one. Then a series of invariants $\alpha_2, \alpha_3, \dots$ of (long) n -knots are defined by

$$\log \Delta_K(t)|_{t=e^h} = \alpha_2(K)h^2 + \alpha_3(K)h^3 + \dots \in \mathbb{Q}[[h]].$$

The following result gives a complete correspondence between finite type invariant of ribbon 2-knots and α_j -invariants.

Proposition 4.8 *Let $k > 1$ and $n > 1$. Then*

- (1) (Habiro–Kanenobu–Shima) α_k is a primitive (= additive) invariant of type k of ribbon n -knots.
- (2) (Habiro–Shima) We have the identification:

$$\{\mathbb{Q}\text{-valued finite type invariants}\} = \mathbb{Q}[\alpha_2, \alpha_3, \dots]$$

of graded Hopf algebras.

In Habiro–Shima [11], the results are stated for $n = 2$ and it is remarked there that their result is generalized to ribbon n -knots. Kazuo Habiro says that the proof for $n > 2$ is exactly the same as for $n = 2$ and we have checked it. By Remark 4.2, Proposition 4.8 also holds for long ribbon n -knots.

Corollary 4.9 *Let n be an odd integer ≥ 3 . For long ribbon n -knots, \hat{z}_k is equal to a polynomial in $\alpha_2, \alpha_3, \dots, \alpha_k$ ($\deg \alpha_j = j$) of degree $\leq k$ and we have*

$$\hat{z}_k \equiv \begin{cases} -\alpha_k & \text{in } \mathcal{I}_k/\mathcal{I}_{k-1} \text{ if } k \text{ is even,} \\ 0 & \text{in } \mathcal{I}_k/\mathcal{I}_{k-1} \text{ if } k \text{ is odd.} \end{cases}$$

For long ribbon 2-knots, $\hat{z}_3 = \alpha_3 + \lambda\alpha_2$ for some constant $\lambda \in \mathbb{R}$.

In particular, Corollary 4.9 implies

Corollary 4.10 *The BCR invariant is nontrivial for $(n, k) = (\text{odd} \geq 3, \text{even} \geq 2)$ or $(2, 3)$.*

Proof of Corollary 4.9 Since \hat{z}_k is an invariant of type k by Theorem 4.1(1), \hat{z}_k is a degree at most k polynomial in $\alpha_2, \dots, \alpha_k$ by Proposition 4.8.

The principal term of \hat{z}_k is determined by Theorem 4.1(2) and Lemma 4.11 below. \square

Lemma 4.11 *The type k invariant α_k satisfies the following conditions.*

- (1) $\alpha_k(\mathcal{J}_i \# \mathcal{J}_j) = 0$ for $i + j = k$, $i, j > 0$.
 (2) $\alpha_k([W_k; c_1, \dots, c_k]) = (-1)^{k-1}$,
 where W_k and the crossings c_1, \dots, c_k on it are defined in [Figure 4\(a\)](#).

Proof (1) is because α_k is additive.

For (2), it suffices to compute the value of α_k at $[K_{W_k}] - 1$ (1: the class of the trivial knot) because $[K_{W_k^{c_j}}] = 1$. Since $[W_k^{c_1}] = 1$ and any other unclaspings at c_j ($j \neq 1$) on it yields again the trivial knot, we have $[W_k^{c_1}; c_2, \dots, c_k] = 0$, and

$$\begin{aligned} [W_k; c_1, \dots, c_k] &= [W_k; c_2, c_3, \dots, c_k] - [W_k^{c_1}; c_2, c_3, \dots, c_k] \\ &= [W_k; c_2, c_3, \dots, c_k] = [W_k, c_3, \dots, c_k] - [W_k^{c_2}; c_3, \dots, c_k] \\ &= [W_k; c_3, \dots, c_k] = \dots = [W_k; c_k] = [K_{W_k}] - 1. \end{aligned}$$

The fundamental group of $[K_{W_k}]$ is generated by meridians of the part from the based disk and of the parts from the disks \check{D}_j in the crossing c_j . We denote the former one by x_0 and the latter ones by x_1, \dots, x_k . Then by the procedure described by Habiro–Kanenobu–Shima [\[10\]](#), the fundamental group of $[K_{W_k}]$ is presented as

$$\langle x_0, x_1, \dots, x_k \mid x_1^{-1} x_2 x_0 x_2^{-1}, x_2^{-1} x_3 x_0 x_3^{-1}, \dots, x_k^{-1} x_1 x_0 x_1^{-1} \rangle.$$

The Alexander polynomial of $[K_{W_k}]$ is computed by using Fox differential as follows:

$$\det \begin{bmatrix} -t^{-1} & t^{-1} - 1 & 0 & 0 & \dots \\ 0 & -t^{-1} & t^{-1} - 1 & 0 & \dots \\ 0 & 0 & -t^{-1} & t^{-1} - 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ t^{-1} - 1 & 0 & 0 & 0 & \dots \end{bmatrix} = (-t^{-1})^k + (-1)^{k-1} (t^{-1} - 1)^k.$$

After the normalization, $\Delta_{[K_{W_k}]}(t) = 1 - (1 - t)^k$. From this we have

$$\log \Delta_{[K_{W_k}]}(e^h) = \log(1 - (1 - e^h)^k) = -(-h)^k + o(h^k).$$

Hence $\alpha_k([K_{W_k}]) = (-1)^{k-1}$ and

$$\alpha_k([W_k; c_1, \dots, c_k]) = \alpha_k([K_{W_k}] - 1) = \alpha_k([K_{W_k}]) - \alpha_k(1) = (-1)^{k-1} - 0. \quad \square$$

Appendix A Invariance proof

In Cattaneo–Rossi [\[7\]](#) and Rossi [\[15\]](#), the invariance of the higher degree z_k is claimed and in some part idea of the proof is given, while explicit proofs are given in the other

parts even for higher degrees. The explicit descriptions of z_k for higher degrees are not given there, while the complete definitions and proofs are given for degrees up to 3. In this section, we give a proof of [Theorem 2.2](#) filling the details in their idea with our definition of z_k , conventions on the diagrams and the orientations on configuration spaces. We also see that z_k can be obtained naturally from a general framework of diagrams and of the geometry of configuration spaces.

The proof of [Theorem 2.2](#) is described in [Section A.2](#). The proof splits into the four cases described in [Section A.1.2](#). More detailed checks corresponding to the four cases, referred in the proof are described in the subsequent subsections starting from [Section A.3](#).

A.1 Descriptions of faces in ∂C_Γ

A.1.1 Codimension one faces It is known that C_Γ is a smooth manifold with corners and ∂C_Γ admits a stratification (see Fulton–MacPherson [\[8\]](#) and Bott–Taubes [\[5\]](#)). We will need only to consider the codimension one faces in ∂C_Γ for our purpose. So we review here how each codimension one face can be described.

As observed in [\[8\]](#) and Axelrod–Singer [\[2\]](#), the set of codimension one faces in ∂C_Γ is in bijective correspondence with the set of all subsets of the vertices of Γ , which are obtained by blowing-up along the corresponding diagonals. Let Δ_A denote the diagonal set consisting of configurations such that all points in A coincide, and let $\Delta_{\infty,A}$ be the diagonal set consisting of configurations such that all points in A coincide at ∞ . Denote by $\mathcal{S}_{\Gamma,A}$ and $\mathcal{S}_{\Gamma,\infty,A}$ the codimension one faces corresponding to blow-ups along Δ_A and $\Delta_{\infty,A}$ respectively.

Let us briefly see how $\mathcal{S}_{\Gamma,A}$ is presented. Details are found in Rossi [\[15\]](#). (This is somewhat analogous to the case of knots in 3–dimension. This case is described by Poirier [\[14\]](#) very explicitly.) Let $V(\Gamma)$ denote the set of vertices of Γ . There are two cases:

Case 1 If $A \subset V(\Gamma)$ does not have internal vertices, then $\mathcal{S}_{\Gamma,A}$ is the pullback bundle $\varphi^* B_A = B_A \times C_{\Gamma/A}$ in the following commutative diagram.

$$(16) \quad \begin{array}{ccc} \mathcal{S}_{\Gamma,A} & \xrightarrow{\widehat{\varphi}} & B_A \\ \widehat{\pi}^\partial \downarrow & & \downarrow \pi^\partial \\ C_{\Gamma/A} & \xrightarrow{\varphi} & * \end{array}$$

Here

- B_A is the space of configurations of points on \mathbb{R}^{n+2} , modulo overall translations and dilations.
- $C_{\Gamma/A}$ is the space of configurations obtained from configurations in C_{Γ} by collapsing points in A ,
- π^{∂} and φ are the maps to a point.

We consider each piece in (16) as a fiber over a point in $\text{Emb}(\mathbb{R}^n, \mathbb{R}^{n+2})$.

Case 2 If $A \subset V(\Gamma)$ has internal vertices, then $S_{\Gamma,A}$ is the pullback bundle $\varphi^* \widehat{B}_A$ in the following commutative diagram.

$$(17) \quad \begin{array}{ccc} S_{\Gamma,A} & \xrightarrow{\widehat{\varphi}} & \widehat{B}_A \\ \widehat{\pi}^{\partial} \downarrow & & \downarrow \pi^{\partial} \\ C_{\Gamma/A} & \xrightarrow{\varphi} & I_n(\mathbb{R}^{n+2}) \end{array}$$

Here

- φ is the composition of the embedding and the generalized Gauss map (the tangent map together with an assignment of the n -frame in \mathbb{R}^{n+2} determined by the embedding) at the point where the points in A coincide,
- \widehat{B}_A is the space of configurations of points on $(\mathbb{R}^{n+2}, \iota \mathbb{R}^n)$, ($\iota \in I_n(\mathbb{R}^{n+2})$ is an n -frame in \mathbb{R}^{n+2}) modulo translations and dilations, together with $\iota \in I_n(\mathbb{R}^{n+2})$,
- π^{∂} is the map giving the underlying n -frame in \mathbb{R}^{n+2} .

For the face $S_{\Gamma,\infty,A}$, the fiber $B_{\infty,A}$ of the unit normal bundle $SN\Delta_{\infty,A}$ over a point in $\Delta_{\infty,A} \subset (S^{n+2})^k$ where the points in A coincide at $\infty \in S^n \subset S^{n+2}$, is identified with the set of points in $T_{\infty}S^n$ (resp. $T_{\infty}S^{n+2}$) modulo overall translations and dilations along $T_{\infty}S^n$. Thus $S_{\Gamma,\infty,A}$ may be identified with $C_{\Gamma \setminus A} \times B_{\infty,A}$. We will call a point in $B_A, \widehat{B}_A, B_{\infty,A}$ a *relative configuration*.

The forms on C_{Γ}^0 defined in Section 2.3 extend smoothly to C_{Γ} as the naturally defined forms by Gauss maps via the coordinates determined by the infinitesimal embedding ι .

A.1.2 Classification of codimension one faces Theorem 2.2 will be proved by looking at the integrals restricted to the codimension one faces classified as follows.

Principal faces The faces corresponding to the diagonal where exactly two of the $|V(\Gamma)|$ points coincide in $S^{n+2} \setminus \{\infty\}$.

Hidden faces The faces corresponding to the diagonal where at least 3 (but not all) of the $|V(\Gamma)|$ points coincide in $S^{n+2} \setminus \{\infty\}$.

Infinite faces The faces corresponding to the diagonal where some of $|V(\Gamma)|$ points coincide at $\infty \in S^n$.

Anomalous faces The faces corresponding to the diagonal where all the $|V(\Gamma)|$ points coincide in $S^n \setminus \{\infty\}$.

A.2 Proof of Theorem 2.2

Let $\pi: E \rightarrow B$ be a bundle with n dimensional fiber F . Then the *push-forward* (or *integral along the fiber*) $\pi_*\omega$ of an $(n+p)$ -form ω on E is a p -form on B defined by

$$\int_c \pi_*\omega = \int_{\pi^{-1}(c)} \omega,$$

where c is a p -dimensional chain in B .

Let $\pi^\partial: \partial_F E \rightarrow B$ be the restriction of π to ∂F -bundle with the orientation induced from $\text{int}(F)$, that is, $\Omega(\partial F) = i_n \Omega(F)$ where n is the in-going normal vector field over ∂F . Then the generalized Stokes theorem for the pushforward is

$$(18) \quad d\pi_*\omega = \pi_*d\omega + (-1)^{\deg \pi^\partial \omega} \pi_*^\partial \omega.$$

The derivation of (18) is, for example, in Bott–Taubes [5, Appendix].

Proof of Theorem 2.2 Let ψ_0 and ψ_1 be two embeddings $\mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$ that are standard near ∞ . The formula (18) is used to prove the invariance of z_k (or \hat{z}_3) as follows. For each Γ , consider

$$(19) \quad \{C_\Gamma(\psi)\}_{\psi \in \text{Emb}(\mathbb{R}^n, \mathbb{R}^{n+2})} \rightarrow \text{Emb}(\mathbb{R}^n, \mathbb{R}^{n+2})$$

as a bundle over the space $\text{Emb}(\mathbb{R}^n, \mathbb{R}^{n+2})$ of embeddings with fiber the configuration space C_Γ and consider z_k as a 0-form on $\text{Emb}(\mathbb{R}^n, \mathbb{R}^{n+2})$, which is defined as a sum of pushforwards of the form $\frac{\omega(\Gamma)w_k(\Gamma)}{|\text{Aut } \Gamma|}$ on the C_Γ -fiber of the bundle (19). Then the invariance of z_k relies on the closedness of it because if two embeddings ψ_0 and ψ_1 are connected by a smoothly parametrized embeddings ψ_t ($t \in [0, 1]$), then by Stokes' theorem,

$$z_k(\psi_1) - z_k(\psi_0) = \int_0^1 dz_k(\psi_t).$$

Since $\omega(\Gamma)$ is closed,

$$dI(\Gamma) = \int_{\partial C_\Gamma} \omega(\Gamma)$$

by (18) where the integral of the RHS is the pushforward restricted to ∂C_Γ in (19). In particular, only the codimension one faces in ∂C_Γ contribute to the above integral. Therefore the obstruction to the closedness of z_k is

$$(20) \quad dz_k = \sum_{\Gamma} \frac{w_k(\Gamma)}{|\text{Aut } \Gamma|} \sum_{S \subset \partial C_\Gamma: \text{codim } 1} \int_S \omega(\Gamma).$$

Recall from A.1.2, there are four types of codimension one faces in ∂C_Γ : principal faces, hidden faces, infinite faces, anomalous faces.

By Proposition A.1 below, all the principal face contributions cancel each other in the sum. Namely,

$$\sum_{\Gamma} \frac{w_k(\Gamma)}{|\text{Aut } \Gamma|} \sum_{S \subset \partial C_\Gamma: \text{principal}} \int_S \omega(\Gamma) = 0.$$

Here the rules of the canceling are given by the ST, ST2 and C relations.

The vanishing of the integral restricted to the hidden and the infinite faces will be discussed in Section A.5 and Section A.6. About the contributions for the anomalous faces will be discussed in Proposition A.12 and A.13. \square

A.3 Configuration space integral restricted to the faces

Now we shall see the explicit form of the integral $I(\Gamma)$ extended to the codimension one faces.

In the diagram (16), the form $\omega(\Gamma)$ extended to $\mathcal{S}_{\Gamma, A}$ can be written as

$$(21) \quad \omega(\Gamma) = \hat{\varphi}^* \lambda_1(\Gamma_A) \wedge \hat{\pi}^{\partial*} \lambda_2(\Gamma/A)$$

for some $\lambda_1(\Gamma_A) \in \Omega^*(B_A)$ and $\lambda_2(\Gamma/A) \in \Omega^*(C_{\Gamma/A})$ both determined by graphs and the Gauss maps. Therefore $\hat{\pi}_*^{\partial} \omega(\Gamma) = \hat{\pi}_*^{\partial} \hat{\varphi}^* \lambda_1(\Gamma_A) \wedge \lambda_2(\Gamma/A) = \varphi^* \pi_*^{\partial} \lambda_1(\Gamma_A) \wedge \lambda_2(\Gamma/A)$ by commutativity of (16).

Similarly, in the diagram (17), the form $\omega(\Gamma)$ extended to $\mathcal{S}_{\Gamma, A}$ has the decomposition (21) too for some $\lambda_1(\Gamma_A) \in \Omega^*(\hat{B}_A)$ and $\lambda_2(\Gamma/A) \in \Omega^*(C_{\Gamma/A})$ both determined by graphs. $\hat{\pi}_*^{\partial} \omega(\Gamma_A) = \varphi^* \pi_*^{\partial} \lambda_1(\Gamma_A) \wedge \lambda_2(\Gamma/A)$ also holds.

In other words, in both cases the integral

$$I(\Gamma, A) := \int_{C_{\Gamma/A}} \varphi^* \pi_*^{\partial} \lambda_1(\Gamma) \wedge \lambda_2(\Gamma/A)$$

along the fiber $C_{\Gamma/A}$, with the orientation induced from $\text{int}(C_\Gamma)$, is precisely the integral of $\omega(\Gamma)$ along the fiber $\mathcal{S}_{\Gamma, A}$, as a fiber over the space of embeddings $\text{Emb}(\mathbb{R}^n, \mathbb{R}^{n+2})$.

A.4 Principal faces

In this subsection, we will prove the following proposition.

Proposition A.1 For $A \subset V(\Gamma)$ with $|A| = 2$,

$$\sum_{\Gamma \in \mathcal{G}_k^0} \sum_A \frac{I(\Gamma, A) w_k(\Gamma)}{|\text{Aut } \Gamma|} = 0.$$

This proposition shows that the configuration space integrals restricted to principal faces cancel each other.

A.4.1 Quasi Jacobi diagrams Since each principal face corresponds to a collapsing of an edge in a Jacobi diagram, it is represented by using the following graphs obtained by collapsing an edge. We will call such a graph a *quasi Jacobi diagram*. We say that the vertex where an edge of Γ has been collapsed is *exceptional*. Vertex orientation of a quasi Jacobi diagram is also defined. It is defined as a choice of a bijection

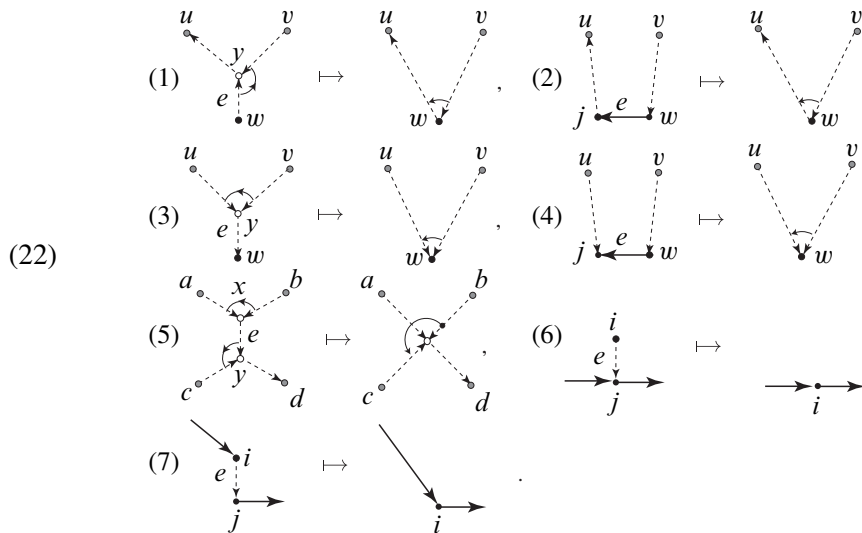
$$o_v: \{\text{two ingoing } \theta\text{-edges incident to } v\} \rightarrow \{1, 2\}$$

for each non-exceptional external vertices v and a bijection

$$\begin{aligned} o_v: \{\text{three ingoing } \theta\text{-edges incident to } v\} &\rightarrow \{1, 2, 3\} && \text{if } v \text{ is 4-valent} \\ o_v: \{\text{two ingoing } \theta\text{-edges incident to } v\} &\rightarrow \{1, 2\} && \text{otherwise} \end{aligned}$$

for the exceptional vertex v having incident θ -edges.

Let Γ be a Jacobi diagram \mathbb{R}^n and e be a θ -edge of Γ . The operator δ_e of Γ giving a quasi Jacobi diagram is defined as follows:



A.4.2 Compactified configuration space for $\delta_e \Gamma$ We define the configuration space $C_{\delta_e \Gamma}^0(\psi)$ so that a point in $C_{\delta_e \Gamma}^0(\psi)$ represents a position on the diagonal $\Delta_{\{ij\}}$ in $(S^{n+2})^k$ where $e = (i, j)$. Let $C_{\delta_e \Gamma}(\psi)$ be its closure in $C_\Gamma(\psi)$. Let

$$\begin{aligned} C_{\delta_e \Gamma}^0(\psi) := & \{(x_1, \dots, x_q, x_{q+1}, \dots, x_{q+s}) \mid \\ & x_1, \dots, x_q \in \mathbb{R}^n, x_{q+1}, \dots, x_{q+s} \in \mathbb{R}^{n+2}, \\ & x_i \neq x_j \quad \text{if } i, j \geq q+1, i \neq j, \\ & \psi(x_i) \neq \psi(x_j) \quad \text{if } i, j \leq q, i \neq j, \\ & \psi(x_i) \neq \psi(x_j) \quad \text{if } i \leq q, j \geq q+1\}. \end{aligned}$$

Then the principal faces corresponding to the contractions in (22) are identified with the following spaces.

type	space	type	space
(1)	$S^{n+1} \times C_{\delta_e \Gamma}$	(5)	$S^{n+1} \times C_{\delta_e \Gamma}$
(2)	$S^{n-1} \times C_{\delta_e \Gamma}$	(6)	$S^{n-1} \times C_{\delta_e \Gamma}$
(3)	$S^{n+1} \times C_{\delta_e \Gamma}$	(7)	$S^{n-1} \times C_{\delta_e \Gamma}$
(4)	$S^{n-1} \times C_{\delta_e \Gamma}$		

Note that if n is even, $C_{\delta_e \Gamma}$ can be canonically oriented. If n is odd, we have the following lemma.

Lemma A.2 *Suppose n is odd. Let Γ_x be a vertex oriented quasi Jacobi diagram with one exceptional vertex x and $\bar{\Gamma}_x$ be Γ_x with its vertex orientation at one external vertex reversed. Then there exists a choice of an orientation $\Omega = \Omega(\Gamma_x)$ on $C_{\Gamma_x}(\psi)$ such that*

$$(24) \quad \Omega(\bar{\Gamma}_x) = -\Omega(\Gamma_x).$$

Proof If the θ part of Γ_x is uni-trivalent except for one external tetravalent vertex x , where four edges e_1, e_2, e_3, e_4 with $\partial e_i = (v_i, x)$ for $i = 1, 2, 3$, $\partial e_4 = (x, v_4)$, $o_x(e_i) = i$ meet, set

$$\begin{aligned} \Omega_x &:= dX_x^1 \wedge \dots \wedge dX_x^{n+2} \wedge \Omega_{\bar{e}_{1-}} \wedge \Omega_{\bar{e}_{2-}} \wedge \Omega_{\bar{e}_{3-}} \wedge \Omega_{\bar{e}_{4+}}, \\ \Omega(\Gamma_x) &:= \Omega_x \wedge \bigwedge_{\substack{e \in E_\theta(\Gamma) \setminus \\ \{e_1, e_2, e_3, e_4\}}} \Omega_e. \end{aligned}$$

If the θ part of Γ_x is uni-trivalent except for one internal bivalent vertex x , where two edges e_1, e_2 with $\partial e_i = (v_i, x)$, $o_x(e_i) = i$ meet, set

$$\begin{aligned}\Omega_x &:= dX_x^1 \wedge \dots \wedge dX_x^n \wedge \Omega_{\bar{e}_{1-}} \wedge \Omega_{\bar{e}_{2-}}, \\ \Omega(\Gamma_x) &:= \Omega_x \wedge \bigwedge_{\substack{e \in E_\theta(\Gamma) \\ \{e_1, e_2\}}} \Omega_e.\end{aligned}$$

The cases of other edge orientations are similar.

If Γ_x has an exceptional vertex x having 2 or 3 incident η -edges, we define

$$\begin{aligned}\Omega_x &:= dX_x^1 \wedge \dots \wedge dX_x^n, \\ \Omega(\Gamma_x) &:= \Omega_x \wedge \bigwedge_{e \in E_\theta(\Gamma)} \Omega_e.\end{aligned}$$

Then the property (24) follows from the definition. \square

A.4.3 Canceling of the integral along principal faces

Lemma A.3 *Suppose n is odd. Let Γ be a vertex oriented Jacobi diagram and e be an edge of Γ . Then the orientation of the principal face induced from the orientation $\Omega(\Gamma)$ is as follows.*

type	space	induced orientation	type	space	induced orientation
(1)	$S^{n+1} \times C_{\delta_e \Gamma}$	$\omega_{n+1} \wedge \Omega(\delta_e \Gamma)$	(5)	$S^{n+1} \times C_{\delta_e \Gamma}$	$-\omega_{n+1} \wedge \Omega(\delta_e \Gamma)$
(2)	$S^{n-1} \times C_{\delta_e \Gamma}$	$-\omega_{n-1} \wedge \Omega(\delta_e \Gamma)$	(6)	$S^{n-1} \times C_{\delta_e \Gamma}$	$\omega_{n-1} \wedge \Omega(\delta_e \Gamma)$
(3)	$S^{n+1} \times C_{\delta_e \Gamma}$	$\omega_{n+1} \wedge \Omega(\delta_e \Gamma)$	(7)	$S^{n-1} \times C_{\delta_e \Gamma}$	$\omega_{n-1} \wedge \Omega(\delta_e \Gamma)$
(4)	$S^{n-1} \times C_{\delta_e \Gamma}$	$\omega_{n-1} \wedge \Omega(\delta_e \Gamma)$			

Proof Let Γ_1 be as in the LHS of case (1) and we consider the orientation induced on the principal face $\mathcal{S}_{\Gamma_1, e_1}$ corresponding to the contraction of the edge $e_1 = (w, y)$. The orientation of C_{Γ_1} is

$$\begin{aligned}\Omega(\Gamma_1) &= (dX_y^1 d^n X_w) \wedge (dX_y^2 \Omega_{\bar{e}_v}) \wedge (\Omega_{\bar{e}_u} dX_y^3 \dots dX_y^{n+2}) \wedge (\text{the rest}) \\ &= dX_y^1 \dots dX_y^{n+2} \wedge d^n X_w \Omega_{\bar{e}_v} \Omega_{\bar{e}_u} \wedge (\text{the rest}).\end{aligned}$$

Let n_1 be the outgoing unit normal vector field on S^{n+1} . Then the ingoing normal vector field on the principal face is given by n_1 . The induced orientation of $\mathcal{S}_{\Gamma_1, e_1}$ is then

$$i_{n_1} \Omega(\Gamma_1) = \omega_{n+1}(\alpha) \wedge d^n X_w \Omega_{\bar{e}_v} \Omega_{\bar{e}_u} \wedge (\text{the rest}) = \omega_{n+1}(\alpha) \wedge \Omega(\delta_{e_1} \Gamma_1)$$

where $\alpha = \frac{X_y - X_w}{|X_y - X_w|}$. For cases (3) and (5), the results follow from the following:

$$\begin{aligned} i_{n_1} \Omega(\Gamma_3) &= \omega_{n+1} d^n X_w \Omega_{\bar{e}_v} \Omega_{\bar{e}_u} \wedge (\text{the rest}) \\ i_{n_1} \Omega(\Gamma_5) &= -\omega_{n+1} d^{n+2} X_x \Omega_{\bar{e}_b} \Omega_{\bar{e}_d} \Omega_{\bar{e}_c} \Omega_{\bar{e}_d} \wedge (\text{the rest}) \end{aligned}$$

Let Γ_2 be as in the LHS of case (2) and we consider the orientation induced on the principal face $\mathcal{S}_{\Gamma_2, e_2}$ corresponding to the contraction of the edge $e_2 = (w, j)$. The orientation of C_{Γ_2} is

$$\Omega(\Gamma_2) = d^n X_w \Omega_{\bar{e}_v} \Omega_{\bar{e}_u} d^n X_j \wedge (\text{the rest}) = d^n X_w d^n X_j \Omega_{\bar{e}_v} \Omega_{\bar{e}_u} \wedge (\text{the rest}).$$

Let n_2 be the outgoing unit normal vector field on S^{n-1} . Then the induced orientation of $\mathcal{S}_{\Gamma_2, e_2}$ is

$$i_{n_2} \Omega(\Gamma_2) = -\omega_{n-1}(\beta) \wedge d^n X_w \Omega_{\bar{e}_v} \Omega_{\bar{e}_u} \wedge (\text{the rest}) = -\omega_{n-1}(\beta) \wedge \Omega(\delta_{e_2} \Gamma_2)$$

where $\beta = \frac{X_j - X_w}{|X_j - X_w|}$. For case (4) the result follows similarly from the following:

$$i_{n_2} \Omega(\Gamma_4) = \omega_{n-1} \Omega_{\bar{e}_w} \Omega_{\bar{e}_v} \Omega_{\bar{e}_u} \wedge (\text{the rest})$$

Let Γ_6 be as in the LHS of case (6) and we consider the orientation induced on the principal face $\mathcal{S}_{\Gamma_6, e_6}$ corresponding to the contraction of the edge $e_6 = (i, j)$. The orientation of C_{Γ_6} is

$$\Omega(\Gamma_6) = d^n X_j d^n X_i \wedge (\text{the rest}).$$

The induced orientation of $\mathcal{S}_{\Gamma_6, e_6}$ is

$$i_{n_2} \Omega(\Gamma_6) = \omega_{n-1}(\gamma) d^n X_i \wedge (\text{the rest}) = \omega_{n-1}(\gamma) \wedge \Omega(\delta_{e_6} \Gamma_6)$$

where $\gamma = \frac{X_j - X_i}{|X_j - X_i|}$. Case (7) is similar. \square

Lemma A.4 *Let (Γ_i, e_i) , for $i = 1, \dots, 7$, be the pair of the Jacobi diagram and the edge to be contracted on it, as in the LHS of case (i) in (22). Then we have*

$$I(\Gamma_1, e_1) = -I(\Gamma_2, e_2), \quad I(\Gamma_3, e_3) = -I(\Gamma_4, e_4), \quad I(\Gamma_6, e_6) = I(\Gamma_7, e_7).$$

Proof By definition,

$$\begin{aligned} I(\Gamma_1, e_1) &= \int_{S^{n+1} \times C_{\delta_{e_1} \Gamma_1}} \theta_{e_1} \wedge \lambda_2(\delta_{e_1} \Gamma_1) \\ &= \int_{S^{n+1}} \theta_{e_1} \int_{C_{\delta_{e_1} \Gamma_1}} \lambda_2(\delta_{e_1} \Gamma_1) = \int_{C_{\delta_{e_1} \Gamma_1}} \lambda_2(\delta_{e_1} \Gamma_1) \end{aligned}$$

where θ_{e_1} denotes the pullback form of ω_{n+1} via the Gauss map u with respect to e_1 . Similarly we have

$$I(\Gamma_2, e_2) = \int_{C_{\delta_{e_2}\Gamma_2}} \lambda_2(\delta_{e_2}\Gamma_2).$$

When n is even, the induced orientations on $C_{\delta_{e_1}\Gamma_1} \cong C_{\delta_{e_2}\Gamma_2}$ are the same, and when n is odd, the induced orientations on $C_{\delta_{e_1}\Gamma_1} \cong C_{\delta_{e_2}\Gamma_2}$ are opposite by Lemma A.3. So it suffices to prove $\lambda_2(\delta_{e_1}\Gamma_1) = (-1)^{n-1}\lambda_2(\delta_{e_2}\Gamma_2)$. In terms of the labels of vertices in (22), we have

$$\omega(\Gamma_1) = \theta_{wy} \wedge \theta_{vy} \wedge \omega((y, u)) \wedge (\text{the rest}).$$

On the principal face corresponding to the contraction of $e_1 = (w, y)$, this form becomes $\theta_{e_1} \wedge \theta_{vw}\omega((w, u)) \wedge (\text{the rest})$ and thus $\lambda_2(\delta_{e_1}\Gamma_1) = \theta_{vw}\omega((w, u)) \wedge (\text{the rest})$. On the other hand, we have

$$\omega(\Gamma_2) = \theta_{vw}\eta_{wj}\omega((j, u)) = (-1)^{n-1}\eta_{wj}\theta_{vw}\omega((j, u)) \wedge (\text{the rest}).$$

Thus $\lambda_2(\delta_{e_2}\Gamma_2) = (-1)^{n-1}\theta_{vw}\omega((w, u)) \wedge (\text{the rest}) \equiv (-1)^{n-1}\lambda_2(\delta_{e_1}\Gamma_1)$.

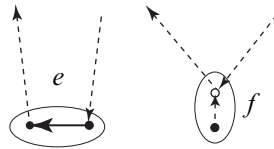
The second and the third identity may be similarly proved from the facts:

- (1) induced orientations on $C_{\delta_{e_3}\Gamma_3} \cong C_{\delta_{e_4}\Gamma_4}$ are the same,
- (2) $\lambda_2(\delta_{e_3}\Gamma_3) = (-1)^n\omega_{n+1}\theta_{vw}\theta_{uw}(\text{rest})$,
- (3) $\lambda_2(\delta_{e_4}\Gamma_4) = (-1)^{n-1}\omega_{n-1}\theta_{vw}\theta_{uw}(\text{rest})$,
- (4) induced orientations on $C_{\delta_{e_6}\Gamma_6} \cong C_{\delta_{e_7}\Gamma_7}$ are the same,
- (5) $\lambda_2(\delta_{e_6}\Gamma_6) = \lambda_2(\delta_{e_7}\Gamma_7)$ □

Proof of Proposition A.1 First we show that if the two vertices in A are not connected by an edge, then $I(\Gamma, A) = 0$. By (16) or (17), $\mathcal{S}_{\Gamma, A}$ is a sphere bundle over $C_{\Gamma/A}$ and from Section A.3, $\omega(\Gamma)$ is basic in this bundle, namely a pullback of some form on $C_{\Gamma/A}$ with less dimension. Hence $I(\Gamma, A) = 0$.

Then it is enough to prove for the faces corresponding to the collapse of an edge of Γ . The proof is divided according to the cases in the definition of δ_e .

Cases (1) and (2) (Appearance of the ST relation) For cases (1) and (2), we will see that the contributions of the faces corresponding to the following pair of contractions of graphs cancel each other:



which we denote by (Γ^a, e) , (Γ^b, f) respectively.

From (23), the face of (Γ^a, e) is identified with $S^{n-1} \times C_{\delta_e \Gamma^a}$, where the factor S^{n-1} is the relative configuration space of 2 points on $T_x \mathbb{R}^n$. The face of (Γ^b, f) is identified with $S^{n+1} \times C_{\delta_f \Gamma^b}$, where the factor S^{n+1} is the relative configuration space of 2 points on \mathbb{R}^{n+2} .

Let n_a be the number of η -edges e' on Γ^a such that $\delta_e \Gamma^a = \delta_{e'} \Gamma^a$. n_b is similarly defined to be the number of η -edges f' on Γ^b such that $\delta_f \Gamma^b = \delta_{f'} \Gamma^b$. Then we have

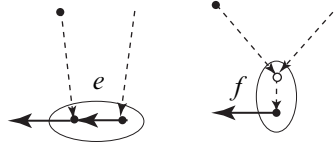
$$\frac{|\text{Aut } \Gamma^a|}{n_a} = \frac{|\text{Aut } \Gamma^b|}{n_b} = |\text{Aut } \delta_e \Gamma^a| = |\text{Aut } \delta_f \Gamma^b|.$$

We assume that each automorphism in $\text{Aut } \Gamma^a$ preserves the orientation of C_{Γ^a} since if not $I(\Gamma^a, e) = 0$. The case of Γ^b is similar. Since the integration along the sphere factors contributes by 1, z_k restricted to these faces is

$$\frac{n_a I(\Gamma^a, e) w_k(\Gamma^a)}{|\text{Aut } \Gamma^a|} + \frac{n_b I(\Gamma^b, f) w_k(\Gamma^b)}{|\text{Aut } \Gamma^b|} = \frac{I(\Gamma^a, e)}{|\text{Aut } \delta_e \Gamma^a|} (w_k(\Gamma^a) - w_k(\Gamma^b)) = 0,$$

where $I(\Gamma^b, f) = -I(\Gamma^a, e)$ by Lemma A.4 and by the ST relation.

Cases (3) and (4) (Appearance of the ST2 relation) For cases (3) and (4), we will see that the contributions of the faces corresponding to the following pair of graphs cancel each other:



which we denote by (Γ^c, e) , (Γ^d, f) respectively. From (23), the faces corresponding to (Γ^c, e) , (Γ^d, f) can be identified with the spaces

$$\begin{aligned} (\Gamma^c, e) &: S^{n-1} \times C_{\delta_e \Gamma^c}, \\ (\Gamma^d, f) &: S^{n+1} \times C_{\delta_f \Gamma^d}. \end{aligned}$$

Let n_c be the number of η -edges e' on Γ^c such that $\delta_e \Gamma^c = \delta_{e'} \Gamma^c$. n_d is similarly defined. Then we have

$$\frac{|\text{Aut } \Gamma^c|}{n_c} = \frac{|\text{Aut } \Gamma^d|}{n_d} = |\text{Aut } \delta_e \Gamma^c| = |\text{Aut } \delta_f \Gamma^d|.$$

We assume that each automorphism in $\text{Aut } \Gamma^c$ preserves the orientation of C_{Γ^c} since if not $I(\Gamma^c, e) = 0$. The case of Γ^d is similar. Note that some of them may not be

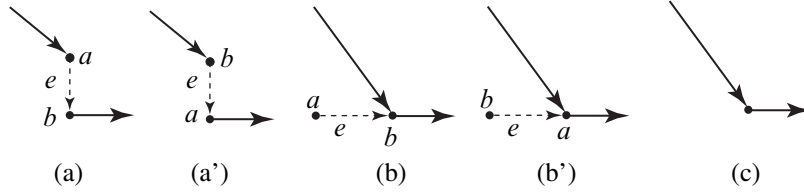


Figure 7: Possible graphs contracted into (c)

Jacobi diagrams. Since the integration along the sphere factors contributes by 1, z_k restricted to these faces is

$$\frac{n_c I(\Gamma^c, e) w_k(\Gamma^c)}{|\text{Aut } \Gamma^c|} + \frac{n_d I(\Gamma^d, f) w_k(\Gamma^d)}{|\text{Aut } \Gamma^d|} = \frac{I(\Gamma^c, e)}{|\text{Aut } \delta_e \Gamma^c|} (w_k(\Gamma^c) - w_k(\Gamma^d)) = 0,$$

where $I(\Gamma^d, f) = -I(\Gamma^c, e)$ by Lemma A.4, and by the ST2 relation.

Cases (6) and (7) (Appearance of the C relation) Then z_k can be written without using $|\text{Aut } \Gamma|$ as

$$z_k = \frac{1}{(2k)!} \sum_{\Gamma: \text{labeled}} I(\Gamma) w_k(\Gamma),$$

since there are $(2k)!/|\text{Aut } \Gamma|$ ways of labeling the $2k$ vertices with distinguished labels in $\{1, 2, \dots, 2k\}$. There are possibly 4 labeled graphs $\Gamma_1^L, \Gamma_2^L, \Gamma_1^R, \Gamma_2^R$ as in Figure 7(a), (a'), (b) and (b') respectively, which are mapped by δ_e to the graph as in Figure 7(c). From (23), the corresponding principal faces can be identified with the following spaces via the tangent framing:

$$S^{n-1} \times C_{\delta_e \Gamma_1^L}, \quad S^{n-1} \times C_{\delta_e \Gamma_2^L}, \quad S^{n-1} \times C_{\delta_e \Gamma_1^R}, \quad S^{n-1} \times C_{\delta_e \Gamma_2^R}.$$

Since the integration along S^{n-1} fiber contributes by some common 2-form μ on $I_n(\mathbb{R}^{n+2})$, the integrals restricted to these faces are

$$\begin{aligned} & \frac{1}{(2k)!} (I(\Gamma_1^L, e) w_k(\Gamma_1^L) + I(\Gamma_2^L, e) w_k(\Gamma_2^L) + I(\Gamma_1^R, e) w_k(\Gamma_1^R) + I(\Gamma_2^R, e) w_k(\Gamma_2^R)) \\ &= \frac{1}{(2k)!} (w_k(\Gamma_1^L) + w_k(\Gamma_2^L) + w_k(\Gamma_1^R) + w_k(\Gamma_2^R)) \int_{C_{\delta_e \Gamma_1^L}} \varphi^* \mu \wedge \lambda_2(\delta_e \Gamma_1^L) = 0, \end{aligned}$$

where $I(\Gamma_i^L, e) = I(\Gamma_i^R, e)$ by Lemma A.4 and by the C relation.

Case (5) There are just 6 labeled graphs $\Gamma_1^I, \Gamma_2^I, \Gamma_1^H, \Gamma_2^H, \Gamma_1^X, \Gamma_2^X$ as in Figure 8(a), (a'), (b), (b'), (c) and (c') respectively, which are mapped by δ_e to the graph Γ^d as in Figure 8(d) (the $-$ signs of (b) and (b') denote the reversion of the vertex orientation at one vertex, and are because the cyclic permutation of 4 edges is an odd permutation).

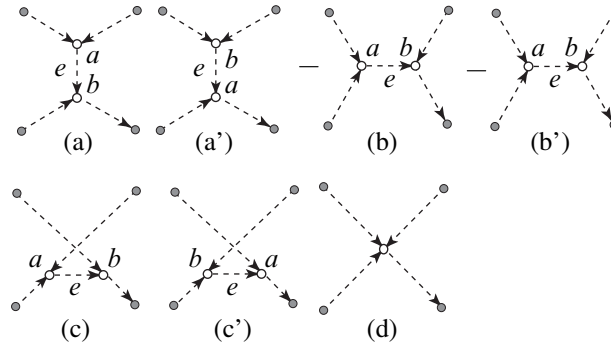


Figure 8: Possible graphs contracted into (d)

Since two of the three edges incident to the tetra-valent vertex of Γ^d are always hairs on a subgraph as in (2)(b), the involution exchanging the two hairs preserves the integrand form and reverses the orientation of the configuration space if n is odd, or reverses the sign of the integrand form and preserves the orientation of the configuration space if n is even. Hence the integral over such faces are self-cancelling. \square

A.5 Hidden faces

Now we briefly review the proof of the vanishing of $I(\Gamma)w_k(\Gamma)$ restricted to the hidden faces, where the proof is explicitly given by Rossi [15]. Let $A \subset V(\Gamma)$, $|A| \geq 3$ be a set of points to be collapsed having external θ -edges (θ -edges connecting points in A and points not in A) and let Γ_A be the subgraph of Γ consisting of edges connecting two vertices in A . The goal of this subsection is to prove the following proposition.

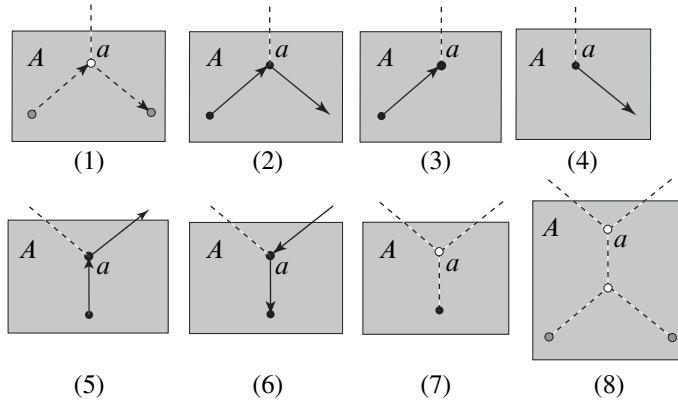
Proposition A.5 *The term $I(\Gamma)w_k(\Gamma)$ restricted to any face corresponding to the diagonal where the points in $A \subset V(\Gamma)$ with $A \neq V(\Gamma)$, $|A| \geq 3$ coincide, vanishes.*

The proof of Proposition A.5 is done by using the Vanishing Lemmata A.8, A.9 and A.10 below, proved in [15] and by Cattaneo, Cotta-Ramusino and Longoni [6].

Lemma A.6 *If $|A| \geq 3$ and A has external θ -edges, then the integral restricted to the corresponding face vanishes.*

Proof By Lemma A.7 below, we need only to consider the cases in which Γ_A is connected.

Let a be a vertex in Γ_A which is an end of an external θ -edge. The possible cases in which Γ_A are connected are as in the following picture:



The reasons for the vanishing of the integral $I(\Gamma)$ along $\mathcal{S}_{\Gamma,A}$ are as follows.

Cases (1) and (2) By [Lemma A.8](#) below.

Cases (3)–(8) By [Lemma A.9](#) below. □

Lemma A.7 *If Γ_A is not connected, then the integral restricted to $\mathcal{S}_{\Gamma,A}$ vanishes.*

Proof By the observations in [Section A.1](#), the face $\mathcal{S}_{\Gamma,A}$ is a bundle $\mathcal{S}_{\Gamma,A} \rightarrow C_{\Gamma/A}$ with fiber F_A the configuration space associated to the subgraph Γ_A modulo translations and dilations. Since Γ_A is not connected, the action T , acting on F_A as translations of one connected component in Γ_A fixing all other components, is defined. The condition $|A| \geq 3$ implies that this action is non trivial on F_A (hence independent of the overall translations acting on F_A) and the quotient map $F_A \rightarrow F_A/T$ gives a bundle $\pi^T: \mathcal{S}_{\Gamma,A} \rightarrow \mathcal{S}_{\Gamma,A}/T$ with n or $(n+2)$ dimensional fiber, depending on whether the translated component has internal vertices or not. Since the T -action does not affect the integrand form $\omega(\Gamma)$ extended to $\mathcal{S}_{\Gamma,A}$, namely the integrand form can be written as $(\pi^T)^*\omega(\Gamma)^T$ for some form $\omega(\Gamma)^T$ on $\mathcal{S}_{\Gamma,A}/T$. Thus

$$\int_{\mathcal{S}_{\Gamma,A}} \omega(\Gamma) = \int_{\mathcal{S}_{\Gamma}} (\pi^T)^*\omega(\Gamma)^T = \int_{\mathcal{S}_{\Gamma/T}} \omega(\Gamma)^T$$

where the integration is the fiber integration. The last integral is over the space with codimension at least $(n+1)$ and so it vanishes. □

Proof of Proposition A.5 By [Lemma A.6](#) and [Proposition 3.1](#), for a non vanishing face the θ part of Γ_A must form a disjoint union of chords, trees and wheels entirely included in Γ_A . So we assume this condition in the rest of the proof.

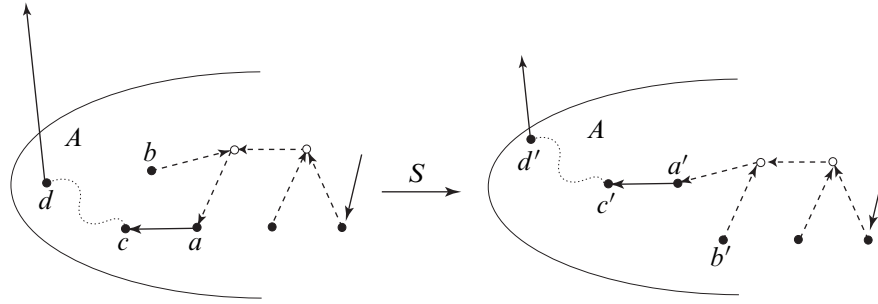


Figure 9: $(x_d, \dots, x_c, x_a, x_b, \dots) \mapsto (x_d + (x_b - x_a), \dots, x_c + (x_b - x_a), x_b, x_a, \dots)$

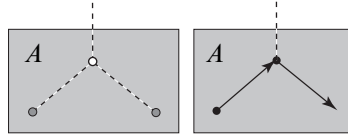
We may assume that there exists an η -edge going from a point in A to a point not in A . Then Γ_A consists of a sequence of subgraphs as in (2)(b) joined by η -edges. If Γ_A has a tree component with at least one external vertex, then we define the following involution on \widehat{B}_A as in Figure 9: let us write the points in the relative configuration as $(x_d, \dots, x_c, x_a, x_b, \dots)$, where the ‘...’ between x_d and x_c are all points which belong to the snake-like piece $c \sim d$ in the graph, and all other ‘...’ denote other points. Then the involution takes the form

$$\begin{aligned}
 &x_e \mapsto x_e + (x_b - x_a) \quad \text{if } e \text{ is a vertex on the piece } c \sim d \text{ of the graph,} \\
 &\text{or } x_a \mapsto x_b, \\
 &x_b \mapsto x_a, \\
 &\text{and } x_f \mapsto x_f \quad \text{otherwise.}
 \end{aligned}$$

This involution reverses (resp. preserves) the orientation of the fiber of \widehat{B}_A and preserves (resp. reverses) the sign of the form if n is odd (resp. even). (Here, the path $c \sim d$ in Figure 9 is disjoint from the depicted tree component and from the invisible part, by the definition of Jacobi diagrams and by the above assumptions, and b may be assumed to be univalent.) Hence the integral vanishes. If Γ has only chords in its θ part, then by the assumption, there has to be at least two chords connected by an η -edge. Hence there must be a subgraph of one of Lemma A.10. \square

The following lemma is an analogue of a lemma of Kontsevich [13], which is proved by Rossi [15].

Lemma A.8 If Γ_A has a subgraph of one of the following forms:

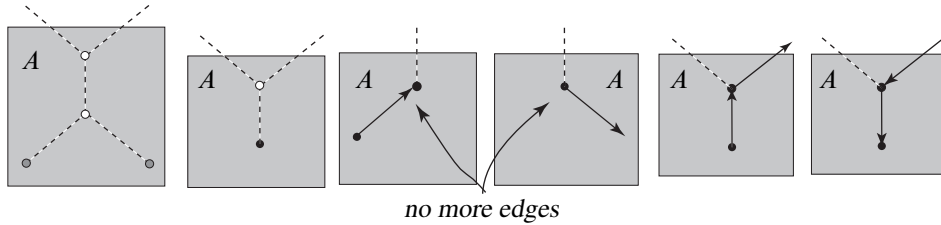


where the gray vertices may both be internal and external, then the integral restricted to $\mathcal{S}_{\Gamma,A}$ vanishes.

Lemma A.8 is proved by using the central symmetry with center the middle of the two gray vertices acting on the white vertex in the first case, and a similar one in the second case. By these symmetries, the integral restricted to $\mathcal{S}_{\Gamma,A}$ is self-cancelling.

The following two lemmas are proved explicitly in [15].

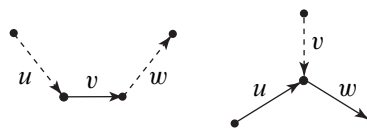
Lemma A.9 If $|A| \geq 3$ and Γ_A has a subgraph of one of the following forms:



then the integral restricted to $\mathcal{S}_{\Gamma,A}$ vanishes.

Lemma A.9 is proved by showing that the quotient space of $\mathcal{S}_{\Gamma,A}$ by the action rescaling the central edge in the box, which does not affect the integrand form, has strictly less dimension than $\mathcal{S}_{\Gamma,A}$.

Lemma A.10 If Γ_A has a subgraph of one of the forms



then the integral restricted to $\mathcal{S}_{\Gamma,A}$ vanishes.

Lemma A.10 is proved by using the symmetry as in **Lemma A.8**, carrying the vectors (u, v, w) to (w, v, u) , cancelling the integral with itself. More precisely, the symmetry for the first case is given as a composition of the reversion of the two boundary vertices

of v and the involution in Lemma A.8 for the two vertices with respect to the remaining vertices. The symmetry for the second case is given as a composition of the central symmetry for the vector v with respect to the target of v , and the involution in Lemma A.8 for the two boundary vertices of the image of v .

A.6 Infinite and anomalous faces

For the infinite faces, the following proposition is proved in [15].

Proposition A.11 *The integral $I(\Gamma)$ restricted to the infinite faces $\mathcal{S}_{\Gamma, \infty, A}$ vanishes.*

This is proved by checking that the infinite faces have codimension at least 2.

A.6.1 Anomaly correction in the case $n = 2$ In the case $n = 2$ and $k = 3$, the following proposition is proved in [7; 15], which is rather difficult part in the proof of the invariance, involving the analysis on invariant forms on the Stiefel and the Grassmann manifolds.

Proposition A.12 *If $n = 2$, then there exists a 2-form $\hat{\rho}(\Gamma)$ on $I_2(\mathbb{R}^4)$ such that*

$$d \int_{C_1} \varphi^* \hat{\rho}(\Gamma) = -I(\Gamma, V(\Gamma)).$$

Hence the addition of the correction term to z_3 :

$$\hat{z}_3(\psi) := z_3(\psi) + \sum_{\Gamma \in \mathcal{G}_3^0} \frac{\tilde{\Theta}(\Gamma)(\psi) w_3(\Gamma)}{|\text{Aut } \Gamma|}$$

where $\tilde{\Theta}(\Gamma) = \int_{C_1} \varphi^* \hat{\rho}(\Gamma)$, is proved to be d -closed.

A.6.2 Anomaly vanishing in the case n odd We show the following proposition as already announced in [15]. Recall that

$$I(\Gamma, V(\Gamma)) = \int_{C_1} \varphi^* \pi_*^\partial \lambda_1(\Gamma).$$

Proposition A.13 *$\pi_*^\partial \lambda_1(\Gamma)$ vanishes for any Γ in the case n is odd.*

Proof Note first that each Γ has even number of vertices and even number of edges. Let x_1, \dots, x_j be internal vertices of Γ mapped into an underlying n dimensional plane with a tangent frame $f \in I_n(\mathbb{R}^{n+2})$ and y_1, \dots, y_k be external vertices of Γ in the ambient space \mathbb{R}^{n+2} so that $(f; x_1, \dots, x_j, y_1, \dots, y_k) \in \widehat{B}_{V(\Gamma)}$. Then consider the involution

$$S: (f; x_1, x_2, \dots, x_j, y_1, \dots, y_k) \\ \mapsto (f; x_1, 2x_1 - x_2, \dots, 2x_1 - x_j, 2f(x_1) - y_1, \dots, 2f(x_1) - y_k)$$

on the fiber. Since $j + k - 1$ is odd if n is odd, S reverses the orientation of the fiber of $\widehat{B}_{V(\Gamma)}$ and preserves the form $\lambda_1(\Gamma)$. Therefore the integration along the fiber vanishes as in the proof of [Lemma A.8](#). \square

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