Almost periodic flows on 3–manifolds

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A 3–manifold which supports a periodic flow is a Seifert fibered space. We define a notion of almost periodic flow and give conditions under which a manifold supporting an almost periodic flow is Seifert fibered. It is well-known that $\mathbb{R}^3$ does not support fixed point free periodic flows, and our results include that $\mathbb{R}^3$ does not support certain almost periodic flows.

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1 Introduction

A *flow* on a manifold M is a continuous action $\Phi\colon M \times \mathbb{R} \to M$ of $\mathbb{R}$ on M. If the action has no global fixed points we say Φ is *fixed point free*. Differentiation of a C^r action yields a C^{r-1} vector field on M. Conversely, the integration of a vector field generates a flow on the manifold. Let Φ_t denote the time t homeomorphism of the manifold $\Phi_t(x) = \Phi(x,t)$. A flow is *periodic* if there exits a $T > 0$ such that Φ_T is the identity map on M. Three dimensional manifolds that support fixed point free periodic flows are Seifert fibered; the orbits under the $\mathbb{R}$–action are the fibers. The following natural question arises, which is the motivating question for this paper.

Question 1 If a Riemannian 3–manifold M supports a "non-trivial" flow Φ such that Φ_1 is close to id_M, is M Seifert fibered?

Given $\epsilon > 0$, any sufficiently small vector field will generate a flow Φ such that $d\big(p, \Phi_1(p)\big) < \epsilon$ for all $p \in M$. Therefore in all of our results we will require that the flow Φ satisfies some "nontriviality" condition.

1.1 Definitions and Results

We begin by recalling some definitions from Riemannian geometry.

Definition 1.1 Let $\gamma\colon [a,b] \to M$ be a geodesic segment. We say γ is a *min-geodesic* if γ is a shortest path between $\gamma(a)$ and $\gamma(b)$.

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The metric ball $B(x;r) = \{y \in M \,|\, d(x,y) < r\}$ is *convex* if any two points in $B(x;r)$ are joined by a unique min-geodesic contained in $B(x;r)$. The largest r such that $B(x;r)$ is convex is called the *convexity radius* at x. The convexity radius for a manifold is defined by,

$$\operatorname{conv\,rad}(M,g) = \inf_{p \in M} \operatorname{conv\,rad}(p)$$

and is positive if M is compact (see Petersen [6]).

In our first result, we assume that our almost periodic flow Φ possesses a flow line that, in some sense, has infinite order in $\pi_1(M)$. Thus Φ is "homotopically non-trivial". Of course all flow lines may be non-compact, so we refer the reader to Section 2 for the definition of a closed up flow line.

Theorem 2.4 *Suppose M is a closed, orientable, connected, irreducible, Riemannian 3–manifold and $\epsilon < \operatorname{conv\,rad}(M)/2$. If M supports a flow $\Phi\colon M \times \mathbb{R} \to M$ such that,*

(1) $d_M\big(x, \Phi(x,1)\big) < \epsilon$ *for all* $x \in M$

(2) *for some* $x \in M$, *the closed up flow line* Γ_x *has infinite order in* $\pi_1(M)$

then M is a Seifert fibered space.

Note that if the convexity radius of M is large, Φ_1 need not be that close to the identity map on M. For the remainder of our results we replace the homotopy condition with a geometric condition (part (2) of Definition 1.2).

Definition 1.2 Given $K \geq 1$ and $\epsilon > 0$, a flow on a manifold M is (K,ϵ)*–almost periodic* if for all $x \in M$:

(1) $d\big(x, \Phi(x,1)\big) < \epsilon$

(2) $d\big(x, \Phi(x,\tfrac{1}{2})\big) > K\epsilon$

It follows from a theorem of P A Smith [9] that S^1 actions on $\mathbb{R}^n$ must have a fixed point. We are able to show that $\mathbb{R}^3$ does not support certain almost periodic flows. We give a specific example of one such result, and refer the reader to Section 4 for the statements of the more general theorems. Recall a vector field X on $\mathbb{R}^3$ is C*–lipschitz* if for all $p, q \in \mathbb{R}^3$,

$$\|X_p - X_q\| \leq C\|p - q\|.$$

Now consider the class of flows that arise from integration of a C–lipschitz vector field.

Theorem 1.3 *Let V be a C–lipschitz vector field on $\mathbb{R}^3$ and Φ the corresponding flow that arises from integration of V. Then Φ is not $(10e^C, \epsilon)$–almost periodic for any $\epsilon > 0$.*

If a manifold supports one of Thurston's eight geometric structures, it is covered by $\mathbb{R}^3$, S^3 or $S^2 \times \mathbb{R}$. In the latter two cases the manifold is Seifert fibered (see Scott [7]). Therefore we consider the case when M is covered by $\mathbb{R}^3$. Theorem 1.3 depends on having a bound on the rate of growth of the length of a particular arc, which the lipschitz condition provides. Using a more general version of this theorem, which is stated in terms of the growth rate of an arc and does not assume the Euclidean metric on $\mathbb{R}^3$, we prove the following. See Section 4 for definitions.

Theorem 5.2 *Let M be a closed, orientable, Riemannian 3–manifold with universal cover homeomorphic to $\mathbb{R}^3$, such that for all $p \in \widetilde{M}$, the exponential $\exp_p\colon T_p\widetilde{M} \to \widetilde{M}$ is a diffeomorphism. Let $0 < 2\epsilon < \operatorname{conv\,rad}(M)$. If M supports a flow $\Phi\colon M \times \mathbb{R} \to M$ that satisfies*

(1) *for some p in M the min-geodesic between p and $\Phi_1(p)$ has time-one growth rate N, and*

(2) *Φ is $(10N, \epsilon)$–almost periodic,*

then M is Seifert fibered.

Remark There are several notions of almost periodic flow. We discuss the definition given by Cartwright [1].

Definition 1.4 A subset E of $\mathbb{R}$ is *dense relative to the number* L if every interval of length L contains an element of E.

Definition 1.5 A flow Φ is *almost periodic* (in the sense of [1] but not this paper) if for every $\epsilon > 0$, there exists an $L > 0$, such that for all $x \in M$, the set

$$E = \{\tau \in \mathbb{R} \mid \text{for all } x \in M, d\big(x, \Phi(x,\tau)\big) < \epsilon\}$$

is dense relative to L.

The above definition, used by Cartwright, is neither stronger nor weaker than the definition used in this paper. Our definition is "coarser" than Definition 1.5, in that we do not require that $d(\mathrm{id}_M, \Phi_1) < \epsilon$ for every ϵ. Also, we do not require that the set $E = \{t \mid d(\mathrm{id}_M(p), \Phi_t(p)) < \epsilon \text{ for all } p \in M\}$ be relatively dense in $\mathbb{R}$, but merely that Φ_1 is close to the identity map. However, Definition 1.5 does not imply condition (2) of Definition 1.2.

1.2 Outline

In Section 2 we prove Theorem 2.4, by showing that the subgroup generated by the flow line of infinite order is central. Theorem 2.4 then follows from the Seifert fibered space Theorem:

Theorem 1.6 (Casson [2], Gabai [4], Scott [8], Mess [5], Tukia [10], Waldhausen [11]) *If M is a compact, irreducible, orientable, connected 3–manifold, and $\pi_1(M)$ contains an infinite cyclic normal subgroup, then M is Seifert fibered.*

Section 3 contains a technical algebraic topology result, which we use repeatedly in Section 4 to prove our results about almost periodic flows on $\mathbb{R}^3$. First we prove that $\mathbb{R}^3$ does not support $(1,\epsilon)$–almost periodic flows with a compact flow line of period 1. We then extend the proof to eliminate the hypothesis of a closed flow line.

In Section 5 we prove Theorem 5.2. If we assume M is covered by $\mathbb{R}^3$, then $\pi_1(M)$ is torsion free, and Theorem 2.4 reduces our question to the case when the closed up flow lines are contractible. If M supports a (K,ϵ)–almost periodic flow, where $\epsilon < \operatorname{conv\,rad}(M)$, and all flow lines contractible, the flow lifts to a (K,ϵ)–almost periodic flow on the universal cover. Using the results about almost periodic flows on $\mathbb{R}^3$, we show that this can not occur.

1.3 Acknowledgements

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2 Almost periodic flows

Let M be a compact Riemannian 3–manifold that supports a flow Φ and x a point in M such that $d\big(x,\Phi_1(x)\big) < \operatorname{conv\,rad}(M)$. We define the closed up flow line starting at x by flowing the point x for unit time to $\Phi_1(x)$, and then moving back to x along the shortest geodesic (which is unique by hypothesis). For two paths $f,g\colon [0,1]\to X$ such that $f(1)=g(0)$, we write $f\cdot g$ to denote the composition of paths.

Definition 2.1 Let $\phi_x\colon [0,1]\to M$ be the flowline for x restricted to the unit interval and let $\lambda_x\colon [0,1]\to M$ be the unique min-geodesic starting at $\Phi(x,1)$ and terminating at x. Then the *closed up flow line* of x is $\Gamma_x = \phi_x\cdot\lambda_x$. See Figure 1.

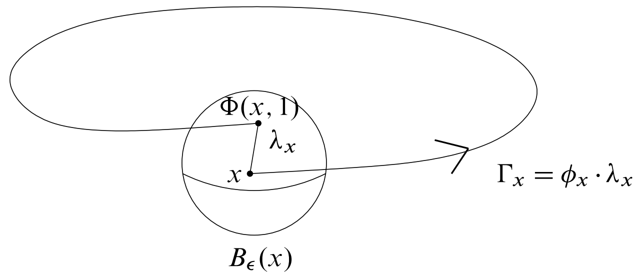

Figure 1: closed up flowline

We use the following lemmas to prove Theorem 2.4. Lemma 2.2 is a a basic fact from Riemannian geometry about the continuity of min-geodesics. For a proof see Do Carmo [3, Theorem 3.7, Remark 3.8].

Lemma 2.2 *Suppose* $\delta < \operatorname{conv\,rad}(M)$. *Let*

$$U = \{(x, y) \in M \times M \colon d(x, y) < \delta\}.$$

Given $p = (x, y) \in U$, *let* $\gamma_p \colon [0,1] \to M$ *be the unique min-geodesic such that* $\gamma_p(0) = x$ *and* $\gamma_p(1) = y$. *Then the function* $F \colon U \times [0,1] \to M$ *defined by*

$$F(p, t) = \gamma_p(t)$$

is continuous.

Lemma 2.3 *Let* M *be a closed, connected, Riemannian* 3*–manifold. Let* $\epsilon = \frac{1}{2}\operatorname{conv\,rad}(M)$ *and* Φ *be a flow on* M *such that* $d(x, \Phi(x,1)) \leq \epsilon$ *for all* x *in* M. *If* Γ_x *and* Γ_y *are the closed up flow lines for* x *and* y, *then* Γ_x *is freely homotopic to* Γ_y.

Proof Let $x, y \in M$ such that $d(x, y) < \epsilon$. Let $\sigma \colon [0,1] \to B(x; \epsilon)$ be a continuous map such that $\sigma(0) = x$ and $\sigma(1) = y$. Define the homotopy $H \colon S^1 \times [0,1] \to M$ by

$$H(s, t) = \Gamma_{\sigma(t)}(s).$$

So for each $t \in [0,1]$, the map $H_t \colon S^1 \to M$ is the closed up flow line of $\sigma(t)$, with $H_0 = \Gamma_x$ and $H_1 = \Gamma_y$. We now show H is continuous. By definition we have $d\big(\sigma(t), x\big) \leq \epsilon$, and the "almost periodicity" property of the flow Φ implies $d\big(\sigma(t), \Phi\big(\sigma(t), 1\big)\big) \leq \epsilon$. Therefore

$$d\big(\Phi\big(\sigma(t), 1\big), x\big) \leq d\big(\Phi\big(\sigma(t), 1\big), \sigma(t)\big) + d\big(\sigma(t), x\big) \leq 2\epsilon.$$

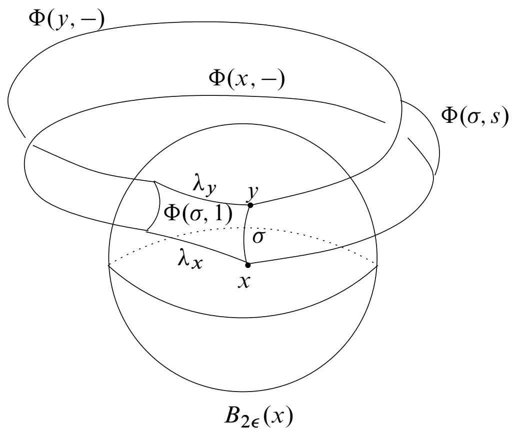

Figure 2

We have just shown that the image of $\Phi_1 \circ \sigma$ is contained in $B(x;2\epsilon)$. See Figure 2. Since σ and Φ are continuous, and the images of both σ and $\Phi(\sigma,1)$ are contained in the convex ball $B_{2\epsilon}(x)$, Lemma 2.2 implies that H is continuous.

Therefore $\Gamma_x \simeq \Gamma_y$. To show that any two closed up flowlines are freely homotopic we define an equivalence relation on M by $x \sim y$ if Γ_x is freely homotopic to Γ_y. These equivalence classes are open therefore M connected implies there is only one equivalence class. □

Theorem 2.4 *Suppose M is a closed, orientable, connected, irreducible, Riemannian 3–manifold and $\epsilon < \operatorname{conv\,rad}(M)/2$. If M supports a flow $\Phi\colon M \times \mathbb{R} \to M$ such that,*

(1) $d_M\big(x, \Phi(x,1)\big) < \epsilon$ *for all* $x \in M$

(2) *for some $x \in M$, the closed up flow line Γ_x has infinite order in $\pi_1(M)$*

then M is a Seifert fibered space.

Proof Let $x \in M$ such that Γ_x has infinite order in $\pi_1(M)$. By Lemma 2.3, all closed up flow lines are freely homotopic, therefore the subgroup generated by Γ_x is central. To see this, let $\beta\colon S^1 \to M$ be a representative of an element of $\pi_1(M,x)$. Then we can slide Γ_x along the path β, through the closed up flow lines $\Gamma_{\beta(s)}$, until Γ_x returns to itself. Therefore $\beta\Gamma_x\beta^{-1} = \Gamma_x$ and $\langle \Gamma_x \rangle$ is an infinite central cyclic subgroup. By Theorem 1.6, the Seifert fibered space Theorem, the result follows. □

3 Reflector homotopies

In this section we prove a technical result which will be used in later proofs.

Definition 3.1 Let D be a 2–disk and $X = \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta$, where Δ denotes the diagonal of $\mathbb{R}^3 \times \mathbb{R}^3$. Let $g\colon X \to X$ be the map that reflects about the diagonal, that is $g\big((x, y)\big) = (y, x)$, and let C be a simple closed g–invariant curve in X. We say the map

$$J\colon (D, \partial D) \times [0, 1] \to (X, C)$$

is a *reflector homotopy* if J satisfies the following:

(1) J is a homotopy of maps of the pair $(D, \partial D) \to (X, C)$.

(2) $J_1 = g \circ J_0$, and therefore $J_0 \simeq g \circ J_0$.

(3) $J_t|_{\partial D}$ is a degree one map.

As motivation for Theorem 3.2, recall that $\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta$ is homotopy equivalent to S^2, and consider the following example.

Example Let D be a closed disk, S^2 the unit sphere in $\mathbb{R}^3$, and N the northern hemisphere $(z \geq 0)$ of S^2. Let $h\colon D \to N$ be a homeomorphism, and $a\colon S^2 \to S^2$ the antipodal map. Then it is easy to see that $h\colon (D, \partial D) \to (S^2, \partial N)$ is not homotopic as a map of pairs to $a \circ h$.

Theorem 3.2 *Reflector homotopies do not exist.*

Proof Assuming J exists, we will determine the induced maps $(J_0)_*$ and $(g \circ J_0)_*$ between the relative homology groups $H_2(D, \partial D; \mathbb{Z}_2)$ and $H_2(X, C; \mathbb{Z}_2)$, and obtain a contradiction.

Using $X \simeq S^2$, a long exact sequence shows $H_2(X, C; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We choose generators for the group $H_2(X, C; \mathbb{Z}_2)$, and determine what the map g_* does on these generators. To do so, we will start with a subspace of X for which we know exactly the behavior of g. Given $r > 0$, let P and E be the spaces

$$P = \{((x, y, z), (-x, -y, -z)) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|(x, y, z)\| = r\}$$
$$E = \{((x, y, 0), (-x, -y, 0)) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|(x, y, 0)\| = r\}$$

Then P is a 2–sphere with equator E, and P is contained in $\Delta^\perp = \{(x, -x) | x \in \mathbb{R}^3\}$. Since C is a compact set we can choose r large enough so that C and E do not intersect. Since $P \hookrightarrow X$ is a homotopy equivalence this sphere is a generator for $H_2(X;\mathbb{Z}_2)$.

The quotient space P/E is the wedge of two spheres. Therefore $H_2(P,E;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the quotients of the northern($z \geq 0$) and southern ($z \leq 0$) hemispheres $[N],[S] \in H_2(P,E;\mathbb{Z}_2)$ are generators for the group. The map g restricted to the sets P and E is the antipodal map. Therefore $g_*\colon H_2(P,E;\mathbb{Z}_2) \to H_2(P,E;\mathbb{Z}_2)$ is the map that switches our generators $[N]$ and $[S]$.

Since P is a generator for $H_2(X)$, the inclusion map $i\colon (P,E) \to (X,E)$ induces an isomorphism $i_*\colon H_2(P,E;\mathbb{Z}_2) \to H_2(X,E;\mathbb{Z}_2)$. Also P and E are both g–invariant, and therefore we have $i \circ g = g \circ i$ on the subspaces where these maps are defined, so the following diagram commutes.

$$\begin{tikzcd} H_2(X,E;\mathbb{Z}_2) \arrow[r, "g_*"] & H_2(X,E;\mathbb{Z}_2) \\ H_2(P,E;\mathbb{Z}_2) \arrow[r, "g_*"] \arrow[u, "i_*"] & H_2(P,E;\mathbb{Z}_2) \arrow[u, "i_*"'] \end{tikzcd}$$

We will now build an isomorphism between $H_2(X,E;\mathbb{Z}_2)$ and $H_2(X,C;\mathbb{Z}_2)$ which commutes with the map g_*. To simplify notation we will use g_* to denote the map induced by the restriction of g on the relative homology groups for the pairs of spaces $(X,E),(X,C),(P,E)$ and the pair (X,Y) which will soon be defined.

Let $X/{\sim_g}$ be the quotient space where $(p,q) \sim_g (q,p)$. Note that g is fixed point free and the map $X \to X/{\sim_g}$ is a 2–fold covering. Therefore $X/{\sim_g} \simeq \mathbb{R}P^2$. Since C and E are embedded g–invariant circles in X, the sets $\bar{C} = C/{\sim_g}$ and $\bar{E} = E/{\sim_g}$ are circles that are doubled covered by C and E respectively. The loop that parameterizes $\bar{C}$ lifts to an arc of C, therefore $\bar{C}$ must be the nontrivial element of $\pi_1(X/{\sim_g})$. Similarly for $\bar{E}$.

Recall the sphere P was chosen to be large enough so that C and E are disjoint, therefore $\bar{C}$ and $\bar{E}$ are also disjoint. Since $\bar{C}$ and $\bar{E}$ represent the same element in the fundamental group of $\pi_1(X/{\sim_g})$, there is a homotopy $f\colon S^1 \times I \to X/{\sim_g}$ between them. The space $X/{\sim_g}$ is six dimensional, and $\bar{C}$ and $\bar{E}$ disjoint, so using a general position argument we may assume f is an embedding with image an annulus in $X/{\sim_g}$, with boundary components $\bar{C}$ and $\bar{E}$. This annulus lifts to a g–invariant embedded annulus Y in X. Therefore $H_1(Y;\mathbb{Z}_2) = \mathbb{Z}_2$, and a long exact sequence argument implies that $H_2(X,Y;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since C generates $H_1(Y;\mathbb{Z}_2) = \mathbb{Z}_2$, we know that the inclusion map $i_C\colon (X,C) \to (X,Y)$ induces an isomorphism between the relative homology groups. Since the annulus Y is g–invariant the following diagram commutes.

$$\begin{array}{ccc} H_2(X,C;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(X,C;\mathbb{Z}_2) \\ {\scriptstyle (i_C)_*}\downarrow & & \downarrow{\scriptstyle (i_C)_*} \\ H_2(X,Y;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(X,Y;\mathbb{Z}_2) \end{array}$$

Similarly E generates $H_1(Y;\mathbb{Z}_2)=\mathbb{Z}_2$ and the inclusion map i_E induces an isomorphism between relative homology groups and we get the commutative diagram

$$\begin{array}{ccc} H_2(X,Y;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(X,Y;\mathbb{Z}_2) \\ {\scriptstyle (i_E)_*^{-1}}\downarrow & & \downarrow{\scriptstyle (i_E)_*^{-1}} \\ H_2(X,E;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(X,E;\mathbb{Z}_2). \end{array}$$

Putting all of the maps together, we have the following, where vertical arrows are isomorphisms.

$$\begin{array}{ccc} H_2(X,C;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(X,C;\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ H_2(X,Y;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(X,Y;\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ H_2(X,E;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(X,E;\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ H_2(P,E;\mathbb{Z}_2) & \xrightarrow{g_*} & H_2(P,E;\mathbb{Z}_2) \end{array}$$

Recall that the map g_* switches the generators $[N]$ and $[S]$ of $H_2(P,E;\mathbb{Z}_2)$. Let Ψ be the isomorphism between $H_2(X,C;\mathbb{Z}_2)$ and $H_2(P,E;\mathbb{Z}_2)$ which is the composition of the isomorphisms on the downward pointing arrows. Then $\widehat{N} := \Psi^{-1}([N])$ and $\widehat{S} := \Psi^{-1}([S])$ are generators for $H_2(X,C;\mathbb{Z}_2)$. Since the above diagram commutes we have proven the following claim.

Claim 1 g_* switches $\widehat{N}$ and $\widehat{S}$.

Recall that $(J_0)_*\colon H_2(D,\partial D;\mathbb{Z}_2) \to H_2(X,C;\mathbb{Z}_2)$. Let $[D]$ denote the generator of $H_2(D,\partial D;\mathbb{Z}_2)$ and suppose that $(J_0)_*([D]) = a\widehat{N} + b\widehat{S}$. Then Claim 1 implies

$$(g\circ J_0)_*([D]) = g_*\circ (J_0)_*([D]) = a\widehat{S} + b\widehat{N}.$$

Since $J_0 \simeq g\circ J_0$ as maps of the pairs $(D,\partial D)\to (X,C)$, the induced maps on relative homology groups $(J_0)_*$ and $(g\circ J_0)_*$ must be equal, and therefore $a=b$.

Claim 2 If $(J_0)_*([D]) = a\widehat{S} + b\widehat{N}$ then $a = b$.

Now consider the exact sequence of the relative homology groups for the pair (X, C). In the following commutative diagram vertical arrows are isomorphisms.

$$\begin{tikzcd} H_2(X;\mathbb{Z}_2) \arrow[r, "j_*"] \arrow[d] & H_2(X,C;\mathbb{Z}_2) \arrow[r, "\partial_*"] \arrow[d] & H_1(C;\mathbb{Z}_2) \arrow[d] \\ \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \end{tikzcd}$$

The map j_* is induced by the inclusion map $j\colon (X,\varnothing) \to (X,C)$. Therefore j_* maps the nontrivial element of $H_2(X;\mathbb{Z}_2)$ onto $\widehat{N} + \widehat{S}$. Since $\partial_* \circ j_*$ is trivial, this implies ∂_* sends $\widehat{N} + \widehat{S}$ to 0. So $\partial_*(\widehat{S} + \widehat{N}) = \partial_*(\widehat{S}) + \partial_*(\widehat{N}) = 0$ or

$$\partial_*(\widehat{S}) = -\partial_*(\widehat{N}).$$

The long exact sequence shows $\widehat{\partial}_*\colon H_2(D,\partial D;\mathbb{Z}_2) \to H_1(\partial D;\mathbb{Z}_2)$ is an isomorphism. By naturality of the connecting homomorphism the following diagram commutes.

$$\begin{tikzcd} {[D]} \in H_2(D,\partial D;\mathbb{Z}_2) \arrow[r, "\widehat{\partial}_*"] \arrow[d, "(J_0)_*"'] \arrow[dr, bend left, "\Gamma"] & H_1(\partial D;\mathbb{Z}_2) \arrow[d, "(J_0|_{\partial D})_*"] \\ H_2(X,C;\mathbb{Z}_2) \arrow[r, "\partial_*"] & H_1(C;\mathbb{Z}_2) \end{tikzcd}$$

Property (3) of the reflector homotopy says that the map $(J_0)|_{\partial D}$ is a degree one map and therefore $\Gamma([D]) \neq 0$. Since the diagram commutes this implies that

$$\begin{aligned} \partial_* \circ (J_0)_*([D]) &= \partial_*(a\,\widehat{S} + b\,\widehat{N}) \\ &= a\,\partial_*(\widehat{S}) + b\,\partial_*(\widehat{N}) \\ &= a\,\partial_*(\widehat{S}) - b\,\partial_*(\widehat{S}) \\ &= (a-b)\,\partial_*(\widehat{S}) \\ &\neq 0 \end{aligned}$$

Therefore $a \neq b$. However this contradicts Claim 2 and we have proven Theorem 3.2. □

4 $\mathbb{R}^3$ does not support almost periodic flows

Before proving that $\mathbb{R}^3$ does not support (K,ϵ)–almost periodic flows for sufficiently large K, we prove Theorem 4.1, which has the additional hypothesis of a closed flow line of period 1.

Theorem 4.1 *Let (M, g) be a Riemannian manifold homeomorphic to $\mathbb{R}^3$ with positive convexity radius. Let $\epsilon = \mathrm{conv\,rad}(M)$. If $\Phi\colon M \times \mathbb{R} \to M$ is a flow with a closed flow line of period 1, then Φ is not $(1, \epsilon)$–almost periodic.*

Note that if M is equipped with the standard Euclidean metric, then M has infinite convexity radius, and Theorem 4.1 is true for any ϵ. We list, for easy reference, several definitions which will be used throughout the remainder of the paper.

- D is a 2–disk.
- $X = M \times M \setminus \Delta$, where Δ is the diagonal of $M \times M$.
- The map $g\colon X \to X$ is reflection about the diagonal and is defined by
$$g\big((x, y)\big) = (y, x).$$
- The *induced flow* on X is $\Psi\colon X \times \mathbb{R} \to X$, where
$$\Psi_t = \Phi_t \times \Phi_t$$
- The *standard embedding* $e\colon M \to X$ is defined by
$$e(p) = \big(p, \Phi_{\frac{1}{2}}(p)\big).$$

Proof of Theorem 4.1 Assume Φ is a $(1, \epsilon)$–almost periodic flow with a closed flow line of period 1. Using Φ we will construct a reflector homotopy, which contradicts Theorem 3.2 and proves our result.

To construct the reflector homotopy, let p be a point in M whose orbit has period 1. Identify S^1 with the unit interval modulo the endpoints and define $f\colon S^1 \to \mathbb{R}^3$ by $f(s) = \Phi(p, s)$. Since $\mathbb{R}^3$ is contractible, we can extend f over the unit disk D so that f is a null homotopy for the closed flow line $\Phi(p, -)$. For any point z in the disk D, let the map $h(z, -)\colon [0, 1] \to M$ be the min-geodesic from $\Phi_1\big(f(z)\big)$ to $f(z)$. Since $d\big(f(z), \phi_1(f(z))\big) < \epsilon$, the map $h\colon D \times [0, 1] \to M$ is continuous by Lemma 2.2. We define the reflector homotopy $J\colon D \times [0, 1] \to X$ by

$$J(z, t) = \begin{cases} \Psi_t \circ e \circ f(z) & 0 \le t \le \frac{1}{2} \\ \big(\Phi_{\frac{1}{2}} \circ f(z), h(z, 2t - 1)\big) & \frac{1}{2} \le t \le 1 \end{cases}$$

During the first time interval the homotopy J uses Ψ to flow the map $e \circ f$ to $\Psi_{\frac{1}{2}} \circ e \circ f$. Note that,

$$J_1(z) = \big(\Phi_{\frac{1}{2}} \circ f(z), h(z, 1)\big) = \big(\Phi_{\frac{1}{2}} \circ f(z), f(z)\big) = g \circ e \circ f(z).$$

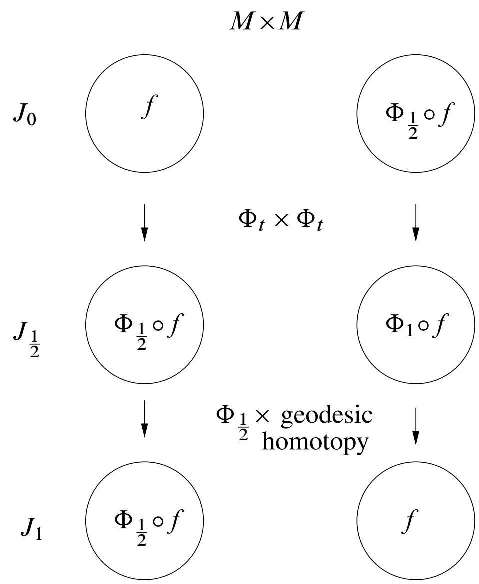

Figure 3: The homotopy J

We see that when $t \in [\frac{1}{2}, 1]$ the map J homotopes $\Psi_{\frac{1}{2}} \circ e \circ f$ to $g \circ e \circ f$ along a short geodesic. See Figure 3. We now verify that J is a reflector homotopy, by showing that J satisfies properties 1–3 of Definition 3.1.

Let $C = J_0(\partial D)$. We will show that C is g–invariant, but first we verify property 1, that J is a homotopy of the pairs $(D, \partial D) \to (X, C)$.

Claim 3 The image of the homotopy J does not intersect Δ, the diagonal of $M \times M$.

Proof Let $z \in D$. For $t \in [0, \frac{1}{2}]$ we have $J(z,t) = \big(\Phi_t \circ f(z), \Phi_{t+\frac{1}{2}} \circ f(z)\big)$. Property 2 of the almost periodic flow implies,

$$d\big(\Phi_t \circ f(z), \Phi_{\frac{1}{2}+t} \circ f(z)\big) = d\big(\Phi_t \circ f(z), \Phi_{\frac{1}{2}}(\Phi_t \circ f(z))\big) > \epsilon.$$

Thus $J(z,t) \notin \Delta$. Now assume $t \in [\frac{1}{2}, 1]$. Since $J_{\frac{1}{2}}(z) = (\Phi_{\frac{1}{2}} \circ f(z), \Phi_1 \circ f(z))$, and $J_1(z) = (\Phi_{\frac{1}{2}} \circ f(z), f(z))$, the first coordinate of the image of J is constant for this time interval. The second coordinate of the image is the min-geodesic between $\Phi_1 \circ f(z)$ and $f(z)$, which are a distance of less than ϵ. That $J_t(z) = \big(\Phi_{\frac{1}{2}}(z), h(z, 2t-1)\big)$ does not belong to Δ is evident from Figure 4, and is verified by the following inequalities.

$$d\big(\Phi_{\frac{1}{2}} \circ f(z), h(z, 2t-1)\big) + d\big(h(z, 2t-1), f(z)\big) \geq d\big(\Phi_{\frac{1}{2}} \circ f(z), f(z)\big) > \epsilon.$$

Since $h(z, 2t-1)$ lies on the min-geodesic between $f(z)$ and $\Phi_1 \circ f(z)$, which has length less than ϵ,

$$d\big(h(z, 2t-1), f(z)\big) < \epsilon.$$

Therefore

$$d\big(\Phi_{\frac{1}{2}} \circ f(z), h(z, 2t-1)\big) \geq d\big(\Phi_{\frac{1}{2}} \circ f(z), f(z)\big) - d\big(h(z, 2t-1), f(z)\big) > 0. \quad \square$$

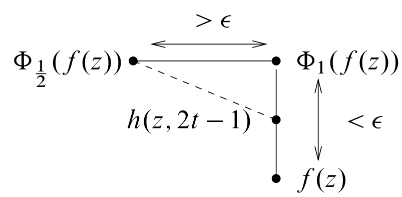

Figure 4: J_t does not intersect Δ

Now we show that $J_t(\partial D) = C$ for all $t \in [0,1]$. Recall f was defined on ∂D by $f(s) = \Phi(p, s)$. Therefore,

$$C = \{\big(\Phi_s(p), \Phi_{s+\frac{1}{2}}(p)\big) | s \in [0,1]\}.$$

Since the image of f is a closed flow line of period 1, the flow line

$$s \mapsto \Psi_s \circ e \circ f(0) = \big(\Phi(p, s), \Phi_{\frac{1}{2}}(p, s)\big)$$

is also closed. Therefore for $t \in [0, \frac{1}{2}]$,

$$J_t(\partial D) = \Psi_t \circ e \circ f(\partial D) = C.$$

Since $f|_{\partial D}$ is a closed flow line with period 1, the map $J_{\frac{1}{2}} = g \circ e \circ f$ on ∂D. Therefore the adjustment that happens during the second time interval occurs only on the interior of the disk, hence the homotopy is constant on ∂D during the interval $[\frac{1}{2}, 1]$. We have shown that the image of $\Psi_t(\partial D)$ is C for $t \in [0,1]$, which completes the verification of property 1.

To verify that the set C is g–invariant, let $t = s + \frac{1}{2}$. Since the flow line for p has period 1 we have

$$\begin{aligned} g(\Phi_s(p), \Phi_{s+\frac{1}{2}}(p)) &= (\Phi_{s+\frac{1}{2}}(p), \Phi_s(p)) \\ &= (\Phi_t(p), \Phi_{t-\frac{1}{2}}(p)) \\ &= (\Phi_t(p), \Phi_{t+\frac{1}{2}}(p)). \end{aligned}$$

Since $(\Phi_t(p), \Phi_{t+\frac{1}{2}}(p))$ is in C, the set is g–invariant.

To verify property 2 of Definition 3.1, note that $J_0 = \Psi_0 \circ e \circ f = e \circ f$ and $J_1 = g \circ e \circ f = g \circ J_0$. Therefore $J_0 \simeq g \circ J_0$.

Lastly, we note that since the map $f|_{\partial D}$ is a homeomorphism onto the closed flowline, the map $J_t|_{\partial D}$ is also a homeomorphism, satisfying the third and final property of a reflector homotopy. □

4.1 Removing the hypothesis of a closed flow line

Let Φ be a flow on a manifold M, and $\alpha\colon [0,1] \to M$ an arc. If we flow the arc α for time t, we denote the length of the resulting arc $\Phi_t \circ \alpha$ by $L_t(\alpha)$.

Definition 4.2 Let M be a Riemannian manifold and Φ a flow on M. Let $\alpha\colon [a,b] \to M$ be a rectifiable arc. Then we define the *time-one growth rate of* α by

$$N(\alpha) := \frac{\max\{L_t(\alpha) \,|\, t \in [0,1]\}}{L_0(\alpha)}.$$

Note that if a min-geodesic α has time-one growth rate N, then $L_t(\alpha) \le N d(p,q)$ for all $t \in [0,1]$.

In this section we show that if Φ is a flow on $M^3 \simeq R^3$ (see hypothesis below) such that for some point $p \in M$ the min-geodesic α between p and $\Phi_1(p)$ has time-one growth rate $N = N(\alpha)$, then Φ is not $(10N, \epsilon)$–almost periodic, for all $\epsilon > 0$. But first a few preliminaries.

Let TM denote the tangent bundle of a Riemannian manifold M. The exponential map, exp: $TM \to M$, is defined as follows. Let $t \mapsto \gamma(t,p,v)$ be the constant speed geodesic in M such that $\gamma(0,p,v) = p$ and $\dot\gamma(0,p,v) = v \in T_p(M)$. Then,

$$\exp(p,v) = \gamma(1,p,v),$$

and $\exp_p = \exp|_{T_pM}$. See Do Carmo [3] for more details.

For a point $x \in M \times M$ let $d(x,\Delta)$ denote the distance between x and Δ. That is,

$$d(x,\Delta) = \min\{d_{M\times M}(x,y) \mid y \in \Delta\}.$$

Lemma 4.3 *Let M be a Riemannian manifold such that for all $p \in M$, the exponential map $\exp_p\colon T_pM \to M$ is a diffeomorphism. Then for any point $(p,q) \in M \times M$,*

$$d\big((p,q),\Delta\big) = \frac{\sqrt{2}}{2}\, d(p,q).$$

Proof Let p and q be points in M and let m be the midpoint of γ, the unique min-geodesic (which exists since $\exp_p$ is a diffeomorphism) between p and q. Then (m,m) is the point in Δ closest to (p,q). To verify this, let x be an arbitrary point in M, and let $\lambda \geq 1$ be the ratio of the lengths $d(p,x)$ and $d(q,x)$. Without loss of generality assume $\lambda = d(p,x)/d(q,x)$. Then,

$$d_{M\times M}\big((x,x),(p,q)\big) = \sqrt{d^2(p,x)+d^2(q,x)} = \sqrt{\lambda^2+1}\; d(q,x). \tag{1}$$

A continuity argument implies there exists a point x' on the image of γ such that $d(p,x')/d(q,x') = \lambda$. Since x' lies on the min-geodesic between p and q, $d(q,x') \leq d(q,x)$. Furthermore, when $x' = m$, we have $\lambda = 1$, and $d_{M\times M}\big((x',x'),(p,q)\big)$ is minimized. Equation (1) implies

$$d_{M\times M}\big((p,q),(m,m)\big) = \sqrt{2}\, d(q,m) = \tfrac{\sqrt{2}}{2} d(p,q). \qquad \square$$

Theorem 4.4 *Let (M,g) be a Riemannian manifold homeomorphic to $\mathbb{R}^3$, such that for all p in M, the exponential map $\exp_p\colon T_pM \to M$ is a diffeomorphism. Let Φ be a (K,ϵ)–almost periodic flow on M such that for some p in M the min-geodesic between p and $\Phi_1(p)$ has time-one growth rate N. Then $K < 10N$.*

If Φ is a flow generated by a C–lipschitz vector field, then a basic fact from differential equations implies that every min-geodesic $\alpha\colon [0,1] \to \mathbb{R}^3$ has a time-one growth rate that is less than e^C. Note that $\mathbb{R}^3$ with the standard Euclidean metric has infinite convexity radius. Therefore Theorem 4.4 implies that a flow Φ on R^3 generated by a C–Lipschitz vector field is not $(10e^C,\epsilon)$–almost periodic, for any $\epsilon > 0$.

The basic strategy of the proof is the same as in Theorem 4.1. We will use Φ to construct a reflector homotopy and then Theorem 3.2 gives us a contradiction. To define the homotopy, we need a g–invariant circle in X. Unlike the previous case we can not use a closed flow line with period one, since we do not assume one exists. Instead we take a segment of a flow line for a period of $[0,1]$ and then close it up to obtain a circle. Using the resulting curve C' we construct the homotopy $J\colon D\times[0,1] \to X$, as before. However, since C' is only approximately g–invariant, the homotopy J is no longer a reflector homotopy. We shall show there is a g–invariant curve C close to the curve C'. Then we take a larger disk D^+ containing D and extend J over D^+ so that $J|_{\partial D^+\times I}$ maps onto C, and the extension of J is a reflector homotopy.

Proof of Theorem 4.4 Let Φ be a flow on M and p a point in M such that the min-geodesic between the points p and $\Phi_1(p)$ has time-one growth rate N. Assume Φ is (K,ϵ)–almost periodic where $K = 10N$. The previous construction of J began with defining a null-homotopy for the closed flow line of period 1. In the absence of this hypothesis, we will construct a map $f\colon S^1 \to X$ such that the image of the map lies on a closed-up flow line for the point p.

Let D be the unit disk in the complex plane and identify $\partial D = \{e^{\pi i s} | s \in \mathbb{R}\}$ with the unit interval $[0,2]$ modulo the endpoints. (Our choice for this parameterization of ∂D makes later calculations less tedious.) Let $\gamma_p\colon [1,2] \to M$ be the min-geodesic with $\gamma_p(1) = \Phi_1(p)$ and $\gamma_p(2) = p$. Define $f\colon \partial D \to X$ by

$$f(s) = \begin{cases} \Phi(p,s) & 0 \le s \le 1 \\ \gamma_p(s) & 1 \le s < 2 \end{cases}$$

Since M is contractible we may extend the map f over the entire disk D. It is important to note that $f(\partial D)$ is no longer invariant under the flow. Using f, define $J\colon D \times [0,1] \to X$ as follows,

$$J(z,t) = \begin{cases} \Psi_t \circ e \circ f(z) & 0 \le t \le \frac{1}{2} \\ \big(\Phi_{\frac{1}{2}} \circ f(z), h(z, 2t-1)\big) & \frac{1}{2} \le t \le 1 \end{cases}$$

This is the same homotopy defined in Theorem 4.1, where $h(z,t)$ is the min-geodesic starting at $\Phi_1(f(z))$ and terminating at $f(z)$. Recall Figure 3 if needed. Under the new hypothesis, the image of J still does not intersect the diagonal of $M \times M$. In fact, since $d(p, \Phi_{\frac{1}{2}}(p)) > K\epsilon$, the image of J misses a neighborhood of the diagonal. First we prove that the map g is approximately equal to $\Psi_{\frac{1}{2}}$ on the image of e. To simplify notation, we use d, rather than $d_{M\times M}$ to denote the induced path metric on $M \times M$.

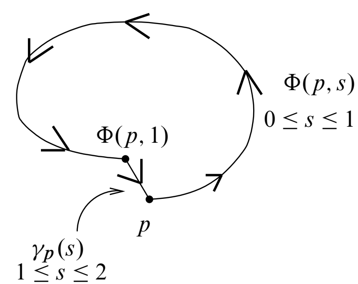

Figure 5: The map $f\colon S^1 \to M$

Claim 4 $d\big(\Psi_{\frac{1}{2}} \circ e(p), g \circ e(p)\big) < \epsilon$ for all p in M.

Proof of Claim 4 This follows immediately from the fact that Φ is almost periodic.

$$d\big(\Psi_{\frac{1}{2}} \circ e(p), g \circ e(p)\big) = d\big((\Phi_{\frac{1}{2}}(p), \Phi_1(p)), (\Phi_{\frac{1}{2}}(p), p)\big) = d_M(\Phi_1(p), p) < \epsilon \quad \square$$

Claim 5 The image of J is contained in $M \times M \setminus N_\delta(\Delta)$ where $\delta = (\frac{\sqrt{2}}{2}K - 1)\epsilon$.

Proof of Claim 5 Let $t \in [0, \frac{1}{2}]$ and $z \in D$. Then

$$J(z,t) = \Psi_t \circ e \circ f(z) = \Psi_t\big(f(z), \Phi_{\frac{1}{2}}(f(z))\big) = \big(\Phi_t(f(z)), \Phi_{t+\frac{1}{2}}(f(z))\big).$$

Property 2 of the flow implies $d_M\big(\Phi_t(f(z)), \Phi_{t+\frac{1}{2}}(f(z))\big) > K\epsilon$. Therefore Lemma 4.3 implies,

$$d\big(J(z,t), \Delta\big) > \frac{\sqrt{2}}{2}K\epsilon.$$

For $t \in [\frac{1}{2}, 1]$ the map $J_t(z)$ is the min-geodesic between the points $J_{\frac{1}{2}}(z) = \Psi_{\frac{1}{2}} \circ e \circ f(z)$ and $g \circ e \circ f(z)$. By Claim 4, these points are less then ϵ apart. Therefore $d\big(J(z,t), J(z, \frac{1}{2})\big) < \epsilon$ for $t \in [\frac{1}{2}, 1]$. Since $d\big(J(z, \frac{1}{2}), \Delta\big) > \frac{\sqrt{2}}{2}K\epsilon$, this implies the claim. □

Since the map $f|_{\partial D}$ is no longer invariant under the flow, for each $t \in [0,1]$ the image of $J_t(\partial D)$ is a different circle in X. However, we will show that $J_t(\partial D)$ is close to a g–invariant circle C. Then we will extend our homotopy over a larger disk D^+, such that ∂D^+ maps to C.

Lemma 4.5 *There exists a g–invariant circle $C \subset X$ and a 1–parameter family of maps $c\colon [0,2]/0{\equiv}2 \times [0, \frac{1}{2}] \to C$ such that c satisfies the following properties.*

(1) $c_{\frac{1}{2}} = g \circ c_0$

(2) $d(c(s,t), J|_{\partial D}(s,t)) < 4N\epsilon$

(3) c_t *is a degree 1 map for all* $t \in [0, \frac{1}{2}]$.

See Section 4.2 for a proof. Let D^+ be the disk of radius 2 centered at the origin of the complex plane. Then D^+ contains the unit disk D. We denote points in the annulus $A = D^+ \setminus D$ by $z = re^{\pi i s}$ with $1 \le r \le 2$ and $s \in [0,2]$. Points in ∂D are therefore denoted by $e^{\pi i s}$. The extension of J over A will map the outer circle of ∂A to the g–invariant circle C constructed in Lemma 4.5 for all $t \in [0,1]$.

For $t \in [0, \frac{1}{2}]$, the map $J_t\colon A \to X$ maps each segment $\{re^{\pi i s} | r \in [1,2]\}$ to the unique min-geodesic starting at the point $J_t|_{\partial D}(s)$ and terminating at the point $c(s,t)$. See Figure 6. Let $v(s,t) = \exp^{-1}\big(J(e^{\pi i s}, t), c(s,t)\big)$. Then $v(s,t)$ is an element of the tangent space of $M \times M$ at the point $J(e^{\pi i s}, t)$. We explicitly define J as follows.

(2) $$J(re^{\pi i s}, t) = \exp\big(J(e^{\pi i s}, t), (r-1)v(s,t)\big).$$

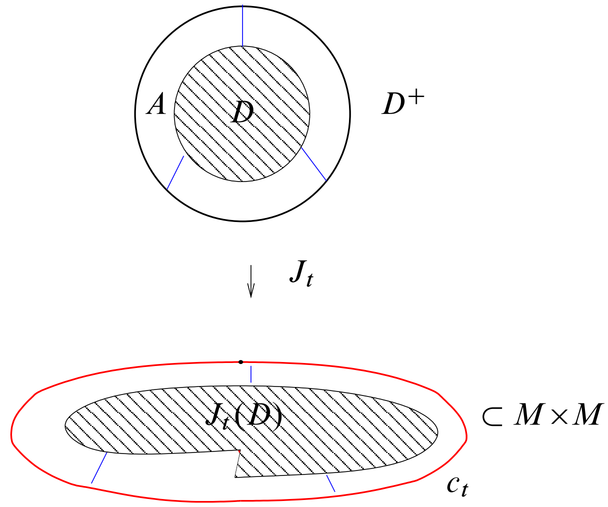

Figure 6: Extending J

By hypothesis, for a point $p \in M$ the exponential map $\exp_p\colon T_p(M) \to M$ is a diffeomorphism. This implies for $(p,q) \in M \times M$ the map

$$\exp_{(p,q)}\colon T_{(p,q)}(M \times M) \to M \times M$$

is a diffeomorphism, and we conclude that J is continuous by Lemma 2.2.

For the second time interval, $t \in [\frac{1}{2}, 1]$, each segment $\{re^{\pi i s} | r \in [1,2]\}$ is mapped by J_t to the min-geodesic between $J_t|_{\partial D}(s)$ and $c(s, \frac{1}{2}) = g \circ c(s, 0)$. The formula is same as in (2), but with $v(s,t) = \exp^{-1}\big(J(e^{\pi i s}, t), c(s, \frac{1}{2})\big)$. Now that we have defined J on all of D^+, we verify that it is a reflector homotopy.

To satisfy property 1 of Definition 3.1, we show that $J_t(D^+) \subseteq X$, and also $J_t(\partial D^+) \subseteq C$. Note that for all t in $[0,1]$ the image of $J_t(\partial D^+)$ is the set C constructed in Lemma 4.5. Claim 5 implies that $J_t(D) \subseteq X$. Therefore we need only to show that $J_t(A)$ does not intersect Δ.

Claim 6 $J_t(A)$ is contained in a $5N\epsilon$ neighborhood of $J_t(\partial D)$.

Proof of Claim 6 First let $t \in [0, \frac{1}{2}]$. On the annulus A, which we parameterized by $re^{\pi i s}$, the map J_t is the min-geodesic with respect to the radius r between the maps $J_t|_{\partial D}$ and c_t. Lemma 4.5 tells us these maps are distance less then $4N\epsilon$ apart. Therefore

$$d\big(J(re^{\pi i s}, t), J(e^{\pi i s}, t)\big) < d\big(c(s,t), J(e^{\pi i s}, t)\big) < 4N\epsilon.$$

For $t \in [\frac{1}{2}, 1]$ the map J_t is the min-geodesic with respect the the radius r between the maps $J_t|_{\partial D}$ and $g \circ c_0$. For $z \in D$, we have $d\big(J(z,t), J(z,\frac{1}{2})\big) < \epsilon$. (This was noted in the proof of Claim 5.) Therefore, for $e^{\pi i s} \in \partial D$ and $re^{\pi i s} \in A$,

$$\begin{aligned} d\big(J(re^{\pi i s}, t), J(e^{\pi i s}, t)\big) &\leq d\big(g \circ c(s,0), J(e^{\pi i s}, t)\big) \\ &= d\big(c(s,\tfrac{1}{2}), J(e^{\pi i s}, t)\big) \\ &\leq d\big(c(s,\tfrac{1}{2}), J(e^{\pi i s}, \tfrac{1}{2})\big) + d\big(J(e^{\pi i s}, \tfrac{1}{2}), J(e^{\pi i s}, t)\big) \\ &\leq 4N\epsilon + \epsilon < 5N\epsilon \end{aligned}$$

which proves the claim. □

Claim 5 states that $J_t(\partial D)$ does not intersect a $\big(\frac{\sqrt{2}}{2}K - 1\big)$ neighborhood of Δ. Since $\big(\frac{\sqrt{2}}{2}K - 1\big)\epsilon > 5N\epsilon$, Claim 6 implies $J_t(A)$ is contained in X.

To verify property 2, that $J_1 = g \circ J_0$, we again need only verify this for points in A, since for $z \in D$, we have $J_1(z) = g \circ J_0(z)$. The segment $S = \{re^{\pi i s} | r \in [1,2]\}$ is mapped by J_0 to the min-geodesic from $J_0|_{\partial D}(s)$ to $c(s,0)$. Call this geodesic λ. On S, the map J_1 is the min-geodesic starting at $J_1|_{\partial D}(s) = g \circ J_0|_{\partial D}(s)$ and ending at $J_1|_{\partial D^+}(s) = g \circ c(s,0)$. Since g is an isometry, this is the geodesic $g \circ \lambda$.

Property 3, that $J_t|_{\partial D^+}$ is a degree one map, follows immediately from Lemma 4.5. Therefore, using Φ we have constructed a reflector homotopy J. This contradicts Theorem 3.2, and the result is proven. □

4.2 Proof of Lemma 4.5

First we prove that on the image of e, the maps g and $\Psi_{\frac{1}{2}}$ are pointwise ϵ–close, and the identity map and Ψ_1 are within $\sqrt{2}\epsilon$.

Lemma 4.6 *For any point p in M,*

$$d(g \circ e(p), \Psi_{\frac{1}{2}} \circ e(p)) < \epsilon \quad \text{and} \quad d\big(e(p), \Psi_1 \circ e(p)\big) < \sqrt{2}\epsilon.$$

Proof

$$d\big(g \circ e(p), \Psi_{\frac{1}{2}} \circ e(p)\big) = d\big((\Phi_{\frac{1}{2}}(p), p), (\Phi_{\frac{1}{2}}(p), \Phi_1(p)\big) = d_M(p, \Phi_1(p)) < \epsilon.$$

Hence

$$\begin{aligned} d\big(e(p), \Psi_1 \circ e(p)\big) &= d\big((p, \Phi_1(p)), (\Phi_1(p), \Phi_{\frac{3}{2}}(p))\big) \\ &= \sqrt{d_M\big(p, \Phi_1(p)\big)^2 + d_M\big(\Phi_{\frac{1}{2}}(p), \Phi_{\frac{3}{2}}(p)\big)^2} \\ &\leq \sqrt{2}\epsilon, \end{aligned}$$

which completes the proof. □

Recall that in the proof of Theorem 4.1, where we assumed the flow line for a point p had period 1, the g–invariant circle C was the set

$$C = e \circ f(\partial D) = \{(\Phi_s(p), \Phi_{s+\frac{1}{2}}(p)) \mid s \in [0,1]\}.$$

We note one other property of the map $e \circ f$. For $s \in [\frac{1}{2}, 1]$,

$$e \circ f(s) = g \circ e \circ f(s - \tfrac{1}{2}).$$

This is easily verified from the definitions of the maps. Note this implies that $g \circ e \circ f(0) = e \circ f(\frac{1}{2})$.

Now we give a sketch of the proof of Lemma 4.5. Although f no longer maps to a flow line with period 1, the bounds that were proven in Lemma 4.6 imply that for $s \in [\frac{1}{2}, 1]$, the point $e \circ f(s)$ is close to $g \circ e \circ f(s - \frac{1}{2})$. So to construct a g–invariant circle C, we define an arc $\alpha\colon [0, \frac{1}{2}] \to M \times M$ such that $\alpha(s)$ is close to $e \circ f(s)$, and $g \circ \alpha(0) = \alpha(\frac{1}{2})$. Then for $s \in [0,1]$,

$$c_0(s) = \begin{cases} \alpha(s) & s \in [0, \frac{1}{2}] \\ g \circ \alpha(s - \frac{1}{2}) & s \in [\frac{1}{2}, 1] \end{cases}$$

The image of c_0 is a g–invariant circle in $M \times M$, and $c_0(s)$ is close to $J_t|_{\partial D}$. For $s \in [1,2]$, $J_t|_{\partial D}(s) = e \circ \gamma_p(s)$, where γ_p was the min-geodesic between $\Phi_1(p)$ and p. Since the time 1 growth rate of γ_p is N, the image of $e \circ \gamma_p$ lies in a neighborhood of $c_0(0) = c_0(1)$. Therefore for $s \in [1,2]$ we define $c_0(s)$ to be the constant map onto $c_0(1)$. By rotating the map c_0, we define the maps $c_t\colon [0,2]/0{\equiv}2 \to C$ so that $c_t(s)$ is close to $J_t|_{\partial D}(s)$.

Lemma 4.5 *There exists a g–invariant circle $C \subset X$ and a 1–parameter family of maps $c\colon [0,2]/0{\equiv}2 \times [0, \frac{1}{2}] \to C$ such that c satisfies the following properties.*

(1) $c_{\frac{1}{2}} = g \circ c_0$

(2) $d(c(s,t), J|_{\partial D}(s,t)) < 4N\epsilon$

(3) c_t *is a degree 1 map for all* $t \in [0, \frac{1}{2}]$.

Proof We construct the arc $\alpha\colon [0, \frac{1}{2}] \to X$, such that $\alpha(s)$ is close to $e \circ f(s)$. Lemma 4.6 implies that $\Psi_{\frac{1}{2}} \circ e \circ f(0) = e \circ f(\frac{1}{2})$ is contained in $B_\epsilon(g \circ e \circ f(0))$. Therefore, by continuity, there exists $\delta < \frac{1}{2}$ such that $e \circ f(s) \in B_\epsilon(g \circ e \circ f(0))$ whenever

$\delta \le s \le \frac{1}{2}$. Let $\sigma\colon [\delta, \frac{1}{2}] \to X$ be the min-geodesic between $e \circ f(\delta)$ and $g \circ e \circ f(0)$. Then

$$\alpha(s) = \begin{cases} e \circ f(s) & 0 \le s \le \delta \\ \sigma(s) & \delta \le s \le \frac{1}{2} \end{cases}.$$

Note that $\alpha(\frac{1}{2}) = g \circ \alpha(0)$. Also, both $\alpha(s)$ and $e \circ f(s)$ are contained in $B_\epsilon(g \circ e \circ f(0))$ for $s \in [\delta, \frac{1}{2}]$, therefore $d(\alpha(s), e \circ f(s)) < 2\epsilon$ for all $s \in [0, \frac{1}{2}]$. We define $c\colon [0,2]/0{\equiv}2 \times [0, \frac{1}{2}] \to X$ by

$$c(s,t) = \begin{cases} \alpha(s+t) & 0 \le s \le \frac{1}{2} - t \\ g \circ \alpha(s+t-\frac{1}{2}) & \frac{1}{2} - t \le s \le 1 - t \\ \alpha(s+t-1) & 1 - t \le s \le 1 \\ \alpha(t) & 1 \le s \le 2 \end{cases}$$

If a_t is the constant map from an interval onto $\alpha(t)$, the map c_0 is simply $\alpha \cdot (g \circ \alpha) \cdot a_0$. As t increases, c_t is equal to $\alpha \cdot (g \circ \alpha) \cdot a_t$ composed with a rotation. Let $C = \{c(s,0) | s \in [0,2]\}$. Then C is a g–invariant circle that is the image of c_t for all t. As defined, c_t may not be a degree one map. If $\Phi(p, \mathbb{R})$ is compact with period less than $\frac{1}{2}$, the degree of the map c_0 will be greater than one. However, by perturbing α arbitrarily small amount, we can make the map an embedding. Then $\alpha \cdot (g \circ \alpha)$ will be an embedding, which implies c_0 is a degree one map, since $c_0 \simeq \alpha \cdot (g \circ \alpha)$.

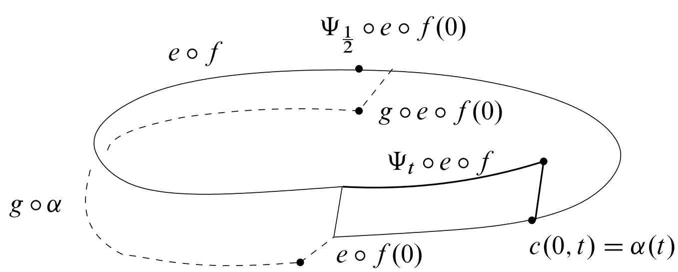

Figure 7: close maps

We now verify that c_t is close to $J_t|_{\partial D} = \Psi_t \circ e \circ f$. As c is piecewise defined, we check each of the four intervals.

Case 1 Let $0 \le s \le \frac{1}{2} - t$, or $t \le s + t \le \frac{1}{2}$. Then

$$d\big(\Psi_t \circ e \circ f(s), \alpha(s+t)\big) = d\big(e \circ f(s+t), \alpha(s+t)\big) < 2\epsilon.$$

This follows immediately from the definition of α.

Case 2 Let $\frac{1}{2}-t \leq s \leq 1-t$, or $\frac{1}{2} \leq s+t \leq 1$. Then

$$\begin{aligned} d\big(\Psi_t \circ e \circ f(s), g \circ \alpha(s+t-\tfrac{1}{2})\big) &= d\big(e \circ f(s+t), g \circ \alpha(s+t-\tfrac{1}{2})\big) \\ &= d\big(\Psi_{\frac{1}{2}} \circ e \circ f(s+t-\tfrac{1}{2}), g \circ \alpha(s+t-\tfrac{1}{2})\big) \\ &\leq d\big(\Psi_{\frac{1}{2}} \circ e \circ f(s+t-\tfrac{1}{2}), g \circ e \circ f(s+t-\tfrac{1}{2})\big) \\ &\qquad + d\big(g \circ e \circ f(s+t-\tfrac{1}{2}), g \circ \alpha(s+t-\tfrac{1}{2})\big) \\ &< \epsilon + 2\epsilon. \end{aligned}$$

The last inequality follows from the fact that g is an isometry, and Lemma 4.6, that $\Psi_{\frac{1}{2}} \approx g$ on the image of e.

Case 3 Let $1-t \leq s \leq 1$, or $1 \leq s+t \leq 1+t$. Then

$$\begin{aligned} d\big(\Psi_t \circ e \circ f(s), \alpha(s+t-1)\big) &= d\big(\Psi_1 \circ e \circ f(s+t-1), \alpha(s+t-1)\big) \\ &\leq d\big(\Psi_1 \circ e \circ f(s+t-1), e \circ f(s+t-1)\big) \\ &\qquad + d\big(e \circ f(s+t-1), \alpha(s+t-1)\big) \\ &\leq \sqrt{2}\epsilon + 2\epsilon. \end{aligned}$$

The last inequality follows from the second conclusion of Lemma 4.6, that $\psi_1 \approx \mathrm{id}$ on the image of e.

Case 4 Let $1 \leq s \leq 2$.

On the interval $[1,2]$, the map f is the min-geodesic between the points p and $\Phi(p,1)$. Therefore for $s \in [1,2]$, we have $d(p, f(s)) < \epsilon$. Since the time-one growth rate of this geodesic is N, this implies $d\big(\Phi(p,t), \Phi(f(s),t)\big) < N\epsilon$ for $t \in [0,1]$. Therefore

$$\begin{aligned} d\big(\Psi_t \circ e \circ f(s), \alpha(t)\big) &\leq d\big(\Psi_t \circ e \circ f(s), e \circ f(t)\big) + \big(e \circ f(t), \alpha(t)\big) \\ &\leq d\big(\Psi_t \circ e \circ f(s), \Psi_t \circ e \circ f(0)\big) + 2\epsilon \\ &= d\Big(\big(\Phi_t(f(s)), \Phi_{t+\frac{1}{2}}(f(s))\big), \big(\Phi_t(p), \Phi_{t+\frac{1}{2}}(p)\big)\Big) + 2\epsilon \\ &\leq \sqrt{2}N\epsilon + 2\epsilon < 4N\epsilon. \end{aligned}$$

This completes the proof. □

5 Almost periodic flows and Seifert fibered spaces

Theorem 5.1 *Let M be a closed, orientable, Riemannian 3–manifold whose universal cover is homeomorphic to $\mathbb{R}^3$. Let $0 < 2\epsilon < \operatorname{conv rad}(M)$. If M supports a $(1,\epsilon)$–almost periodic flow $\Phi\colon M \times \mathbb{R} \to M$ with a closed flowline of period 1, then M is Seifert fibered.*

Proof Let p be a point in M such that $\mu = \Phi(p, \mathbb{R})$ is compact and has period 1. Since M is covered by $\mathbb{R}^3$ its fundamental group is torsion free. If μ has infinite order in $\pi_1(M)$, then by Theorem 2.4 the manifold M is Seifert fibered. This leaves the case that μ is contractible. Closed up flow lines are freely homotopic, by Lemma 2.3, hence all closed up flow lines are homotopic to μ and contractible. Therefore the lifted flow $\widetilde{\Phi}$ on the universal cover $\widetilde{M}$ with the lifted metric is $(1,\epsilon)$–almost periodic. Since the universal cover $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$, this contradicts Theorem 4.1, hence the case when μ is contractible can not occur. □

Theorem 5.2 *Let M be a closed, orientable, Riemannian 3–manifold with universal cover homeomorphic to $\mathbb{R}^3$, such that for all $p \in \widetilde{M}$, the exponential $\exp_p\colon T_p\widetilde{M} \to \widetilde{M}$ is a diffeomorphism. Let $0 < 2\epsilon < \operatorname{conv rad}(M)$. If M supports a flow $\Phi\colon M \times \mathbb{R} \to M$ that satisfies*

1. *for some p in M the min-geodesic between p and $\Phi_1(p)$ has time-one growth rate N, and*
2. *Φ is $(10N, \epsilon)$–almost periodic,*

then M is Seifert fibered.

Proof Let p be the point in M such that the min-geodesic γ between p and $\Phi_1(p)$ has time-one growth rate N. Let Γ_p the closed up flow line for p. As before, Theorem 2.4 implies we need only consider the case when Γ_p is trivial in $\pi_1(M)$.

The lifted flow $\widetilde{\Phi}$ on the universal cover $\widetilde{M}$ with the lifted metric is $(10N, \epsilon)$–almost periodic. Let $\widetilde{p}$ be a lift of p. Then $\widetilde{\gamma}$, the lift of γ, is the min-geodesic between $\widetilde{p}$ and $\widetilde{\Phi}_1(p)$ which also has a time-one growth rate of N under the flow $\widetilde{\Phi}$. Therefore $\widetilde{\Phi}$ satisfies the hypothesis of Theorem 4.4 and we get a contradiction for this case. □

Corollary 5.3 *Let M be a closed, orientable, connected, Riemannian 3–manifold with non-positive sectional curvature. If M supports a flow Φ such that for some p in M the min-geodesic between p and $\Phi_1(p)$ has time-one growth rate N, then Φ is not $(10N, \epsilon)$–almost periodic.*

This Corollary follows immediately from Hadamard's Theorem (see Do Carmo [3] for a proof).

Theorem 5.4 (Hadamard) *Let M^n be a complete Riemannian manifold, simply connected, with all sectional curvature $K < 0$. Then M is diffeomorphic to $\mathbb{R}^n$; more precisely $\exp_p \colon T_p M \to M$ is a diffeomorphism.*

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