Dieudonné modules and p–divisible groups associated with Morava K–theory of Eilenberg–Mac Lane spaces

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We study the structure of the formal groups associated to the Morava K–theories of integral Eilenberg–Mac Lane spaces. The main result is that every formal group in the collection $\{K(n)^* K(\mathbb{Z}, q), q = 2, 3, \ldots\}$ for a fixed n enters in it together with its Serre dual, an analogue of a principal polarization on an abelian variety. We also identify the isogeny class of each of these formal groups over an algebraically closed field. These results are obtained with the help of the Dieudonné correspondence between bicommutative Hopf algebras and Dieudonné modules. We extend P Goerss' results on the bilinear products of such Hopf algebras and corresponding Dieudonné modules.

1 Introduction

The theory of formal groups gave rise to a powerful method for solving various problems of algebraic topology thanks to the fundamental works by Novikov [9] and Quillen [12]. Formal groups in topology arise when one applies a complex oriented cohomology theory to the infinite complex projective space $\mathbb{C}P^\infty$. However the formal groups obtained in this way are all one-dimensional and so far the rich and intricate theory of higher dimensional formal groups remained outside of the realm of algebraic topology. One could hope to get nontrivial examples in higher dimensions by applying a generalized cohomology to an H–space. For most known cohomology theories and H–spaces this hope does not come true, however there is one notable exception. Quite surprisingly, the Morava K–theories applied to the integral Eilenberg–Mac Lane spaces give rise to formal groups in higher dimensions. Moreover, these formal groups are exceptionally good in the sense that they have *finite height*.

This striking result belongs to Ravenel and Wilson [14] who used it to prove the so-called Conner–Floyd conjecture. However until now there has not been a systematic study of the remarkable collection of formal groups discovered by Ravenel and Wilson. This study is our main objective in this paper.

The main tool for Ravenel and Wilson was the notion of a Hopf ring and its behaviour in spectral sequences. The definition of a Hopf ring was recently put in a conceptual framework by Goerss by introducing a suitable symmetric monoidal category for bicommutative Hopf algebras in [4]. We make substantial use of the results of this paper.

It is well-known that the most effective way to study formal groups, particularly those of finite height or, more generally, *p–divisible groups* is via the *Dieudonné functor* which associates to a formal group a module over a certain ring called the Dieudonné ring (see Manin [7] or Demazure [2]. Goerss supplied the category of Dieudonné modules with a monoidal structure and showed that the Dieudonné functor is monoidal. We use this technique to study the structure of Ravenel–Wilson formal groups.

Our main result is that the spectrum multiplication

$$K(\mathbb{Z}/p^{\nu}, q) \wedge K(\mathbb{Z}/p^{\nu}, n-q) \to K(\mathbb{Z}/p^{\nu}, n)$$

induces a kind of Poincaré duality on $K(n)_* K(\mathbb{Z}/p^{\nu}, -)$ where $K(n)$ is the nth Morava K–theory. More precisely, we show that the Hopf algebras $K(n)_* K(\mathbb{Z}/p^{\nu}, q)$ and $K(n)_* K(\mathbb{Z}/p^{\nu}, n-q)$ are dual to each other for n odd and 'twisted dual' for n even (precise formulations are found in the main text). Moreover, the formal groups $K(n)^* K(\mathbb{Z}, q+1)$ and $K(n)^* K(\mathbb{Z}, n-q+1)$ are Serre dual to each other and we identify explicitly the isogeny classes of these formal groups over an algebraic closure of $\mathbb{F}_p$, the field of p elements.

The main ingredient in the proof is the theorem of Ravenel and Wilson which shows that the collection of Hopf algebras $K(n)_* K(\mathbb{Z}/p^{\nu}, -)$ forms an *exterior* Hopf ring on $K(n)_* K(\mathbb{Z}/p^{\nu}, 1)$. The 'Poincaré duality' mentioned above is not a formal consequence of the Ravenel–Wilson theorem, though, but follows from rather exceptional properties of the Hopf algebra $K(n)_* K(\mathbb{Z}/p^{\nu}, 1)$.

The paper is organized as follows. In sections 2 and 3 we introduce the so-called $\boxtimes$–product (or bilinear product) in the category of bicommutative Hopf algebras $\mathcal{H}$. We mostly follow Goerss' paper [4]; however our construction of the bilinear product is more explicit than his and we provide some instructive examples. In section 4 we discuss an appropriate version of the Cartier duality which is more general than the usual one in that we do not restrict our study to finite-dimensional Hopf algebras. We also answer Paul Goerss' question to explicitly describe the internal Hom functor in $\mathcal{H}$. Sections 5 through 8 discuss the monoidal structure on the category of Dieudonné modules $\mathcal{D}$ as well as the Dieudonné correspondence. Again, our main source is [4], but we also consider the internal Hom functor and duality in $\mathcal{D}$. In section 9 we discuss exterior Hopf rings, exterior Dieudonné algebras and their relations to generalized

homology of Ω–spectra. Sections 10 and 11 are devoted to the study of the structure of the exterior Hopf algebra on $K(n)_* K(\mathbb{Z}/p^\nu, 1)$ and the associated Dieudonné exterior algebra. The main results are formulated and proved in Section 12.

1.1 Notation and conventions

We consider bialgebras over a fixed field $\mathbf{k}$, ie collections of data $(H, \Delta_H, m_H, \epsilon_H, i_H)$. Here H is a vector space over $\mathbf{k}$, $m_H\colon H \otimes_{\mathbf{k}} H \to H$ and $\Delta_H\colon H \to H \otimes_{\mathbf{k}} H$ are associative and coassociative multiplication and comultiplication, $\epsilon_H\colon H \to \mathbf{k}$ and $i_H\colon \mathbf{k} \to H$ are the counit and the unit respectively. The comultiplication and the counit are required to be homomorphisms of unital algebras. We will usually omit the subscript H when it is clear from the context. A bialgebra having an antipode will be called a Hopf algebra.

Additionally all bialgebras and Hopf algebras will be assumed to be commutative and cocommutative. The antipode $H \to H$ will be denoted by $[-1]_H$. Thus, for two Hopf algebras A, B the set of all Hopf algebra homomorphisms $A \to B$ is an abelian group. The addition of two homomorphisms $f, g\colon A \to B$ is defined as the composition

$$ A \xrightarrow{\Delta_A} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{m_B} B \ . $$

The zero homomorphism is the composite map

$$ A \xrightarrow{\epsilon_A} \mathbf{k} \xrightarrow{i_B} B \ . $$

The additive inverse to a homomorphism $f\colon A \to B$ is given by precomposing f with $[-1]_B\colon B \to B$ or, equivalently, postcomposing it with $[-1]_A\colon A \to A$. According to a well-known theorem of Grothendieck, see [2, page 30], the category of Hopf algebras form an abelian category which will be denoted by $\mathcal{H}$. The zero object in $\mathcal{H}$ is the ground field $\mathbf{k}$.

If $A = B$ then the set of endomorphisms of A is also a ring with respect to the composition of endomorphisms. The ring of integers $\mathbb{Z}$ maps canonically into this ring and we denote by $[n]_A$ or simply by $[n]$ the image of $n \in \mathbb{Z}$ under this map.

For an element x in a Hopf algebra we will write $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$. Similarly the n–fold diagonal $\Delta^n\colon H \to H^{\otimes n}$ will be written as $\sum x^{(1)} \otimes \ldots \otimes x^{(n)}$. Our ground field $\mathbf{k}$ will have characteristic p unless indicated otherwise; from Section 8 onwards p will be odd. In Sections 2 through 4 the symbol $\otimes$ will stand for $\otimes_{\mathbf{k}}$; later on all unmarked tensor products are assumed to be taken over the p–adic integers $\mathbb{Z}_p$. We will denote by $\mathbb{F}_p$ the field consisting of p elements, and by $\mathbb{Q}_p$ the field of p–adic rational numbers. For a Hopf algebra H we will denote by $F\colon H \to H$ the Frobenius

morphism. The dual vector space H^* to H has the structure of an algebra and so the raising to the pth power determines a linear map $H^* \to H^*$. The (continuous) dual to this map gives a linear endomorphism $V \colon H \to H$ called the *Verschiebung*.

2 Bilinear products of Hopf algebras

In this section we will introduce the operation in the category of Hopf algebras which models the tensor product of abelian groups.

Definition 2.1 Let H_1, H_2, K be Hopf algebras. Let ϕ be a morphism of coalgebras

$$\phi \colon H_1 \otimes H_2 \to K.$$

We will write $x \circ y$ for $\phi(x, y)$. Then ϕ is called a *bilinear map* if the following axioms hold for all $x, y \in H_1, z, w \in H_2$:

(1) $xy \circ z = \sum (x \circ z^{(1)})(y \circ z^{(2)})$

(2) $x \circ wz = \sum (x^{(1)} \circ w)(x^{(2)} \circ z)$

(3) $x \circ 1 = \epsilon(x) \cdot 1$

(4) $1 \circ z = \epsilon(z) \cdot 1$

The bilinear product of two Hopf algebras is defined with the help of a suitable universal property with respect to bilinear maps. More precisely:

Definition 2.2 For two Hopf algebras H_1, H_2 their *bilinear product* $H_1 \boxtimes H_2$ is the unique Hopf algebra together with a bilinear map

$$\gamma \colon H_1 \otimes H_2 \to H_1 \boxtimes H_2$$

such that for any bilinear map $H_1 \otimes H_2 \to K$ there exists a unique Hopf algebra map $H_1 \boxtimes H_2 \to K$ making the following diagram commute:

$$
\begin{array}{ccc}
H_1 \otimes H_2 & \xrightarrow{\ \gamma\ } & H_1 \boxtimes H_2 \\
\downarrow & & \swarrow \\
K &
\end{array}
$$

Of course, one still needs to prove that $H_1 \boxtimes H_2$ satisfying the afore-mentioned universal property exists. A proof of this result is contained in [4]. Below we will give another, hopefully more explicit description of the bilinear product.

Let $S(H_1 \otimes H_2)$ be the symmetric algebra on $H_1 \otimes H_2$. For $a_1 \in H_1, a_2 \in H_2$ we will write the elements $a_1 \otimes a_2 \in S(H_1 \otimes H_2)$ as $a_1 \circ a_2$. The algebra $S(H_1 \otimes H_2)$ has a unique structure of a bialgebra defined by the requirement that the canonical inclusion $H_1 \otimes H_2 \to S(H_1 \otimes H_2)$ be a map of coalgebras. Explicitly, for $a_1 \in H_1, a_2 \in H_2$ we have

$$(2\text{--}1) \qquad \Delta(a_1 \circ a_2) = \sum a_1^{(1)} \circ a_2^{(1)} \otimes a_1^{(2)} \circ a_2^{(2)}.$$

Consider the ideal J in $S(H_1 \otimes H_2)$ generated by the elements:

(1) $a_1(x, y, z) = xy \circ z - \sum (x \circ z^{(1)})(y \circ z^{(2)})$

(2) $a_2(x, w, z) = x \circ wz - \sum (x^{(1)} \circ w)(x^{(2)} \circ z)$

(3) $b_1(x) = x \circ 1 - \epsilon(x) \cdot 1$

(4) $b_2(z) = 1 \circ z - \epsilon(z) \cdot 1$

Let us show that J is a coideal with respect to the coalgebra structure in $S(H_1 \otimes H_2)$. We will restrict ourselves with checking the elements $a_1(x, y, z)$ and $b_1(x)$ only; the proof for $a_2(x, w, z)$ and $b_2(z)$ is similar. We have:

$$\Delta b_1(x) = \sum x^{(1)} \circ 1 \otimes x^{(2)} \circ 1 - \epsilon(x) 1 \otimes 1$$
$$= \sum (b_1(x^{(1)}) + \epsilon(x^{(1)})1) \otimes (b_1(x^{(2)}) + \epsilon(x^{(2)})1) - \epsilon(x) 1 \otimes 1$$
$$= \sum b_1(x^{(1)}) \otimes b_1(x^{(2)}) + b_1(x^{(1)}) \otimes \epsilon(x^{(2)})1 + \epsilon(x^{(1)}1) \otimes b_1(x^{(2)})$$

We see that $\Delta b_1(x) \in J \otimes S + S \otimes J$. Further,

$$\Delta(xy \circ z) = (\Delta x \Delta y) \circ \Delta z = \sum (x^{(1)} y^{(1)}) \circ z^{(1)} \otimes (x^{(2)} y^{(2)}) \circ z^{(2)}.$$

On the other hand:

$$\Delta \sum (x \circ z^{(1)})(y \circ z^{(2)}) = \sum \Delta(x \circ z^{(1)}) \Delta(y \circ z^{(2)})$$
$$= \sum [(x^{(1)} \circ z^{(1)(1)}) \otimes (x^{(2)} \circ z^{(1)(2)})][(y^{(1)} \circ z^{(2)(1)}) \otimes (y^{(2)} \circ z^{(2)(2)})]$$
$$= \sum (x^{(1)} \circ z^{(1)(1)})(y^{(1)} \circ z^{(2)(1)}) \otimes (x^{(2)} \circ z^{(1)(2)})(y^{(2)} \circ z^{(2)(2)})$$

Using cocommutativity of the comultiplication on H_2 we can rewrite the last expression as follows:

$$\sum (x^{(1)} \circ z^{(1)(1)})(y^{(1)} \circ z^{(1)(2)}) \otimes (x^{(2)} \circ z^{(2)(1)})(y^{(2)} \circ z^{(2)(2)} \equiv \Delta(xy \circ z)$$
$$\mod (S \otimes J + J \otimes S)$$

We have the following result.

Proposition 2.3 *The algebra $S(H_1 \otimes H_2)/J$ has the structure of a Hopf algebra where the diagonal is given by the formula* (2–1)*. The antipode is given by the formula*

$$[-1](x \circ y) = ([-1]x) \circ y.$$

Proof It follows from the foregoing discussion that $S(H_1 \otimes H_2)/J$ is a bialgebra; thus it only remains to check that the endomorphism $[-1]$ on $S(H_1 \otimes H_2)/J$ defined by the above formula is indeed the antipode. Denoting by m the multiplication map on $S(H_1 \otimes H_2)/J$ we have

$$x \circ y \xrightarrow{\Delta} \sum x^{(1)} \circ y^{(1)} \otimes x^{(2)} \circ y^{(2)} \xrightarrow{[-1] \otimes [1]} \sum ([-1]x^{(1)}) \circ y^{(1)} \otimes x^{(2)} \circ y^{(2)}$$
$$\xrightarrow{m} \sum [([-1]x^{(1)})x^{(2)}] \circ y = \sum \epsilon(x) 1 \circ y = \epsilon(x)\epsilon(y) 1$$

as required. $\square$

Remark 2.4 Similarly we can show that $[-1](x \circ y) = x \circ ([-1]y)$. Furthermore, using the equality $[-1]^2 = [1] = id$ it is easy to see that the following identity holds for any $x \in H_1$, $y \in H_2$:
$$x \circ y = [-1]x \circ [-1]y.$$

Finally we have the following result which is a direct consequence of the above constructions.

Corollary 2.5 *The Hopf algebra $S(H_1 \otimes H_2)/J$ satisfies the universal property of* Definition 2.2 *and thus realizes the bilinear product $H_1 \boxtimes H_2$.*

Example 2.6 Let G_1, G_2 be abelian groups. Then we have the following isomorphism of Hopf algebras:
$$\mathbf{k}[G_1] \boxtimes \mathbf{k}[G_2] \cong \mathbf{k}[G_1 \otimes_{\mathbb{Z}} G_2].$$
Indeed, it is easy to see, using the universal property of the bilinear product, that a bilinear map $\mathbf{k}[G_1] \otimes \mathbf{k}[G_2] \to H$ is equivalent to (the usual notion of) a bilinear map from $G_1 \times G_2$ to the group of group-like elements of H.

Example 2.7 For any Hopf algebra H we have natural isomorphisms:
$$\mathbf{k}[\mathbb{Z}] \boxtimes H \cong H \cong H \boxtimes \mathbf{k}[\mathbb{Z}].$$

The canonical bilinear map $\mathbf{k}[\mathbb{Z}] \otimes H \to H$ is constructed as follows. Let $t^n \in \mathbf{k}[\mathbb{Z}] = \mathbf{k}[t, t^{-1}]$ and $a \in H$. Then $t^n \circ a \mapsto [n](a)$. This determines a homomorphism of Hopf algebras $\mathbf{k}[\mathbb{Z}] \boxtimes H \to H$. The inverse map is specified by $a \mapsto t \circ a$. The second isomorphism is constructed similarly.

Example 2.8 Let $H_1 = \mathbf{k}[x_1, \ldots, x_n], H_2 = \mathbf{k}[y_1, \ldots, y_k]$ where x_i and y_i are primitive. Then $H_1 \boxtimes H_2$ is isomorphic to the polynomial algebra on primitive generators $x_i \circ y_j$ where $i = 1, 2, \ldots n, j = 1, 2, \ldots k$.

Example 2.9 Let $H = \mathbf{k}[x]/x^p$. Then $H \boxtimes H \cong \mathbf{k}[x]$ where x is primitive.

Remark 2.10 The isomorphisms of Examples 2.8 and 2.9 could be obtained directly using our explicit construction of $H_1 \boxtimes H_2$. For example, a simple calculation shows that in $\mathbf{k}[x] \boxtimes \mathbf{k}[x]$ one has $x^n \circ x^m = \delta_{mn} n! (x \circ x)^n$ and thus $x \circ x$ is a polynomial generator in $\mathbf{k}[x] \boxtimes \mathbf{k}[x]$. It is simpler, however, to do this using the Dieudonné correspondence which will be discussed later on.

Remark 2.11 We are only interested in the case when the field $\mathbf{k}$ has characteristic $p \neq 0$. Note, however, that the construction of the bilinear product goes through also in the characteristic zero case. The isomorphisms of examples 2.6, 2.7 and 2.8 continue to hold. Example 2.9 has no analogue in characteristic zero.

Summing up the above discussion we have the following result.

Theorem 2.12 *The category $\mathcal{H}$ together with the bilinear product $\boxtimes$ is a symmetric monoidal category. The unit for this monoidal structure is the Hopf algebra $\mathbf{k}[\mathbb{Z}]$.*

Proof The existence of the natural commutativity and associativity isomorphisms follows from the corresponding properties of the usual tensor product together with the universality of $\boxtimes$. The same arguments give rise to the commutativity of the hexagon and pentagon diagrams. We will refer to Mac Lane [6] for the definition and basic properties of symmetric monoidal categories.　　　　　　　□

3　Further properties of the bilinear product

We list a few basic properties of the product $\boxtimes$.

Proposition 3.1 *Let $x \in H_1$, $y \in H_2$. Then the following formulas hold in $H_1 \boxtimes H_2$:*

(1)　$V(x \circ y) = Vx \circ Vy$

(2)　$F(Vx \circ y) = x \circ Fy$

(3)　$F(x \circ Vy) = Fx \circ y.$

(4)　$([n]x) \circ y = [n](x \circ y) = x \circ [n]y$

Proof   The stated formulas follow directly from the definition of the product $\boxtimes$. Equations (1), (2) and (3) are essentially proved in [14, Lemma 7.1]. For (4) we have the following identities:

$$
\begin{aligned}
([n]x) \circ y &= \sum (x^{(1)} \ldots x^{(n)}) \circ y \\
&= \sum (x^{(1)} \circ y^{(1)}) \ldots (x^{(n)} \circ y^{(n)}) \\
&= [n](x \circ y) \\
&= \sum x \circ (y^{(1)} \ldots y^{(n)}) \\
&= x \circ ([n]y) \qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Furthermore, for any two Hopf algebras H_1, H_2 there exists an internal Hom object $\underline{\mathrm{Hom}}(H_1, H_2)$, cf [4, Corollary 5.8]. It is characterized by the property that for any K in $\mathcal{H}$

$$\mathrm{Hom}_{\mathcal{H}}(H_1 \boxtimes K, H_2) \cong \mathrm{Hom}_{\mathcal{H}}(H_1, \underline{\mathrm{Hom}}(K, H_2)).$$

We will denote by $\mathcal{H}_\nu$ the subcategory of $\mathcal{H}$ consisting of the Hopf algebras H such that $[p^\nu](a) = \epsilon(a)$ for any $a \in H$ (the equivalent condition is that $[p^\nu]$ is the zero element in the ring $\mathrm{Hom}_{\mathcal{H}}(H, H)$). Using the language of algebraic geometry one can say that H represents a p^ν–torsion group scheme. The union of all subcategories $\mathcal{H}_\nu$ will be denoted by $\mathcal{H}_\infty$.

Similarly we denote by $\mathcal{H}(N)$ the subcategory of $\mathcal{H}$ formed by those Hopf algebras for which the N th iteration of the Verschiebung V^N is the zero endomorhism. The union of all $\mathcal{H}(N)$ will be denoted by $\mathcal{H}(\infty)$. We will call the objects in $\mathcal{H}(\infty)$ *irreducible Hopf algebras* (in algebraic geometry the corresponding objects are called unipotent group schemes.)

Note that the category $\mathcal{H}_\nu$ as well as $\mathcal{H}_\infty$ forms an ideal inside the symmetric monoidal category $\mathcal{H}$ in the sense that if $H \in \mathcal{H}_\nu$ then for any $A \in \mathcal{H}$ the Hopf algebras $H \boxtimes A$ and $A \boxtimes H$ belong to $\mathcal{H}_\nu$. This follows from formula (4) of Proposition 3.1.

We also have the corresponding statement for the internal Hom functor:

Proposition 3.2   *Let H be a Hopf algebra in $\mathcal{H}_\nu$. Then for any $A \in \mathcal{H}$ the Hopf algebra $\underline{\mathrm{Hom}}(A, H)$ as well as $\underline{\mathrm{Hom}}(H, A)$ also belong to $\mathcal{H}_\nu$.*

Proof   Let us prove that $\underline{\mathrm{Hom}}(A, H) \in \mathcal{H}_\nu$, the remaining case is treated similarly. Consider $\underline{\mathrm{Hom}}(A, H)$ as a functor of the second argument; we claim that for any n the map $[n]: \underline{\mathrm{Hom}}(A, H) \longrightarrow \underline{\mathrm{Hom}}(A, H)$ is induced by the map $[n]: H \longrightarrow H$. The claim obviously implies the statement of the proposition.

We have two maps: $[n]$ and $\underline{\mathrm{Hom}}(A, [n])$: $\underline{\mathrm{Hom}}(A, H) \longrightarrow \underline{\mathrm{Hom}}(A, H)$; to show that they coincide it suffices to check that the induced two maps

$$\phi, \psi \colon \mathrm{Hom}(C, \underline{\mathrm{Hom}}(A, H)) \longrightarrow \mathrm{Hom}(C, \underline{\mathrm{Hom}}(A, H))$$

coincide for any $C \in \mathcal{H}$. Identifying $\mathrm{Hom}(C, \underline{\mathrm{Hom}}(A, H))$ with $\mathrm{Hom}(C \boxtimes A, H)$ we see that ϕ and ψ are both equal to the map $C \longrightarrow \underline{\mathrm{Hom}}(A, H)$ which is induced by $[n] \colon C \longrightarrow C$. $\qquad\square$

The category $\mathcal{H}_\nu$ (but not $\mathcal{H}(N)$) has a unit making it a symmetric monoidal category:

Proposition 3.3 *The unit in the monoidal category $\mathcal{H}_\nu$ is the Hopf algebra $\mathbf{k}[\mathbb{Z}/p^\nu]$.*

Proof Let $t \in \mathbf{k}[\mathbb{Z}/p^\nu]$ be the generator in $\mathbb{Z}/p^\nu$. The unit map $\mathbf{k}[\mathbb{Z}/p^\nu] \boxtimes H \to H$ is specified by $t^n \circ a \mapsto [n]a$ for $a \in H$. This is a well-defined map since

$$\epsilon(a) = [p^\nu]a = t^{p^\nu} \circ a = 1 \circ a.$$

The inverse map is defined as $a \mapsto t \circ a$. $\qquad\square$

We will finish this section by introducing another subcategory $\mathcal{H}^f$ inside $\mathcal{H}$ consisting of *finite dimensional* Hopf algebras. In the algebro–geometric literature they are known under the name of *finite abelian group schemes*. It is well-known (Demazure and Gabriel, [3]) that the category $\mathcal{H}^f$ splits as a direct product:

$$\mathcal{H}^f = \mathcal{H}_{ll} \times \mathcal{H}_{lr} \times \mathcal{H}_{rl} \times \mathcal{H}_{rr}.$$

Here $\mathcal{H}_{ll}$ consists of Hopf algebras which are local with local dual, $\mathcal{H}_{lr}$ are the Hopf algebras which are local and whose duals are reduced (have no nilpotent elements), $\mathcal{H}_{rl}$ are the Hopf algebras which are reduced with local duals and $\mathcal{H}_{rr}$ stands for those Hopf algebras which are reduced together with their duals. Let us give some typical examples of Hopf algebras in each of the four categories listed above.

(1) $\mathcal{H}_{ll}$: $\mathbf{k}[x]/x^p$ where x is primitive;

(2) $\mathcal{H}_{lr}$: a group algebra of a finite abelian p–group:

(3) $\mathcal{H}_{rl}$: a $\mathbf{k}$–dual to the group algebra of a finite abelian p–group;

(4) $\mathcal{H}_{rr}$: a group algebra of a finite abelian group whose order is coprime to p.

Remark 3.4 Note that $\mathcal{H}_{ll} \times \mathcal{H}_{lr} \times \mathcal{H}_{rl}$ coincides with $\mathcal{H}_\infty^f := \mathcal{H}^f \bigcap \mathcal{H}$. The reason for considering the category $\mathcal{H}_\infty^f$ is that it is closed with respect to the Cartier duality which will be discussed in the next section and that it behaves well with

respect to the Dieudonné correspondence. However observe, that $\mathcal{H}^f$ or $\mathcal{H}^f_\infty$ are not closed with respect to $\boxtimes$. Consider, for example the Hopf algebra $H = \mathbf{k}[x]/x^p$ with $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then $H \boxtimes H \cong \mathbf{k}[x]$ according to Example 2.9, in particular, $H \boxtimes H$ is not finite-dimensional.

4 Duality for Hopf algebras

We will now describe a version of the Cartier duality in the category $\mathcal{H}$ and some of its subcategories. We start by defining for any Hopf algebra $H \in \mathcal{H}$ its dual Hopf algebra H^0. This construction makes sense for an arbitrary (not necessarily commutative or cocommutative) Hopf algebra over a field.

Note that if H is not finite dimensional over $\mathbf{k}$ then $H^* = \mathrm{Hom}(H, \mathbf{k})$ is a *topological* Hopf algebra rather than a usual Hopf algebra. That means that the comultiplication $\Delta \colon H^* \longrightarrow H^* \widehat{\otimes} H^* := (H \otimes H)^*$ does not take its values in $H^* \otimes H^*$ as required for a Hopf algebra but in a bigger space $H^* \widehat{\otimes} H^*$. We say that $A \subset H^*$ is a Hopf subalgebra of H^* if A is a subalgebra and $\Delta(A) \in A \otimes A \subset H^* \widehat{\otimes} H^*$. In addition, we require that A be closed under the antipode. Clearly then, A itself is a Hopf algebra.

Definition 4.1 Let $H \in \mathcal{H}$. Define the (Cartier) dual Hopf algebra H^0 to be the union of all Hopf subalgebras inside H^*.

Remark 4.2 Note that if A, B are Hopf subalgebras in H^* then so is $A \cdot B$, the set of all linear combinations of products of elements in A and B. It implies that the union of all Hopf subalgebras in H^* is again a Hopf algebra.

Clearly, if H is finite-dimensional then H^0 coincides with H^*. However, in general H^0 could be very complicated. Consider, for example the Hopf algebra $\mathbf{k}[t]$ where the generator t is primitive. It is well-known that the graded dual to $\mathbf{k}[t]$ (where t is taken to be a homogeneous element of degree 2) is Γ, the algebra of divided powers. However, $\mathbf{k}[t]^0$ is much bigger than Γ, in particular it always contains $\mathbf{k}[\mathbf{k}]$, the $\mathbf{k}$–group ring of $\mathbf{k}$ considered as an additive group. To see that observe that there is a natural evaluation map $\mathbf{k}[t] \longrightarrow \mathrm{Map}(\mathbf{k}, \mathbf{k})$ where $\mathrm{Map}(\mathbf{k}, \mathbf{k})$ stands for the set of all maps of sets $\mathbf{k} \longrightarrow \mathbf{k}$. It is clear that $\mathrm{Map}(\mathbf{k}, \mathbf{k})$ is a topological Hopf algebra which is dual to $\mathbf{k}[\mathbf{k}]$. Taking the continuous dual to the above map we arrive at the inclusion $\mathbf{k}[\mathbf{k}] \longrightarrow H^*$. Related questions are discussed in authors' paper [1].

Definition 4.3 The dualizing object D_h in $\mathcal{H}$ is defined as $D_h := \mathbf{k}[\mathbb{Z}]^0$, the dual Hopf algebra to the unit object $\mathbf{k}[\mathbb{Z}]$. Similarly define the dualizing object in $\mathcal{H}_\nu$ to be $D(\nu)_h := \mathbf{k}[\mathbb{Z}/p^\nu]^*$, the $\mathbf{k}$–linear dual to the unit object in $\mathcal{H}_\nu$.

Remark 4.4 The subscript h in the notation for the dualizing object stands for 'Hopf' and its purpose is to distinguish it from the dualizing object in the category of Dieudonné modules. We will suppress this subscript as well as the dependence on ν in cases when no confusion is possible.

Definition 4.5 Let A, B be Hopf algebras in $\mathcal{H}$. A map $\langle, \rangle\colon A \otimes B \to \mathbf{k}$ is called a *bilinear pairing* if the following axioms hold for any $a, a_1, a_2 \in A, b, b_1, b_2 \in B$:

(1) $\langle a, b_1 b_2 \rangle = \langle \Delta a, b_1 \otimes b_2 \rangle$

(2) $\langle a_1 a_2, b \rangle = \langle a_1 \otimes a_2, \Delta b \rangle$

(3) $\langle 1, b \rangle = \epsilon(b)$

(4) $\langle a, 1 \rangle = \epsilon(a)$

Clearly, a bilinear pairing $A \otimes B \to \mathbf{k}$ is equivalent to a map of Hopf algebras $A \to B^0$. In what follows we treat the Cartier duality in the category $\mathcal{H}_\nu$. However Lemma 4.6, Definition 4.7 and Theorem 4.8 have obvious analogues, with similar proofs in the category $\mathcal{H}$.

Lemma 4.6 *A bilinear pairing $A \otimes B \to \mathbf{k}$ for $A, B \in \mathcal{H}_\nu$ determines and is determined by a map of Hopf algebras $A \boxtimes B \to D(\nu)$.*

Proof Let $\phi\colon A \boxtimes B \to D(\nu)$ be a Hopf algebra map. Taking its linear dual we obtain a map of algebras $D(\nu)^* = \mathbf{k}[\mathbb{Z}/p^\nu] \to (A \boxtimes B)^*$ (since ϕ is a map of coalgebras). The latter gives rise to an invertible element in $(A \boxtimes B)^*$ whose p^νth power is 1. We can consider this element as a map $A \otimes B \to \mathbf{k}$. The axioms 1–4 for the bilinear pairing follow from the corresponding axioms for a bilinear map.

Conversely, a bilinear pairing $\langle, \rangle\colon A \otimes B \to \mathbf{k}$ could be considered as an element $f \in (A \otimes B)^*$. We need to show that $f^{p^\nu} = 1$ in $(A \otimes B)^*$. Let $a \otimes b \in A \otimes B$. We have:

$$
\begin{aligned}
f^{p^\nu}(a \otimes b) &= \sum f(a^{(1)} \otimes b^{(1)}) \dots f(a^{(p^\nu)} \otimes b^{(p^\nu)}) \\
&= \sum \langle a^{(1)}, b^{(1)} \rangle \dots \langle a^{(p^\nu)}, b^{(p^\nu)} \rangle \\
&= \langle a^{(1)} \dots a^{(p^\nu)}, b \rangle \\
&= \langle [p^\nu](a), b \rangle \\
&= \langle \epsilon(a), b \rangle \\
&= \epsilon(a)\epsilon(b)
\end{aligned}
$$

as required. $\square$

Definition 4.7 For $H \in \mathcal{H}_\nu$ define its dual $D_h H$ as $D_h H := \underline{\mathrm{Hom}}(H, D(\nu))$.

Following our customary abuse of notation we will usually shorten $D_h H$ to DH. Then we have the following result.

Theorem 4.8 *There is a natural isomorphism of Hopf algebras $DH \cong H^0$ in $\mathcal{H}_\nu$.*

Proof Let A be a Hopf algebra in $\mathcal{H}$. We have a natural isomorphism $\mathrm{Hom}_{\mathcal{H}_\nu}(A, DH)$ $\cong \mathrm{Hom}_{\mathcal{H}_\nu}(A \boxtimes H, D)$. By the previous lemma the set $\mathrm{Hom}_{\mathcal{H}_\nu}(A \boxtimes H, D)$ is in natural $1-1$ correspondence with the set of bilinear pairings $A \otimes H \to \mathbf{k}$ and the latter bijects with the set of topological Hopf algebra maps $A \to H^*$. Thus, there is a natural bijection of sets

$$\mathrm{Hom}_{\mathcal{H}_\nu}(A, DH) \approx \mathrm{Hom}_{\mathcal{H}_\nu}(A, H^0)$$

and therefore $DH \cong H^0$. $\square$

Corollary 4.9 *The Cartier duality functor restricts to the subcategory $\mathcal{H}^f$.*

Proof *Indeed, it interchanges the categories $\mathcal{H}_{lr}$ and $\mathcal{H}_{rl}$ and maps the category $\mathcal{H}_{ll}$ to itself.* $\square$

Corollary 4.10 *For any finite-dimensional Hopf algebra H in $\mathcal{H}_\nu$ there is a natural isomorphism $H \cong DDH$.* $\square$

Corollary 4.11 *Let H_1, H_2 be Hopf algebras in $\mathcal{H}_\nu$ and assume that H_2 is finite dimensional. Then*

$$\underline{\mathrm{Hom}}(H_1, H_2) \cong (H_1 \boxtimes H_2^*)^0.$$

Proof We have the following sequence of isomorphisms of Hopf algebras:

$$\begin{aligned}
\underline{\mathrm{Hom}}(H_1, H_2) &\cong \underline{\mathrm{Hom}}(H_1, DDH_2) \\
&\cong \underline{\mathrm{Hom}}(H_1 \boxtimes DH_2, D) \\
&\cong D(H_1 \boxtimes DH_2) \\
&\cong (H_1 \boxtimes H_2^*)^0 \qquad\qquad \square
\end{aligned}$$

The above formula for the internal Hom functor answers the question of Paul Goerss [4, Section 5] at least for finite dimensional p–torsion Hopf algebras. In the next section we will give a corresponding formula for the internal Hom of Dieudonné modules.

5 The bilinear product of Dieudonné modules

In this section we specialize $\mathbf{k} = \mathbb{F}_p$. There is little doubt that all our constructions could be generalized to the case of an arbitrary perfect field, however our topological examples do not require this level of generality and we restrict ourselves with considering the case of a prime field only.

Definition 5.1 The category $\mathcal{D}$ is the category of modules over the Dieudonné ring $R = \mathbb{Z}_p[V, F]/(VF - p)$. The objects in $\mathcal{D}$ will be called Dieudonné modules. The subcategory $\mathcal{D}(N)$ consists of those Dieudonné modules for which V^N acts trivially. The subcategory $\mathcal{D}_\nu$ of $\mathcal{D}$ consists of those Dieudonné modules for which p^ν acts trivially. The union of $\mathcal{D}(N)$ will be denoted by $\mathcal{D}(\infty)$.

Clearly $\mathcal{D}, \mathcal{D}(N), \mathcal{D}_\nu$ are abelian categories. We will next introduce the notion of a *bilinear map* in $\mathcal{D}$ similar to the bilinear map of Hopf algebras discussed in the second section of the paper.

Definition 5.2 Let M, N, L be R–modules. A map $f \colon M \otimes N \longrightarrow L$ is called a bilinear map if it is $\mathbb{Z}_p$–bilinear and

(1) $Ff(Vm \otimes n) = f(m \otimes Fn)$

(2) $Ff(m \otimes Vn) = f(Fm \otimes n)$

(3) $Vf(m \otimes n) = f(Vm \otimes Vn)$.

Just as before, the notion of a bilinear map leads naturally to the notion of a *bilinear product* in the category of Dieudonné modules.

Definition 5.3 For two Dieudonné modules M and N we define their bilinear product $M \boxtimes N$ as the unique Dieudonné module supplied with a bilinear map $M \otimes N \to M \boxtimes N$ such that for any bilinear map $M \otimes N \to L$ there exists a unique R–module map $M \boxtimes N \to L$ making the following diagram commute:

$$
\begin{array}{ccc}
M \otimes N & \longrightarrow & M \boxtimes N \\
\downarrow & \swarrow & \\
L & &
\end{array}
$$

An explicit description of $M \boxtimes N$ is given in the following construction.

Let $M, N \in \mathcal{D}$ and consider $M \otimes N$ as an $\mathbb{Z}_p[V]$–module so that

$$V(x \otimes y) = Vx \otimes Vy.$$

Then $R \otimes_{\mathbb{Z}_p[V]} (M \otimes N)$ has an obvious structure of a left R–module. Set

$$M \boxtimes N := R \otimes_{\mathbb{Z}_p[V]} (M \otimes N)/ \sim$$

where $\sim$ is the R–submodule generated by the following elements:

(5–1) $$F \otimes Vm \otimes n - 1 \otimes m \otimes Fn$$

(5–2) $$F \otimes m \otimes Vn - 1 \otimes Fm \otimes n$$

Hence the structure of an R–module on $M \boxtimes N$ as follows:

$$F(r \otimes m \otimes n) = Fr \otimes m \otimes n$$
$$V(r \otimes m \otimes n) = r \otimes Vm \otimes Vn$$

The element $1 \otimes m \otimes n \in M \boxtimes N$ will be denoted by $m \circ n$.

The following result is an analogue of [Theorem 2.12](#). Its proof is a direct inspection of definitions.

Theorem 5.4 *The category $\mathcal{D}$ together with the bilinear product $\boxtimes$ is a symmetric monoidal category. The unit for this monoidal structure is R–module $\mathbb{1}_{\mathcal{D}} = \mathbb{Z}_p$ where F acts as a multiplication by p and V is the identity automorphism.* $\square$

The product $\boxtimes$ also determines a monoidal structure in $\mathcal{D}_\nu$ and $\mathcal{D}(N)$. The unit in $\mathcal{D}_\nu$ is the module $\mathbb{1}_{\mathcal{D}(\nu)} = \mathbb{Z}_p/p^\nu$ where, as above, F acts as a multiplication by p and V is the identity automorphism. The category $\mathcal{D}(N)$ has no unit.

Remark 5.5 The structure of the bilinear product of two R–modules is not so obvious even in the simplest cases. For example one can prove that $R \boxtimes R$ is isomorphic to the direct sum of copies of R with generators $1 \circ 1, V^l \circ 1, 1 \circ V^l$ where $l = 1, 2, \ldots$.

6 Duality for Dieudonné modules

We will start our treatment of duality in the category $\mathcal{D}$ by introducing the internal Hom functor.

Definition 6.1 Let $M, N \in \mathcal{D}$ or $M, N \in \mathcal{D}_\nu$. Define $\underline{\mathrm{Hom}}(M, N)$ to be the subgroup of $\mathrm{Hom}(R \otimes M, N)$ consisting of such $f \colon R \otimes M \longrightarrow N$ for which:

(1) $Ff(Vr \otimes m) = f(r \otimes Fm)$

(2) $Ff(r \otimes Vm) = f(Fr \otimes m)$

(3) $Vf(r \otimes m) = f(Vr \otimes Vm)$

Furthermore, define the structure of an R–module on $\underline{\mathrm{Hom}}(M, N)$ by the following formula:

$$(6\text{–}1) \qquad\qquad (r \cdot f)(a \otimes m) = f(ra \otimes m)$$

Remark 6.2 It is clear that $\underline{\mathrm{Hom}}(M, N)$ is isomorphic to $\mathrm{Hom}_R(R \boxtimes M, N)$ as an abelian group. It implies, in particular, that $\underline{\mathrm{Hom}}(M, N)$ is a $\mathbb{Z}_p$–module. However the structure of an R–module on $\underline{\mathrm{Hom}}(M, N)$ is different from that on $\mathrm{Hom}_R(R \boxtimes M, N)$

Proposition 6.3 *Formula* (6–1) *determines the structure of an R–module on* $\underline{\mathrm{Hom}}(M, N)$.

Proof One needs to check that $(F \cdot f)(a \otimes m)$ and $(V \cdot f)(a \otimes m)$ belong to $\underline{\mathrm{Hom}}(M, N)$, ie, that they satisfy the formulas (1)–(3) of Definition 6.1 and that the relations $VF = FV = p$ hold. This verification is completely straightforward. $\square$

Proposition 6.4 *There is a natural isomorphism in $\mathcal{D}$:*

$$(6\text{–}2) \qquad\qquad \mathrm{Hom}_R(M \boxtimes N, L) \cong \mathrm{Hom}_R(M, \underline{\mathrm{Hom}}(N, L))$$

Proof The R–module $\underline{\mathrm{Hom}}(N, L)$ is, by definition, a subgroup in $\mathrm{Hom}(R \otimes N, L)$. Therefore $\mathrm{Hom}_R(M, \underline{\mathrm{Hom}}(N, L))$ is identified with a certain subgroup inside

$$\mathrm{Hom}_R(M, \mathrm{Hom}(R \otimes N, L)) \cong \mathrm{Hom}(M \otimes_R (R \otimes N), L)$$
$$\cong \mathrm{Hom}(M \otimes N, L).$$

More precisely, an easy inspection shows that this subgroup consists of $f \in \mathrm{Hom}(M \otimes N, L)$ for which the identities (1)–(3) of Definition 5.2 hold. It follows that the collection of such f is isomorphic to $\mathrm{Hom}_R(M \boxtimes N, L)$.

So we showed that (6–2) is an isomorphism of abelian groups. Let us now check that this is an R–module isomorphism. Let $f \in \mathrm{Hom}_R(M \boxtimes N, L)$. We will consider f as a map $M \otimes N \longrightarrow L$. Denote by $\bar{f}$ the corresponding homomorphism

$$M \longrightarrow \underline{\mathrm{Hom}}(N, L) \subset \mathrm{Hom}(R \otimes N, L).$$

Thus, $\bar{f}(m)(r \otimes n) = f(rm \otimes n)$. Let $a \in R$. We have:

$$\bar{f}(am)(r \otimes n) = f(ram \otimes n)$$
$$= \bar{f}(m)(ar \otimes n)$$
$$= (a \cdot \bar{f})(m)(r \otimes n) \qquad\qquad \square$$

Finally, we introduce the notion of the Cartier duality in the category $\mathcal{D}_v$.

Definition 6.5 The dualizing module in the category $\mathcal{D}_\nu$ is the group $D_d(\nu) := \mathbb{Z}/p^\nu\mathbb{Z}$ where V acts as multiplication by p and F is the identity automorphism. For $M \in \mathcal{D}_\nu$ define its dual Dieudonné module $D_d M$ as $\underline{\mathrm{Hom}}(M, D_d(\nu))$.

Remark 6.6 The subscript d in the definition of the dualizing module is supposed to distinguish it from the dualizing object for Hopf algebras. We will suppress this subscript as well as the dependence on ν whenever practical.

We now introduce the notion of a *bilinear pairing* of Dieudonné modules analogous to the corresponding notion in the category of Hopf algebras.

Definition 6.7 A bilinear pairing of two Dieudonné modules $M, N \in \mathcal{D}_\nu$ is a map

$$\{,\}\colon M \otimes N \to \mathbb{Z}/p^\nu$$

such that for any $m \in M$ and $n \in N$:

(1) $\{m, Fn\} = \{Vm, n\}$

(2) $\{Fm, n\} = \{m, Vn\}$

Remark 6.8 Obviously a bilinear pairing $M \otimes N \to \mathbb{Z}/p^\nu$ nothing but a bilinear map $M \otimes N \to D$ or, equivalently, a map of Dieudonné modules $M \boxtimes N \to D$.

We have the following result whose proof is a simple check.

Proposition 6.9 *For $M \in \mathcal{D}_\nu$ the module DM can be identified with the abelian group $\mathrm{Hom}(M, \mathbb{Z}/p^\nu)$. The action of the operators F and V is specified by the formulas:*

$$(F \cdot f)(m) = f(Vm)$$

$$(V \cdot f)m = f(Fm) \qquad \square$$

Remark 6.10 It is also possible to consider the Cartier duality in the whole category $\mathcal{D}$. In this case the dualizing module D will be the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$ where V and F act as before. The analogue of Proposition 6.9 continues to hold in this context.

7 The Dieudonné correspondence

Recall that the Witt Hopf algebra $W_n^{\mathbb{Z}} = \mathbb{Z}[x_0, \ldots, x_n]$ admits a unique Hopf algebra structure over $\mathbb{Z}$ for which the Witt polynomials $P_k = x_0^{p^k} + px_1^{p^{k-1}} + \ldots + p^k x_k$ are primitive. We will write W_n for $W_n^{\mathbb{Z}} \otimes \mathbb{F}_p$.

There is a map of Hopf algebras $\bar{V}_n: W_{n+1} \longrightarrow W_n$ defined as $\bar{V}_n(x_0) = 0$ and $\bar{V}_n(x_i) = x_{i-1}$ for $i > 0$. Then the classical Dieudonné theorem, cf [3], states:

Theorem 7.1 *The functor $\mathcal{F}: \mathcal{H} \longrightarrow \mathcal{D}$:*

$$H \mapsto \varinjlim_{n} \mathrm{Hom}_{\mathcal{H}}(W_n, H)$$

establishes an equivalence of the subcategory $\mathcal{H}(\infty)$ of $\mathcal{H}$ and the subcategory $\mathcal{D}(\infty)$ of $\mathcal{D}$.

The proof of the above theorem uses the fact that W_n is a projective generator of the abelian category $\mathcal{H}(n)$.

Next recall that the categories $\mathcal{H}(\infty)$ and $\mathcal{D}$ are monoidal. The following result shows that $\mathcal{F}$ is a monoidal functor.

Theorem 7.2 *There is a natural isomorphism*

$$\mathcal{F}(H_1 \boxtimes H_2) \cong \mathcal{F}(H_1) \boxtimes \mathcal{F}(H_2).$$

Proof A detailed proof in the graded case is contained in Goerss' paper [4]. Goerss' scheme carries over to the ungraded case and we will show briefly how this is done.

The first step is to construct a homomorphism of R–modules

(7–1) $$\phi: \mathcal{F}(H_1) \boxtimes \mathcal{F}(H_2) \longrightarrow \mathcal{F}(H_1 \boxtimes H_2).$$

Arguing as in Goerss' paper one can show that for a Hopf algebra H over $\mathbb{Z}_p$ having a lifting of the Frobenius the Dieudonné module $\mathcal{F}(H \otimes \mathbb{F}_p)$ is isomorphic to $R \otimes_{\mathbb{Z}_p[V]} Q(H)$ where $Q(H)$ is the space of indecomposables of H. Next one proves that for such Hopf algebras H_1, H_2 there is an isomorphism

$$Q(H_1 \boxtimes H_2) \cong Q(H_1) \otimes Q(H_2).$$

Since the Hopf algebra $W_n \boxtimes W_n$ does have a lifting of the Frobenius we conclude that

$$\mathcal{F}(W_n \boxtimes W_n) \cong R \otimes_{\mathbb{Z}_p[V]} Q(W_n^{\mathbb{Z}} \boxtimes W_n^{\mathbb{Z}})$$
$$\cong R \otimes_{\mathbb{Z}_p[V]} Q(W_n^{\mathbb{Z}}) \otimes Q(W_n^{\mathbb{Z}}).$$

Consider the element

$$\kappa_n \circ \kappa_n = 1 \otimes x_n \otimes x_n \in R \otimes_{\mathbb{Z}_p[V]} Q(W_n^{\mathbb{Z}}) \otimes Q(W_n^{\mathbb{Z}}).$$

This element represents a map of Hopf algebras

$$\underline{\Delta}_n \colon W_n \longrightarrow W_n \boxtimes W_n.$$

We will use $\underline{\Delta}_n$ to construct the homomorphism (7–1) as follows. Without loss of generality assume that in H_1 and H_2 the operator V^n is trivial. Then $\mathcal{F}(H_1) = \mathrm{Hom}_{\mathcal{H}}(W_n, H_1)$ and $\mathcal{F}(H_2) = \mathrm{Hom}_{\mathcal{H}}(W_n, H_2)$. Consider the composite map:

$$\widetilde{\phi} \colon \mathcal{F}(H_1) \otimes \mathcal{F}(H_2) \cong \mathrm{Hom}_{\mathcal{H}}(W_n, H_1) \otimes \mathrm{Hom}_{\mathcal{H}}(W_n, H_2)$$

$$\downarrow$$

$$\mathrm{Hom}_{\mathcal{H}}(W_n \boxtimes W_n, H_1 \boxtimes H_2) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(W_n, H_1 \boxtimes H_2) \cong \mathcal{F}(H_1 \boxtimes H_2)$$

Here the last map is induced by $\underline{\Delta}_n$. Then one can show that $\widetilde{\phi}$ is a bilinear map and therefore induces a map (7–1).

Finally, one shows that ϕ is an isomorphism for $H_1 = W_k$, $H_2 = W_l$ for any k and l and then derives that ϕ is an isomorphism in general. $\qquad\square$

Observe that the Dieudonné functor is defined on the category $\mathcal{H}(\infty)$ of irreducible Hopf algebras. This category is not closed with respect to the Cartier duality. There is another version of the Dieudonné equivalence which is defined on the category $\mathcal{H}_\infty^f$ of finite dimensional p–torsion Hopf algebras. We will now describe this version. Note that this category *is* closed with respect to the Cartier duality.

The categories $\mathcal{H}_{ll}$ and $\mathcal{H}_{lr}$ consist of irreducible Hopf algebras and therefore the Dieudonné functor is defined on them as above. We will define the Dieudonné functor on $\mathcal{H}_{rl}$ using the fact that $D(\mathcal{H}_{rl}) = \mathcal{H}_{lr}$. Namely, for $H \in \mathcal{H}_{rl}$ set $\mathcal{F}(H) = D\mathcal{F}(DH)$.

Thus, the functor $\mathcal{F}$ maps the category $\mathcal{H}_\infty^f$ onto the subcategory $\mathcal{D}^f$ of R–modules consisting of Dieudonné module of finite length. This subcategory is the product of three subcategories $\mathcal{D}_{ll}$, $\mathcal{D}_{lr}$ and $\mathcal{D}_{rl}$. Here $\mathcal{D}_{ll}$ consists of those R–modules for which V and F act nilpotently, $\mathcal{D}_{lr}$ is the R–modules with nilpotent F and invertible V and $\mathcal{D}_{rl}$ is the R–modules with nilpotent V and invertible F. We will sum this up in the following theorem, cf [2]:

Theorem 7.3 *The functor $\mathcal{F}$ establishes an equivalence of categories $\mathcal{H}_\infty^f$ and $\mathcal{D}^f$. The functor $\mathcal{F}$ respects the Cartier duality in $\mathcal{H}^f$ and $\mathcal{D}^f$.*

8   Twisted duality

Over the prime field $\mathbb{F}_p$ there are other candidates for a dualizing object in the categories $\mathcal{H}$ and $\mathcal{D}$ all of which become isomorphic upon passing to the algebraic closure of $\mathbb{F}_p$. In this section we consider one particular choice of a dualizing object since it will arise naturally in our study of the Morava K–theories of Eilenberg–Mac Lane spaces.

From now on our ground field **k** will have characteristic $p > 2$. We will work here in the categories $\mathcal{H}_\nu$ and $\mathcal{D}_\nu$ of Hopf algebras and Dieudonné modules which are annihilated by the νth power of p although one could make similar constructions in other categories.

Definition 8.1

(1)   The twisted dualizing module D'_d in $\mathcal{D}_\nu$ is the abelian group $\mathbb{Z}/p^\nu$ where F and V act as the multiplication by -1 and $-p$ respectively. For a Dieudonné module M its twisted dual $D'_d M$ is defined as $D'_d M = \underline{\mathrm{Hom}}(M, D'_d)$.

(2)   The twisted dualizing object in $\mathcal{H}_\nu$ is the Hopf algebra D'_h whose Dieudonné module is D'_d. For a Hopf algebra H we define its twisted dual $D'_h H$ as $D'_h H := \underline{\mathrm{Hom}}(M, D'_h)$.

Remark 8.2   Recall that the usual dualizing object in $\mathcal{H}_\nu$ is the dual to the group algebra of the group $\mathbb{Z}/p^\nu$. It is unlikely that one can give such a simple explicit description of D'_h and so we have to resort to the Dieudonné correspondence instead.

The notion of a twisted dualizing object leads one to introduce *twisted bilinear pairing*.

Definition 8.3   A twisted bilinear pairing of two Dieudonné modules $M, N \in \mathcal{D}_\nu$ is a map

$$[,]: M \otimes N \to \mathbb{Z}/p^\nu$$

such that for any $m \in M$ and $n \in N$:

(1) $[m, Fn] = -[Vm, n]$

(2) $[Fm, n] = -[m, Vn]$

Remark 8.4   Obviously a twisted bilinear pairing $M \otimes N \to \mathbb{Z}/p^\nu$ is nothing but a bilinear map $M \otimes N \to D'_d$ or, equivalently, a map of Dieudonné modules $M \boxtimes N \to D'_d$.

To relate two types of duality in $\mathcal{H}_\nu$ we need a more general form of the Dieudonné correspondence, cf [3], which we will now recall.

Let $\mathbf{k}$ be an algebraic extension of $\mathbb{F}_p$ obtained by adjoining a root of some irreducible polynomial $h(x)$ so that $\mathbf{k} \cong \mathbb{F}_p[x]/(h(x))$. Denote by $\overline{h}(x)$ an integral lifting of $h(x)$. Then the ring of Witt vectors $W(\mathbf{k})$ is the ring $\mathbb{Z}_p[x]/\overline{h}(x)$. It possesses a lifting of the Frobenius automorphism on $\mathbf{k}$ which will be denoted by σ. Then the Dieudonné ring $R(\mathbf{k})$ consists of all finite sums of the form

$$\sum_{i>0} \alpha_{-i} V^i + \alpha_0 + \sum_{i>0} \alpha_i F^i$$

where $\alpha_i \in W(\mathbf{k})$. The multiplication law is determined by the commutation relations

$$VF = FV = p$$
$$F\alpha = \sigma(\alpha)F$$
$$V\sigma(\alpha) = \alpha V$$

where $\alpha \in W(\mathbf{k})$. Furthermore for any Dieudonné module M the $R(\mathbf{k})$–module $M(\mathbf{k})$ is defined as $W(\mathbf{k}) \otimes M$ where F and V act according to the commutation rules above. Note that the definition of the ring $W(\mathbf{k})$ can easily be extended to the case of an infinite algebraic extension by simply taking the union of $W(L)$ over subfields L of $\mathbf{k}$ having finite degree over $\mathbb{F}_p$.

The Dieudonné correspondence in this case reads as follows:

Theorem 8.5 *The category $\mathcal{H}(\infty)$ of irreducible Hopf algebras over $\mathbf{k}$ is equivalent to the subcategory of modules over $R(\mathbf{k})$ for which V acts nilpotently. For a Hopf algebra H over $\mathbb{F}_p$ the Dieudonné module corresponding to $\mathbf{k} \otimes H$ is isomorphic to $\mathcal{F}(H)(\mathbf{k})$.*

We now have the following result.

Proposition 8.6 *Let $\mathbf{k}$ be a quadratic extension of $\mathbb{F}_p$. Then $\mathbf{k} \otimes D'_h$ and $\mathbf{k} \otimes D_h$ are isomorphic Hopf algebras.*

Proof The Dieudonné module corresponding to $\mathbf{k} \otimes D_h$ is $W(\mathbf{k}) \otimes \mathbb{Z}/p^\nu$ where F and V act on the factor $\mathbb{Z}/p^\nu$ as the identity and multiplication by p respectively. Similarly the Dieudonné module of $\mathbf{k} \otimes D'_h$ is the same abelian group $W(\mathbf{k}) \otimes \mathbb{Z}/p^\nu$ where now F and V act on the factor $\mathbb{Z}/p^\nu$ as the minus identity and multiplication by $-p$ respectively.

Observe that since $p > 2$ the extension **k** is obtained by adding to $\mathbb{F}_p$ the square root of a certain element $q \in \mathbb{F}_p$. Note that the Frobenius automorphism $\sigma \colon \mathbf{k} \to \mathbf{k}$ permutes the roots of the quadratic polynomial $x^2 - q$, ie $\sigma(\sqrt{q}) = -\sqrt{q}$. It follows that the map $1 \otimes x \mapsto \sqrt{q} \otimes x \colon \mathbf{k} \otimes D'_h \to \mathbf{k} \otimes D_h$ establishes an isomorphism between these two Dieudonné module structures. By Theorem 8.5 the corresponding Hopf algebras are isomorphic. $\qquad\square$

Corollary 8.7 *Let H be a Hopf algebra in $\mathcal{H}_\nu$. Then for a quadratic extension **k** of $\mathbb{F}_p$ the Hopf algebras $\mathbf{k} \otimes D_h H$ and $\mathbf{k} \otimes D'_h H$ are isomorphic.* $\qquad\square$

9 Hopf rings, Dieudonné algebras and generalized homology of Ω–spectra

In this section we introduce the notion of a Hopf ring, the corresponding notion of a Dieudonné algebra and relate them to generalized homology of multiplicative spectra. Note that Hopf rings originally appear in the work of Milgram [8] and were later studied by Ravenel and Wilson [13] in the context of their calculation of the homology of Ω–spectrum MU.

Definition 9.1 A Hopf ring A is a Hopf **k**–algebra together with a map $\phi \colon A \boxtimes A \longrightarrow A$ which is required to be associative: $\phi(\phi \boxtimes 1) = \phi(1 \boxtimes \phi)$. A commutative Hopf ring is a Hopf ring A for which the product ϕ is commutative. In other words a (commutative) Hopf ring is a (commutative) monoid in the symmetric monoidal category $\mathcal{H}$.

Remark 9.2 Sometimes it is convenient to require that a Hopf ring A have a unit. In other words there is a map $\mathbf{k}[\mathbb{Z}] \longrightarrow A$ subject to the obvious conditions. (Of course in the category $\mathcal{H}_\nu$ the appropriate notion of a unit is a map $\mathbf{k}[\mathbb{Z}/p^\nu] \longrightarrow A$).

The main example of a Hopf ring is the *exterior* Hopf ring on a Hopf algebra H (Ravenel and Wilson use the term 'free Hopf ring').

Definition 9.3 Let H be a Hopf algebra. Then its exterior Hopf ring $\Lambda_\boxtimes(H)$ is defined as

$$\Lambda_\boxtimes(H) = \mathbb{I} \oplus H \oplus (H \boxtimes H)/\Sigma_2 \oplus \ldots \oplus H^{\boxtimes n}/\Sigma_n \oplus \ldots .$$

Here $\mathbb{I}$ stands for the appropriate unit ($\mathbf{k}[\mathbb{Z}]$ in $\mathcal{H}$ and $\mathbf{k}[\mathbb{Z}/p^\nu]$ in $\mathcal{H}_\nu$) and Σ_n is the symmetric group on n symbols which operates on $H^{\boxtimes n}$ according to the rule

$$\sigma(h_1 \circ \ldots \circ h_n) = [-1]^{\operatorname{sgn}\sigma} h_{\sigma(1)} \circ \ldots \circ h_{\sigma(n)}$$

where $h_1, \ldots, h_n \in H$.

The nonunital version $\Lambda_{\boxtimes+}(H)$ is defined as

$$\Lambda_{\boxtimes+}(H) = H \oplus (H \boxtimes H)/\Sigma_2 \oplus \ldots \oplus H^{\boxtimes n}/\Sigma_n \oplus \ldots.$$

Clearly $\Lambda_{\boxtimes}(H)$ and $\Lambda_{\boxtimes+}(H)$ are both Hopf rings with respect to the operation $\circ$, and $\Lambda_{\boxtimes}(H)$ also has a unit. The Hopf rings $\Lambda_{\boxtimes+}(H)$ and $\Lambda_{\boxtimes}(H)$ are not commutative, but rather skew-commutative Hopf rings in the sense that the following relation holds for any $h_1, h_2 \in H$:

$$h_1 \circ h_2 = [-1]h_2 \circ h_1.$$

The corresponding notion in the category of Dieudonné modules is called the Dieudonné algebra:

Definition 9.4 A (commutative) Dieudonné algebra is a (commutative) monoid in the category $\mathcal{D}$ or $\mathcal{D}_\nu$.

The definition of an exterior Dieudonné algebra is likewise clear:

Definition 9.5 Let M be an R–module. Then its exterior algebra in $\mathcal{D}$ or $\mathcal{D}_\nu$ is defined as

$$\Lambda_{\boxtimes}(M) = \mathbb{I} \oplus M \oplus (M \boxtimes M)/\Sigma_2 \oplus \ldots \oplus M^{\boxtimes n}/\Sigma_n \oplus \ldots.$$

Here $\mathbb{I}$ is the unit in $\mathcal{D}$ or $\mathcal{D}_\nu$, ie, the Dieudonné module $\mathbb{Z}_p$ or $\mathbb{Z}/p^\nu$ with appropriate actions of F and V. The symmetric group Σ_n operates on $M^{\boxtimes n}$ according to the rule

$$\sigma(m_1 \circ \ldots \circ m_n) = (-1)^{\operatorname{sgn}\sigma} m_{\sigma(1)} \circ \ldots \circ m_{\sigma(n)}$$

where $m_1 \ldots m_n \in m$. The nonunital version $\Lambda_+(M)$ is defined as

$$\Lambda_{\boxtimes+}(M) = M \oplus (M \boxtimes M)/\Sigma_2 \oplus \ldots \oplus M^{\boxtimes n}/\Sigma_n \oplus \ldots.$$

Again, the exterior Dieudonné algebra $\Lambda_{\boxtimes}(M)$ is not commutative, but rather skew-commutative in the obvious sense.

Remark 9.6 Let H be an irreducible Hopf algebra. Then $H^{\boxtimes n}$ as well as $(H^{\boxtimes n})/\Sigma_n$ will be irreducible as well for any $n > 0$. By the Dieudonné correspondence the R–module $\mathcal{F}(\Lambda_{\boxtimes+}(H))$ is isomorphic to $\Lambda_{\boxtimes+}(\mathcal{F}(H))$. In particular, it is a Dieudonné algebra. If H is a finite-dimensional Hopf algebra we can use the version of the Dieudonné equivalence given in Theorem 7.3 and extend $\mathcal{F}$ to the unital Hopf ring $\Lambda(H)$. In this case we obtain $\mathcal{F}(\Lambda_{\boxtimes}(H)) = \Lambda_{\boxtimes}(\mathcal{F}(H))$.

Now let E be an Ω–spectrum, ie, a sequence $\{E_q,\ q = 0, 1, \ldots\}$ of based spaces together with weak equivalences $\Omega(E_{q+1}) \simeq E_q$. We assume that E is a ring spectrum, ie there exist maps of based spaces

$$(9\text{–}1) \qquad\qquad E_q \wedge E_l \longrightarrow E_{q+l}$$

satisfying the usual associativity axioms.

Furthermore, let $h_*(-)$ be a generalized (multiplicative) homology theory which satisfies the perfect Künneth formula, ie, $h_*(X \times Y) \cong h_*(X) \otimes_{h_*} h_*(Y)$ for any spaces X and Y. In addition, we will assume that $h_*(E_n)$ is concentrated in even degrees for all n. The last requirement is imposed because in our context all Hopf algebras are evenly graded (ungraded, in fact). To deal with a more general case the theory should be reworked to incorporate graded Hopf algebras.

According to [13] the graded h_*–module $\bigoplus_n h_*(E_n)$ has the structure of a Hopf ring where the $\circ$–product

$$\circ\colon h_*(E_n) \otimes h_*(E_m) \longrightarrow h_*(E_{n+m})$$

is induced by the mutiplication (9–1) in E. Now take $E = H\mathbb{Z}/p^\nu$, the Eilenberg–Mac Lane spectrum mod p^ν. In this case $E_q = K(\mathbb{Z}/p^\nu, q)$. Next, set $h_*(-) = K(n)_*(-)$, the nth Morava K–theory at an odd prime p. It will be convenient for us to consider its ungraded version $\bar{K}(n)$. It is defined as $\bar{K}(n)_*(X) = K(n)_*(X) \otimes_{K(n)_*} \mathbb{F}_p$ where $K(n)_*$ acts on $\mathbb{F}_p$ through the map $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}] \to \mathbb{F}_p\colon v_n \mapsto 1$. Since we do not consider the graded Morava K–theory we will use the notation $K(n)_*(-)$ for $\bar{K}(n)_*(-)$.

Now recall the following fundamental result of Wilson and Ravenel [14].

Theorem 9.7 *The Hopf ring $K(n)_* K(\mathbb{Z}/p^\nu, -)$ is the exterior Hopf ring on $K(n)_* K(\mathbb{Z}/p^\nu, 1)$.*

The Hopf algebra $H_\nu = K(n)_* K(\mathbb{Z}/p^\nu, 1)$ is a well-understood object. It is a finite dimensional Hopf algebra whose dual has the form $H_\nu^* = \mathbb{F}_p[[t]]/[p^\nu](t)$ where the diagonal is induced by the formal group law of the Morava K–theory and $[p^\nu](t)$ is the corresponding p–series. If one uses Hazewinkel's generators to construct the spectrum $K(n)$ then its p–series has the form $[p](t) = t^{p^n}$.

Note that H_ν is a finite-dimensional irreducible Hopf algebra. Let us now find its Dieudonné module. Consider instead the Hopf algebra

$$H = \varprojlim_\nu H_\nu^* = K(n)^* K(\mathbb{Z}, 2).$$

Then $H = \mathbb{F}_p[[t]]$ and $\Delta(t) \in H \hat{\otimes} H$ is determined by the formal group law of the Morava K–theory. It has finite height n and it is easy to see that $\Phi(\lambda)$, the characteristic polynomial of the Frobenius endomorphism $t \mapsto t^p$ has the form $\lambda^n - p$.

Recall (see for example [2]) that there is a contravariant one-to-one correspondence between formal groups of finite height and Dieudonné modules of finite type which are free as $\mathbb{Z}_p$–modules. We claim that the Dieudonné module corresponding to H has the form $M = R/(V^{n-1} - F)$. Indeed, M clearly has height n and the dimension of the formal group corresponding to M equals the length of the R–module M/VM which is equal to one. Now a 1–dimensional formal group of a given height is determined uniquely by the characteristic polynomial of the Frobenius which is $\lambda^n - p$ in our case. Under the contravariant correspondence between formal groups and Dieudonné modules the Frobenius endomorphism on the formal group side corresponds to the Verschiebung on the Dieudonné module side. Since this characteristic polynomial of the Verschiebung on M is $\lambda^n - p$ we conclude that the Dieudonné module of H is indeed M.

Finally, the Dieudonné module $\mathcal{F}(H_\nu)$ is obtained by reducing M modulo p^ν and we arrive at the following result:

Lemma 9.8 *The Dieudonné module $\mathcal{F}(H_\nu)$ is isomorphic to $\mathbb{Z}/p^\nu[F, V]/\sim$ where $\sim$ is generated by the relations $V^{n-1} = F$ and $VF = p$.*   □

This lemma allows one to obtain results about the Hopf ring $\Lambda_{\boxtimes} H_\nu$ via the corresponding Dieudonné algebra $\Lambda_{\boxtimes}(M_\nu)$.

Remark 9.9 The ring of coefficients of $K(n)$ is isomorphic to $\mathbb{F}_p[v_n^{\pm 1}]$ where v_n has degree $2(p^n - 1)$. For this version of the Morava K–theory its multiplicative structure is determined uniquely and leads to a unique formal group law whose Dieudonné module is as described. Over the field $\mathbb{F}_p$ there exist many nonisomorphic 1–dimensional formal groups of height n (all of which become isomorphic after passing to the algebraic closure of $\mathbb{F}_p$). These formal groups could be realized by the 2–periodic version of $K(n)$ (which supports many inequivalent product structures).

10   Structure of the Dieudonné algebra $\Lambda_{\boxtimes}[R/(V^{n-1} - F)]$

In this section we investigate the structure of the exterior Dieudonné algebra on the module $M = R/(V^{n-1} - F)$. It turns out that with this particular choice of a module this exterior algebra is isomorphic to the conventional exterior algebra on M. Note that

the formula for the Dieudonné module of $K(n)_* K(\mathbb{Z}/p^\nu, q)$ also appears in Sadofsky and Wilson [15].

To fix the notation, observe that M is a free $\mathbb{Z}_p$–module with basis $1, V, V^2, \ldots, V^{n-1} = F$. Denote V^k by a_{n-k-1} where $k = 0, \ldots, n-1$. In the basis $a_0, \ldots, a_{n-1}$ the R–module structure on M is specified by the formulas:

$$Va_0 = pa_{n-1}; Va_i = a_{i-1}, i \geq 1$$
$$Fa_i = V^{n-1}a_i, i \geq 0$$

Let us consider the (conventional) exterior algebra $\Lambda(M)$ of the free $\mathbb{Z}_p$–module M. Thus,

$$\Lambda(M) = \Lambda^0 \oplus \Lambda^1 \oplus \ldots \oplus \Lambda^n$$

where Λ^q is a free $\mathbb{Z}_p$–module with the basis $a_I = a_{i_1} \wedge \ldots a_{i_q}, 0 \leq i_1 < \ldots < i_q < n$. We will denote the generator of the 1–dimensional $\mathbb{Z}_p$–module Λ^0 by 1, and that of Λ^n, by $a_* = a_0 \wedge \ldots \wedge a_{n-1}$.

We will regard $\Lambda^0 = \mathbb{Z}_p$ as the R–module $\mathbb{I}$. Note that $\Lambda^1 = M$ has a structure of an R–module. Define the structure of an R–module on Λ^q for $q > 1$ inductively by the formulas:

(10–1) $$Va_I = V(a_{i_1} \wedge \ldots \wedge a_{i_{q-1}}) \wedge a_{i_q-1}$$

(10–2) $$Fa_I = a_{i_1+1} \wedge F(a_{i_2} \wedge \ldots \wedge a_{i_q})$$

Note that the right hand side of (10–1) and (10–2) is well defined since $i_1 < n-1$ and $i_q > 0$ when $q > 1$. Let us check the consistency of the above action. We have:

$$\begin{aligned}
FVa_I &= F(V(a_{i_1} \wedge \ldots \wedge a_{i_{q-1}}) \wedge a_{i_q-1}) \\
&= (-1)^{q-1} F(a_{i_q-1} \wedge V(a_{i_1} \wedge \ldots \wedge a_{i_{q-1}})) \\
&= (-1)^{q-1} a_{i_q} \wedge FV(a_{i_1} \wedge \ldots \wedge a_{i_{q-1}}) \\
&= (-1)^{q-1} pa_{i_q} \wedge (a_{i_1} \wedge \ldots \wedge a_{i_{q-1}}) \\
&= pa_{i_1} \wedge \ldots \wedge a_{i_q} \\
&= pa_I
\end{aligned}$$

The condition $VF = p$ is checked similarly. Observe that formulas (10–1) and (10–2) imply the following identities:

(10–3) $$V(a_{i_1} \wedge \ldots \wedge a_{i_q}) = Va_{i_1} \wedge \ldots \wedge Va_{i_q}$$

(10–4)
$$F(V(a_{i_1} \wedge \ldots \wedge a_{i_s}) \wedge a_{i_{s+1}} \wedge \ldots \wedge a_{i_q}) = a_{i_1} \wedge \ldots \wedge a_{i_s} \wedge F(a_{i_{s+1}} \wedge \ldots \wedge a_{i_q})$$

Note that for these formulas to hold the indices do not necessarily have to be ordered.

The structure of a Dieudonné module on Λ^n turns out to be especially simple as the following result demonstrates.

Lemma 10.1 *The following formulas hold:*

(10–5) $$Va_* = (-1)^{n-1} p a_*$$

(10–6) $$Fa_* = (-1)^{n-1} a_*$$

Proof

$$
\begin{aligned}
F(a_0 \wedge \ldots \wedge a_{n-1}) &= a_1 \wedge F(a_1 \wedge \ldots \wedge a_{n-1}) \\
&= \ldots = a_1 \wedge a_2 \ldots \wedge a_{n-1} \wedge Fa_{n-1} \\
&= a_1 \wedge a_2 \wedge \ldots \wedge a_{n-1} \wedge a_0 \\
&= (-1)^{n-1} a_*
\end{aligned}
$$

Next, using formula (10–6) and the identity $VF = p$ we arrive at formula (10–5). $\square$

We will introduce a $\mathbb{Z}_p$–linear scalar product on $\Lambda(M)$ as follows. For $A \in \Lambda^q(L)$, $B \in \Lambda^p(L)$ the product $\langle A, B \rangle = 0$ if $p + q \neq n$; the case $p + q = n$ is determined by the formula

(10–7) $$A \wedge B = \langle A, B \rangle a_*.$$

Lemma 10.2 *The scalar product $\langle , \rangle \colon \Lambda^q \otimes \Lambda^{n-q} \to \mathbb{Z}_p$ satisfies the following property:*

$$\langle Va_I, a_J \rangle = (-1)^{n-1} \langle a_I, Fa_J \rangle$$

Proof We have for $q > 0$:

$$Va_I \wedge a_J = \langle Va_I, a_J \rangle a_*$$

Applying the operator F to the last formula we obtain:

$$F(Va_I \wedge a_J) = \langle Va_I, a_J \rangle Fa_* = (-1)^{n-1} \langle Va_I, a_J \rangle a_*$$

On the other hand according to formula (10–4) we have:

$$F(Va_I \wedge a_J) = a_I \wedge Fa_J = \langle a_I, Fa_J \rangle a_*$$

as required.

Now let $q = 0$. For the generator $1 \in \mathbb{Z}_p = \Lambda^0$ we have $V \cdot 1 = 1$. Therefore

$$\langle V \cdot 1, a_* \rangle = \langle 1, a_* \rangle.$$

Using (10–6) we get

$$\langle 1, F a_* \rangle = (-1)^{n-1} \langle 1, a_* \rangle. \qquad \square$$

As a final preparation to our main theorem in this section we will formulate and prove the following general result.

Lemma 10.3 *Let M, N be two R–modules which are free as $\mathbb{Z}_p$–modules. Then the canonical map*

$$M \otimes N \to M \boxtimes N \colon m \otimes n \mapsto m \circ n = 1 \otimes m \otimes n \in M \boxtimes N$$

is a monomorphism.

Proof Since M, N have no p–torsion it suffices to prove that the map $M \otimes N \to M \boxtimes N$ is monomorphic after tensoring with $\mathbb{Q}$. In other words we have to show that the localization map

$$M \otimes M \to \mathbb{Q}[V, V^{-1}] \otimes_{\mathbb{Q}[V]} M \otimes M$$

is an inclusion. The latter condition is equivalent to the $\mathbb{Q}[V]$–module $M \otimes N$ having no V–torsion. This holds since $M \otimes N$ has no p–torsion. $\qquad \square$

Corollary 10.4 *Let M be a free $\mathbb{Z}_p$–module. Then the canonical map*

$$\Lambda(M) \to \Lambda_{\boxtimes}(M)$$

is an inclusion. $\qquad \square$

We are now ready to relate $\Lambda(M)$ to the exterior Dieudonné algebra $\Lambda_{\boxtimes}(M)$.

Theorem 10.5 *For any $q = 0, 1, \ldots, n$ there is an isomorphism of R–modules $\phi_q \colon \Lambda^q(M) \to \Lambda^q_{\boxtimes}(M)$ given by the formulas*

$$\mathbb{I} = \Lambda^0 \to \Lambda^0_{\boxtimes} = \mathbb{I} \colon 1 \mapsto 1$$
$$\Lambda^q \to \Lambda^q_{\boxtimes} \colon a_{i_1} \wedge \ldots \wedge a_{i_q} \mapsto a_{i_1} \circ \ldots \circ a_{i_q} \quad \text{for} \quad q > 0.$$

Proof A straightforward inspection shows that ϕ is indeed a map of R–modules. It follows from Corollary 10.4 that ϕ is an inclusion. It remains to prove that it is an epimorphism. Consider the elements $a_{i_1} \circ \ldots \circ a_{i_q} \in \Lambda_{\boxtimes}^q$ where $i_1 < \ldots < i_q$. We will call such elements *admissible*. Therefore we are reduced to proving that admissible elements additively span $\Lambda_{\boxtimes}^q$. Suppose by induction that this is so and show that admissible elements span $\Lambda_{\boxtimes}^{q+1}$.

We are assuming that $q > 1$ since the cases $q \leq 1$ are obvious. It follows from the inductive assumption that the collection $\{F^k \otimes a_{i_1} \circ \ldots \circ a_{i_{q+1}}\}$ span $\Lambda_{\boxtimes}^{q+1}$. Here $k = 0, 1, 2 \ldots$ and $i_1 < \ldots < i_{q+1}$. Since $q > 1$ for any given $x = F^k \otimes a_{i_1} \circ \ldots \circ a_{i_{q+1}}$ the elements $a_{i_1}, \ldots, a_{i_q}$ are in the image of V, namely $a_{i_1} = V(a_{i_1+1}), \ldots, a_{i_q} = V(a_{i_q+1})$. Using repeatedly the relation (5–1) we conclude that x can be rewritten as a multiple of an admissible element. This completes the inductive step. $\square$

Now fix a positive integer ν and consider the Dieudonné module $M_\nu := M \otimes \mathbb{Z}/p^\nu$ and the corresponding Hopf algebra H_ν. Then the corresponding exterior algebra $\Lambda(M_\nu) \cong \Lambda_{\boxtimes}(M_\nu)$ has a scalar product induced from $\Lambda(M)$ and we arrive at the following result.

Corollary 10.6

(1) *Let $n \geq 1$ be odd, $0 \leq q \leq n$. Then the Dieudonné modules $\Lambda_{\boxtimes}^q(M_\nu)$ and $\Lambda_{\boxtimes}^{n-q}(M_\nu)$ as well as Hopf algebras $\Lambda_{\boxtimes}^q H_\nu$ and $\Lambda_{\boxtimes}^{n-q} H_\nu$ are dual to each other.*

(2) *Let $n > 1$ be even, $0 \leq q \leq n$. Then the Dieudonné modules $\Lambda_{\boxtimes}^q(M_\nu)$ and $\Lambda_{\boxtimes}^{n-q}(M_\nu)$ as well as Hopf algebras $\Lambda_{\boxtimes}^q H_\nu$ and $\Lambda_{\boxtimes}^{n-q} H_\nu$ are twisted dual to each other. Thus, they become dual in the usual sense after a quadratic extension of $\mathbb{F}_p$.*

Proof It follows from Lemma 10.2 that the scalar product on $\Lambda_{\boxtimes}(M)$ induces a bilinear pairing between $\Lambda_{\boxtimes}^q(M_\nu)$ and $\Lambda_{\boxtimes}^{n-q}(M_\nu)$ for n odd. Similarly for n even this pairing is twisted bilinear. It is clearly nondegenerate and the result follows. $\square$

We conclude this section with the useful observation that the R–module $\Lambda^q(M) \cong \Lambda_{\boxtimes}^q(M)$ is actually a module over $R/(V^{n-q} - F^q)$.

Proposition 10.7 *In the R–module $\Lambda^q(M)$ the following relation holds: $V^{n-q} = F^q$.*

Proof Note that since $VF = p$ and $\Lambda^q(M)$ is a free $\mathbb{Z}_p$–module the operator V is monomorphic on $\Lambda^q(M)$. Therefore it suffices to prove the relation $V^q \cdot V^{n-q} = V^q \cdot F^q$ or $V^n = p^q$. For a generator $a_{i_1} \wedge \ldots \wedge a_{i_q}$, according to formula (10–3), we have

$$V^n(a_{i_1} \wedge \ldots \wedge a_{i_q}) = V^n a_{i_1} \wedge \ldots \wedge V^n a_{i_q}$$
$$= pa_{i_1} \wedge \ldots \wedge pa_{i_q}$$
$$= p^q a_{i_1} \wedge \ldots \wedge a_{i_q}$$

as required. $\square$

Remark 10.8 We want to stress that the isomorphism of $\Lambda_{\boxtimes}(M)$ with the usual exterior algebra $\Lambda(M)$ relies essentially on the peculiar properties of the module $M = R/(V^{n-1} - F)$. Calculations show that bilinear products as well as exterior Dieudonné algebras of cyclic R–modules which are free of finite rank over $\mathbb{Z}_p$ usually contain p–torsion and are often no longer finitely generated over $\mathbb{Z}_p$.

11 Decomposition of $\Lambda_{\boxtimes}[R/(V^{n-1} - F)]$ up to isogeny over $\overline{\mathbb{F}}_p$

Here $\mathbf{k}$ will be an algebraic extension of $\mathbb{F}_p$, in fact we will be most interested in the case when $\mathbf{k} = \overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$. Recall the definition of the Dieudonné ring $R(\mathbf{k})$ and $R(\mathbf{k})$–module $N(\mathbf{k})$ for an R–module N from Section 8. We will denote by $\widetilde{W}(\mathbf{k})$ the field of fractions of $W(\mathbf{k})$. The ring $\widetilde{W}(\mathbf{k}) \otimes_{W(\mathbf{k})} R \cong \mathbb{Q} \otimes R(\mathbf{k})$ will be denoted by $\widetilde{R}(\mathbf{k})$ and $\widetilde{N}(\mathbf{k})$ will stand for the $\widetilde{R}(\mathbf{k})$–module $\widetilde{W}(\mathbf{k}) \otimes_{W(\mathbf{k})} N \cong \mathbb{Q} \otimes N(\mathbf{k})$.

Definition 11.1 An F–space is a $\widetilde{R}(\mathbf{k})$–module which is finite-dimensional as a $\widetilde{W}(\mathbf{k})$–vector space. F–spaces form a category whose morphisms are simply the morphisms of $\widetilde{R}(\mathbf{k})$–modules.

The $R(\mathbf{k})$–modules we will be interested in here will be torsion-free and finite rank over $W(\mathbf{k})$. Any such module N determines a lattice in the F–space $\widetilde{N} = \mathbb{Q} \otimes N$.

Definition 11.2 Let N_1 and N_2 be two torsion-free $R(\mathbf{k})$ modules of finite rank over $W(\mathbf{k})$. We say that N_1 and N_2 are *isogenous* if the corresponding F–spaces $\widetilde{N}_1$ and $\widetilde{N}_2$ are isomorphic.

Now let us introduce the Dieudonné modules $R_{n,q} = R/(V^{n-q} - F^q)$. Observe that

$$\widetilde{R}_{n,q} = \widetilde{W}(\mathbf{k})[T]/(p^{n-q} - T^n)$$

and F acts as a multiplication by T. It is interesting to note that for p, q coprime and $\mathbf{k}$ an algebraic extension of $\mathbb{F}_p$ of degree n the module $\widetilde{R}_{n,q}(\mathbf{k})$ is a division algebra over $\mathbb{Q}_p$ with Hasse invariant q/n, see Pierce [11, Chapter 17]. The module $R_{n,q}(\mathbf{k})$ is in this case a maximal order in $\widetilde{R}_{n,q}(\mathbf{k})$.

From now on we assume that $\mathbf{k} = \mathbb{F}_p$. It is known that the category of F–spaces is semisimple and all simple objects are of the form $\widetilde{R}_{n,q}$ where n, q are nonnegative relatively prime integers and $n > 0$. Recall that we denoted by M the R–module $R_{n,1} = R/(V^{n-1} - F)$. Let (n, q) be the greatest common factor of n and q and set $n_0 := \frac{n}{(n,q)}$, $q_0 := \frac{q}{(n,q)}$ We have the following result.

Theorem 11.3 *The $R(\mathbf{k})$–module $\Lambda^q(M)$ is isogenous to the direct sum of $\frac{1}{n_0}\binom{n}{q}$ copies of the $R(\mathbf{k})$–module R_{n_0,q_0}.*

Proof Observe that the operator V permutes (up to multiplication by a scalar factor) the basis vectors a_I in the $\widetilde{R}$–module $\mathbb{Q} \otimes \Lambda^q(M)$. Since $V^n = p^q$ in $\mathbb{Q} \otimes \Lambda^q(M)$ by Proposition 10.7 we see that this module decomposes into a direct sum of submodules isomorphic to quotients of $\widetilde{R}_{n,q}$. Similarly $\mathbb{Q} \otimes \Lambda^q(M)(\mathbf{k})$ decomposes into a direct sum of quotients of $\widetilde{R}_{n,q}(\mathbf{k})$. Therefore it suffices to show that $\widetilde{R}_{n,q}(\mathbf{k})$ is isomorphic to the direct sum of copies of $\widetilde{R}_{n_0,q_0}(\mathbf{k})$. We claim that if there exists a nontrivial map of R–modules

$$(11\text{--}1) \qquad\qquad \widetilde{R}_{r,s}(\mathbf{k}) \to \widetilde{R}_{n,q}(\mathbf{k})$$

then $\frac{r}{s} = \frac{n}{q}$. This claim will clearly imply what we need.

The proof of the claim is similar to Proposition D in [2, page 79]. A map (11–1) is equivalent to having an element $x \in \widetilde{R}_{n,q}(\mathbf{k})$ for which $F^r x = p^{r-s}x$. Note that $\widetilde{R}_{n,q}(\mathbf{k})$ has a basis f_j such that if $x = \sum b_j f_j$ then

$$F^n x = \sum \sigma^n(b_j) p^{n-q} f_j$$

and so

$$F^{rn} x = \sum \sigma^{rn}(b_j) p^{r(n-q)} f_j.$$

On the other hand if $F^r x = p^{r-s}x$ then

$$F^{rn} x = p^{(r-s)n} x$$
$$= \sum b_j p^{(r-s)n} f_j.$$

It follows that $\sigma^n(b_j)p^{(r-s)n} = b_j\,p^{r(n-q)}$ and since the Frobenius σ preserves p–adic valuation on $\widetilde{W}(\mathbf{k})$ we conclude that $(r-s)n = r(n-q)$. It follows that $\frac{r}{s} = \frac{n}{q}$ as required. □

12 p–divisible groups associated with $K(n)^* K(\mathbb{Z}, q)$

12.1 Basic definitions

We start by recalling some standard definitions and facts from the theory of p–divisible groups referring the reader to Demazure [2], Tate [17] or Schatz [16] for details. Here $\mathbf{k}$ is an algebraic extension of $\mathbb{F}_p$.

Definition 12.1 A p–divisible group of height h over a field $\mathbb{F}_p$ is a sequence $G = (H_\nu, i_\nu)$, $\nu = 0, 1, 2, \ldots$ of Hopf algebras over $\mathbf{k}$ with $\dim H_\nu = \nu h$ and $i_\nu\colon H_{\nu+1} \to H_\nu$ is a Hopf algebra homomorphism such that for each ν the sequence

$$H_{\nu+1} \xrightarrow{\;[p^\nu]\;} H_{\nu+1} \xrightarrow{\;i_\nu\;} H_\nu \longrightarrow 0$$

is exact in $\mathcal{H}$.

Since $[p^{\nu+1}] = [p] \cdot [p^\nu]$ is the trivial endomorphism of $H_{\nu+1}$ we conclude that there exists a map $j_\nu\colon H_\nu \longrightarrow H_{\nu+1}$ making the following diagram commute:

$$
\begin{array}{ccc}
H_{\nu+1} & & \\
{\scriptstyle [p]}\big\uparrow & \searrow^{\,j_\nu} & \\
H_{\nu+1} & \xrightarrow[\;i_\nu\;]{} & H_\nu
\end{array}
$$

The topological Hopf algebra $H = \varprojlim H_\nu$ represents a formal group from which the sequence (H_ν, i_ν) could be recovered by setting $H_\nu := \mathrm{coker}\{[p^\nu]\colon H \to H\}$. We will use the term '$p$–divisible group' also for the corresponding formal group. The dimension of the p–divisible group G is the Krull dimension of H.

Next, H will be isomorphic to a ring of formal power series (one say that in this case the corresponding formal group is *smooth*) if and only if each H_ν is a local ring.

There is a suitable version of duality for p–divisible groups. As far as we know twisted duality has not been considered before.

Definition 12.2 If $G = (H_\nu, i_\nu)$ is a p–divisible group then its *Serre dual p–divisible group* is defined as $DG = (H_\nu^*, j_\nu^*)$. It has the same height as (H_ν, i_ν). Its *twisted Serre dual* is the collection $D'G = (D'H_\nu, D'j_\nu)$ where D' is the functor of twisted duality on the category of p–torsion Hopf algebras.

Remark 12.3 Of course, the notions of the Serre dual and twisted Serre dual p–divisible group coincide when **k** contains a quadratic extension of $\mathbb{F}_p$.

Example 12.4

(1) A one-dimensional p–divisible group of height 1 is represented by the topological Hopf algebra $H = \mathbf{k}[[x]]$ with $\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x$, the so-called multiplicative formal group. This p–divisible group is smooth. Its Serre dual p–divisible group is represented by the Hopf algebra $\mathbf{k}[\mathbb{Z}]^*$, the dual group ring of the infinite cyclic group. The dimension of this p–divisible group is zero and it is not smooth.

(2) Let X be an abelian scheme over **k**. Then the kernel of the multiplication by p^ν on X is a finite group scheme which is represented by a Hopf algebra whose dimension is a power of p. The resulting inverse system of Hopf algebras constitutes a p–divisible group.

To conclude our review of the background material we note that the height of a p–divisible group is equal to the sum of its dimension and the dimension of its dual:

$$\text{height}(G) = \text{height}(DG) = \dim(G) + \dim(DG)$$

12.2 Dieudonné correspondence

We now briefly review the Dieudonné correspondence adapted to p–divisible groups following [2]. Let $G = (H_\nu, i_\nu)$ be a p–divisible group. Its Dieudonné module is defined as $\mathcal{F}(G) = \varprojlim \mathcal{F}(H_\nu)$. Then we have a theorem:

Theorem 12.5 *The correspondence $G \mapsto \mathcal{F}(G)$ induces an equivalence between the categories of p–divisible groups over **k** and $R(\mathbf{k})$–modules which are finite rank free $W(\mathbf{k})$–modules. The height of G is equal to the dimension of $\mathcal{F}(G)$. If the Dieudonné module corresponding to G is actually a module over $\widehat{R}(\mathbf{k}) := \varprojlim_n R(\mathbf{k})/V^n$ then G is smooth. The dimension of G equals $\dim_{\mathbf{k}}[(\mathcal{F}(G)/V\mathcal{F}(G)]$.*

Example 12.6 Let $R_{n,q} = R(\mathbf{k})/(V^{n-q} - F^q)$. The corresponding p–divisible group G has height n. It is smooth of dimension q for $q = 1, 2 \ldots, n$. The Serre dual to G has Dieudonné module $R_{n,n-q} = R(\mathbf{k})/(V^q - F^{n-q})$.

For any p–divisible group G one can define the F–space $\mathcal{F}(G) \otimes \mathbb{Q}$. Then two p–divisible groups are isogenous (ie, there exists a monomorphism between their representing Hopf algebras with a finite-dimensional cokernel) if and only if the corresponding F–spaces are isomorphic.

12.3 Main theorems

Now let $H_\nu(q) := K(n)^* K(\mathbb{Z}/p^\nu, q)$. Also denote $H_\nu^*(q) = K(n)_* K(\mathbb{Z}/p^\nu, q)$ by $H^\nu(q)$. The inclusion

$$\mathbb{Z}/p^\nu \longrightarrow \mathbb{Z}/p^{\nu+1}$$

induces a map of spaces

$$K(\mathbb{Z}/p^\nu, q) \longrightarrow K(\mathbb{Z}/p^{\nu+1}, q)$$

which in turn gives rise to a map of Hopf algebras $i_\nu \colon H_{\nu+1}(q) \to H_\nu(q)$.

We can formulate now our main result.

Theorem 12.7

(1) *The sequence $(H_\nu(q), i_\nu)$ forms a p–divisible group of height $\binom{n}{q}$.*

(2) *For $q = 1, 2, \ldots, n-1$ the p–divisible group $(H_\nu(q), i_\nu)$ is smooth. The corresponding formal group is represented by a formal power series ring on $\binom{n-1}{q-1}$ variables and could be identified with $K(n)^* K(\mathbb{Z}, q+1)$.*

(3) *If n is odd then there is an isomorphism*

$$H^\nu(n) \cong D(\nu) := \mathbb{F}_p[\mathbb{Z}/p^\nu]^*.$$

If n is even then $H^\nu(n) \cong D'(\nu)$ where $D'(\nu)$ is the twisted dualizing Hopf algebra.

(4) (a) *Let n be odd and $0 < q < n$. Then the $\circ$–pairing*

$$H^\nu(q)] \boxtimes H^\nu(n-q) \longrightarrow H^\nu(n)$$

induces an isomorphism of the formal group of $K(n)^* K(\mathbb{Z}, q+1)$ with the Serre dual of the formal group of $K(n)^* K(\mathbb{Z}, n-q+1)$. In particular the Hopf algebras $H^\nu(q)$ and $H_\nu(n-q)$ are isomorphic.

(b) *For n even the above statement holds provided one replaces 'dual' with 'twisted dual'.*

Proof We saw in Section 10 that the Dieudonné modules corresponding to $K(\mathbb{Z}/p^\nu, q)$ are obtained by reducing modulo p^ν from the modules $\Lambda_{\boxtimes}^q(M)$ where $M = R/(V^{n-1} - F)$. These modules torsion-free and their rank over $\mathbb{Z}_p$ is equal to $\dim \Lambda^q(M) = \binom{n}{q}$. This proves part (1).

Further note that since the identity $V^{n-q} = F^q$ holds in $\Lambda_{\boxtimes}(M) \cong \Lambda(M)$ the corresponding p–divisible group is smooth for $q < n$. From the formula (10–3) for the action of V on $\Lambda_{\boxtimes}(M)$ we deduce that the image of V is spanned modulo p

by the monomials $a_{i_1} \wedge \ldots \wedge a_{i_q}$ where $i_q \neq n-1$. It follows that the monomials $a_{i_1} \wedge \ldots \wedge a_{i_{q-1}} \wedge a_{n-1} \mod V$ form a basis in $\Lambda^q(M)/V\Lambda^q(M)$ and so its dimension over $\mathbb{F}_p$ equals $\binom{n-1}{q-1}$. Therefore the dimension of the formal group in question is as claimed.

Next, observe that there is a homotopy equivalence of spaces

$$K(\mathbb{Q}/\mathbb{Z}, q) \simeq \text{hocolim } K(\mathbb{Z}/p^\nu, q)$$

where the homotopy colimit is taken over all ν and prime numbers p. This homotopy equivalence induces an isomorphism of Hopf algebras

$$\varprojlim_\nu K(n)^* K(\mathbb{Z}/p^\nu, q) \cong K(n)^* K(\mathbb{Q}/\mathbb{Z}, q).$$

The obvious map $K(\mathbb{Q}/\mathbb{Z}, q)) \to K(\mathbb{Z}, q+1)$ induces an isomorphism on $K(n)$–theory. Therefore the p–divisible group $H_\nu = K(n)^* K(\mathbb{Z}/p^\nu, q)$ is indeed representable by the Hopf algebra $K(n)^* K(\mathbb{Z}, q+1)$ as claimed. This proves part (2). The remaining claims (3) and (4) follow from Corollary 10.6. □

Remark 12.8 It is curious to note that the Serre duality between $K(n)^* K(\mathbb{Z}, q+1)$ and $K(n)^* K(\mathbb{Z}, n-q+1)$ breaks down for $q=0$. Indeed, $K(\mathbb{Z}, 1)$ is simply the circle S^1 and so $K(n)^* K(\mathbb{Z}, 1)$ cannot give rise to a formal group. However the duality between $K(n)_* K(\mathbb{Z}/p^\nu, 0)$ and $K(n)_* K(\mathbb{Z}/p^\nu, n)$ continues to hold. The point is that $K(n)^* K(\mathbb{Z}, q+1)$ can no longer be related to $K(n)_* K(\mathbb{Z}/p^\nu, q)$ for $q=0$.

Remark 12.9 Our results are formulated under the assumption that **k** has characteristic $p \neq 2$. Note, however, that an appropriate version of Theorem 12.7 holds for $p=2$ as well. The point is that although in this case $K(n)$ is not a commutative ring spectrum, $K(n)_* K(\mathbb{Z}/2^\nu, q)$ is still a bicommutative Hopf algebra. See Johnson and Wilson [5] for details. In the exterior Dieudonné algebra we should impose the additional condition that the circle product of two equal elements is zero. The notion of twisted duality is extraneous; the twisted dualizing object is the same as the untwisted one.

The identification of the isogeny class of $K(n)^* K(\mathbb{Z}, q+1)$ is an immediate consequence of Theorem 11.3. Recall that for two nonnegative integers q, n the pait n_0, q_0 is specified by the condition that $\frac{q}{n} = \frac{q_0}{n_0}$ and that q_0, n_0 be coprime. Thus, we obtain the following' result.

Theorem 12.10 *For $0 < q < n$ the formal group of $K(n)^* K(\mathbb{Z}, q+1)$ is isogenous to the product of $\frac{1}{n_0}\binom{n}{q}$ copies of the p–divisible group corresponding to the Dieudonné module R_{n_0, q_0}.* □

Remark 12.11 One might wonder whether the formal groups corresponding to $K(n)^*K(\mathbb{Z}, q+1)$ are *algebraicizable*, ie, whether there exists abelian schemes of which they are formal completions. Since abelian schemes are always isogenous with their dual the same is true for their completions. This is the so-called Manin symmetry condition, see Manin [7] or Oort [10]. It follows that if n is odd or if n is even but $q \neq \frac{n}{2}$ the formal group of $K(n)^*K(\mathbb{Z}, q+1)$ cannot be algebraicized. Note that for n even the formal group of $K(n)^*K(\mathbb{Z}, \frac{n}{2}+1)$ is *supersingular*, ie, it is isogenous (over $\overline{\mathbb{F}}_p$) to the product of copies of one-dimensional formal group of height 2.

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