Some results on vector bundle monomorphisms

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In this paper we use the singularity method of Koschorke [2] to study the question of how many different nonstable homotopy classes of monomorphisms of vector bundles lie in a stable class and the percentage of stable monomorphisms which are not homotopic to stabilized nonstable monomorphisms. Particular attention is paid to tangent vector fields. This work complements some results of Koschorke [3; 4], Libardi–Rossini [7] and Libardi–do Nascimento–Rossini [6].

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1 Introduction

For $a < n$ let $\alpha^a$ and $\beta^n$ be vector bundles of dimension $a$ and $n$, respectively, over a closed smooth connected $n$-dimensional manifold $M$. For simplicity they are denoted by $\alpha$ and $\beta$, respectively. The following two related problems have been considered by several authors. The first one is to know if there is a stable (resp. nonstable) monomorphism between the vector bundles $\alpha$ and $\beta$. This is a quite general problem, which has been extensively studied. In particular it includes the problem of the span of a manifold (the maximal number of vector fields over a manifold which are linearly independent over every point). Although we are not particularly concerned with this problem here, we would like to point out the following relevant result by Koschorke [4, Theorem 1] for the problem above. He gives a complete answer to the question of existence of a stable (nonstable) monomorphism from $\alpha$ to $\beta(\alpha \oplus \ell^l$ to $\beta \oplus \ell^l, l > 0$) in the so-called metastable range $n > 2a$, in spite of the fact that it is not an easy task to verify the conditions on which the answer is based, even for $a = 1, 2$. The second problem is: whenever there is a stable (resp. nonstable) vector bundle monomorphism from $\alpha$ to $\beta$, find how many stable (resp. nonstable) monomorphisms there are, and the relation between them. In order to study this problem, we recall Koschorke [2, Theorem 4.14], that there is a bijection between the set of homotopy classes of monomorphisms and the normal bordism group $\Omega_d(M \times P^\infty; \Phi)$ in the stable case (resp. $\Omega_d(P(\alpha); \Phi_1)$ in the nonstable case), for $n > 2a + 1$. Here $P(\alpha)$ is the projectification of the vector
bundle $\alpha$, and $\Phi = \lambda \otimes p_1^* (\beta - \alpha) - p_1^* \tau_M$. $\Phi_1 = \lambda \otimes p^*(\beta - \alpha) - p^* \tau_M$, which we denote simply by $\lambda \otimes (\beta - \alpha) - \tau_M$, are virtual vector bundles over $M \times P^\infty$ and $P(\alpha)$, respectively, where $p_1: M \times P^\infty \to M$ and $p: P(\alpha) \to M$ are the projections. The line bundle $\lambda$ in the case of $\Phi$ is the pullback by $p_2$ of the canonical line bundle over $P^\infty$, and in the case of $\Phi_1$ denotes the canonical line bundle over $P(\alpha)$.

Then we have the stabilizing homomorphism

$$\text{st}_{\alpha}: \Omega_{\alpha}(P(\alpha); \Phi_1) \to \Omega_{\alpha}(M \times P^\infty; \Phi),$$

where the cardinalities of the kernel and cokernel of this homomorphism measure, respectively, the number of different homotopy classes of nonstable monomorphisms, which stabilize to the same homotopy class of stable monomorphisms, and the percentage of stable monomorphisms which are not homotopic to (stabilized) nonstable monomorphisms. The second problem has been studied by Koschorke [4] for the case $a = 2$, $n$ odd, by Libardi and Rossini [7], for the nonstable case $a = 2$, $n$ even and $H_1(M; \mathbb{Z}) = 0$ and by Libardi, Nascimento and Rossini [6], for the stable case, $a = 2$ and $n$ odd.

In this paper we study the kernel and the cokernel of $\text{st}_{\alpha}$ in the cases $a = 1, n > 3$ and $a = 2, n > 5$ and even, which complements some results of [4; 7] and [6].

Throughout this paper we will make use of the auxiliary virtual vector bundle $\Phi_0 = \lambda \otimes p_1^* \beta - p_1^* \tau_M$, which we denote simply by $\lambda \otimes \beta - \tau_M$, over $M \times P^\infty$, $\eta = \beta - \alpha - \tau_M$ and of its orientation line bundle $\xi_\eta$ over $M$.

We denote $\rho: H_*(X; \mathbb{Z}) \to H_*(X; \mathbb{Z}_2)$ the modulo two reduction homomorphism of the integral local coefficient $\mathbb{Z}$ and we point out that for $a = 1$, $P(\alpha) = M$ and the virtual vector bundle $\Phi_1$ over $P(\alpha) = M$ becomes $\alpha \otimes \beta - \tau_M$.

We state our main results.

**Theorem 1.1** Let $a = 1$, $n > 3$ odd and suppose there is a monomorphism $u_0: \alpha \hookrightarrow \beta$ over $M$. Then the cokernel $\text{coker} \text{st}_1$ is isomorphic to $\mathbb{Z}_2$ and the kernel $\text{ker} \text{st}_1$ is either zero or isomorphic to $\mathbb{Z}_2$.

The $\text{ker} \text{st}_1$ is zero if one of the conditions below holds.

(a$_1$) \quad $w_2(\Phi_1)[\rho(H_2(M; \mathbb{Z}_\eta))] \neq 0$,

(a$_2$) \quad $w_2(\Phi_1) = 0$ and $n \equiv 1(4)$ and $w_1(\alpha) = w_1(\beta)$,

(a$_3$) \quad $w_2(\Phi_1) = 0$ and $n \equiv 3(4)$, $w_1(M) = 0$ and ($w_1(\alpha) = 0$ or $w_1(\beta) = 0$).

The $\text{ker} \text{st}_1 \cong \mathbb{Z}_2$ if one of the conditions below holds.
Then, we obtain the following consequences from the two theorems above.

Denote by $|X|$ the cardinality of the set $X$ and $\eta_0 = \beta - \tau_M$ a virtual vector bundle over $M$. For $|H_1(M; \mathbb{Z}_{\eta_0})|$ finite, we define $k = k(\alpha, \beta)$, such that $|H_1(M; \mathbb{Z}_2)| \cdot |\ker st_1| = k \cdot |H_1(M; \mathbb{Z}_{\eta_0})|$. Therefore $|\ker st_1|$ is determined by the number $k$. We state the next result in terms of $k$.

**Theorem 1.2** Let $\alpha = 1$, $n > 3$ be even and suppose there is a monomorphism $u_0: \alpha \leftrightarrow \beta$ over $M$.

Then, $\ker st_1 = 0$ if $w_1(\beta) = w_1(M)$ and $\ker st_1 \simeq \mathbb{Z}_2$ if $w_1(\beta) \neq w_1(M)$.

(a) If $|H_1(M; \mathbb{Z}_{\eta_0})|$ is infinite, $|\ker st_1|$ is infinite.

(b) If $|H_1(M; \mathbb{Z}_{\eta_0})|$ is finite then $|\ker st_1|$ is determined by $k$ as follows:

(b1) For $w_2(\Phi_1)[\rho(H_2(M; \mathbb{Z}_{\eta_0}))] \neq 0$, $k = 1$ if $w_1(\beta) = w_1(M)$ and $k = 2$ if $w_1(\beta) \neq w_1(M)$.

(b2) For $w_2(\Phi_1) = 0$ and $n \equiv 0(4)$, $k = 2$ if $w_1(\beta) = w_1(M)$ and $k = 4$ if $w_1(\beta) \neq w_1(M)$.

(b3) For $w_2(\Phi_1) = 0$ and $n \equiv 2(4)$, $k = 1$ if $w_1(\beta) = w_1(M)$ and $(w_1(M))^2 = 0 \neq w_1(\alpha)$, $k = 2$ if $w_1(\beta) = w_1(M)$ and $w_1(\beta)w_1(M) = 0 = w_1(\alpha)$, $k = 2$ if $w_1(\beta) = w_1(M)$ and $(w_1(M))^2 \neq (w_1(\alpha))^2$, $k = 4$ if $w_1(\beta) \neq w_1(M)$ and $(w_1(\alpha))^2 \neq w_1(\beta)w_1(M)$.

In the special case of the tangent vector fields, ie, when $\alpha$ is the trivial line bundle and $\beta$ is the tangent bundle $\tau_M$ of $M$, the second Stiefel–Whitney class $w_2(\Phi_1)$ is zero. Then we obtain the following consequences from the two theorems above.

For $n$ odd, we have:

(a) $\ker st_1 \simeq \mathbb{Z}_2$.
(b1) if $w_1(M) = 0$, then $\ker st_1 = 0$ and
(b2) if $w_1(M) \neq 0$ then $\ker st_1 \simeq \mathbb{Z}_2$.

For $n$ even, we have:

(a) $\ker st_1 = 0$.
(b) if $|H_1(M; \mathbb{Z}_M)|$ is infinite, where $\mathbb{Z}_M$ is the $\mathbb{Z}$-local system given by the orientation of the manifold $M$, $|\ker st_1|$ is infinite, otherwise
(b1) $k = 2$ if $n \equiv 0(4)$ or $(n \equiv 2(4)$ and $(w_1(M))^2 \neq 0)$ and

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(b_2) \ k = 1 \text{ if } n \equiv 2(4) \text{ and } (w_1(M))^2 = 0.

Now let a = 2 and n even. Recall that the case n odd has been studied in [4].

**Theorem 1.3** Let a = 2, n > 5 be even and suppose there is a monomorphism \( u_0 : \alpha \hookrightarrow \beta \) over \( M^n \). Assume that one of conditions below is valid.

\[ w_2(\eta_0)[\rho(H_2(M; \overline{Z}_{\eta_0})] \neq 0 \text{ or } (w_1(\beta) \neq 0 \text{ for } n \equiv 0(4) \text{ and } w_1(M) \neq 0 \text{ for } n \equiv 2(4)). \]

Let \( A \) be the subset of \( H_1(M; \overline{Z}_{\eta_0}) \) given by

\[ \{ z \mid \forall y \in H^1(M; \overline{Z}_2), y(\rho(z)) = (yw_2(\alpha))\rho(c_3) + yw_1(\beta - \tau_M)(c_2), c_3 \in H_3(M; \overline{Z}_n), c_2 \in H_2(M; \overline{Z}_2) \}, \]

then

(a_1) \ \text{coker } \text{st}_2 \simeq \overline{Z}_2 \text{ if also } w_1(\beta) = w_1(M),

(a_2) \ \text{coker } \text{st}_2 \simeq 0 \text{ if also } w_1(\beta) = w_1(M) \text{ and } w_2(\alpha)\rho(H_2(M; \overline{Z}_\alpha)) = 0,

(a_3) \ \text{coker } \text{st}_2 \simeq \mathbb{Z} \text{ if also } w_1(\beta) = w_1(M) \text{ and } w_2(\alpha)\rho(H_2(M; \overline{Z}_\alpha)) \neq 0 \text{ and }\]

(b) \ \text{ker } \text{st}_2 \simeq H_1(M; \overline{Z}_{\eta_0})/A \oplus \overline{Z}_2, \text{ if also } w_2(\Phi) = 0.

**Corollary 1.4** In the special case of tangent plane fields, ie when \( \alpha \) is the trivial bundle, \( \beta \) is the tangent bundle of \( M \) and \( w_1(M) \neq 0 \) then \( \text{st}_2 \) is surjective and \( \text{ker } \text{st}_2 \simeq H_1(M; \mathbb{Z}) \oplus \overline{Z}_2 \).

## 2 Preliminaries and notations

Given \( \alpha^n \) and \( \beta^n \) vector bundles over \( M \) of dimension \( a \) and \( n \), denoted by \( \alpha \) and \( \beta \), we will consider the virtual bundles \( \Phi = \lambda \otimes p_1^*(\beta - \alpha) - p_1^*\tau_M, \Phi_1 = \lambda \otimes p_2^*(\beta - \alpha) - p_2^*\tau_M \), which we denote simply by \( \lambda \otimes (\beta - \alpha) - \tau_M \), virtual vector bundles over \( M \times P^\infty \) and \( P(\alpha) \), respectively, where \( p_1: M \times P^\infty \rightarrow M, p: P(\alpha) \rightarrow M \) are the projections. The line bundle \( \lambda \) in the case of \( \Phi \) is the pullback by \( p_2 \) of the canonical line bundle over \( P^\infty \) and in the case of \( \Phi_1 \) denotes the canonical line bundle over \( P(\alpha) \).

We recall that for \( a = 1 \), \( P(\alpha) = M \) and the virtual vector bundle \( \Phi_1 \) over \( P(\alpha) = M \) becomes \( \alpha \otimes \beta - \tau_M \). Throughout this paper we will make use of the auxiliary virtual vector bundle \( \eta = \beta - \alpha - \tau_M \) and of its orientation line bundle \( \xi_\eta \) over \( M \). Let

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$x = w_1(\lambda)$ and $\Phi_0 = \lambda \otimes p_1^* \beta - p_1^* \tau_M$, which we denote simply by $\lambda \otimes \beta - \tau_M$, over $M \times P^\infty$ which we many times denote simply $M^\infty$.

For $a = 1$, we have

$$w_1(\Phi) = (n + 1)x + w_1(\eta)$$

$$w_2(\Phi) = \begin{cases} x(w_1(\beta) + w_1(\alpha)) + w_2(\eta), & n \equiv 1(4) \\ xw_1(M) + w_2(\eta), & n \equiv 2(4) \\ x^2 + x(w_1(\beta) + w_1(\alpha)) + w_2(\eta), & n \equiv 3(4) \\ x^2 + xw_1(M) + w_2(\eta), & n \equiv 0(4) \end{cases}$$

$$w_1(\Phi_0) = nx + w_1(\beta - \tau_M).$$

For $a = 2$ and $n$ even, we have

$$w_1(\Phi) = w_1(\eta) = w_1(\beta) + w_1(\alpha) + w_1(M)$$

$$w_2(\Phi) = \begin{cases} x(w_1(M) + w_1(\eta)) + w_2(\eta), & \text{if } n \equiv 2(4) \\ x^2 + x(w_1(M) + w_1(\eta)) + w_2(\eta) & \text{if } n \equiv 0(4) \end{cases}$$

$$w_2(\eta) = w_2(\beta) + w_2(\alpha) + w_1^2(\alpha) + w_1(\beta)w_1(\alpha) + w_2(M) + w_1^2(M) + w_1(\beta)w_1(M) + w_1(\alpha)w_1(M).$$

Given a vector bundle $\theta^\ell$, of dimension $\ell$, we denote $\xi_\theta = \Lambda^\ell \theta$ the corresponding orientation line bundle. If $\tau_K$ is the tangent bundle of a smooth manifold $K$ we also write $\xi_K$ for $\xi_{\tau_K}$. If $\Phi$ is a virtual vector bundle, $\xi_\Phi$ denotes the orientation line bundle determined by $w_1(\xi_\Phi) = w_1(\Phi)$. For more details see Randall–Daccach [8, Section III.6].

Let $\mathbb{Z}_\theta$ (or $\mathbb{Z}_{w_1(\theta)}$) denote the group $\mathbb{Z}$ if $w_1(\theta) = 0$ or $\mathbb{Z}_2$ if $w_1(\theta) \neq 0$, and let $\tilde{\mathbb{Z}}_\theta$ (or $\tilde{\mathbb{Z}}_{w_1(\theta)}$) denote the twisted integer coefficient system associated to the orientation line bundle $\xi_\theta$.

For a topological space $X$ and a virtual bundle $\Phi = \Phi^+ - \Phi^-$ over $X$, let $\Omega_k(X; \xi_\Phi)$ be the group of bordism classes $[K, g, or]$, where $K$ is a smooth closed $k$ dimensional manifold, $g: K \to X$ is a continuous map and $or: \xi_K \to g^*(\xi_\Phi)$ is an isomorphism.

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We also consider the map \( f_k : \Omega_k(X; \Phi) \to \Omega_k(X; \xi_\Phi) \) which forgets the vector bundle isomorphism \( \tau_k \oplus g^*(\Phi^+) \overset{\sim}{\to} g^*(\Phi^-) \) and retains the orientation information.

If \( X \) is connected we have that \( \Omega_0(X; \theta) \simeq \mathbb{Z}_\theta \simeq \Omega_0(X; \xi_\theta) \).

For each \( n \), the homomorphism

\[
\Delta_\theta : \Omega_n(X; \Phi) \to \Omega_{n-\ell}(X; \Phi + \theta)
\]

is defined by considering

\[
w = \left[ W^n \xrightarrow{g} X, h : \tau W \oplus g^*(\Phi^+) \overset{\sim}{\to} g^*(\Phi^-) \right]
\]

in \( \Omega_n(X; \Phi) \), \( Z^{n-\ell} \subset W^n \) (the zero set of a generic section of the vector bundle \( g^*(\theta) \) over \( W^n \)), and the isomorphism \( \nu(Z^{n-\ell}, W^n) \simeq g^*(\theta)|_{Z^{n-\ell}} \) by the formula

\[
\Delta_\theta(w) = \left[ Z^{n-\ell} \overset{g|_Z}{\to} X, \tau Z \oplus g|_Z^*(\Phi^+ + \theta) \overset{\sim}{\to} g|_Z^*(\Phi^-) \right].
\]

The homomorphism \( \Delta_\theta \) and its weak analogue \( \overline{\Delta}_\theta : \overline{\Omega}_n(X; \xi_\Phi) \to \overline{\Omega}_{n-\ell}(X; \xi_{\Phi + \theta}) \) and the homomorphism \( \text{st}_a \) and its weak analogue \( \overline{\text{st}}_a : \overline{\Omega}_a(P(\alpha); \xi_{\Phi_1}) \to \overline{\Omega}_a(M \times P^\infty; \xi_{\Phi}) \) fit into the long exact Gysin sequences (1) and (2) below:

(1) \[
\cdots \to \Omega_j(M^\infty; \Phi) \xrightarrow{\tau_j + 1} \Omega_{j-a}(M^\infty; \Phi_0) \xrightarrow{\delta_j^i} \Omega_{j-1}(P(\alpha); \Phi_1) \xrightarrow{\text{st}_a} \Omega_{j-1}(M^\infty; \Phi) \to \cdots
\]

(2) \[
\cdots \to \overline{\Omega}_j(M^\infty; \xi_{\Phi}) \xrightarrow{\overline{\tau}_j + 1} \overline{\Omega}_{j-a}(M^\infty; \xi_{\Phi_0}) \xrightarrow{\overline{\delta}_j^i} \overline{\Omega}_{j-1}(P(\alpha); \xi_{\Phi_1}) \xrightarrow{\overline{\text{st}}_a} \overline{\Omega}_{j-1}(M^\infty; \xi_{\Phi}) \to \cdots
\]

Here, \( \overline{\tau}_2 = \overline{\Delta}_a \circ \overline{\delta}_1^i \) and \( \overline{\delta}_1^i \) come from Gysin sequences and \( \overline{\tau}_2 = \overline{\tau}_2 \circ f_2^\infty \), where \( f_2^\infty : \Omega_2(M \times P^\infty; \Phi) \to \overline{\Omega}_2(M \times P^\infty; \xi_{\Phi}) \) is the forgetful map and \( M^\infty \) denote \( M \times P^\infty \) (see [4, Sections 1 and 2] for details).

The sequences above and other two singularity sequences defined in [2, Theorem 9.3], for \( a = 2 \), fit together into the next commutative diagram, where we have exactness in all sequences if the two “pinching conditions”, below, hold:

(i) \( \Omega_1(M \times P^\infty; \Phi_0) \simeq \overline{\Omega}_1(M \times P^\infty; \xi_{\Phi_0}) \)

(ii) \( \text{st}_1 : \Omega_1(P(\alpha) \times BO(2); \Phi + \Gamma) \overset{\sim}{\to} \Omega_1(M \times P^\infty \times BO(2); \Phi + \Gamma) \) is an isomorphism.
In the diagram, we have $\Phi_2 = \Phi + \Gamma$ where $\Gamma$ is $\gamma_2 \otimes \xi_2 + \xi \gamma_2 - \gamma_2$ and $\gamma_2$ denotes the canonical plane bundle over $BO(2)$ (see [4, Theorem 3.1]). Also for $g_2 = st_2 \circ f_2 = f_2^\infty \circ st_2$, $Z_2$ is in place of $\Omega_0(M \times P^\infty \times BO(2); \Phi_2)$ (see [2, Section 9.2, Theorem 9.3]), $Z_{\Phi_0}$ is in place of $\Omega_0(M \times P^\infty; \xi_{\Phi_0})$ and $M^\infty$ in the place of $M \times P^\infty$.

![Diagram](image)

Figure 1: Diagram for (a=2)

For more details see [2, Section 9] and [4, Section 3].

We recall from [4, Proposition 3.3], that if $a = 2$ and $n$ is even, the two pinching conditions above are equivalent to each one of the conditions:

$$w_2(\beta - \tau_M)[\rho(H_2(M; \tilde{Z}_{\beta - \tau_M})] \neq 0$$

or

$$\begin{cases} w_1(\beta) \neq 0 & \text{if } n \equiv 0(4) \\ w_1(M) \neq 0 & \text{if } n \equiv 2(4). \end{cases}$$

For $\Phi = \lambda \otimes (\beta - \alpha) - \tau_M$ and $\eta = \beta - \alpha - \tau_M$, as defined before, we have from [4, Propositions 1.2 and 1.3], the following Proposition.

**Proposition 2.1** For $i \in \mathbb{Z}$

$$\overline{\Omega}_i(M \times P^\infty; \xi_{\Phi}) \simeq \begin{cases} \mathcal{N}_i(M) & \text{if } a \neq n(2) \\ \overline{\Omega}_i(M; \xi_{\eta}) \oplus \mathcal{N}_{i-1}(M) & \text{if } a \equiv n(2). \end{cases}$$
3 Proofs of the theorems

We will study the kernel and cokernel of the stabilizing homomorphism

\[ \text{st}_a: \Omega_a(\alpha; \Phi_1) \rightarrow \Omega_a(M \times P^\infty; \Phi). \]

We recall that

\[ \Omega_0(M^\infty; \Phi_0) \cong \overline{\Omega}_0(M^\infty; \xi_{\Phi_0}) \]

\[ \cong \begin{cases} \mathbb{Z} & \text{if } w_1(\Phi_0) = nx + w_1(\beta) + w_1(M) = 0 \\ \mathbb{Z}_2 & \text{if } w_1(\Phi_0) = nx + w_1(\beta) + w_1(M) \neq 0. \end{cases} \]

\[ \Omega_0(\alpha; \Phi_1) \cong \overline{\Omega}_0(\alpha; \xi_{\Phi_1}) \]

\[ \cong \begin{cases} \mathbb{Z} & \text{if } w_1(\Phi_1) = nw_1(\alpha) + w_1(\beta) + w_1(M) = 0 \\ \mathbb{Z}_2 & \text{if } w_1(\Phi_1) = nw_1(\alpha) + w_1(\beta) + w_1(M) \neq 0. \end{cases} \]

Let us consider also the following commutative diagram, for \( a = 1 \), where the horizontal sequences are defined in [2, Theorem 9.3] and the vertical sequences come from the Gysin sequences (1) and (2), also for \( a = 1 \). We also recall that in this case \( \alpha_1 = M \).

![Diagram for (a = 1)](image)

Figure 2: Diagram for \((a = 1)\)

**Lemma 1** If \( \sigma_{j_2} \) and \( \sigma_{j_2}^\infty \) are surjective or both null maps then \( \ker \text{st}_1 \cong \ker \overline{s}_{j_1} \).

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If \( \sigma_{j_2} \) is the null map and \( \sigma_{j_2}^\infty \) is a surjective map then \( \ker st_1 \) is an extension of \( \ker \bar{st}_1 \) by \( \mathbb{Z}_2 \).

**Proof** The proof is obtained by diagram chasing in Figure 2 above. \( \square \)

**Lemma 2** Let \( n \) be odd and \( w_2(\Phi_1) = 0 \) then for \( n \equiv 1(4) \), \( \sigma_{j_2}^\infty = 0 \), if \( w_1(\beta - \alpha) = 0 \) and \( \sigma_{j_2}^\infty \) is surjective if \( w_1(\beta - \alpha) \neq 0 \).

For \( n \equiv 3(4) \), \( \sigma_{j_2}^\infty = 0 \), if \( w_1(M) = 0 \) and \( (w_1(\alpha) = 0 \) or \( w_1(\beta) = 0 \)) and \( \sigma_{j_2}^\infty \) is surjective if \( w_1(\bar{M}) \neq 0 \).

**Proof** Let \( w = [K^2, (h_1, h_2), \bar{h}] \in \widetilde{\Omega}_2(M \times P^\infty; \xi_\Phi) \simeq \widetilde{\Omega}_2(M; \xi_\eta) \oplus H_1(M; \mathbb{Z}_2) \simeq H_2(M; \mathbb{Z}_2) \).

We observe that \( w \) corresponds to \( ([K^2, h_1, h_1, y[L^1]]) \), where \( L^1 \subset K^2 \) is the zero set of a generic section of the pullback vector bundle \( h^*_2(\lambda) \) over \( K^2 \), and \( h \) is equivalent to an isomorphism \( h^*_2(\lambda) \simeq \xi_L \oplus \xi^*_1(\xi_\eta) \). Then \( w_1(\lambda) = w_1(L) + w_1(\eta) \) (see [4, Section 1.8]), for simplicity \( L^1 \) and \( K^2 \) are denoted by \( L \) and \( K \), respectively.

It follows from [4, Section 0.18] that for all \( y \in H^1(M; \mathbb{Z}_2) \), \( y([L]) = yw_1(\lambda)([K]) \).

Using [2, Theorem 9.3], we have that

\[
\begin{align*}
\sigma_{j_2}^\infty(w) &= (w_1(\lambda)w_1(\beta - \alpha) + w_2(\eta))(\xi) \quad \text{if} \quad n \equiv 1(4) \\
\sigma_{j_2}^\infty(w) &= ((w_1(\lambda))^2 + w_1(\lambda)(w_1(\beta - \alpha) + w_2(\eta)))(\xi) \quad \text{if} \quad n \equiv 3(4).
\end{align*}
\]

In the first case we have

\[
\sigma_{j_2}^\infty(w) = w_1(\beta - \alpha)([L]) + w_2(\eta)([K])
\]

and in the second case,

\[
\begin{align*}
\sigma_{j_2}^\infty(w) &= (w_1(\lambda) + w_1(\beta - \alpha))(\xi) + w_2(\eta)(\xi) \\
&= (w_1(L) + w_1(\eta) + w_1(\beta - \alpha))(\xi) + w_2(\eta)(\xi) \\
&= w_1(M)(\xi) + w_2(\eta)(\xi).
\end{align*}
\]

It follows that for \( s \in H_2(M; \mathbb{Z}_2) \) and \( t \in H_1(M; \mathbb{Z}_2) \) we have

\[
\begin{align*}
\sigma_{j_2}^\infty(w) &= w_1(\beta - \alpha)(t) + w_2(\eta)(s) \quad \text{if} \quad n \equiv 1(4) \\
\sigma_{j_2}^\infty(w) &= w_1(M)(t) + w_2(\eta)(s) \quad \text{if} \quad n \equiv 3(4)
\end{align*}
\]

The result follows by observing that if \( n \equiv 1(4) \), \( w_2(\Phi_1) = w_1(\alpha)w_1(\beta - \alpha) + w_2(\eta) \) and if \( n \equiv 3(4) \), \( w_2(\Phi_1) = w_1(\alpha)w_1(\beta) + w_2(\eta) \). \( \square \)
Proof of Theorem 1.1 Let $n > 3$ be odd and $a = 1$.

We have that $w_1(\Phi_0) = x + w_1(\beta - \tau_M)$ and $w_1(\Phi_1) = w_1(\eta)$. Note that $H_1(M \times P^\infty) \cong H_1(M) \oplus H_1(P^\infty) \cong H_1(M) \oplus \mathbb{Z}_2$ and, if we call $b$ the generator of the $\mathbb{Z}_2$ factor, we have $x(b) \neq 0$, $w_1(\beta)(b) = 0 = w_1(M)(b)$ and so we have $w_1(\Phi_0) \neq 0$, which gives us $\mathbb{Z}_{\Phi_0} \cong \mathbb{Z}_2 \cong \Omega_0(M \times P^\infty; \xi_{\Phi_0})$ and from [4, Proposition 2.1], $\overline{\text{st}}_1$ and $\overline{\text{st}}_0$ are injective. Therefore, $\overline{\tau}_1$ is onto $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \cong \mathbb{Z}_2$, see Figure 2.

But $\overline{\text{coker st}}_1 \cong \mathbb{Z}_2$. It follows also from Diagram IV and the fact that $\overline{\text{st}}_1$ is injective that $\ker \text{st}_1 \subset \text{im } \delta_1$, the image of $\delta_1$, which is zero or isomorphic to $\mathbb{Z}_2$.

If $w_2(\Phi_1)[\rho(H_2(M; \mathbb{Z}))] \neq 0$, $\sigma_{j_2}$ is surjective. Therefore we can conclude that $\Omega_1(P(\alpha); \Phi_1) \cong \Omega_1(P(\alpha); \xi_{\Phi_1}) \cong H_1(M; \mathbb{Z}_2)$ and recalling that $\overline{\text{st}}_1$ is injective we obtain in Diagram IV that $\text{st}_1$ is also injective.

If $w_2(\Phi_1) = 0$, $\sigma_{j_2}$ is the null map, and then $\Omega_1(P(\alpha); \Phi_1)$ is an extension of $H_1(M; \mathbb{Z}_2)$ by $\mathbb{Z}_2$. The result follows from Lemmas 1 and 2 above. \hfill \Box

Lemma 3 Let $n > 3$ be even and $a = 1$.

If $w_1(\beta) = w_1(M)$, $\overline{\text{st}}_1$ is surjective and $H_1(M; \mathbb{Z})$ is an extension of $H_1(M; \mathbb{Z}_2)$ by $\ker \text{st}_1$. We have also that $|H_1(M; \mathbb{Z})| = |\ker \text{st}_1|.|H_1(M; \mathbb{Z}_2)|$.

If $w_1(\beta) \neq w_1(M)$, $\text{coker} \overline{\text{st}}_1 \cong \mathbb{Z}_2$ and $H_1(M; \mathbb{Z}_2)$ is an extension of $\mathbb{Z}_2$ by $H_1(M; \mathbb{Z}_2_{\eta_0})$ ker $\overline{\text{st}}_1$. We have also $2.|H_1(M; \mathbb{Z}_2_{\eta_0})| = |\ker \overline{\text{st}}_1|.|H_1(M; \mathbb{Z}_2)|$.

Proof Let us consider the long exact Gysin sequence (2).

If $w_1(\beta) = w_1(M)$, $w_1(\Phi_0) = 0$ and so $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \cong \mathbb{Z}$ is a free abelian group. Since $n$ is even, $\overline{\Omega}_1(M \times P^\infty; \xi_{\Phi}) \cong H_1(M; \mathbb{Z}_2)$ is a torsion group so $\overline{\tau}_1$ is zero and then $\overline{\text{st}}_1$ is surjective. It follows that $H_1(M; \mathbb{Z})/\ker \overline{\text{st}}_1 \cong H_1(M; \mathbb{Z}_2)$.

If $w_1(\beta) \neq w_1(\beta)$, $\overline{\text{st}}_0$ is injective from [4, Proposition 2.1] and $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \cong \mathbb{Z}_2$. Therefore, $\mathbb{Z}_2 \cong \text{coker} \overline{\text{st}}_1 \cong H_1(M; \mathbb{Z}_2)/\ker \overline{\tau}_1$, where $\ker \overline{\tau}_1 \cong H_1(M; \mathbb{Z}_2_{\eta_0})/\ker \overline{\text{st}}_1$. \hfill \Box

Proof of Theorem 1.2 Let $n > 3$ be even and $a = 1$. In this case we recall that $w_1(\Phi_0) = w_1(\eta_0) = w_1(\Phi_1)$. If $w_1(\beta) = w_1(M)$ it follows from Lemma 3 that $\overline{\text{st}}_1$ is surjective. Therefore, by Figure 2, $\text{st}_1$ is also surjective. If $w_1(\beta) \neq w_1(M), w_1(\Phi_0) \neq 0$ and so $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \cong \mathbb{Z}_2$. We have again that $\overline{\text{st}}_0$ is injective and so $\overline{\tau}_1$ is onto $\mathbb{Z}_2 \cong \overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0})$. 

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We recall that if $w_2(\Phi_1) = 0$, the map $\sigma_j = 0$, and so $\Omega_1(M; \Phi_1)$ is an extension of $H_1(M; \mathbb{Z})$.

With the condition $w_2(\Phi_1) = 0$ we have that if $n \equiv 2(4)$ and $w_1(\alpha) = 0 = w_1(\beta)w_1(M)$, then in case $s \in \mathbb{Z}$.

Theorem 9.3) $S$ has index at most $\mathbb{Z}_2$.

Therefore, the results follow by remarking that if $\sigma_j^\infty$ is surjective, then

$$\Omega_1(M \times P^\infty; \Phi) \cong H_1(M; \mathbb{Z})$$

and if $\sigma_j^\infty = 0$, then $\Omega_1(M \times P^\infty; \Phi)$ is an extension of $H_1(M; \mathbb{Z})$ by $\mathbb{Z}_2$.

Let us consider $\phi$ and $\phi(z) = (s, t)$. Then

$$\mu(\tau_2(z)) = w_2(\alpha)\rho(s) + w_1(\beta - \tau_M(t))$$

Let us consider $w_1(\beta) = w_1(M)$. Then if $w_2(\alpha)\rho(\tau_2(M; \mathbb{Z}_2)) = 0$, $\tau_2 = 0$, and it follows by Lemma 4 that $\tau_2 = 0$ and then $\text{st}_2$ is surjective. If $w_2(\alpha)\rho(\tau_2(M; \mathbb{Z}_2)) \neq 0$, $\tau_2 \neq 0$ and so coker $\text{st}_2 \cong \mathbb{Z}$.
If \( w_1(\beta) \neq w_1(M) \), \( \tau_2 \) is onto \( \mathbb{Z}_2 \) by [4, Proposition 2.1] and so is \( \tau_2 \). If \( w_2(\Phi) = 0 \) then \( f_2^\infty \) is surjective. Therefore \( \coker st_2 \simeq \mathbb{Z}_2 \). If \( w_2(\Phi) \neq 0 \), we have that \( \sigma f_2^\infty \) is onto \( \mathbb{Z}_2 \). Since \( \operatorname{forg}_2 = f_2^\infty \circ st_2 \), we have that \( \coker \operatorname{forg}_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). If \( st_2 \) is surjective, \( \operatorname{forg}_2 \) and \( f_2^\infty \) have the same image and then the index of this image in \( \overline{\Omega}_2(P^\infty \times M; \xi_\Phi) \) is 2, which is a contradiction. It follows that \( \tau_2 \neq 0 \) and again \( \coker st_2 \simeq \mathbb{Z}_2 \).

Let us now analyze \( \ker st_2 \) when \( w_2(\Phi) = 0 \). We have that \( \delta_2 \) and \( \delta_3^\infty \) are injective (see footnote of [2, Theorem 9.3]). In the Figure 1 we see that \( \sigma f_3^\infty \) is surjective. As a consequence, \( \im \sigma f_3^\infty \) follows from Theorem 1.3 that st is surjective, \( \coker st_2 \) is surjective. Therefore \( \im \sigma f_3^\infty \) is zero set of a generic section of the vector bundle \( \ker st_2 \simeq H_1(M; \mathbb{Z}_2) \oplus \mathbb{Z}_2 / \overline{\tau_3}(\ker \sigma f_3^\infty) \), where \( \ker \sigma f_3^\infty = \overline{\Omega}_3(M \times P^\infty; \xi_\Phi) \).

We compute now the image of \( \overline{\Omega}_3(M \times P^\infty; \xi_\Phi) \) by \( \overline{\tau}_3 \). Let 
\[
\begin{align*}
w &= \left[ K^3 \xrightarrow{\varepsilon_1, \varepsilon_2} M \times P^\infty; \mathbb{g} \right] \in \overline{\Omega}_3(M \times P^\infty; \xi_\Phi) \\
&\simeq H_3(M; \mathbb{Z}_2) \oplus \mathbb{H}_2(M) \\
&\simeq H_3(M; \mathbb{Z}_2) \oplus H_2(M; \mathbb{Z}_2) \oplus \mathbb{Z}_2.
\end{align*}
\]

We remark that \( w \) corresponds to \((\left[ K^3 \xrightarrow{\varepsilon_1} M, \mathbb{g} \right], [L^2 \xrightarrow{\varepsilon_1, \varepsilon_2} M])\), where \( L^2 \) is the zero set of a generic section of the vector bundle \( \lambda \) over \( K \), and
\[
L^2 \xrightarrow{\varepsilon_1, \varepsilon_2} M
\]
corresponds to \((\left[ g_1 | L^2 \right] \mu(L^2), [P^2 \xrightarrow{c} M])\), where \( c \) is a constant map. Notice that for all \( y \in H^1(M; \mathbb{Z}_2), y([P^2]) = 0 \) and \( y([\mu(\overline{\tau}_3(w))]) = y(w_2(\lambda \otimes \alpha^2))([M]) \) and the image of \( \overline{\tau}_3 \subset H_1(M; \mathbb{Z}_2) \). Therefore, \( \ker st_2 \simeq H_1(M; \mathbb{Z}_2) / A \oplus \mathbb{Z}_2 \), where
\[
A = \{ z | y(\rho(c)) = (y w_2(\alpha)) \rho(c_3) + y w_1(\beta - \tau_M)(c_2) \},
\]
\[
c_3 \in H_3(M; \mathbb{Z}_2), c_2 \in H_2(M; \mathbb{Z}_2), \forall y \in H^1(M; \mathbb{Z}_2)
\]
This completes the proof.

**Proof of Corollary 1.4** Since \( w_1(\beta) = w_1(M) \neq 0 \) and \( \alpha \) is the trivial bundle, it follows from Theorem 1.3 that \( st_2 \) is surjective.

Using Figure 1 we obtain that \( \ker st_2 \simeq H_1(M; \mathbb{Z}) \oplus \mathbb{Z}_2 / \overline{\tau}_3(f_3^\infty(\overline{\Omega}_3(M \times P^\infty; \Phi))) \).

But under our hypothesis \( \overline{\tau}_3 = 0 \), then \( \ker st_2 \simeq H_1(M; \mathbb{Z}) \oplus \mathbb{Z}_2 \).
4 Examples

Example 1 Let $p_1, p_2, p_3$ and $q_1, q_2, q_3$ be arbitrary integers and consider the vector bundles $\alpha^2 = \gamma_1' \otimes \gamma_2' \otimes \gamma_3'$ and $\beta^6 = \gamma_1 \times \gamma_2 \times \gamma_3$ over $M^6 = S_1^2 \times S_2^2 \times S_3^2$. Each factor $S_i^2$ is identified with $CP(1), \gamma_i'$ and $\gamma_i$ are complex line bundles characterized respectively by $c_1(\gamma_i') = p_i \cdot g^i$ and by $c_1(\gamma_i) = q_i \cdot g^i$, where $g^i$ denotes the generator of $H^2(S_i^2; \mathbb{Z})$ and $c_1$ is the first Chern class. Therefore, we have that

$$w_2(\alpha) = p_1 \rho(g^1) + p_2 \rho(g^2) + p_3 \rho(g^3),$$
$$w_2(\beta) = q_1 \rho(g^1) + q_2 \rho(g^2) + q_3 \rho(g^3),$$
$$w_2(M) = 0.$$

From [5, Proposition 4.3] we know that there is a non-stable monomorphism $u_0 : \alpha \mapsto \beta$ in the following cases

(i) $p_1 + p_2 + p_3 = 0$ and at least one of the $q_i$’s is zero and
(ii) $p = p_1 + p_2 + p_3 \neq 0$ and $(q_1 - p)(q_2 - p)(q_3 - p)$ is divisible by $4p$.

Since $w_1(M), w_1(\beta)$ and $w_2(M)$ are all zero, to satisfy the pinching conditions we need $w_2(\beta) \neq 0$, that is, at least one of the $q_i$’s is $\neq 0$.

Different choices of the parameters give rise to different examples.

(1) If $p = 0, p_i \equiv 0(2)$, for some $k, q_k \equiv 1(2)$ and for some $j, q_j = 0$, we have $w_2(\alpha) = 0$ and $w_2(\beta) \neq 0$ and so $\text{coker } st_2$ is surjective.

(2) If for some $k, p_k \equiv 1(2), p = 0$, for some $l, q_l \equiv 1(2)$ and for some $j, q_j = 0$, then $w_2(\alpha) \neq 0$ and $w_2(\beta) \neq 0$ and so $\text{coker } st_2 \simeq \mathbb{Z}$.

Note that

$$w_2(\Phi) = w_2(\eta) = w_2(\alpha) + w_2(\beta)$$
$$= (p_1 + q_1) \rho(g^1) + (p_2 + q_2) \rho(g^2) + (p_3 + q_3) \rho(g^3).$$

(3) If $p_i \equiv q_i(2) i = 1, 2$ and $3, p = 0, q_k = 0$ for some $k$ and for some $l, q_l \equiv 1(2)$, then $w_2(\alpha) = w_2(\beta)$ and so $w_2(\Phi) = 0$ and $\text{coker } st_2 \simeq \mathbb{Z}_2$.

Example 2 Let $M = P^2 \times S^4$, and let $\alpha$ and $\beta$ be trivial bundles over $M$. Since $w_1(M) \neq 0, w_1(\beta) \neq w_1(\beta)$. In this case

$$w_2(\Phi) = 0,$$
$$\text{ker } st_2 \simeq H_1(P^2 \times S^4; \mathbb{Z}_2) \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2,$$
$$\text{coker } st_2 \simeq \mathbb{Z}_2.$$
Example 3  We consider some examples of the special case of tangent vector fields, that is, when $\alpha$ is the trivial line bundle and $\beta$ is the tangent bundle $\tau_M$ over a $n$–dimensional manifold $M$ with $n > 3$.

(1) $M = S^1 \times S^{2k}$. We have $\text{coker } st_1 \simeq \mathbb{Z}_2$, that is, half of the stable vector fields over $S^1 \times S^{2k}$ come from nonstable vector fields. Also, $\ker st_1 = 0$, so we have at most one nonstable vector field stabilizing to any stable one.

(2) $M = S^1 \times S^{4k+3}$. In this case, we have infinitely many nonstable vector fields and two stable ones. The homomorphism $st_1$ is surjective, so, infinitely many nonstable vector fields are associated to each of the two stable ones.

(3) $M = P^{2k+1} \times S^{2k+1}$. In this case $st_1$ is an isomorphism, so to any stable vector field there is a unique nonstable one stabilizing to it.

(4) $M = P^{2k+1}$. In this case, $w_1(M) = 0$ and $\text{coker } st_1 \simeq \mathbb{Z}_2$. We have 4 stable and 4 nonstable vector fields. So only 2 stable ones come from nonstable ones and there are two of them for each stable one.

Example 4  We give now an example where $\alpha$ is a non trivial orientation line bundle $\xi_{M^n}$ over a manifold $M^n$ ($n \geq 3$) and $\beta$ is $\tau_{M^n}$. We observe that if $M^n$ is orientable $\xi_{M^n}$ is trivial, a case already considered above. So, we are taking $M^n$ to be a nonorientable manifold.

$M = S^1 \times P^{4k}$. Then $H_1(M; \mathbb{Z}) \simeq \mathbb{Z}_2$, (see Borsari–Gonçalves [1, Theorem 2.5]) and $st_1$ and $\tilde{st}_1$ are injective. In the Diagram IV we get $|\Omega_1(M; \Phi_1)| = |\Omega_1(P(\alpha); \Phi_1)| = 4$ and $|\Omega_1(M \times P^{4k}; \Phi)| = 8$; so $\text{coker } st_1 \simeq \mathbb{Z}_2$, that is, only half of the homotopy classes of stable monomorphism contains unstable ones, and if so, each class contains exactly one element.

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References


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Some results on vector bundle monomorphisms


