Hochschild homology, Frobenius homomorphism and Mac Lane homology

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We prove that $\mathsf{H}_i(A, \Phi(A)) = 0$, $i > 0$. Here A is a commutative algebra over the prime field $\mathbb{F}_p$ of characteristic $p > 0$ and $\Phi(A)$ is A considered as a bimodule, where the left multiplication is the usual one, while the right multiplication is given via Frobenius endomorphism and $\mathsf{H}_\bullet$ denotes the Hochschild homology over $\mathbb{F}_p$. This result has implications in Mac Lane homology theory. Among other results, we prove that $\mathsf{HML}_\bullet(A, T) = 0$, provided A is an algebra over a field K of characteristic $p > 0$ and T is a strict homogeneous polynomial functor of degree d with $1 < d < \mathsf{Card}(K)$.

55P43, 16E40; 19D55, 55U10

1 Introduction

In this short note we study Hochschild and Mac Lane homology of commutative algebras over the prime field $\mathbb{F}_p$ of characteristic $p > 0$. Let us recall that Mac Lane homology is isomorphic to the topological Hochschild homology (Pirashvili–Waldhausen [13]) and to the stable K–theory as well (Franjou et al [4]).

Let A be a commutative algebra over the prime field $\mathbb{F}_p$ of characteristic $p > 0$ and let $\Phi(A)$ be an A–A–bimodule, which is A as a left A–module, while the right multiplication is given via Frobenius endomorphism. We prove that the Hochschild homology vanishes $\mathsf{H}_i(A, \Phi(A)) = 0$, $i > 0$. The proof makes use a simple result on homotopy groups of simplicial rings, which says that if $R_\bullet$ is a simplicial ring such that all rings involved in $R_\bullet$ satisfy $x^m = x$, $m \geq 2$ identity then $\pi_i(R_\bullet) = 0$ for all $i > 0$. These results has implications in Mac Lane homology theory. We extend the computation of Franjou–Lannes and Schwartz [6] of Mac Lane (co)homology of finite fields with coefficients in symmetric S^d and divided Γ^d powers to arbitrary commutative $\mathbb{F}_p$–algebras, provided that $d > 1$. As a consequence of our computations we show that $\mathsf{HML}_\bullet(A, T) = 0$, provided T is a strict homogeneous polynomial functor of degree $d > 1$ and A is an algebra over a field K of characteristic $p > 0$ with $\mathsf{Card}(K) > d$.

Published: 2 August 2007 DOI: 10.2140/agt.2007.7.1071

Many thanks to referee for his valuable comments and to Petter Andreas Bergh for his remarks. The author was partially supported by the grant from MEC, MTM2006-15338-C02-02 (European FEDER support included).

2 When it is too easy to compute homotopy groups

It is well known that the homotopy groups of a simplicial abelian group $(A_\bullet, \partial_\bullet, s_\bullet)$ can be computed as the homology of the normalized chain complex $(N_\bullet(A_\bullet), d)$, where

$$N_n(A_\bullet) = \{x \in A_n | \partial_i(x) = 0, i > 0\}$$

and the boundary map $N_n(A_\bullet) \to N_{n-1}(A_\bullet)$ is induced by ∂_0. Our first result shows that if $A_\bullet$ has a simplicial ring structure and the rings involved in $A_\bullet$ satisfy extra conditions then homotopy groups are zero in positive dimensions. This fact is an easy consequence of the following result which is probably well known.

Lemma 1 *Let $R_\bullet$ be a simplicial object in the category of not necessarily associative rings and let $x, y \in N_n(R_\bullet)$ be two elements. Assume $n > 0$ and x is a cycle. Then the cycle $xy \in N_n(R_\bullet)$ is a boundary.*

Proof Consider the element

$$z = s_0(xy) - s_1(x)s_0(y).$$

Then we have

$$\partial_0(z) = xy - (s_0\partial_0(x))y = xy.$$

Moreover,

$$\partial_1(z) = xy - xy = 0.$$

We also have

$$\partial_2(z) = (s_0\partial_1(x))(s_0\partial_1(y)) - x(s_0\partial_1(y)) = 0.$$

Similarly for all $i > 2$ we have

$$\partial_i(z) = (s_0\partial_{i-1}(x))(s_0\partial_{i-1}(y)) - (s_1\partial_{i-1}(x))(s_0\partial_{i-1}(y)) = 0.$$

Hence z is an element of $N_{n+1}(R_\bullet)$ with $\partial(z) = xy$. □

Corollary 2 *Let $R_\bullet$ be a simplicial ring. If the rings involved in $R_\bullet$ satisfy $x^m = x$ identity for $m \geq 2$, then*

$$\pi_n(R_\bullet) = 0, \quad n > 0.$$

Proof Take a cycle $x \in N_n(R_\bullet)$, $n > 0$. Then the class of $x = xx^{m-1}$ in $\pi_n(R_\bullet)$ is zero. □

Remark A more general fact is true. Let $\mathbf{T}$ be a pointed algebraic theory (Schwede [15]) and let $X_\bullet$ be a simplicial object in the category of $\mathbf{T}$–models [15]. Then $\pi_1(X_\bullet)$ is a group object in the category of $\mathbf{T}$–models, while $\pi_i(X_\bullet)$ are abelian group objects in the category of $\mathbf{T}$–models for all $i > 1$. Thus $\pi_i(X_\bullet) = 0$, $i \geq 1$ provided all group objects are trivial. This is what happens for the category of rings satisfying the identity $x^m = x$, $m \geq 2$. Another interesting case is the category of Heyting algebras (Esakia [3]).

3 Hochschild homology with twisted coefficients

In what follows the ground field is the prime field $\mathbb{F}_p$ of characteristic $p > 0$. All algebras are taken over $\mathbb{F}_p$ and they are assumed to be associative. For an algebra R and an R–R–bimodule B we let $\mathsf{H}_\bullet(R, B)$ and $\mathsf{H}^\bullet(R, B)$ be the Hochschild homology and cohomology of R with coefficients in B. Let us recall that

$$\mathsf{H}_\bullet(R, B) = \mathsf{Tor}_\bullet^{R\otimes R^{op}}(R, B)$$

and

$$\mathsf{H}^\bullet(R, B) = \mathsf{Ext}^\bullet_{R\otimes R^{op}}(R, B).$$

Moreover, let $C_\bullet(R, B)$ be the standard simplicial vector space computing Hochschild homology

$$\pi_\bullet(C_\bullet(R, B)) \cong \mathsf{H}_\bullet(R, B).$$

Recall that $C_n(R, B) = B \otimes R^{\otimes n}$, while

$$\partial_0(b, r_1, \dots, r_n) = (br_1, \dots, r_n),$$
$$\partial_i(b, r_1, \dots, r_n) = (b, r_1, \dots, r_i r_{i+1}, \dots, r_n), \quad 0 < i < n$$

and

$$\partial_n(b, r_1, \dots, r_n) = (r_n b, r_1, \dots, r_{n-1}).$$

Here $b \in B$ and $r_1, \dots, r_n \in R$.

Let $n \geq 1$ be a natural number and let A be a commutative $\mathbb{F}_p$–algebra. The Frobenius homomorphism gives rise to the functors Φ^n from the category of A–modules to the category of A–A–bimodules, which are defined as follows. For an A–module M the bimodule $\Phi^n(M)$ coincides with M as a left A–module, while the right A–module structure on $\Phi^n(M)$ is given by

$$ma = a^{p^n} m, \quad a \in A, \quad m \in M.$$

Having A–A–bimodule $\Phi^n(M)$ we can consider the Hochschild homology $\mathsf{H}_\bullet(A, \Phi^n(M))$. In this section we study these homologies. In order to state our results we need some notation. We let $\psi^n(A)$ be the quotient ring $A/(a-a^{p^n})$, $n \geq 1$ which is considered as an A–module via the quotient map $A \twoheadrightarrow \psi^n(A)$. Thus ψ^n is the left adjoint of the inclusion of the category of commutative $\mathbb{F}_p$–algebras with identity $x^m = x, m = p^n$ to the category of all commutative $\mathbb{F}_p$–algebras.

Example 3 Let $n \geq 1$. If K is a finite field with $q = p^d$ element then $\psi^n(K) = K$ if $n = dt$, $t \in \mathbb{N}$ and $\psi^n(K) = 0$ if $n \neq dt$, $t \in \mathbb{N}$.

Lemma 4 *Let A is a commutative algebra over a field K of characteristic $p > 0$ with $\mathsf{Card}(K) > p^n$. Then $\psi^n(A) = 0$, $n \geq 1$.*

Proof By assumption there exists $k \in K$ such that $k^{p^n} - k$ is an invertible element of K. It follows then that the elements of the form $a^{p^n} - a$ generates whole A. □

Theorem 5 *Let A be a commutative $\mathbb{F}_p$–algebra and $n \geq 1$. Then*

$$\mathsf{H}_i(A, \Phi^n(A)) = 0$$

for all $i > 0$ and

$$\mathsf{H}_0(A, \Phi^n(A)) \cong \psi^n(A).$$

Proof The proof consists of three steps.

Step 1 The theorem holds if $A = \mathbb{F}_p[x]$ In this case we have the following projective resolution of A over $A \otimes A = \mathbb{F}_p[x, y]$:

$$0 \to \mathbb{F}_p[x, y] \xrightarrow{\eta} \mathbb{F}_p[x, y] \xrightarrow{\epsilon} \mathbb{F}_p[x] \to 0.$$

Here $\epsilon(x) = \epsilon(y) = x$ and η is induced by multiplication by $(x - y)$. Hence for any A–A–bimodule B, we have $\mathsf{H}_i(A, B) = 0$ for $i > 1$ and

$$\mathsf{H}_0(A, B) \cong \mathsf{Coker}(u) \text{ and } H_1(A, B) \cong \ker(u),$$

where $u\colon B \to B$ is given by $u(b) = xb - bx$. If $B = \Phi^n(\mathbb{F}_p[x])$, then $u\colon \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ is the multiplication by $(x^{p^n} - x)$ and we obtain $\mathsf{H}_1(A, \Phi^n(A)) = 0$ and $\mathsf{H}_0(A, \Phi^n(A)) = \psi^n(A)$

Step 2 The theorem holds if A is a polynomial algebra Since Hochschild homology commutes with filtered colimits it suffices to consider the case when $A = \mathbb{F}_p[x_1, \dots, x_d]$. By the Künneth theorem for Hochschild homology (see Mac Lane

[10, Theorem X.7.4]) we have $\mathsf{H}_\bullet(A,\Phi^n(A)) = \mathsf{H}_\bullet(\mathbb{F}[x],\Phi^n(\mathbb{F}[x]))^{\otimes d}$ and the result follows.

Step 3 The theorem holds for arbitrary A We use the same method as used in the proof by Loday [9, Theorem 3.5.8]. First we choose a simplicial commutative algebra $L_\bullet$ such that each L_n is a polynomial algebra, $n \geq 0$ and $\pi_i(L_\bullet) = 0$ for all $i > 0$, $\pi_0(L_\bullet) = A$. Such a resolution exists thanks to (Quillen [14]). Now consider the bisimplicial vector space $C_\bullet(L_\bullet,\Phi^n(L_\bullet))$. The sth horizontal simplicial vector space is the simplicial vector space $L_\bullet^{\otimes s+1}$. By the Eilenberg–Zilber–Cartier and Künneth theorems it has zero homotopy groups in positive dimensions and $\pi_0(L_\bullet^{\otimes s+1}) = A^{\otimes s+1}$. On the other hand the tth vertical simplicial vector space of $C_\bullet(L_\bullet,\Phi^n(L_\bullet)))$ is isomorphic to the Hochschild complex $C_\bullet(L_t,\Phi^n(L_t)))$ which has zero homology in positive dimensions by the previous step. Hence both spectral sequences corresponding to the bisimplicial vector space $C_\bullet(L_\bullet,\Phi^n(L_\bullet))$ degenerate and we obtain the isomorphism

$$\mathsf{H}_\bullet(A,\Phi^n(A)) \cong \pi_\bullet(\psi^n(L_\bullet)).$$

Now we can use Corollary 2 to finish the proof. □

Corollary 6 *Let A be a commutative $\mathbb{F}_p$–algebra, M be an A–module and $n \geq 1$. Then there exist functorial isomorphisms*

$$\mathsf{H}_\bullet(A,\Phi^n(M)) \cong \mathsf{Tor}_\bullet^A(\psi^n(A),M), \quad n \geq 0$$

and

$$\mathsf{H}^\bullet(A,\Phi^n(M)) \cong \mathsf{Ext}_A^\bullet(\psi^n(A),M), \quad n \geq 0.$$

In particular, if A is a commutative algebra over a field K of characteristic $p > 0$ with $\mathsf{Card}(K) > p^n$, then

$$\mathsf{H}_\bullet(A,\Phi^n(M)) = 0 = \mathsf{H}^\bullet(A,\Phi^n(M)).$$

Proof Observe that $C_\bullet(A,\Phi^n(A))$ is a complex of left A–modules. By Theorem 5 it is a free resolution of $\psi^n(A)$ in the category of A–modules. Hence it suffices to note that

$$C_\bullet(A,\Phi^n(M)) \cong M \otimes_A C_\bullet(A,\Phi^n(A)),$$
$$C^\bullet(A,\Phi^n(M)) \cong \hom_A(C_\bullet(A,\Phi^n(A)),M),$$

where C^* denotes the standard complex for Hochschild cohomology. The last assertion follows from Lemma 4. □

Example 7 It follows for instance that $\mathsf{H}^i(A,\Phi^n(M))=0$, $i>0$, provided M is an injective A–module and $n\geq 1$. In particular $\mathsf{H}^i(A,\Phi^n(A))=0$ if A is a selfinjective algebra. On the other hand if $A=\mathbb{F}_p[x_1,\dots,x_d]$ then $\mathsf{H}^i(A,\Phi^n(A))=0$, $i\neq d$, $n\geq 1$ and $\mathsf{H}^d(A,\Phi^n(A))=\psi^n(A)$, $n\geq 1$.

4 Application to Mac Lane cohomology

We recall the definition of Mac Lane (co)homology. For an associative ring R we let $\mathbf{F}(R)$ be the category of finitely generated free left R–modules. Moreover, we let $\mathfrak{F}(R)$ be the category of all covariant functors from the category $\mathbf{F}(R)$ to the category of all R–modules. The category $\mathfrak{F}(R)$ is an abelian category with enough projective and injective objects. By definition (Jibladze–Pirashvili [8]) the *Mac Lane cohomology* of R with coefficient in a functor $T\in\mathfrak{F}(R)$ is given by

$$\mathsf{HML}^\bullet(R,T):=\mathsf{Ext}^\bullet_{\mathfrak{F}(R)}(I,T),$$

where $I\in\mathfrak{F}(R)$ is the inclusion of the category $\mathbf{F}(R)$ into the category of all left R–modules. One defines Mac Lane homology in a dual manner (see Pirashvili–Waldhausen [13, Proposition 3.1]). For an R–R–bimodule B, one considers the functor $B\otimes_R(-)$ as an object of the category $\mathfrak{F}(R)$. For simplicity we write $\mathsf{HML}_\bullet(R,B)$ instead of $\mathsf{HML}_\bullet(R,B\otimes_R(-))$. There is a binatural transformation

$$\mathsf{HML}_\bullet(R,B)\to\mathsf{H}_\bullet(R,B)$$

which is an isomorphism in dimensions 0 and 1.

In the rest of this section we consider Mac Lane (co)homology of commutative $\mathbb{F}_p$–algebras.

Lemma 8 *For any commutative $\mathbb{F}_p$–algebra A one has an isomorphism*

$$\mathsf{HML}_{2i}(A,\Phi^n(A))=\psi^n(A),\quad i\geq 0, n\geq 1,$$

and

$$\mathsf{HML}_{2i+1}(A,\Phi^n(A))=0,\quad i\geq 0, n\geq 1.$$

Proof According to (Pirashvili [12, Proposition 4.1]) there exists a functorial spectral sequence

$$E^2_{pq}=\mathsf{H}_p(A,\mathsf{HML}_q(\mathbb{F}_p,B))\Longrightarrow\mathsf{HML}_{p+q}(A,B).$$

Here B is an A–A–bimodule. By the well-known computation of Breen [2], Bökstedt [1] (see also Franjou–Lannes–Schwartz [6]) we have

$$\mathsf{HML}_{2i}(\mathbb{F}_p,B)=B$$

and

$$\mathsf{HML}_{2i+1}(\mathbb{F}_p, B) = 0.$$

Now we put $B = \psi^n(A)$ and use Theorem 5 to get $E^2_{pq} = 0$ for all $p > 0$. Hence the spectral sequence degenerates and the result follows. □

We now consider Mac Lane cohomology with coefficients in strict polynomial functors (Friedlander–Suslin [7]). Let us recall that the strict homogeneous polynomial functors of degree d form an abelian category $\mathfrak{P}_d(A)$ and there exist an exact functor $i\colon \mathfrak{P}_d(A) \to \mathfrak{F}(A)$ (Franjou et al [5]). For an object $T \in \mathfrak{P}_d(A)$ we write $\mathsf{HML}_\bullet(A, T)$ instead of $\mathsf{HML}_\bullet(A, i(T))$. Projective generators of the category $\mathfrak{P}_d$ are tensor products of the divided powers, while the injective cogenerators are symmetric powers. Let us recall that the dth divided power functor $\Gamma^d \in \mathfrak{F}(A)$ and d-th symmetric functors S^n are defined by

$$\Gamma^d(M) = (M^{\otimes d})^{\Sigma_d}, \quad S^n(M) = (M^{\otimes d})_{\Sigma_d}.$$

Here tensor products are taken over A, Σ_d is the symmetric group on d letters, which acts on the d-th tensor power by permuting of factors, $M \in \mathbf{F}(A)$ and X^G (resp. X_G) denotes the module of invariants (resp. coinvariants) of a G–module X, where G is a group.

For a functor $T \in \mathfrak{F}(A)$ we let $\widetilde{T} \in \mathfrak{F}(\mathbb{F}_p)$ be the functor defined by

$$\widetilde{T}(V) = T(V \otimes A).$$

According to Pirashvili–Waldhausen [13, Theorem 4.1] the groups $\mathsf{HML}_i(\mathbb{F}_p, \widetilde{T})$ have an A–A–bimodule structure. The left action comes from the fact that T has values in the category of left A–modules, while the right action comes from the fact that T is defined on $\mathbf{F}(A)$. In particular it uses the action of T on the maps $l_a\colon X \to X$, where $a \in A$, $X \in \mathbf{F}(A)$ and l_a is the multiplication on a. Since $T(l_a) = l_{a^d}$ if T is a strict homogeneous polynomial functor of degree d Friedlander–Suslin [7], the bimodule $\mathsf{HML}_i(\mathbb{F}_p, \widetilde{T})$ is of the form $\Phi^n(M)$ provided $d = p^n$.

Theorem 9 *Let $d > 1$ be an integer and let A be a commutative $\mathbb{F}_p$–algebra. Then $\mathsf{HML}_\bullet(A, \Gamma^d) = 0$ if d is not a power of p. If $d = p^n$ and $n > 0$, then*

$$\mathsf{HML}_i(A, \Gamma^d) = 0 \text{ if } i \neq 2p^n t, t \geq 0$$

and

$$\mathsf{HML}_i(A, \Gamma^d) = \psi^n(A) \text{ if } i = 2p^n t, t \geq 0.$$

In particular $\mathsf{HML}_\bullet(A, \Gamma^d) = 0$ provided A is an algebra over a field K of characteristic $p > 0$ with $\mathsf{Card}(K) > d$.

Proof According to Pirashvili–Waldhausen [13, Theorem 4.1] and Pirashvili [12] there exists a functorial spectral sequence:

$$E^2_{pq} = \mathsf{H}_p(A, \mathsf{HML}_q(\mathbb{F}_p, \widetilde{T})) \Longrightarrow \mathsf{HML}_{p+q}(A, T).$$

For $T = \Gamma^n_A$ one has $\widetilde{T} = \Gamma^n_{\mathbb{F}_p} \otimes A$. Here we used the notation Γ^n_A in order to emphasize the dependence on the ring A. By the result of Franjou, Lannes and Schwartz [6], $\mathsf{HML}_i(\mathbb{F}_p, \widetilde{T})$ vanishes unless $d = p^n$ and $i = 2p^n t$, $t \geq 0$. Moreover in these exceptional cases $\mathsf{HML}_i(\mathbb{F}_p, \widetilde{T})$ equals to $\Phi^n(A)$ (as an A–A–module). Hence the spectral sequence together with Theorem 5 gives the result. □

Corollary 10 *Let A be a commutative algebra over a field K of characteristic $p > 0$ with $\mathsf{Card}(K) > d > 1$. If T is a strong homogeneous polynomial functor of degree d. Then*

$$\mathsf{HML}_\bullet(A, T) = 0 = \mathsf{HML}^\bullet(A, T).$$

Proof We already proved that the result is true if T is a divided power. By the well-known vanishing result (Pirashvili [11]) the result is also true if $T = T_1 \otimes T_2$ with $T_1(0) = 0 = T_2(0)$. Since any object of $\mathfrak{P}_d$ has a finite resolution which consists with finite direct sums of tensor products of divided powers [7] the result follows. □

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Received: 14 March 2007