Contact Ozsváth–Szabó invariants and Giroux torsion

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In this paper we prove a vanishing theorem for the contact Ozsváth–Szabó invariants of certain contact 3–manifolds having positive Giroux torsion. We use this result to establish similar vanishing results for contact structures with underlying 3–manifolds admitting either a torus fibration over S^1 or a Seifert fibration over an orientable base. We also show – using standard techniques from contact topology – that if a contact 3–manifold (Y, ξ) has positive Giroux torsion then there exists a Stein cobordism from (Y, ξ) to a contact 3–manifold (Y, ξ') such that (Y, ξ) is obtained from (Y, ξ') by a Lutz modification.

57R17; 57R57

1 Introduction

In [12] Giroux introduced the important invariant $\mathrm{Tor}(Y, \xi)$ of a contact 3–manifold (Y, ξ), which is now called the *Giroux torsion*, and is defined as follows: $\mathrm{Tor}(Y, \xi)$ is the supremum of the integers $n \geq 1$ for which there is a contact embedding of

$$\mathbb{T}_n := (T^2 \times [0,1], \ker(\cos(2\pi nz)dx - \sin(2\pi nz)dy))$$

into (Y, ξ). We say that $\mathrm{Tor}(Y, \xi) = 0$ if no such embedding exists.

Closed, toroidal 3–manifolds carry infinitely many universally tight contact structures obtained by inserting copies of $\mathbb{T}_n$ around incompressible tori (see Colin [1; 2], Colin–Giroux–Honda [4] and Honda–Kazez–Matić [19]). Remarkably, as the following result shows, embedded copies of $\mathbb{T}_n$ are the only source of infinite families of distinct tight contact structures on a closed 3–manifold.

Theorem 1.1 (Colin–Giroux–Honda [4, Theorem 1.4]) *Let Y be a closed 3–manifold. For every natural number n the 3–manifold Y carries at most finitely many isomorphism classes of tight contact structures with Giroux torsion bounded above by n.*

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Contact Ozsváth–Szabó invariants are very useful tools in studying contact structures on closed 3–manifolds. The nature of these invariants is still unclear though, and it is natural to ask how the invariants change under the introduction of Giroux torsion. Based on the analogy between Seiberg–Witten and Heegaard Floer theories the following is expected:

Conjecture 1.2 If $\mathrm{Tor}(Y,\xi) > 0$ then the untwisted contact Ozsváth–Szabó invariant $c(Y,\xi)$ vanishes.[1]

Conjecture 1.2 has been verified by Paolo Ghiggini [8] for a class of contact Seifert fibered 3–manifolds. In the following we will extend his result and verify Conjecture 1.2 for a much wider family of contact 3–manifolds.

A recent result of D Gay [5] asserts that a contact structure with positive Giroux torsion is not strongly fillable. Since strongly fillable contact structures have nonvanishing contact Ozsváth–Szabó invariants, Conjecture 1.2 is consistent with Gay's result. Indeed, starting from any contact 3–manifold with positive Giroux torsion, Gay constructs a symplectic cobordism which contains homologically essential 2–spheres with self–intersection zero. This suggests that such a cobordism could be used to find constraints on certain Seiberg–Witten invariants, thus re–proving Gay's nonfillability result. On the other hand, at present we do not understand well enough how the contact invariants of Ozsváth and Szabó's behave under the maps induced between the relevant Heegaard Floer groups by general symplectic cobordisms. This is the main obstacle which prevents us from proving Conjecture 1.2 using Gay's construction.

In this paper we build a different type of cobordism on a contact 3–manifold with positive Giroux torsion. Our cobordism is better suited than the one of Gay's to study the contact Ozsváth–Szabó invariants because it is a union of Stein 2–handles, and the behaviour of the invariants under the corresponding maps is well understood. It follows that the contact invariant is always in the image of such a map, which is very useful. In fact, in the cases considered in this paper we prove that the invariant is equal to zero by showing that a certain map induced by the cobordism vanishes.

Throughout the paper, every 3–manifold will be considered to be oriented and every contact structure positive and cooriented. Recall that in Ozsváth–Szabó [27] a variety of homology groups – the Ozsváth–Szabó homologies – and a natural map

$$\varphi_{(Y,\mathbf{t})}\colon HF^{\infty}(Y,\mathbf{t}) \to HF^{+}(Y,\mathbf{t})$$

[1] After the completion of this work, P Ghiggini, K Honda and J van Horn-Morris posted [10] containing a proof of Conjecture 1.2. Their approach — relying on a newly invented invariant of Honda, Kazez and Matić [18] — is different from the surgery theoretic approach we adopt in this paper.

are defined for closed, oriented spin^c 3–manifolds. (For more about Ozsváth–Szabó homologies see Section 2.)

Definition 1.3 We say that the closed 3–manifold Y has *simple Ozsváth–Szabó homology* at the spin^c structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$ if the map $\varphi_{(Y,\mathbf{t})}$ is surjective. Y is called *OSz–simple* if Y has simple Ozsváth–Szabó homology for every spin^c structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$.

A rational homology sphere is called an *L–space* in Ozsváth–Szabó [29] provided that the map $\varphi_{(Y,\mathbf{t})}$ is surjective for every $\mathbf{t}$. Examples of L–spaces can be produced by considering plumbings of spheres along trees with no "bad vertices" (see Ozsváth–Szabó [24]) or by taking double branched covers of S^3 along nonsplit, alternating links (see Ozsváth–Szabó [30, Section 3]). If one uses $\mathbb{Z}/2\mathbb{Z}$–coefficients then Seifert fibered 3–manifolds over an orientable base with sufficiently large background Chern numbers are OSz–simple, see Section 2. Also, it is not hard to see that Y is OSz–simple if and only if $-Y$ is.

Given a contact 3–manifold (Y,ξ), we shall denote by $\mathbf{t}_\xi$ the spin^c structure induced by ξ. Our first result is:

Theorem 1.4 *Let (Y,ξ) be a contact 3–manifold such that Y is OSz–simple. If $\mathrm{Tor}(Y,\xi) > 1$ then $c(Y,\xi) = 0$. If $b_1(Y) \le 1$ then $\mathrm{Tor}(Y,\xi) > 0$ already implies $c(Y,\xi) = 0$.*

Remark 1.5 The proof of Theorem 1.4 works under the weaker assumption that Y has simple Ozsváth–Szabó homology at the spin^c structure $\mathbf{t}_\xi$. One way to check that Y has simple Ozsváth–Szabó homology at $\mathbf{t}$ is to prove that Y is OSz–simple, see Proposition 2.4.

The following two results deal with many cases where the underlying manifolds are not OSz–simple.

Theorem 1.6 *Let Y be a closed 3–manifold which admits a torus fibration over S^1. Then there exists an integer $n_Y \ge 0$ such that for every contact structure ξ on Y with $\mathrm{Tor}(Y,\xi) > n_Y$ we have $c(Y,\xi) = 0$. If the monodromy A of the torus fibration is trivial then $n_Y = 0$. If A is elliptic ($|\operatorname{tr}(A)| < 2$) or parabolic ($|\operatorname{tr}(A)| = 2$) then $n_Y \le 1$.*

Theorem 1.7 *Let (Y,ξ) be a contact 3–manifold such that Y admits a Seifert fibration over an orientable base. If $\mathrm{Tor}(Y,\xi) > 2$ then using $\mathbb{Z}/2\mathbb{Z}$–coefficients the contact Ozsváth–Szabó invariant $c(Y,\xi)$ vanishes.*

In a slightly different direction, these vanishing results can be used to study strong and Stein fillability of contact structures. In [6] Ghiggini found the first examples of strongly fillable contact structures which are not Stein fillable. His examples live on the Brieskorn 3–spheres $-\Sigma(2,3,12n+5)$, $n \geq 1$. The following consequence of Theorem 1.4, pointed out to us by Paolo Ghiggini, slightly generalizes [6, Theorem 1.5].

Theorem 1.8 *Let Σ_n be the Brieskorn homology 3–sphere of type $(2,3,6n+5)$, oriented as the link of the corresponding isolated singularity. For every $n \geq 2$, the oriented 3–manifold $-\Sigma_n$ carries a strongly fillable contact structure which is not Stein fillable.*

Finally, using standard techniques from contact topology we establish the following Theorem 1.9, which lies within the circle of ideas of this paper and appears to be of independent interest. It is worth pointing out that we did not use Theorem 1.9 to prove any of the previous results, except for the second part of the statement of Theorem 1.4. Given a smoothly embedded torus $T \subset (Y,\xi)$ with characteristic foliation made of simple closed curves, following Colin [3] and Colin–Giroux–Honda [4], we call the insertion of a copy of $\mathbb{T}_1$ around T a *Lutz modification* of ξ along T (not to be confused with the so–called *Lutz twist* along a knot transverse to ξ).

Theorem 1.9 *Let $n \geq 1$, and suppose that $\mathbb{T}_n$ embeds inside the contact 3–manifold (Y,ξ). Then there is a sequence of Legendrian surgeries on (Y,ξ) which yields a contact 3–manifold (Y,ξ') such that (Y,ξ) is obtained from (Y,ξ') by a Lutz modification along an embedded copy of $\mathbb{T}_{n-1}$ [2].*

The paper is organized as follows. Section 2 is devoted to the recollection of basic facts regarding Ozsváth–Szabó homologies and contact Ozsváth–Szabó invariants. We also compute the Ozsváth–Szabó homology groups of some of the 3–manifolds which will appear in later arguments. In Section 3 we prove a few auxiliary results which will be used in the proofs of the results stated above. In Section 4 we prove all the results except Theorem 1.9, which is proved in Section 5.

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[2] When $n=1$, this is to be interpreted as an embedded 2–torus with characteristic foliation made of simple closed curves.

2 Contact Ozsváth–Szabó invariants

2.1 Ozsváth–Szabó homologies

In the seminal papers of Ozsváth and Szabó [27; 26] a collection of homology groups – the Ozsváth–Szabó homologies – $\widehat{HF}(Y,\mathbf{t})$, $HF^{\pm}(Y,t)$ and $HF^{\infty}(Y,\mathbf{t})$ have been assigned to any closed, oriented spin^c 3–manifold $(Y,\mathbf{t})$. A spin^c cobordism $(W,\mathbf{s})$ from $(Y_1,\mathbf{t}_1)$ to $(Y_2,\mathbf{t}_2)$ induces $\mathbb{Z}[U]$–equivariant homomorphisms $\widehat{F}_{W,\mathbf{s}}$, $F^{\pm}_{W,\mathbf{s}}$ and $F^{\infty}_{W,\mathbf{s}}$ between the corresponding groups. We shall use the symbol $F_{W,\mathbf{s}}$ to denote any of these maps. For a fixed spin^c structure $\mathbf{t}\in \mathrm{Spin}^c(Y)$ these groups fit into long exact sequences

$$\ldots \to HF^-_d(Y,\mathbf{t}) \to HF^{\infty}_d(Y,\mathbf{t}) \xrightarrow{\varphi(Y,\mathbf{t})} HF^+_d(Y,\mathbf{t}) \to \ldots$$

$$\ldots \to \widehat{HF}_d(Y,\mathbf{t}) \to HF^+_d(Y,\mathbf{t}) \xrightarrow{\cdot U} HF^+_{d-2}(Y,\mathbf{t}) \to \ldots$$

These exact sequences are functorial with respect to the maps induced by spin^c cobordisms. Throughout the paper we shall use Ozsváth–Szabó homology groups with $\mathbb{Z}$–coefficients, with the exceptions of Theorem 1.7 and Proposition 2.6, where $\mathbb{Z}/2\mathbb{Z}$–coefficients are used.

Ozsváth–Szabó homology groups and the maps induced by the cobordisms form a TQFT in the sense that the composition of two spin^c cobordisms $(W_1,\mathbf{s}_1)$ and $(W_2,\mathbf{s}_2)$ induce a map which can be given by the composition of the maps. There is, however, a subtlety following from the fact that the spin^c structures $\mathbf{s}_i$ on W_i $(i=1,2)$ do not uniquely determine a spin^c structure on the union $W_1\cup W_2$. Consequently the composition formula reads as follows:

Theorem 2.1 (Ozsváth–Szabó [31, Theorem 3.4]) *Suppose $(W_1,\mathbf{s}_1)$ and $(W_2,\mathbf{s}_2)$ are spin^c cobordisms with $\partial W_1=-Y_1\cup Y_2$, $\partial W_2=-Y_2\cup Y_3$ and set $W=W_1\cup_{Y_2}W_2$. Let $\mathcal{S}$ denote the set of spin^c structures on W which restrict to W_i as $\mathbf{s}_i$ for $i=1,2$. Then*

$$F_{W_2,\mathbf{s}_2}\circ F_{W_1,\mathbf{s}_1}=\sum_{\mathbf{s}\in\mathcal{S}}\pm F_{W,\mathbf{s}}.$$

An important ingredient in our subsequent discussions is

Proposition 2.2 (Ozsváth–Szabó [31, Lemma 8.2]) *Suppose that $(W,\mathbf{s})$ is a 4–dimensional spin^c cobordism between $(Y_1,\mathbf{t}_1)$ and $(Y_2,\mathbf{t}_2)$. If $b_2^+(W)>0$ then the map*

$$F^{\infty}_{W,\mathbf{s}}\colon HF^{\infty}(Y_1,\mathbf{t}_1)\to HF^{\infty}(Y_2,\mathbf{t}_2)$$

is zero.

Corollary 2.3 *Suppose that the 4–dimensional cobordism W between Y_1 and Y_2 has $b_2^+(W) > 0$ and Y_1 has simple OSz–homology at $\mathbf{t}_1$. Then for every spin^c structure $\mathbf{s} \in \mathrm{Spin}^c(Y)$ with $\mathbf{s}|_{Y_1} = \mathbf{t}_1$ the maps $F^+_{W,\mathbf{s}}$ and $\widehat{F}_{W,\mathbf{s}}$ vanish.*

Proof Proposition 2.2 implies the vanishing of $F^\infty_{W,\mathbf{s}}$. Combining this with the assumption that Y_1 has simple OSz–homology at $\mathbf{t}_1$, the fact that

$$F^+_{W,\mathbf{s}} \circ \varphi_{(Y_1,\mathbf{s}|Y_1)} = \varphi_{(Y_2,\mathbf{s}|_{Y_2})} \circ F^\infty_{W,\mathbf{s}}$$

immediately implies the vanishing of $F^+_{W,\mathbf{s}}$. The vanishing of $\widehat{F}_{W,\mathbf{s}}$ now follows from the naturality of the exact sequence connecting the groups $\widehat{HF}$ and HF^+. □

Examples of OSz–simple manifolds are provided by certain torus bundles over S^1.

Proposition 2.4 *A torus bundle $Y \to S^1$ with elliptic or parabolic monodromy $A \in SL(2,\mathbb{Z})$ (that is, $|\operatorname{tr}(A)| < 2$ or $|\operatorname{tr}(A)| = 2$) is OSz–simple.*

Proof Suppose first that Y has elliptic monodromy. By the classification of torus bundles over S^1 (see, for example, Hatcher [16]) it follows that, up to changing its orientation, Y is the boundary of one of the three plumbings described in Figure 1. In

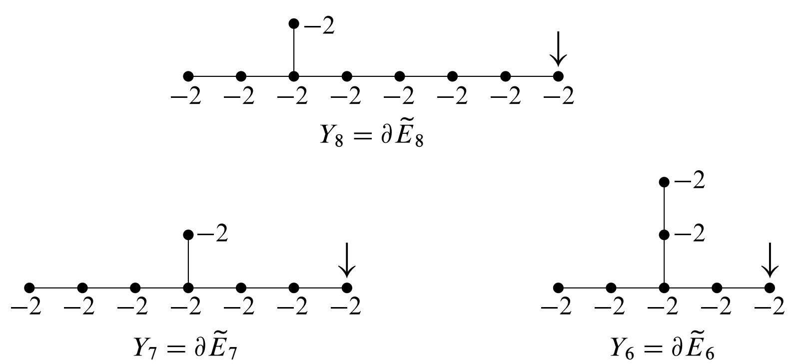


Figure 1: Torus bundles with elliptic monodromy

fact, these plumbings are regular neighbourhoods of the elliptic singular fibers $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$, see Harer–Kas–Kirby [15]. It is easy to check that by deleting the vertices indicated by the arrows one gets the 3–manifolds $S^3_{i-9}(K)$, where K denotes the left–handed trefoil knot ($i = 6, 7, 8$). On the other hand, by assigning weight (-1) to the vertices indicated by the arrows, we get the lens spaces $L(9-i, 1)$ ($i = 6, 7, 8$). Since lens spaces, and all r–surgeries on K with $r \le -1$ are L–spaces (see Ozsváth–Szabó

[26, Section 3] and [23, Lemma 7.12 and Section 8]), the surgery exact triangle [26] for the $\widehat{HF}$–theory implies that rk $\widehat{HF}(Y_i) \leq 2(9-i)$ $(i=6,7,8)$. Since

$$H_1(Y_i;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/(9-i)\mathbb{Z} \qquad (i=6,7,8),$$

and for a torsion spin^c structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$ with $b_1(Y)=1$ we have rk $\widehat{HF}(Y,\mathbf{t}) \geq 2$ (see Lisca–Stipsicz [21]), we conclude that rk $\widehat{HF}(Y_i) = 2(9-i)$, which, in view of [21, Proposition 2.2] shows that each Y_i $(i=6,7,8)$ is OSz–simple. (For Y_8 the same fact is proved in [23, Section 8.1].)

Let us now consider the case of parabolic monodromy. By the classification of torus bundles (see, for example, Hatcher [16]), we know that Y is either a circle bundle over a torus or a Klein bottle, or it is diffeomorphic to the Seifert fibered 3–manifold described by the diagram of Figure 2.

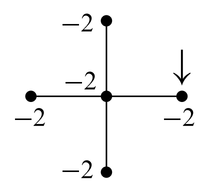


Figure 2: The Seifert fibered manifold $M\left(0; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

The trivial circle bundle T^3 over the 2–torus is OSz–simple by Ozsváth–Szabó [23, Section 8.4]. If the circle bundle $Y_n \to T^2$ has Euler number n then Y_n is diffeomorphic to $M\{0,0,n\}$ of [23, Subsection 8.2]. For $n=1$ the Ozsváth–Szabó homology group $\widehat{HF}(Y_1)$ is shown in the proof of [23, Proposition 8.4] to be $\mathbb{Z}^4$, verifying the statement. For $n>1$ we can proceed by a simple induction on n: By the surgery triangle written for the n–framed unknot in the surgery diagram for $M\{0,0,n\}$ described in [23, Subsection 8.2] we get

$$\mathrm{rk}\ \widehat{HF}(Y_{n+1}) \leq \mathrm{rk}\ \widehat{HF}(S^1\times S^2 \# S^1 \times S^2) + \mathrm{rk}\ \widehat{HF}(Y_n) = 4 + 4n = 4(n+1).$$

On the other hand,

$$H_1(Y_{n+1};\mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/(n+1)\mathbb{Z}$$

implies that rk $\widehat{HF}(Y_{n+1}) \geq 4(n+1)$, hence Y_n is OSz–simple for $n \geq 0$. Since by reversing the orientation if necessary, we may assume the Euler number n to be positive, we conclude the proof for circle bundles over T^2.

Circle bundles over the Klein bottle K can be handled similarly. A surgery description of such a 3–manifold Z_n with Euler number n is given by Figure 3 (see Gompf and Stipsicz [14, Figure 6.4 with $k=0$ and $l=2$]). Simple Kirby calculus shows that this

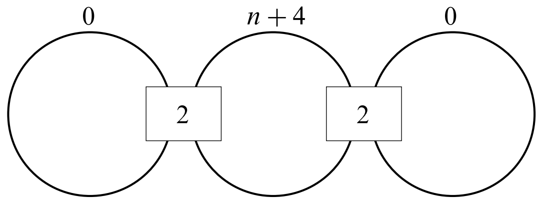
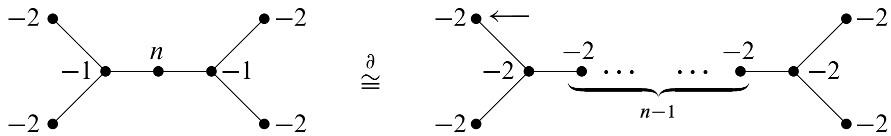


Figure 3: Circle bundle over the Klein bottle with Euler number n



(a) (b)

Figure 4: Alternative plumbing diagrams for circle bundle over the Klein bottle

diagram provides the same 3–manifold as the plumbing of Figure 4(a).

For $n > 0$ this is equivalent to the plumbing of Figure 4(b), and for $n = 0$ (after turning the diagram into a surgery picture and sliding one (-1)–circle over the other and cancelling the 0–framed unknot against the (-1)–circle) we get that Figure 4(a) gives the same 3–manifold as Figure 2. As before, we can assume that $n \geq 0$ by possibly reversing orientation. Consider the surgery exact sequence for the vertex indicated by the arrow in Figure 4(b) (and Figure 2 for $n = 0$). Notice that the two other manifolds in the surgery triangle are both L–spaces: one is diffeomorphic to the link L_{n+4} of the D_{n+4} singularity, while the other to L_n for $n \geq 4$, to $L(4,3)$ for $n = 3$, to $L(2,1)\#L(2,1)$ for $n = 2$, to $L(4,1)$ for $n = 1$ and finally to $-L_4$ for $n = 0$. Since L_n is well–known to have elliptic geometry, by Ozsváth–Szabó [29, Proposition 2.3] it is an L–space. Thus, since $|H_1(L_n;\mathbb{Z})| = 4$, we have

$$\widehat{HF}(L_n) \cong \widehat{HF}(L(4,i)) \cong \widehat{HF}(L(2,1)\#L(2,1)) \cong \mathbb{Z}^4.$$

This implies that $\operatorname{rk} \widehat{HF}(Z_n) \leq 8$. On the other hand, $H_1(Z_n;\mathbb{Z})$ is either $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ (depending on the parity of n), hence we conclude that $\widehat{HF}(Z_n) = \mathbb{Z}^8$, verifying the statement. □

Remark 2.5 Notice that torus bundles with elliptic monodromies are boundaries of neighbourhoods of type II, II^*, III, III^*, IV, IV^* fibers in elliptic fibrations (see Harer–Kas–Kirby [15]). Torus bundles with parabolic monodromies can be regarded (up to orientation) as boundaries of neighbourhoods of elliptic I_n–fibers (when the torus bundle is a circle bundle over T^2, $n \geq 1$) and of elliptic I_n^*–fibers (which are S^1–fibrations over the Klein bottle, $n \geq 0$), see [15].

Further examples of OSz–simple 3–manifolds are provided by certain Seifert fibered 3–manifolds. If Y is a Seifert fibered 3–manifold over S^2 with nonnegative background Chern number then $-Y$ is the boundary of a starshaped plumbing with no bad vertices (in the sense of Ozsváth–Szabó [24]). By [24] this implies that $-Y$, and therefore Y is OSz–simple. If Y is a Seifert fibered 3–manifold over an orientable base and the background Chern number is large enough then Y is OSz–simple, provided we use $\mathbb{Z}/2\mathbb{Z}$–coefficients in the definition of the Ozsváth–Szabó homology groups. Most probably the statement holds true for $\mathbb{Z}$–coefficients as well, but since the computational tool we will use in the proof has been verified in Ozsváth–Szabó [22] with $\mathbb{Z}/2\mathbb{Z}$–coefficients, we restrict our attention to this case. Before stating the result we need to fix our notations on Seifert fibered spaces. We do this following the conventions of [22]. We say that a Seifert fibered 3–manifold Y over a genus g surface has *Seifert invariants* $\left(a, \frac{r_1}{q_1}, \ldots, \frac{r_n}{q_n}\right)$ if the Seifert fibration on Y is obtained in the canonical way by performing $\left(-\frac{q_1}{r_1}\right)$–, …, $\left(-\frac{q_n}{r_n}\right)$–surgeries along n fibers of the circle bundle $Y_{g,a} \to \Sigma_g$ over an orientable genus–g surface Σ_g with Euler number $e(Y_{g,a}) = a$. In the above definition we assume $\frac{r_i}{q_i} \in (0,1) \cap \mathbb{Q}$, $i = 1, \ldots, n$.

Proposition 2.6 *Let Y be a Seifert fibered 3–manifold over a genus g surface with Seifert invariants $\left(a, \frac{r_1}{q_1}, \ldots, \frac{r_n}{q_n}\right)$. If $a > 2g$ and we consider Ozsváth–Szabó homology groups with $\mathbb{Z}/2\mathbb{Z}$–coefficients, then Y is OSz–simple.*

Proof Let us fix a spin^c structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$ on Y. According to [22, Theorem 10.1] we only need to check that the function $h_t \colon \mathbb{Z} \to \mathbb{Z}$ has a unique local minimum for every t satisfying $-g \le t \le g$. By definition,

$$h_t(s) = \begin{cases} \sum_{i=0}^{s-1} \delta_t(i) & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -\sum_{i=s}^{-1} \delta_t(i) & \text{if } s < 0, \end{cases}$$

where

$$\delta_t(s) = (-1)^{s+1} t + \xi_0 + a \cdot s + \sum_{i=1}^{n} \left\lfloor \frac{\xi_i + r_i \cdot s}{q_i} \right\rfloor. \tag{2–1}$$

In the formula above we adopt the notations of [22, Section 10]; in particular

$$H_1(Y;\mathbb{Z}) \cong \frac{H_1(\Sigma_g;\mathbb{Z}) \oplus \mathbb{Z}/m_0\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/m_n\mathbb{Z}}{(a \cdot m_0 + \sum_{i=1}^{n} r_i \cdot m_i = 0, r_i \cdot m_0 - q_i \cdot m_i = 0)}$$

and under the identification $\mathrm{Spin}^c(Y) \cong H_1(Y;\mathbb{Z})$ provided by [22, Theorem 10.1] the spin^c structure $\mathbf{t}$ corresponds to $\xi_0 \cdot m_0 + \ldots + \xi_n \cdot m_n$.

Therefore it suffices to show that, for every $-g \le t \le g$, the function δ_t changes sign only once, that is,

$$\delta_t(s) > 0 \quad \text{implies} \quad \delta_t(s+1) > 0.$$

Notice that we have a freedom in choosing ξ_0 (and [22, Theorem 10.1] shows, in particular, that different choices giving the same spin^c structure yield the same Ozsváth–Szabó homology groups). Since we are only concerned with torsion spin^c structures, we can fix ξ_0 to be arbitrarily large in absolute value and negative. It then follows from Formula (2–1) that we can assume $\delta_t(s) < 0$ for $s \le 0$. Now suppose that $-g \le t \le g$, $s > 0$ and $\delta_t(s) > 0$. To finish the proof it clearly suffices to verify that $\delta_t(s) < \delta_t(s+1)$. Since

$$\sum_{i=1}^{n} \left\lfloor \frac{\xi_i + r_i \cdot s}{q_i} \right\rfloor < \sum_{i=1}^{n} \left\lfloor \frac{\xi_i + r_i \cdot (s+1)}{q_i} \right\rfloor$$

and $2|t| \le 2g < a$, we have

$$\delta_t(s) = (-1)^{s+1} t + \xi_0 + a \cdot s + \sum_{i=1}^{n} \left\lfloor \frac{\xi_i + r_i \cdot s}{q_i} \right\rfloor <$$
$$- (-1)^{s+1} t + \xi_0 + a \cdot s + a + \sum_{i=1}^{n} \left\lfloor \frac{\xi_i + r_i \cdot (s+1)}{q_i} \right\rfloor = \delta_t(s+1),$$

as required. □

2.2 Contact Ozsváth–Szabó invariants

A contact structure ξ on Y determines an element $c(Y,\xi) \in \widehat{HF}(-Y, \mathbf{t}_\xi)$ (and similarly in $HF^+(-Y, \mathbf{t}_\xi)$) up to sign, which has the following crucial properties (see Ozsváth–Szabó [28; 25]):

- $\pm c(Y,\xi)$ is an isotopy invariant of the contact 3–manifold (Y,ξ);
- $c(Y,\xi) = 0$ if the contact structure ξ is overtwisted;
- $c(Y,\xi) \neq 0$ if (Y,ξ) is strongly fillable;
- if (Y_L, ξ_L) is given by contact (-1)–surgery along the Legendrian knot $L \subset (Y,\xi)$, inducing the Stein cobordism W with canonical spin^c structure $\mathbf{s}_0$ then by $\overline{W}$ denoting W when turned upside down we have

  $$F_{\overline{W}, \mathbf{s}_0}(c(Y_L, \xi_L)) = c(Y,\xi) \qquad \text{and} \qquad F_{\overline{W}, \mathbf{s}}(c(Y_L, \xi_L)) = 0$$

  for all other spin^c structures $\mathbf{s} \neq \mathbf{s}_0$, see Ghiggini [9] and Plamenevskaya [32].

Remark 2.7 The above statements hold true using both $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients. Ozsváth and Szabó defined twisted versions of their contact invariants in such a way that every weakly fillable contact structure admits a nontrivial twisted contact Ozsváth–Szabó invariant for some appropriate twisting. In this paper, however, we concentrate on *untwisted* invariants.

3 Auxiliary results

In this section we establish two auxiliary results which will be used in the proofs of the next section.

Suppose that $T^2\times[0,1]$ is embedded into a 3–manifold Y. Consider the tori $T_i = T^2\times\{t_i\}$ with $0<t_1<t_2<\ldots<t_k<0$ and for every $i=1,\ldots,k$ let $\{C_i^j\}_{j=1}^{s_i}\subset T_i$ be a finite collection of parallel and disjoint simple closed curves. Perform 3–dimensional Dehn surgery along each C_i^j with framing -1 with respect to the framing induced by the torus T_i, and call Y' the resulting 3–manifold. In the following D_C will denote the right–handed Dehn twist along the curve $C\subset T^2$ in the mapping class group Γ_1 of the torus T^2. To keep notation short, a Dehn twist along a curve C_i^j isotopic to C_i will be denoted by D_{C_i}.

Proposition 3.1 *The 3–manifold Y' is obtained from Y by cutting it along $T^2\times\{0\}$ and regluing via the diffeomorphism*

$$D_{C_k}^{s_k}\circ D_{C_{k-1}}^{s_{k-1}}\circ\ldots\circ D_{C_1}^{s_1}.$$

Proof It is an easy exercise to check that the surgery along each C_i^j results in cutting Y along $T\times\{t_i\}$ and regluing with the map $D_{C_i^j}$. To prove the statement we only need to check that performing all the surgeries is equivalent to cutting and regluing via the composition of diffeomorphisms in the order stated. In order to see this, modulo an easy induction argument it suffices to show that if F, G and H are closed, oriented surfaces and $\varphi\colon F\to G$, $\psi\colon G\to H$ are orientation–preserving diffeomorphisms, then the two quotients

$$\frac{(F\times[0,\frac13])\sqcup(G\times[\frac13,\frac23])\sqcup(H\times[\frac23,1])}{(x,\frac13)\sim(\varphi(x),\frac13),\ (y,\frac23)\sim(\psi(y),\frac23)}\quad\text{and}\quad\frac{(F\times[0,\frac23])\sqcup(H\times[\frac23,1])}{(x,\frac23)\sim((\psi\circ\varphi)(x),\frac23)}$$

are orientation–preserving diffeomorphic. In fact, an orientation–preserving diffeomorphism is induced by the map

$$(\mathrm{id}_F\times\mathrm{id}_{[0,\frac13]})\sqcup(\varphi^{-1}\times\mathrm{id}_{[\frac13,\frac23]})\sqcup(\mathrm{id}_H\times\mathrm{id}_{[\frac23,1]}).$$ □

Let us now fix an identification of $T^2 \times \{0\}$ with $\mathbb{R}^2/\mathbb{Z}^2$, and let a and b denote the linear curves with slopes 0 and ∞, respectively, obtained by mapping the coordinate axes of $\mathbb{R}^2$ to $\mathbb{R}^2/\mathbb{Z}^2$. For short, let us also denote by a and b the right–handed Dehn twists along the curves a and b. It is a well–known fact that a and b generate the mapping class group Γ_1 of the torus T^2, which has presentation

$$\Gamma_1 = \langle a, b \mid aba = bab,\ (ab)^6 = 1 \rangle.$$

Using the relation $aba = bab$ it easily follows from $(ab)^6 = 1$ that $(a^3b)^3 = (b^3a)^3 = 1$. Consider the element

$$\gamma = (a^3b)^3 b = a^3ba^3ba^3b^2 \tag{3–1}$$

in Γ_1 viewed as a product of six factors, each of which is a power of either a or b. Let $0 < t_1 < \cdots < t_6 < 1$, and consider simple closed curves $C_i \subset T^2 \times \{t_i\}$ with C_i isotopic to b for i odd and to a for i even. By adding the right number of parallel copies of the same curve on each torus $T^2 \times \{t_i\}$ we can ensure that the diffeomorphism associated via Proposition 3.1 to performing (-1)–surgery along each of the curves is the above elemenent $\gamma \in \Gamma_1$. Attach 4–dimensional 2–handles along the above knots with framing (-1) with respect to the surface framings induced by the tori $T^2 \times \{t_i\}$, and denote the resulting 4–dimensional cobordism built on Y by W.

Proposition 3.2 *The 4–dimensional cobordism W defined above satisfies $b_2^+(W) > 0$.*

Proof Consider the first six a–curves C_i and their corresponding 2–handles. By sliding each of the first five 2–handles over the next one it is easy to see that W contains the 4–manifold obtained by attaching 2–handles to a chain of five (-2)–framed unknots contained in a 3–ball inside Y. Now consider the four b–curves and slide their corresponding 2–handles in the same way. This gives a framed link still contained in a 3–ball inside Y with intersection graph given by Figure 5.

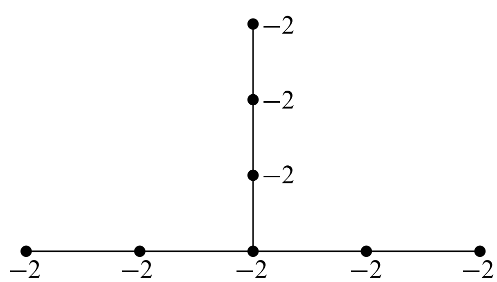


Figure 5: The 4–dimensional plumbing $P \subseteq W$

Viewing Figure 5 as representing a smooth 4–dimensional plumbing $P \subseteq W$, it is easy to check that the associated 8×8 intersection matrix has determinant -3, hence the matrix has at least one positive eigenvalue. This implies that $b_2^+(P) > 0$ and therefore $b_2^+(W) > 0$. □

4 Proofs of Theorems 1.4, 1.6, 1.7 and 1.8

Proof of Theorem 1.4 The assumption that (Y, ξ) has torsion at least 2 implies that there is a contact embedding $i\colon \mathbb{T}_2 \hookrightarrow (Y, \xi)$. Since the germ of a contact structure around a surface is determined by the induced characteristic foliation, for some small $\epsilon > 0$ there is a contact embedding

$$(T^2 \times [-\epsilon, 1+\epsilon], \ker(\cos(4\pi z)dx - \sin(4\pi z)dy)) \hookrightarrow (Y, \xi) \tag{4–1}$$

extending i above. Fix an identification of the torus T^2 with $\mathbb{R}^2/\mathbb{Z}^2$ such that the characteristic foliation induced on $T^2 \times \{0\}$ has slope ∞ and let a and b be simple closed curves on T^2 with slopes 0 and ∞, respectively. As before, by abuse of notation, we shall denote by a and b the elements of the mapping class group of the torus Γ_1 determined by positive Dehn twists around the curves a and b. Since in the group Γ_1 we have $(a^3b)^3 = (b^3a)^3 = 1$, it follows that

$$1 = (a^3b)^3(b^3a)^3 = a^3ba^3ba^3b^4ab^3ab^3a,$$

therefore

$$1 = a(a^3b)^3(b^3a)^3a^{-1} = a^4ba^3ba^3b^4ab^3ab^3. \tag{4–2}$$

Then conjugating the last word in (4–2) by b^n we get the word

$$(b^nab^{-n})^4b(b^nab^{-n})^3b(b^nab^{-n})^3b^4(b^nab^{-n})b^3(b^nab^{-n})b^3, \tag{4–3}$$

which is easily checked to be a composition of powers of Dehn twists along simple closed curves of slopes

$$-n, \infty, -n, \infty, -n, \infty, -n, \infty, -n, \infty. \tag{4–4}$$

If n is sufficiently large, by (4–1) we can locate inside (Y, ξ) embedded tori whose characteristic foliations are made of simple closed curves having slopes given by (4–4). In view of Proposition 3.1 we can perform Legendrian surgery on a suitable number of parallel Legendrian curves on such tori, so that the resulting smooth 3–manifold is obtained by cutting Y along T and regluing via a diffeomorphism whose isotopy class is specified by the word (4–3). But by construction this word represents $1 \in \Gamma_1$, therefore the resulting 3–manifold is Y again, so the construction yields a Stein cobordism W from (Y,ξ) to (Y,ξ') for some contact structure ξ'. Moreover, we know that $c(Y,\xi)=\widehat{F}_{\overline{W},\mathbf{s}_0}(c(Y,\xi'))$. Since the word (4–2) contains the word given by (3–1), by Proposition 3.2 we have $b_2^+(W)>0$. Therefore if Y is OSz–simple then Corollary 2.3 implies that the map $\widehat{F}_{\overline{W},\mathbf{s}_0}$ vanishes, verifying that $c(Y,\xi)=0$.

We shall now prove the second part of the statement assuming Theorem 1.9, which will be proved in Section 5. Suppose that $b_1(Y)\le 1$ and $\mathrm{Tor}(Y,\xi)>0$, that is, we have a contact embedding $\mathbb{T}_1\hookrightarrow(Y,\xi)$. In Section 5 we will show that one can build a Stein cobordism W_1 from (Y,ξ) to a contact 3–manifold (Y,ξ') using the word $(b^3a)^3$, in such a way that (Y,ξ) is obtained from (Y,ξ') by a Lutz modification. It follows by Colin [1] that the contact structures ξ and ξ' are homotopic as 2–plane fields, and therefore if $c(Y,\xi)\neq 0$ the two invariants $c(Y,\xi)$ and $c(Y,\xi')$ are elements of the same Ozsváth–Szabó group $\widehat{HF}_d(-Y,\mathbf{t}_\xi)$. Moreover, since Y is OSz–simple and $b_1(Y)\le 1$, this group is of rank 1 (see Ozsváth–Szabó [23, Definition 4.9] when $H_1(Y;\mathbb{Z})\cong\mathbb{Z}$, the remark on [23, page 250] in general, and also Lisca–Stipsicz [21, Proposition 2.2]). Thus, the restriction of the map $F_{\overline{W}_1,\mathbf{s}_0}$ to $\widehat{HF}_d(-Y,\mathbf{t}_\xi)$ is multiplication by some $k\in\mathbb{Z}$. Now observe that the smooth cobordism $W_2=W_1\circ W_1$ obtained by "composing" W_1 with itself can be constructed using the word $(b^3a)^6$, which contains (up to conjugation) the word given by (3–1). Therefore by Proposition 3.2 $b_2^+(W_2)>0$. As before, by Corollary 2.3 this implies $F^\infty_{\overline{W_2},\mathbf{s}}=0$ for any $\mathbf{s}\in\mathrm{Spin}^c(\overline{W_2})$, and so by Theorem 2.1

$$(F_{\overline{W_1},\mathbf{s}_0}\circ F_{\overline{W_1},\mathbf{s}_0})(c(Y,\xi'))=\sum\pm F_{\overline{W_2},\mathbf{s}}(c(Y,\xi'))=0,$$

and it follows that $k=0$. This shows that $c(Y,\xi)=F_{\overline{W_1},\mathbf{s}_0}(c(Y,\xi'))=0$, concluding the proof. □

Proof of Theorem 1.6 If the monodromy is trivial then Y is the 3–torus T^3. Suppose that ξ is a contact structure on T^3 with $\mathrm{Tor}(T^3,\xi)>0$. By the classification of tight contact structures on T^3 (see Kanda [20]), up to applying a diffeomorphism of T^3 we may assume that there is a contact embedding $\mathbb{T}_1\hookrightarrow(T^3,\xi)$ such that $T^2\times\{0\}\subseteq\mathbb{T}_1$ maps to $T^2\times\{s\}\subseteq T^2\times S^1=T^3$ for some $s\in S^1$. Fix an identification of $T^2\times\{s\}$ with $\mathbb{R}^2/\mathbb{Z}^2$, and denote, as before, by a and b, respectively, the right–handed Dehn twists along simple closed curves with slopes 0 and ∞. Arguing as in the proof of Theorem 1.4 we can use the word $(a^3b)^3b=b$ in the mapping class group to build a Stein cobordism W from (T^3,ξ) to (Y_1,ξ'), where Y_1 is a torus bundle over S^1 with monodromy b. By Proposition 2.4 the 3–manifold Y_1 is OSz–simple and by Proposition 3.2 we have $b_2^+(W)>0$, therefore by Corollary 2.3 it follows that $c(T^3,\xi)=F_{\overline{W},\mathbf{s}_0}(c(Y_1,\xi'))=0$.

The proof of the statement when $|\operatorname{tr}(A)| < 2$ or $|\operatorname{tr}(A)| = 2$ follows from Proposition 2.4 combined with Theorem 1.4.

Now suppose that $|\operatorname{tr}(A)| > 2$. In this case any incompressible torus is isotopic to the fiber of the fibration (see Hatcher [16, Lemma 2.7]). Let the monodromy of the fibration be denoted by $A \in SL_2(\mathbb{Z}) \cong \Gamma_1$ and fix a decomposition of A^{-1} into the product of right–handed Dehn twists. Since $\mathbb{T}_1$ contains tori with linear characteristic foliations of any rational slope, if $\mathbb{T}_n \hookrightarrow (Y, \xi)$ with n sufficiently large, by performing suitable Legendrian surgeries as before we can construct a Stein cobordism W_A from (Y, ξ) to (T^3, ξ_A) for some contact structure ξ_A. Moreover, up to choosing a larger n we may assume that $\operatorname{Tor}(T^3, \xi_A) > 0$, and therefore by the first part of the proof $c(T^3, \xi_A) = 0$. It follows that $c(Y, \xi) = F_{\overline{W_A}, \mathbf{s}_0}(c(T^3, \xi_A)) = 0$. Notice that in this way a bound for the optimal n_Y can be easily deduced from the decomposition of A^{-1} into the product of right–handed Dehn twists. (This bound is still far from the value $n_Y = 0$ predicted by Conjecture 1.2.) $\square$

Proof of Theorem 1.7 By, for example, [16, page 30], unless Y is an elliptic or parabolic torus bundle over S^1, an incompressible torus $T \hookrightarrow Y$ can be isotoped to be the union of regular fibers of the Seifert fibration. Therefore, in view of Theorem 1.6 we may assume that T consists of regular fibers. By assumption there is a contact embedding $\mathbb{T}_n \hookrightarrow (Y, \xi)$ with $n > 2$, and we can write $\mathbb{T}_n = \mathbb{T}_{n-1} \cup \mathbb{T}_1$. Since $\mathbb{T}_1$ contains tori with linear characteristic foliations with any rational slope, we may assume that for every integer $k \geq 0$ one of those tori contains k disjoint Legendrian knots $L_1, \ldots, L_k$ each of which is smoothly isotopic to a regular fiber of the fibration, and such that the contact framings and the framings induced by the torus (that is, by the fibration) coincide. Performing Legendrian surgeries along $L_1, \ldots, L_k$ gives a Stein cobordism W_L from (Y, ξ) to a contact Seifert fibered 3–manifold (Y', ξ') such that, when choosing k sufficiently large, Y' is a Seifert fibered 3–manifold over an orientable base with background Chern number sufficiently high. By Proposition 2.6 the 3–manifold Y' is OSz–simple (with $\mathbb{Z}/2\mathbb{Z}$–coefficients). By the construction we have $\mathbb{T}_{n-1} \hookrightarrow (Y', \xi')$ and by assumption $n - 1 > 1$, therefore Theorem 1.4 implies that $c(Y', \xi') = 0$. Thus $c(Y, \xi) = F_{\overline{W_L}, \mathbf{s}_0}(c(Y', \xi')) = 0$. $\square$

Proof of Theorem 1.8 In this proof we assume familiarity with the work of Ghiggini [6; 9]. Ghiggini considers a family $\{\zeta_i\}$ (denoted $\{\eta_i\}$ in [6; 9]) of contact structures on $-\Sigma_n$, where the index i varies in the set

$$\mathcal{P}_n = \{-n+1, -n+3, \ldots, n-3, n-1\}.$$

Let (M_0, ξ_1) denote the Stein fillable contact 3–manifold obtained by Legendrian surgery on the Legendrian right–handed trefoil with $\mathrm{tb} = +1$ in (S^3, ξ_{st}). Each contact structure ζ_i is constructed by performing Legendrian surgery along a Legendrian knot inside (M_0, ξ_1). Ghiggini also considers a different tight contact structure ξ_n on M_0, and defines a contact structure η_0 on $-\Sigma_n$ by Legendrian surgery along a Legendrian knot in (M_0, ξ_n). Denoting by $\overline{\xi}$ the contact structure ξ with reversed orientation, Ghiggini shows that $\overline{\zeta}_i$ is isotopic to ζ_{-i} for every $i \in \mathcal{P}_n$ and η_0 is isotopic to $\overline{\eta}_0$. All of the above holds regardless of the parity of n. Since the statement has been already proved in [6] for every even n, from now on we shall assume that n is odd. Arguing as in [6, Lemma 4.4] and [6, Proof of Theorem 2.4], it follows that

$$c^+(\eta_0) = \alpha_0 c^+(\zeta_0) + \sum_{i \in \mathcal{P} \setminus \{0\}} \alpha_i (c^+(\zeta_i) + c^+(\zeta_{-i})), \tag{4–5}$$

where we may assume $\alpha_i \in \{0, 1\}$ (it suffices to work with $\mathbb{Z}/2\mathbb{Z}$–coefficients). Recall that each of the Legendrian surgeries from (M_0, ξ_1) to $(-\Sigma_n, \zeta_i)$ as well as the Legendrian surgery from (M_0, ξ_n) to $(-\Sigma_n, \eta_0)$ induce the same smooth 4–dimensional cobordism V. Thus, arguing as in [6, Proof of Theorem 2.4] we have

$$F^+_{\overline{V}}(c^+(\zeta_i) + c^+(\zeta_{-i})) = 0 \pmod 2$$

for every $i \in \mathcal{P} \setminus \{0\}$. Therefore, in view of Equation (4–5) we have

$$\alpha_0 c^+(\xi_1) = \alpha_0 F^+_{\overline{V}}(c^+(\zeta_0)) = F^+_{\overline{V}}(c^+(\eta_0)) = c(\xi_n) = 0,$$

where the last equality follows from Theorem 1.6 because M_0 is a torus bundle with elliptic monodromy and by construction $\mathrm{Tor}(M_0, \xi_n) \geq n-1 > 1$ because $n \geq 3$. Since (M_0, ξ_1) is Stein fillable, we have $c^+(\xi_1) \neq 0$, therefore we conclude that $\alpha_0 = 0$, and from this point on the argument proceeds as in [6, Proof of Theorem 2.4]. □

5 Proof of Theorem 1.9

The proof of Theorem 1.4 relied on the construction of a particular cobordism W from Y to Y which, provided the contact structure ξ on Y had torsion $\mathrm{Tor}(Y, \xi) > 1$, also supported a Stein structure. The chosen Stein cobordism might seem to be *ad hoc*, but as we explain below, the contact surgery on (Y, ξ) corresponding to this Stein cobordism has a clear contact topological interpretation: it is the inverse of a Lutz modification.

In this section we shall assume familiarity with results, notation and terminology from Giroux [11] and Honda [17]. Let T^2 be a 2–torus with an identification $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Let $B_0 \cong T^2 \times [0, 1]$ and $B_1 \cong T^2 \times [0, 1]$ be basic slices of the same sign with boundary slopes respectively (s_0, s) and (s, s_1). Let B denote $B_0 \cup B_1$, the contact 3–manifold obtained by gluing together (via the identity map) B_0 and B_1 along their boundary components of slope s. Let $T \subset B$ be a minimal convex torus parallel to the boundary having slope s, and let $C \subset T$ be one of its Legendrian divides.

Lemma 5.1 *The contact 3–manifold B' obtained from B by Legendrian surgery along C is isomorphic to the contact 3–manifold $B_0 \cup B_1'$, where B_1' is a basic slice with the same sign as B_0 and boundary slopes $(s, D_C^{-1}(s_1))$.*

Proof It is easy to see that since B is the union of two basic slices, it is contactomorphic to a toric layer sitting inside a neighborhood of a Legendrian knot in the standard contact 3–sphere (S^3, ξ_{st}). It follows that B' contact embeds into a closed contact 3–manifold (Y, ζ) given by a Legendrian surgery on (S^3, ξ_{st}) in such a way that the image of any torus in B' parallel to the boundary bounds a solid torus in Y. Since (Y, ζ) is Stein fillable and hence tight, B' must be both tight and minimally twisting, otherwise one could easily find an overtwisted disk inside (Y, ζ). We can choose the identification $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ so that $s_0 = 1$ and $s = 0$. Then $s_1 = -\frac{1}{n}$ for some integer $n \geq 0$ and the action of D_C^{-1} on s_1 is determined by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n \\ -1 \end{pmatrix} = \begin{pmatrix} n+1 \\ -1 \end{pmatrix},$$

which shows that the boundary slopes of B' are 1 and $-\frac{1}{n+1}$, therefore B' decomposes as $B_0 \cup B_1'$. It remains to check that the signs of B_1' and B_0 are equal. Observe that $B_1 \subset \overline{B}_1$, where $\overline{B}_1$ is a basic slice with boundary slopes $(0, 1)$ and $B_0 \cup \overline{B}_1$ embeds in $(T^3, \ker(\cos(4\pi z)dx - \sin(4\pi z)dy))$, which is symplectically fillable (see Giroux [13]). By doing Legendrian surgery on $B_0 \cup \overline{B}_1$ along C and computing as before we see that $B' \subset B'' = B_0 \cup \overline{B}_1'$, where $\overline{B}_1'$ is a basic slice with boundary slopes $(0, \infty)$. On the other hand, B'' is tight, minimally twisting and has boundary slopes $(1, \infty)$, hence it is a basic slice as well. Therefore by Honda's Gluing Theorem [17, Section 4.7.4] the sign of $\overline{B}_1'$ must be the same as the sign of B_0. But the inclusion $B_1' \subset \overline{B}_1'$ implies, again by the Gluing Theorem, that the sign of B_1' must be the same as the sign of $\overline{B}_1'$. This concludes the proof. □

Suppose that $a, b \in \mathbb{R}$, $a < b$, and define

$$\mathbb{T}_n[a,b] = (T^2 \times [a,b], \ker(\cos(2\pi nz)dx - \sin(2\pi nz)dy)).$$

In this notation, we have $\mathbb{T}_n[0,1] = \mathbb{T}_n$, where $\mathbb{T}_n$ is defined in Section 1. Suppose that $a < c < b$, the characteristic foliation $\mathcal{F}$ on the torus $T^2 \times \{c\} \subset \mathbb{T}_n[a,b]$ is a union of simple closed curves, and let $C \subset T^2 \times \{c\}$ be such a closed curve. Then, there is a diffeomorphism

$$D_C \colon T^2 \times \{c\} \to T^2 \times \{c\}$$

representing the right–handed Dehn twist along C and such that $D_C(\mathcal{F}) = \mathcal{F}$.

Lemma 5.2 *The contact 3–manifold obtained from $\mathbb{T}_n[a,b]$ by Legendrian surgery along C is isomorphic to the contact 3–manifold*

$$\mathbb{T}_n[a,c] \cup_{D_C} \mathbb{T}_n[c,b]$$

obtained by gluing $\mathbb{T}_n[a,c]$ to $\mathbb{T}_n[c,b]$ via the diffeomorphism D_C.

Proof Suppose that the torus $T = T^2 \times \{c\}$ has slope s. Then T can be slightly perturbed to become a convex torus with minimal dividing set of slope s in such a way that a closed leaf C of the characteristic foliation becomes a Legendrian divide (see Ghiggini [7, Lemma 3.4]). We can choose $c_0 \in (a,c)$ and $c_1 \in (c,b)$ so that the tori $T_0 = T^2 \times \{c_0\}$ and $T_1 = T^2 \times \{c_1\}$ can be perturbed to minimal convex tori with boundary slopes s_0 and s_1, respectively, making sure at the same time that the resulting layers B_0 between T_0 and T and B_1 between T and T_1 are both basic slices. Since

$$\mathbb{T}_n[a,b] \subset (T^2 \times \mathbb{R}, \ker(\cos(2\pi nz)dx - \sin(2\pi nz)dy)),$$

using the Gluing Theorem as in the proof of Lemma 5.1 one can easily check that B_0 and B_1 must have the same sign. We have the decomposition

$$\mathbb{T}_n[a,b] = N_0 \cup B_0 \cup B_1 \cup N_1,$$

where each of N_0 and N_1 is a toric layer with only one convex boundary component. In view of Lemma 5.1, the result of Legendrian surgery along C can be decomposed as

$$N_0 \cup B_0 \cup B_1' \cup_{D_C} N_1,$$

where B_0 and B_1' have the same sign and B_1' is glued to N_1 via the diffeomorphism D_C. But it is easy to check that this is exactly the decomposition of the contact 3–manifold $\mathbb{T}_n[a,c] \cup_{D_C} \mathbb{T}_n[c,b]$ obtained by perturbing $T^2 \times \{c_0\}$, $T^2 \times \{c\}$ and $T^2 \times \{c_1\}$ to become minimal convex tori. □

Proof of Theorem 1.9 Arguing as in the proof of Theorem 1.4 we see that for some small $\epsilon > 0$ there is a contact embedding

$$(T^2 \times [-\epsilon, 1+\epsilon], \ker(\cos(2\pi nz)dx - \sin(2\pi nz)dy)) \hookrightarrow (Y, \xi)$$

extending $\mathbb{T}_n \hookrightarrow (Y,\xi)$. We can choose $\delta > 0$, $-\epsilon < -\delta < 0$, and an identification $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ so that the characteristic foliations on $T_{-\delta} = T^2 \times \{-\delta\}$ and $T_0 = T^2 \times \{0\}$ are made of simple closed curves and have slope, respectively, 0 and ∞. Then up to reparametrizing the interval [0, 1] we may assume that the characteristic foliations on

$T_{1/4} = T^2 \times \{1/4\}$, $T_{1/2} = T^2 \times \{1/2\}$ and $T_{3/4} = T^2 \times \{3/4\}$ are made of simple closed curves and have slopes, respectively, 0, ∞ and 0. Let

$$C_{-\delta} \subset T_{-\delta}, \quad C_{1/4} \subset T_{1/4}, \quad C_{3/4} \subset T_{3/4}$$

and

$$D_0^1, D_0^2, D_0^3 \subset T_0, \quad D_{1/2}^1, D_{1/2}^2, D_{1/2}^3 \subset T_{1/2}, \quad D_1^1, D_1^2, D_1^3 \subset T_1$$

be disjoint closed leaves of the respective characteristic foliations. Observe that if we perform Legendrian surgery on (Y, ξ) along (the images of) each of the curves C's and D's we obtain a contact 3–manifold of the form (Y, ξ'). In fact, each C–curve has slope 0, while each D–curve has slope ∞. Therefore, if we denote by A, respectively B, the corresponding Dehn twists up to isotopy, since in the mapping class group of the torus

$$B^3 A B^3 A B^3 A = (B^3 A)^3 = 1, \tag{5–1}$$

if follows from Proposition 3.1 that the 3–manifold underlying the result of the Legendrian surgeries is still Y. To see that (Y, ξ) is obtained from (Y, ξ') by a Lutz modification, observe that the tori $T_{-\delta}$, T_0, $T_{1/4}$, $T_{1/2}$, $T_{3/4}$ and T_1 induce, for some $\delta' > 0$, $\epsilon > \delta' > 0$, a decomposition

$$(T^2 \times [-\delta, 1+\delta'], \xi) = N_1 \cup N_2 \cup N_3 \cup N_4 \cup N_5 \cup N_6,$$

where the boundary components of N_6 have characteristic foliations of slopes (∞, n) for some $n \geq 1$. When we perform the Legendrian surgeries along D_1^1, D_1^2, D_1^3 and $C_{3/4}$, according to Lemma 5.2 the above decomposition becomes

$$N_1 \cup N_2 \cup N_3 \cup N_4 \cup_A N_5 \cup_{B^3} N_6.$$

Since

$$A^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

the boundary slopes of $N_1 \cup N_2 \cup N_3 \cup N_4 \cup_A N_5$ are $(0, -1)$. Similarly, after Legendrian surgery along $D_{1/2}^1$ we get $N_1 \cup N_2 \cup N_3 \cup_B N_4 \cup_A N_5$ with boundary slopes $(0, 0)$, and after Legendrian surgery along $D_{1/2}^2$ and $D_{1/2}^3$ we get, respectively, $N_1 \cup N_2 \cup N_3 \cup_{B^2} N_4 \cup_A N_5$ and $N_1 \cup N_2 \cup N_3 \cup_{B^3} N_4 \cup_A N_5$ with boundary slopes $(0, 1)$ and $(0, 2)$. It is easily checked that

$$N_3 \cup_{B^3} N_4 \cup_A N_5 = N_3 \cup_{B^3} N_4' = N_3 \cup N_4'',$$

where N_4' is a basic slice with boundary slopes $(\infty, -1)$ and N_4'' is a basic slice with boundary slopes $(\infty, 2)$. After Legendrian surgery along $C_{1/4}$ we get

$$N_2 \cup_A N_3 \cup N_4'' = N_2',$$

where N_2' is a basic slice with boundary slopes $(\infty, -2)$. Similarly,

$$N_1 \cup_{B^3} N_2' = N_1',$$

where N_1' is a basic slice with boundary slopes $(0, 1)$. Performing the remaining Legendrian surgery along $C_{-\delta}$ amounts to replacing N_1' back with N_1, which, in view of Equation (5–1), is glued via the identity map to the original N_6. Since $N_2 \cup N_3 \cup N_4 \cup N_5 \cong \mathbb{T}_1$, this concludes the proof. □

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