

Extending Johnson’s and Morita’s homomorphisms to the mapping class group

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We extend certain homomorphisms defined on the higher Torelli subgroups of the mapping class group to crossed homomorphisms defined on the entire mapping class group. In particular, for every $k \geq 2$, we construct a crossed homomorphism ϵ_k which extends Morita’s homomorphism $\tilde{\tau}_k$ to the entire mapping class group. From this crossed homomorphism we also obtain a crossed homomorphism extending the k th Johnson homomorphism τ_k to the mapping class group.

D Johnson and S Morita obtained their respective homomorphisms by considering the action of the mapping class group on the nilpotent truncations of the surface group; our approach is to mimic Morita’s construction topologically by using nilmanifolds associated to these truncations. This allows us to take the ranges of these crossed homomorphisms to be certain finite-dimensional real vector spaces associated to these nilmanifolds.

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1 Introduction

Let $\Sigma = \Sigma_{g,1}$ be a compact, connected, oriented smooth surface of genus $g \geq 3$ with a single boundary component and a basepoint $*$ on the boundary component. The mapping class group $Mod_{g,1}$ of Σ is the group $\text{Diff}(\Sigma, \partial\Sigma)$ of diffeomorphisms of Σ fixing $\partial\Sigma$ pointwise, modulo isotopies fixing $\partial\Sigma$ pointwise.

Let $\pi = \pi_1(\Sigma, *)$ be the fundamental group of Σ , which is a free group on $2g$ generators. The obvious action of $\text{Diff}(\Sigma, \partial\Sigma)$ on the based loops of Σ induces an action of $Mod_{g,1}$ on π . Consider the lower central series of π , given by $\pi^{(0)} = \pi$ and $\pi^{(k+1)} = [\pi, \pi^{(k)}]$. The k th nilpotent truncation of π is the group $\Gamma_k = \pi/\pi^{(k-1)}$. This is a nilpotent group of class $(k-1)$. Since the group $\pi^{(k)}$ is characteristic in π , the action of $Mod_{g,1}$ on π descends to an action on Γ_k for each k . So we have a representation $Mod_{g,1} \rightarrow \text{Aut}(\Gamma_k)$. The k th Torelli group $I_{g,1}(k)$ is the kernel of this representation. These groups have been studied by Johnson and others (see [12]). Andreadakis defined analogues of these groups in $\text{Aut}(\pi)$ in [1].

Let $H = H_1(\Sigma; \mathbb{Z})$ be the integral first homology group of the surface and let \mathcal{L}_{k+1} be the abelian group $\pi^{(k-1)}/\pi^{(k)}$. There is an exact sequence (see Morita [17, Proposition 2.3]):

$$0 \rightarrow \text{Hom}(H, \mathcal{L}_{k+1}) \rightarrow \text{Aut}(\Gamma_{k+1}) \rightarrow \text{Aut}(\Gamma_k) \rightarrow 1$$

The image of $I_{g,1}(k)$ under the representation $\text{Mod}_{g,1} \rightarrow \text{Aut}(\Gamma_{k+1})$ lies in the image of $\text{Hom}(H, \mathcal{L}_{k+1}) \hookrightarrow \text{Aut}(\Gamma_{k+1})$. So by restriction, we have a homomorphism $\tau_{g,1}(k)$ for any $k \geq 2$:

$$\tau_{g,1}(k): I_{g,1}(k) \rightarrow \text{Hom}(H, \mathcal{L}_{k+1})$$

This is called the k th Johnson homomorphism. It was defined by Johnson in [11].

Consider the standard (bar) group cohomology chains $C_*(\pi)$ of π . The action of $\text{Mod}_{g,1}$ on π induces an action on $C_*(\pi)$. The abelian group $C_1(\pi)$ is the free abelian group with generating set π , so we have an element $[\ell] \in C_1(\pi)$, where $\ell \in \pi$ is the class of the boundary loop in Σ . We denote the normal closure of $\langle \ell \rangle$ in π by $\langle \bar{\ell} \rangle$. Select a 2-chain $C \in C_2(\pi)$ such that $\partial C = -[\ell]$ and the image of C in $C_2(\pi/\langle \bar{\ell} \rangle)$ is a cycle representing the fundamental class of $\pi/\langle \bar{\ell} \rangle$. For $[\phi] \in \text{Mod}_{g,1}$, the chain $[\phi] \cdot C - C$ is in $Z_2(\pi)$ because $[\ell]$ is fixed by $[\phi]$, and it is in $B_2(\pi)$ because $H_2(\pi) = 0$. Pick a chain $D \in C_3(\pi)$ that bounds $[\phi] \cdot C - C$. Suppose further that $[\phi] \in I_{g,1}(k)$. Then $[\phi]$ fixes the image of C in $C_3(\Gamma_k)$ and the image of D in $C_3(\Gamma_k)$ is a cycle. The k th Morita homomorphism $\tilde{\tau}_{g,1}(k)$ is the map taking $[\phi]$ to the homology class of this cycle:

$$\tilde{\tau}_{g,1}(k): I_{g,1}(k) \rightarrow H_3(\Gamma_k)$$

Morita proved that this is a well-defined homomorphism in [16, Theorem 3.1], where he also proved that it refines $\tau_{g,1}(k)$ (see Theorem 5.1 below). When the genus is suppressed, we use the notation τ_k for $\tau_{g,1}(k)$ and $\tilde{\tau}_k$ for $\tilde{\tau}_{g,1}(k)$.

Since π is a free group, each Γ_k is a finitely generated, torsion-free nilpotent group, and is therefore, by a theorem of Mal'cev, a lattice in a unique simply-connected nilpotent Lie group G_k over the reals. Let $X_k = G_k/\Gamma_k$, the homogeneous space of Γ_k , and let \mathfrak{g}_k be the Lie algebra of G_k . The standard Lie algebra chain complex $C_*(\mathfrak{g}_k)$ on \mathfrak{g}_k is a chain complex with chains $C_n(\mathfrak{g}_k)$ equal to the exterior power $\Lambda^n \mathfrak{g}_k$ and a boundary determined by the Lie bracket. Note that the vector spaces $C_n(\mathfrak{g}_k)$ are finite-dimensional for every n and k . By a theorem of Nomizu [20], the homology $H_*(\mathfrak{g}_k)$ of this complex is isomorphic to $H_*(X_k; \mathbb{R})$. This is explained in Section 2 below.

A *crossed homomorphism* from a group G to a G -module M is a map $G \rightarrow M$ satisfying a product rule that is twisted by the action of G on M :

$$f(gh) = g \cdot f(h) + f(g)$$

This is the same thing as a group cohomology 1-cocycle in $Z^1(G; M)$, so we may speak of the cohomology class of a crossed homomorphism in $H^1(G; M)$.

The issue of extending Johnson's homomorphisms is first addressed in Morita's paper [17], where he defines a crossed homomorphism extending $\tau_{g,1}(2)$. Later, Morita constructed an extension of $\tau_{g,1}(3)$ in [18], and Perron in [21] built an extension of $\tau_{g,1}(3)$ by different methods. Hain showed in [6, Section 14.6] that $\tau_{g,1}(k)$ extends to a representation of $Mod_{g,1}$ in a semi-direct product of a unipotent algebraic group over \mathbb{Q} with $Sp_{2g}(\mathbb{Q})$. We note that while semi-direct products and crossed homomorphisms are related, the results of this paper do not appear to follow directly from Hain's results. Kawazumi constructed a different class of maps in [13] that extend the higher Johnson homomorphisms to $Aut(\pi)$, and showed in [14] that these maps are not crossed homomorphisms.

Main Theorem A *Fix $k \geq 2$. There is an action of $Mod_{g,1}$ on the finite-dimensional real vector space $C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$, and with respect to this action, there is a crossed homomorphism*

$$\epsilon_k: Mod_{g,1} \rightarrow C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$$

that extends the Morita homomorphism:

$$\begin{array}{ccc} I_{g,1}(k) & \xrightarrow{\tilde{\tau}_{g,1}(k)} & H_3(\Gamma_k) \\ \downarrow & & \downarrow \\ Mod_{g,1} & \xrightarrow{\epsilon_k} & C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k) \end{array}$$

The unlabeled map on the right is the composition:

$$H_3(\Gamma_k) \xrightarrow{\cong} H_3(X_k; \mathbb{Z}) \hookrightarrow H_3(X_k; \mathbb{R}) \xrightarrow{\cong} H_3(\mathfrak{g}_k) \hookrightarrow C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$$

We construct this crossed homomorphism in Section 3.2 below. The construction is a generalization of a topological construction for $\tau_{g,1}(2)$ (see Johnson [11] or Hain [5]) and is similar to a different topological construction of $\tilde{\tau}_{g,1}(k)$ due to A Heap (see [8, Theorem 4] and [8, Theorem 22]).

The projection $\Gamma_{k+1} \rightarrow \Gamma_k$ induces $G_{k+1} \rightarrow G_k$ on Mal'cev completions, which then induces $\mathfrak{g}_{k+1} \rightarrow \mathfrak{g}_k$ on Lie algebras. Let \mathfrak{l}_{k+1} be the kernel of $\mathfrak{g}_{k+1} \rightarrow \mathfrak{g}_k$, and let $\mathfrak{g}_{k+1}^{(1)}$ denote the commutator subalgebra of \mathfrak{g}_{k+1} . From Main Theorem A, we obtain

Main Theorem B Fix $k \geq 2$. There is a $Mod_{g,1}$ -equivariant map

$$\tilde{d}^2: C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k) \rightarrow C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$$

such that the crossed homomorphism

$$\tilde{d}^2 \circ \epsilon_k: Mod_{g,1} \rightarrow C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$$

extends the Johnson homomorphism:

$$\begin{array}{ccc} I_{g,1}(k) & \xrightarrow{\tau_{g,1}(k)} & \text{Hom}(H, \mathcal{L}_{k+1}) \\ \downarrow & & \downarrow \\ Mod_{g,1} & \xrightarrow{\tilde{d}^2 \circ \epsilon_k} & C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1}) \end{array}$$

The inclusion on the right is explained in the proof of the theorem.

Remark It is a fact of group cohomology theory (see Brown [2, chapter III.5]) that a crossed homomorphism on a subgroup may be extended to the entire group by expanding the target module to a larger module, called a co-induced module. As an abelian group, the co-induced module of $H_3(\Gamma_k)$ from $I_{g,1}(k)$ to $Mod_{g,1}$ is a direct sum of copies of $H_3(\Gamma_k)$ indexed over the cosets of $I_{g,1}(k)$ in $Mod_{g,1}$. Our approach finds a crossed homomorphism extending $\tilde{\tau}_{g,1}(k)$ with range in a finite-dimensional vector space, while the co-induced module approach would make the range an infinite-rank $Mod_{g,1}$ -module with no additional structure.

Remark As is common for crossed homomorphisms, the definitions of these crossed homomorphisms involve an essential choice. Varying this choice changes their values on mapping classes outside of $I_{g,1}(k)$, but it does not change the cohomology classes

$$[\epsilon_k] \in H^1(Mod_{g,1}; C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k))$$

and
$$[\tilde{d}^2 \circ \epsilon_k] \in H^1(Mod_{g,1}; C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})).$$

This is explained in [Proposition 3.2](#) below.

As an application of [Main Theorem A](#) and [Main Theorem B](#), we obtain the following Corollary:

Main Corollary A We have the following injections for all $k \geq 2$:

$$\begin{aligned}
 & \text{Mod}_{g,1}/I_{g,1}(2k-1) \hookrightarrow (\text{Mod}_{g,1}/I_{g,1}(k)) \times (C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)) \\
 \text{by} \quad & \alpha I_{g,1}(2k-1) \mapsto (\alpha I_{g,1}(k), \epsilon_k(\alpha)) \\
 \text{and} \quad & \text{Mod}_{g,1}/I_{g,1}(k+1) \hookrightarrow (\text{Mod}_{g,1}/I_{g,1}(k)) \times (C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})) \\
 \text{by} \quad & \alpha I_{g,1}(k+1) \mapsto (\alpha I_{g,1}(k), \tilde{d}^2 \circ \epsilon_k(\alpha))
 \end{aligned}$$

The semi-direct products are taken with respect to actions of $\text{Mod}_{g,1}/I_{g,1}(k)$ that are induced from the usual actions of $\text{Mod}_{g,1}$ on the respective vector spaces.

Section 2 develops the ideas from the theory of Nilpotent Lie groups that are relevant to our construction. Section 3 contains the construction of ϵ_k and Section 4 completes the proof of Main Theorem A. Main Theorem B and Main Corollary A are proven in Section 5.

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2 Preliminaries about nilmanifolds

This section develops some ideas about nilmanifolds, including some consequences of Mal'cev's rigidity theory and some notions relating to the homology of compact homogeneous spaces of nilpotent Lie groups.

The 3-dimensional Heisenberg group is a good example to keep in mind throughout this section.

2.1 Homology of Lie algebras and nilpotent homogeneous spaces

One of the key ingredients in the construction of ϵ_k is the homology of Lie algebras. Koszul [15] is a classical reference for this theory; Weibel [23, chapter 7] is a modern one. All vector spaces and Lie algebras in this paper are defined over the reals.

Suppose that G is a Lie group with Lie algebra \mathfrak{g} . Let $X = G/\Gamma$ be the homogeneous space of G with respect to a lattice Γ in G . Let \mathfrak{g}^* be the dual vector space to \mathfrak{g} , i.e. the vector space of linear functionals on \mathfrak{g} . We use these to define the following graded vector spaces:

$$C_*(\mathfrak{g}) = \bigoplus_n \Lambda^n \mathfrak{g}, \quad \text{and} \quad C^*(\mathfrak{g}) = \bigoplus_n \Lambda^n \mathfrak{g}^*$$

The vector space $C_*(\mathfrak{g})$ has a degree-one boundary map ∂ given by

$$(2-1) \quad \partial(V_1 \wedge \dots \wedge V_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} [V_i, V_j] \wedge V_1 \wedge \dots \wedge \widehat{V}_i \wedge \dots \wedge \widehat{V}_j \wedge \dots \wedge V_n$$

where $V_1, \dots, V_n \in \mathfrak{g}$ and where the caret ($\widehat{}$) indicates omission.

By definition, we have an evaluation pairing between $C_n(\mathfrak{g})$ and $C^n(\mathfrak{g})$ for any n . The degree -1 differential d on $C^*(\mathfrak{g})$ is defined by

$$d\alpha(\bar{V}) = \alpha(\partial\bar{V})$$

where $\alpha \in C^n(\mathfrak{g})$ and $\bar{V} \in C_n(\mathfrak{g})$ for some n .

It is a straightforward consequence of the Jacobi identity that the pair $(C_*(\mathfrak{g}), \partial)$ is a chain complex and the pair $(C^*(\mathfrak{g}), d)$ is a cochain complex. A proof can be found in Koszul [15].

We use left-propagation to connect Lie algebras to Lie groups and homogeneous spaces. Let $C^*(G; \mathbb{R})$ and $C^*(X; \mathbb{R})$ be the De Rham cochain complexes on G and X respectively.

Definition 2.1 For $\alpha \in C^*(\mathfrak{g})$, define $L(\alpha)$ to be the unique differential form on G that is invariant under the left action of G and restricts to α at the identity e . This defines a map $L: C^*(\mathfrak{g}) \rightarrow C^*(G; \mathbb{R})$ called the *left-propagation map*.

For $\alpha \in C^*(\mathfrak{g})$, the differential form $L(\alpha)$ descends to a form on X , so we also get a left propagation map $L: C^*(\mathfrak{g}) \rightarrow C^*(X; \mathbb{R})$.

One can write down an explicit formula for the value of $L(\alpha)$ at a point g in G ; from this we deduce that L is well-defined.

The following theorem of Nomizu is from [20].

Theorem 2.1 (Nomizu) *If $X = G/\Gamma$ is the homogeneous space of a connected, simply-connected nilpotent Lie group G with respect to a lattice Γ and \mathfrak{g} is the Lie algebra of G , then the left-propagation map*

$$L: C^*(\mathfrak{g}) \hookrightarrow C^*(X; \mathbb{R})$$

induces an isomorphism

$$L_*: H^*(\mathfrak{g}) \xrightarrow{\cong} H^*(X; \mathbb{R})$$

on cohomology groups.

The goal of this subsection is to construct an adjoint to the map L_* . Let $C_*(X)$ and $C_*(X; \mathbb{R})$ denote the piecewise-smooth singular homology chains of X with coefficients in \mathbb{Z} and \mathbb{R} respectively.

Definition 2.2 For each n , the vector space $C_n(\mathfrak{g})$ is finite dimensional and is canonically identified with its double dual, $(C^n(\mathfrak{g}))^*$. We can therefore think of the transpose to L as a map:

$$L^*: (C^*(X; \mathbb{R}))^* \rightarrow C_*(\mathfrak{g})$$

Define

$$v: C_*(X; \mathbb{R}) \rightarrow C_*(\mathfrak{g})$$

to be the composition of L^* with the map $C_*(X; \mathbb{R}) \rightarrow (C^*(X; \mathbb{R}))^*$ given by

$$C \mapsto \left(\alpha \mapsto \int_C \alpha \right).$$

We can think of v as taking singular chains on X and smoothing them out into left-invariant multi-vector fields.

Claim 2.1 *The map v is a chain complex map.*

Proof We only need to prove that v respects the boundary maps. The map L^* obviously respects the boundary, and the integration map respects the boundary by Stokes's theorem. \square

Lemma 2.1 *For each n , for every $\alpha \in C^n(\mathfrak{g})$ and $C \in C_n(X; \mathbb{R})$, we have:*

$$(2-2) \quad \alpha(v(C)) = \int_C L(\alpha)$$

Proof This is immediate from the definition. \square

Lemma 2.2 *Suppose we have a basis B for $C_n(\mathfrak{g})$. This basis determines a dual basis B^* for $C^n(\mathfrak{g})$ and a map $()^*: B \rightarrow B^*$. This gives us the following coordinate formula for v_n :*

$$v_n(C) = \sum_{\bar{V} \in B} \left(\int_C \bar{V}^* \right) \bar{V}$$

where $C \in C_n(X; \mathbb{R})$.

Proof This is also immediate from the definition. \square

Corollary 2.1 *Suppose $X = G/\Gamma$ is a compact homogeneous space of a connected, simply-connected nilpotent Lie group G with respect to a discrete subgroup Γ and \mathfrak{g} is the Lie algebra of Γ . Then the map*

$$v: C_*(X; \mathbb{R}) \rightarrow C_*(\mathfrak{g})$$

induces an isomorphism

$$v_*: H_*(X; \mathbb{R}) \xrightarrow{\cong} H_*(\mathfrak{g})$$

on homology groups.

Proof Fix an n . For $\alpha \in Z^n(\mathfrak{g})$ and $C \in Z_n(X; \mathbb{R})$, [Equation \(2–2\)](#) applies and induces the following evaluation identity on homology and cohomology:

$$[\alpha]([v(C)]) = [L(\alpha)]([C])$$

This means that with respect to the evaluation pairings, v_* is adjoint to L_* . The hypotheses are the same as those for [Theorem 2.1](#), so we have that L_* is an isomorphism. The adjoint of an isomorphism is an isomorphism, so v_* is also an isomorphism. \square

2.2 Actions on nilmanifolds

The starting point of this subsection is Mal'cev rigidity theory, for which Raghunathan [\[22, chapter II\]](#) is a good reference. Throughout this subsection, Γ is finitely generated, torsion-free nilpotent group. The existence statement of [Theorem 2.2](#) follows from [\[22, Theorem 2.18\]](#) and the uniqueness statement of [Theorem 2.2](#) and [Theorem 2.3](#) follow from [\[22, Theorem 2.11\]](#).

Theorem 2.2 (Mal'cev) *There is a unique connected, simply-connected, nilpotent Lie group G such that $\Gamma \hookrightarrow G$ as a lattice. This Lie group is called the Mal'cev completion of Γ .*

Theorem 2.3 (Mal'cev) *Every map $\Gamma \rightarrow \Gamma'$ of torsion-free, nilpotent groups extends to a unique map of $G \rightarrow G'$ of their Mal'cev completions. In particular, if $\Gamma' = \Gamma$, this describes an injective homomorphism $\text{Aut}(\Gamma) \hookrightarrow \text{Aut}(G)$.*

Let G be the Mal'cev completion of Γ and consider the homogeneous space $X = G/\Gamma$. This is a manifold with the coset of the identity $[e]$ as a natural basepoint. Since torsion-free nilpotent Lie groups are aspherical, the space X is a $K(\Gamma, 1)$.

Proposition 2.1 *There is an action of $\text{Aut}(\Gamma)$ on $(X, [e])$, such that the induced action on the fundamental group is the usual action of $\text{Aut}(\Gamma)$ on Γ .*

Proof This follows immediately from [Theorem 2.3](#). □

The action of $\text{Aut}(\Gamma)$ on G induces an action on \mathfrak{g} by taking derivatives at the identity. The actions on \mathfrak{g} and $C_*(\mathfrak{g})$ are denoted with push-forward notation (e.g. $\phi_* \bar{V}$). The right actions on their duals are denoted with pull-back notation on the left (e.g. $\phi^* \alpha$).

Claim 2.2 *The left-propagation map $L: C^*(\mathfrak{g}) \rightarrow C^*(X; \mathbb{R})$ is $\text{Aut}(\Gamma)$ -equivariant.*

Proof One can show by a computation that the map

$$C^*(X; \mathbb{R}) \rightarrow \mathfrak{g}^*$$

by $\alpha \mapsto \alpha|_{[e]}$

is equivariant. The claim then follows from the definition of L . □

Claim 2.3 *The map $v: C_*(X; \mathbb{R}) \rightarrow C_*(\mathfrak{g})$ is $\text{Aut}(\Gamma)$ -equivariant.*

Proof This is immediate from the fact that L is equivariant and [Lemma 2.1](#). □

Proposition 2.2 *There is an $\text{Aut}(\Gamma)$ -equivariant map*

$$s: \Gamma \rightarrow C^\infty((S^1, *), (X, [e]))$$

*that is a section to the projection $C^\infty((S^1, *), (X, [e])) \rightarrow \pi_1(X) = \Gamma$.*

Here $C^\infty((S^1, *), (X, [e]))$ is the set of smooth, based loops of X .

Proof For $\gamma \in \Gamma$, let $f(\gamma): \mathbb{R} \rightarrow G$ be the one-parameter family in G such that $f(\gamma)(1) = \gamma$. This exists and is unique by [Theorem 2.3](#). The map $f(\gamma)$ descends to a map $s(\gamma): (S^1, *) \rightarrow (X, [e])$. The map $s(\gamma)$ is obviously smooth, giving us $s: \Gamma \rightarrow C^\infty((S^1, *), (X, [e]))$. Let $\rho: \Gamma \rightarrow \text{Aut}(G)$ denote the action of $\text{Aut}(\Gamma)$ on G . If $\phi \in \text{Aut}(\Gamma)$, then $\rho(\phi) \circ f(\gamma)$ is a one-parameter family with $\rho(\phi) \circ f(\gamma)(1) = \phi \cdot \gamma$. By uniqueness, $\rho(\phi) \circ f(\gamma) = f(\phi \cdot \gamma)$, so s is $\text{Aut}(\Gamma)$ -equivariant. The fact that s is a section to $C^\infty((S^1, *), (X, [e])) \rightarrow \Gamma$ follows immediately from [Proposition 2.1](#). □

3 The main construction

3.1 Preliminary definitions

Fix a $k \geq 2$. Since g is forever fixed and no comparisons to surfaces of other genera will be made, we will drop it from the notation. We recall the definitions of the following objects, which were mentioned in the introduction: the group $\Gamma_k = \pi/\pi^{(k-1)}$ is the class $(k-1)$ nilpotent truncation of the surface group π , the Lie group G_k is the Mal'cev completion of Γ_k , the Lie algebra \mathfrak{g}_k is the tangent space at the identity of G_k , and the space X_k is the homogeneous space G_k/Γ_k . Also recall that we have an action of Mod on Γ_k .

We obtain actions of Mod on G_k , \mathfrak{g}_k and X_k via $Mod \rightarrow \text{Aut}(\Gamma_k)$, since $\text{Aut}(\Gamma_k)$ acts on these same objects by [Theorem 2.3](#) and [Proposition 2.1](#). The induced action of Mod on $C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$ gives it the module structure we use to prove [Main Theorem A](#).

To continue, we need to define a space of maps that relate Σ and X_k . As the first step, we need to find an analogue of the boundary loop $\partial\Sigma$ in the space X_k .

Proposition 3.1 *If $p: \pi \rightarrow \Gamma_k$ is the canonical projection, and $[\partial\Sigma] \in \pi_1(\Sigma, *)$ is the boundary loop class, then there is a smooth representative ℓ_k of $p_*[\partial\Sigma] \in \pi_1(X_k, [e])$ that is pointwise fixed by the action of Mod on X_k .*

Proof By [Proposition 2.2](#), we have a section s from Γ_k to the space of based smooth loops in X_k , such that the image of s is invariant under the action of Mod . Since the boundary loop class $[\partial\Sigma]$ is fixed by the action of Mod , so is its image $p_*[\partial\Sigma]$ in Γ_k . So if we let ℓ_k be $s(p_*[\partial\Sigma])$, then ℓ_k is pointwise fixed. \square

Let ℓ_k be as defined by [Proposition 3.1](#). We also use the symbol ℓ_k to refer to the image $\ell_k(S^1)$. Fix a piecewise-smooth map

$$i_{\partial\Sigma}: (\partial\Sigma, *) \rightarrow (\ell_k, [e])$$

that sends $[\partial\Sigma]$ to $[\ell_k]$ at the fundamental group level.

Definition 3.1 Define S_k to be the space of all piecewise-smooth maps $\Sigma \rightarrow X_k$ that restrict to $i_{\partial\Sigma}$ on $\partial\Sigma$ and induce the canonical projection $\pi \rightarrow \Gamma_k$ on fundamental groups. We topologize S_k by giving it the compact–open topology.

For $\phi \in \text{Diff}(\Sigma, \partial\Sigma)$ and $i \in S_k$, we define $\phi \cdot i$ by the formula

$$(\phi \cdot i)(p) = \phi \cdot (i(\phi^{-1}(p)))$$

for $p \in \Sigma$. Here $\text{Diff}(\Sigma, \partial\Sigma)$ acts on X_k through its projection to Mod .

If $F: \Sigma \times [0, 1] \rightarrow X_k$ is a homotopy through maps in S_k , we define $\phi \cdot F$ by

$$(\phi \cdot F)_t = \phi \cdot F_t$$

for $t \in [0, 1]$.

Claim 3.1 *The action of $\text{Diff}(\Sigma, \partial\Sigma)$ on S_k is well-defined.*

Proof We only need to show that for $i \in S_k$ and $\phi \in \text{Diff}(\Sigma, \partial\Sigma)$, we have $\phi \cdot i \in S_k$. It follows from [Proposition 2.1](#) that $\phi \cdot i$ induces the canonical projection on fundamental groups, and it follows from [Proposition 3.1](#) that $(\phi \cdot i)|_{\partial\Sigma} = i|_{\partial\Sigma}$. \square

Lemma 3.1 *The space S_k is nonempty.*

Proof Let $h: \Sigma \rightarrow X_k$ be a smooth map inducing the canonical projection on fundamental groups. Such a map exists because Σ and X_k are both smooth, aspherical manifolds.

Since $h|_{\partial X}$ and $i|_{\partial\Sigma}$ induce the same map on fundamental groups, we have a homotopy $F: \partial\Sigma \times [0, 1] \rightarrow X_k$ from $i|_{\partial\Sigma}$ to $h|_{\partial X}$. By standard approximation theory (for example, see Hirsch [\[9\]](#)), we may assume F is smooth in the time direction and the fixed-time maps are smooth for time not equal to 0.

By using a boundary collar in Σ we can modify h to get a map in S_k . Let N be a tubular neighborhood of $\partial\Sigma$ in Σ . Let $\alpha: N \rightarrow \partial\Sigma \times [0, 1]$ be a diffeomorphism such that $\alpha(x) = (x, 0)$ for $x \in \partial\Sigma$. Let $\beta: \overline{\Sigma - N} \rightarrow \Sigma$ be a diffeomorphism such that $\alpha(x) = (\beta(x), 1)$ for $x \in \partial N - \partial\Sigma$. Let i be defined by:

$$i(x) = \begin{cases} h(\beta(x)) & \text{if } x \in \overline{\Sigma - N} \\ F(\alpha(x)) & \text{if } x \in N \end{cases}$$

The reader can easily verify that $i \in S_k$. \square

Lemma 3.2 *The space S_k is contractible.*

Proof Let $\widetilde{\Sigma}$ be the universal cover of Σ and pick a basepoint $* \in \widetilde{\Sigma}$ that projects to $* \in \Sigma$. Let \widetilde{S}_k be

$$\widetilde{S}_k = \{ \tilde{i}: \widetilde{\Sigma} \rightarrow G_k \mid \tilde{i}(*) = e \text{ and } \tilde{i} \text{ is a lift of some } i \in S_k \}$$

with the compact–open topology.

Fix a map $\tilde{t} \in \tilde{S}_k$. Define $\tilde{F}: \tilde{S}_k \times [0, 1] \rightarrow \tilde{S}_k$ by

$$\tilde{F}(\tilde{j}, t)(\tilde{x}) = \tilde{j}(\tilde{x})(\tilde{j}(\tilde{x})^{-1}\tilde{t}(\tilde{x}))^t$$

where $\tilde{j} \in \tilde{S}_k$, $t \in [0, 1]$, and $\tilde{x} \in \tilde{\Sigma}$. Here the exponent is given by the Lie group logarithm and exponential maps.

Let $p: \pi \rightarrow \Gamma_k$ denote the projection. Since the maps in S_k induce p on fundamental groups, we have:

$$\tilde{j}(\alpha \cdot \tilde{x}) = p(\alpha)\tilde{j}(\tilde{x})$$

for any $\tilde{j} \in \tilde{S}_k$, $\tilde{x} \in \tilde{\Sigma}$, and $\alpha \in \pi$.

The homotopy $\tilde{F}(\tilde{j}, t)$ obeys the same rule:

$$\begin{aligned} \tilde{F}(\tilde{j}, t)(\alpha \cdot \tilde{x}) &= \tilde{j}(\alpha\tilde{x})(\tilde{j}(\alpha\tilde{x})^{-1}\tilde{t}(\alpha\tilde{x}))^t \\ &= p(\alpha)\tilde{j}(\tilde{x})(\tilde{j}(\tilde{x})^{-1}p(\alpha)^{-1}p(\alpha)\tilde{t}(\tilde{x}))^t \\ &= p(\alpha)\tilde{F}(\tilde{j}, t)(\tilde{x}) \end{aligned}$$

Since $\tilde{F}(\tilde{j}, t)$ observes this rule, it descends to a continuous map $F(j, t): \Sigma \rightarrow X_k$. Here $j \in S_k$ is the result of projecting \tilde{j} down to S_k . This notation is justified since $F(j, 0) = j$ and F clearly varies continuously in j and t . Further, for each j , $(F(j, t))_* = p$ on fundamental groups. One can easily see that $\tilde{F}(\tilde{j}, t)$ is constant on $\partial\tilde{\Sigma}$ (with respect to t and \tilde{j}), so for all t and j , we have that $F(j, t)|_{\partial\Sigma}$ is indeed $i_{\partial\Sigma}$. So the map F is actually a map

$$F: S_k \times [0, 1] \rightarrow S_k$$

and $F(j, 1) = i$ for any j . So F is a contraction for S_k . □

3.2 The main construction

Now we are ready to give the definition of the crossed homomorphism extending the k th Johnson homomorphism.

At this point we fix a chain $T \in C_3(\Sigma \times [0, 1])$ representing the fundamental class of $\Sigma \times [0, 1]$ relative to $\partial(\Sigma \times [0, 1])$. This fundamental class is the one corresponding to the orientation of $\Sigma \times [0, 1]$ induced from the orientation on Σ .

The map $v: C_*(X_k; \mathbb{R}) \rightarrow C_*(\mathfrak{g}_k)$ is as defined in [Definition 2.2](#).

Definition 3.2 (The extension of the k th Morita homomorphism) For $i \in S_k$, the extended k th Morita map relative to i is the crossed homomorphism

$$\epsilon_{k,i}: \text{Mod} \rightarrow C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$$

defined by

$$[\phi] \mapsto [v((F_\phi)_*T)]$$

where ϕ is any representative of $[\phi]$ and the map $F_\phi: \Sigma \times [0, 1] \rightarrow X_k$ is any piecewise-smooth homotopy from i to $\phi \cdot i$ through maps in S_k .

The homotopy F_ϕ exists by Lemma 3.2. Although $\epsilon_{k,i}$ depends on the choice of $i \in S_k$, we will suppress this and write ϵ_k when there is no confusion. We proceed by proving that the value of ϵ_k is independent of the other choices in its definition, and then by proving that it is, in fact, a crossed homomorphism. The other choices in the definition are the choice of representative T of $[\Sigma \times [0, 1]]$, the choice of representative ϕ of $[\phi]$ and the choice of homotopy F_ϕ .

Lemma 3.3 Suppose M is a surface, and $f: M \rightarrow X_k$ is piecewise-smooth. Then for any chain $C \in C_3(M)$, we have $v(f_*C) = 0$. In particular, if N is a 3-manifold, $D \in C_3(N)$ and the piecewise-smooth map $F: N \rightarrow X_k$ factors through f , then $v(F_*D) = 0$.

Proof By the coordinate form in Lemma 2.2, the chain $v(f_*C)$ is a linear combination of basis elements, where the coefficients are integrals of differential 3-forms over f_*C . Since f is piecewise-smooth, we may pull back these differential 3-forms to M and compute the integrals there. Of course, every 3-form on M is degenerate, so $v(f_*C) = 0$. The lemma follows. \square

Lemma 3.4 Suppose $F: \Sigma \times [0, 1] \rightarrow X_k$ is a piecewise-smooth homotopy through elements of S_k . Then $[v(F_*T)] \in C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$ does not depend on the choice T of representative of the fundamental class of $\Sigma \times [0, 1]$ relative to its boundary. In particular, the value of $\epsilon_k([\phi])$ does not depend on the choice of representative of the fundamental class of $\Sigma \times [0, 1]$ relative to its boundary.

Proof Let T and T' represent the fundamental class of $\Sigma \times [0, 1]$ relative to its boundary. Then ∂T and $\partial T'$ represent the same class in $H_2(\partial(\Sigma \times [0, 1]))$. Pick a class $C \in C_3(\partial(\Sigma \times [0, 1]))$ with $\partial C = \partial T - \partial T'$. Then the chain $T - T' - C$ is a cycle in $Z_3(\Sigma \times [0, 1])$. Furthermore, since $H_3(\Sigma \times [0, 1]) = H_3(\Sigma) = 0$, we know $T - T' - C \in B_3(\Sigma \times [0, 1])$.

The chain C is supported on $\partial(\Sigma \times [0, 1])$, so by Lemma 3.3:

$$v(F_*C) = 0$$

The claim follows from:

$$v(F_*T) - v(F_*T') = v(F_*(T - T' - C)) \in v(B_3(X_k)) \subset B_3(\mathfrak{g}_k) \quad \square$$

Lemma 3.5 *The value of $\epsilon_k([\phi])$ does not depend on the choice of homotopy F_ϕ in its definition, for a fixed choice of representative ϕ of $[\phi]$.*

Proof Let $i' = \phi \cdot i$. Suppose that $F, F': \Sigma \times [0, 1] \rightarrow X_k$ are two homotopies through maps in S_k , both from i to i' . It follows from [Lemma 3.2](#) that there is a homotopy $K: \Sigma \times [0, 1]^2 \rightarrow X_k$ through maps in S_k , from F to F' relative to their endpoints.

We can pick a representative $C \in C_4(\Sigma \times [0, 1]^2)$ of the fundamental class of $\Sigma \times [0, 1]^2$ relative to its boundary such that:

$$\partial K_*C = F'_*T + \underline{i}'_*T - F_*T - \underline{i}_*T + (\underline{i}_{\partial\Sigma})_*(\partial T \times [0, 1]^2)$$

Here $\underline{i}, \underline{i}'$ and $\underline{i}_{\partial\Sigma}$ denote the constant homotopies and $\partial T \times [0, 1]^2$ is a representative of the fundamental class of $\partial\Sigma \times [0, 1]^2$ relative to boundary. By [Lemma 3.3](#):

$$v(\underline{i}'_*T) = v(\underline{i}_*T) = v((\underline{i}_{\partial\Sigma})_*(\partial T \times [0, 1]^2)) = 0$$

Then we have

$$v(F'_*T) - v(F_*T) = v(\partial K_*C) = \partial v(K_*C)$$

which is in $B_3(\mathfrak{g}_k)$. □

Lemma 3.6 *Let F and F' be piecewise-smooth homotopies through maps in S_k such that $F_1 = F'_0$. Let F'' be the concatenation of F with F' . Then F'' is a piecewise-smooth homotopy through maps in S_k and we have:*

$$[v(F_*T)] + [v(F'_*T)] = [v(F''_*T)]$$

Proof It is obvious that F'' is a piecewise-smooth homotopy through maps in S_k .

Choose a representative $T' \in C_3(\Sigma \times [0, 1])$ of the fundamental class relative to boundary, such that:

$$F''_*T' = F_*T + F'_*T$$

By [Lemma 3.4](#), we have:

$$[v(F''_*T)] = [v(F''_*T')]$$

The lemma follows. □

Lemma 3.7 *Suppose that F is a piecewise-smooth homotopy through maps in S_k and $\phi \in \text{Diff}(\Sigma, \partial\Sigma)$. Then:*

$$[v((\phi \cdot F)_*T)] = \phi \cdot [v(F_*T)]$$

Proof By the definitions of the actions, we have:

$$(\phi \cdot F)_* T = \phi \cdot (F_* \phi_*^{-1} T)$$

However, since ϕ is orientation preserving, $\phi_*^{-1} T$ is also a representative of the fundamental class of $\Sigma \times [0, 1]$ relative to the boundary. The lemma follows from [Lemma 3.4](#). \square

Claim 3.2 *The value of $\epsilon_k([\phi])$ does not depend on the choice of representative ϕ of $[\phi]$.*

Proof Suppose $\phi, \psi \in \text{Diff}(\Sigma, \partial\Sigma)$ are isotopic. Then there is a smooth homotopy K on Σ , relative to $*$, taking the identity map to $\phi\psi^{-1}$. Then $i \circ K$ is a smooth homotopy from i to $\phi\psi^{-1} \cdot i$ through maps in S_k . [Lemma 3.3](#) tells us $v((i \circ K)_* T) = 0$.

Let F_ψ be a piecewise-smooth homotopy from i to $\psi \cdot i$ through maps in S_k , and let F' be the concatenation of $i \circ K$ with $\phi\psi^{-1} \cdot F_\psi$. Then F' is a piecewise-smooth homotopy from i to $\phi \cdot i$, and by [Lemma 3.6](#) and [Lemma 3.7](#) we have:

$$[v(F'_* T)] = [v((i \circ K)_* T)] + [v((F_\psi)_* T)]$$

Since $v((i \circ K)_* T) = 0$, we get the same value for $\epsilon_k([\phi])$ using either representative. \square

Claim 3.3 *The map ϵ_k is a crossed homomorphism.*

Proof Let $\phi, \psi \in \text{Diff}(\Sigma, \partial\Sigma)$. Let F_ϕ, F_ψ be piecewise-smooth homotopies through maps in S_k from i to $\phi \cdot i$ and $\psi \cdot i$ respectively. Let F' be the concatenation of F_ϕ with $\phi \cdot F_\psi$. Then F' is a piecewise-smooth homotopy through maps in S_k from i to $\phi\psi \cdot i$. By [Lemma 3.6](#) and [Lemma 3.7](#), we have:

$$[v(F'_* T)] = [v((F_\phi)_* T)] + \phi \cdot [v((F_\psi)_* T)]$$

The claim follows immediately. \square

The previous claim completes the proof that ϵ_k is well-defined.

Proposition 3.2 *The cohomology class*

$$[\epsilon_k] = [\epsilon_{k,i}] \in H^1(\text{Mod}, C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k))$$

does not depend on the choice of $i \in S_k$, or on the choice of $i_{\partial\Sigma}$ in the definition of S_k .

Proof First we show the independence from the choice of $i \in S_k$. Suppose $i, i' \in S_k$. Let K be any smooth homotopy from i to i' through maps in S_k . For $\phi \in \text{Diff}(\Sigma, \partial\Sigma)$, let F'_ϕ be a piecewise-smooth homotopy from i' to $\phi \cdot i'$, through maps in S_k . Let F_K be the concatenation of K with F'_ϕ and then with the homotopy $\phi \cdot K$ travelled in reverse.

This F_K is a piecewise-smooth homotopy from i to $\phi \cdot i$ through maps in S_k . By the definition of ϵ_k , and [Lemma 3.6](#) and [Lemma 3.7](#):

$$\begin{aligned} \epsilon_{k,i}([\phi]) &= [v((F_K)_*T)] \\ &= [v(K_*T)] + [v((F'_\phi)_*T)] - [v((\phi \cdot K)_*T)] \\ &= \epsilon_{k,i'}([\phi]) - ([\phi] \cdot [v(K_*T)] - [v(K_*T)]) \\ &= \epsilon_{k,i'}([\phi]) - (\delta[v(K_*T)])([\phi]) \end{aligned}$$

Since $\delta[v(K_*T)] \in B^1(\text{Mod}, C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k))$, we have shown that $[\epsilon_k]$ does not depend on i if $i_{\partial\Sigma}$ is fixed.

Now suppose that $i_{\partial\Sigma}$ and $i'_{\partial\Sigma}: \partial\Sigma \rightarrow \ell_k$ are two different maps as in the definition of S_k , and suppose that S_k and S'_k are the spaces of maps they respectively define. Let $i \in S_k$ and let $K: \partial\Sigma \times [0, 1] \rightarrow \ell_k$ be a homotopy relative to $*$ from $i_{\partial\Sigma}$ to $i'_{\partial\Sigma}$. We can get an element $i' \in S'_k$ by extending i by K on a collar of the boundary of Σ . This is the same trick as in the proof of [Lemma 3.1](#), so we omit the details.

For $\phi \in \text{Diff}(\Sigma, \partial\Sigma)$, let F_ϕ be a piecewise-smooth homotopy through maps in S_k , with F_ϕ going from i to $\phi \cdot i$. Let $\underline{K}: \partial\Sigma \times [0, 1]^2 \rightarrow \ell_k$ be the constant homotopy from K to itself. Let N be a tubular neighborhood of $\partial\Sigma$. Let F'_ϕ be a piecewise-smooth homotopy from i' to $\phi \cdot i'$ through maps in S_k , such that $F'_\phi|_{N \times [0, 1]}$ factors through \underline{K} and $F'_\phi|_{\overline{\Sigma - N} \times [0, 1]}$ factors through F_ϕ . Let $T' \in C_3(\Sigma \times [0, 1])$ and $C \in C_3(\partial\Sigma \times [0, 1]^2)$ be representatives for the respective fundamental classes relative to boundary such that:

$$(F'_\phi)_*T' = (F_\phi)_*T + \underline{K}_*C$$

By [Lemma 3.3](#), we know $v(\underline{K}_*C) = 0$. Then we get the same value for $\epsilon_k([\phi])$ using $i_{\partial\Sigma}$ and i as we get using $i'_{\partial\Sigma}$ and i' . The proposition follows. \square

3.3 The main construction restricted to the k th Torelli group

The goal of this section is to show that the main construction works nicely with simplicial homology when we restrict to the k th Torelli group $\mathcal{I}(k)$.

Let C_Σ be a representative of the fundamental class of Σ relative to $\partial\Sigma$. By abuse of notation, let $C_\Sigma \times \{0\}$ and $C_\Sigma \times \{1\}$ represent the images of C_Σ under the identifications

of Σ with $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. Similarly, let $\partial C_\Sigma \times [0, 1]$ and $C_\Sigma \times [0, 1]$ be representatives of the fundamental classes of $\partial\Sigma \times [0, 1]$ and $\Sigma \times [0, 1]$ relative to boundaries, respectively, so that we have:

$$\partial(C_\Sigma \times [0, 1]) = \partial C_\Sigma \times [0, 1] + C_\Sigma \times \{1\} - C_\Sigma \times \{0\}$$

In this section $H_*(X_k)$ denotes the piecewise-smooth simplicial homology with integer coefficients.

Definition 3.3 The topological version of the k th Morita homomorphism is the map

$$\bar{\epsilon}_k: \mathcal{I}(k) \rightarrow H_3(X_k)$$

given by:

$$[\phi] \mapsto [(F_\phi)_*(C_\Sigma \times [0, 1]) - i_* D_\phi]$$

Here $i \in S_k$, ϕ is a representative of $[\phi]$, the map $F_\phi: \Sigma \times [0, 1] \rightarrow X_k$ is a piecewise-smooth homotopy from i to $\phi \cdot i$ through maps in S_k and the chain $D_\phi \in C_3(\Sigma)$ bounds the chain

$$\phi_*^{-1} C_\Sigma - C_\Sigma + (\text{proj}_1)_*(\partial C_\Sigma \times [0, 1])$$

where $\text{proj}_1: \partial\Sigma \times [0, 1] \rightarrow \partial\Sigma$ is the first coordinate projection.

Claim 3.4 The chain $(F_\phi)_*(C_\Sigma \times [0, 1]) - i_* D_\phi$ in Definition 3.3 exists and is a cycle.

Proof The chain D_ϕ exists because its specified boundary is a cycle in $Z_2(\Sigma)$ and $H_2(\Sigma) = 0$. The claim then follows from this computation:

$$\begin{aligned} \partial((F_\phi)_* C_\Sigma \times [0, 1]) &= (F_\phi)_*(\partial C_\Sigma \times [0, 1] + C_\Sigma \times \{1\} - C_\Sigma \times \{0\}) \\ &= i_*(\text{proj}_1)_*(\partial C_\Sigma \times [0, 1]) + i_* \phi_*^{-1} C_\Sigma - i_* C_\Sigma \\ &= \partial i_* D_\phi \end{aligned} \quad \square$$

Lemma 3.8 For any $i \in S_k$, the maps $v_* \circ \bar{\epsilon}_k$ and $\epsilon_{k,i}|_{\mathcal{I}(k)}$ are equal.

Proof Fix $\phi \in \text{Diff}(\Sigma, \partial\Sigma)$ with $[\phi] \in \mathcal{I}(k)$. We have $v(i_* D_\phi) = 0$ by Lemma 3.3. Then

$$v_*(\bar{\epsilon}_{k,i}([\phi])) = [v((F_\phi)_*(C_\Sigma \times [0, 1]))] = \epsilon_{k,i}([\phi])$$

where F_ϕ is as in definition of $\bar{\epsilon}_{k,i}$. □

Lemma 3.9 The map $H_3(X_k) \rightarrow H_3(X_k; \mathbb{R})$ is an injection.

Proof A theorem of Igusa and Orr [10, Theorem 5.9] states that $H_3(\Gamma_k)$ is torsion free. The claim follows. □

Claim 3.5 *The map $\bar{\epsilon}_k$ is a well-defined homomorphism and does not depend on the choices in its definition, including the choices of $i \in S_k$ and $i_{\partial\Sigma}$ in the definition of S_k .*

Proof By Lemma 3.9 and Corollary 2.1, the map $v_*: H_3(X_k) \rightarrow H_3(\mathfrak{g}_k)$ is an injection. Then by Lemma 3.8, and the fact that ϵ_k is well-defined, we see that $\bar{\epsilon}_k$ does not depend on any of the choices except possibly i and $i_{\partial\Sigma}$. The fact that v is equivariant implies that $\bar{\epsilon}_k$ is a crossed homomorphism, but the action of $\mathcal{I}(k)$ on $C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$ is trivial, making $\bar{\epsilon}_k$ a homomorphism. By Proposition 3.2, varying i or $i_{\partial\Sigma}$ only changes $\bar{\epsilon}_k$ by adding a coboundary, but $B^1(\mathcal{I}(k), C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k))$ is trivial. \square

Remark It is a difficult fact that third homology of X_k is isomorphic to the 3-dimensional relative (non-spin) bordism group $\Omega_3(\Gamma_k)$. In light of this fact, this $\bar{\epsilon}_k$ can be identified with a homomorphism on $\mathcal{I}(k)$ defined by Heap in [8]. We will not pursue this connection in the current paper.

4 Equivalence of $\bar{\epsilon}_k$ and Morita's homomorphism

4.1 Morita's homomorphism

We recall some ideas from group homology theory, for which Brown [2] is a reference. Let G be a group and let $F_*(G)$ and $C_*(G)$ denote the standard (bar) resolution and standard group chains, respectively. Of course, $H_*(G) = H_*(C_*(G))$. The G -complex $F_*(G)$ has a \mathbb{Z} -basis of elements of the form (g_0, \dots, g_n) . This descends to a basis for $C_*(G)$ of elements of the form $[(g_0, \dots, g_n)]$, where the brackets denote the co-invariance class. A homomorphism f from G to a group G' induces a map $f_*: F_*(G) \rightarrow F_*(G')$, given by $[(\gamma_0, \dots, \gamma_n)] \mapsto [(f(\gamma_0), \dots, f(\gamma_n))]$. This gives the action of $\text{Aut}(G)$ on $F_*(G)$, which induces the action on $C_*(G)$ and in turn induces the usual action on $H_*(G)$.

Now consider the complex $C_*(\pi)$. Since Mod injects in $\text{Aut}(\pi)$, we have Mod acting on $C_*(\pi)$. Let $\ell \in \pi$ be the class of the positive boundary loop in Σ . Then clearly, $[\ell] \in B_1(\pi)$ is fixed by the action of Mod . Let $p: \pi \rightarrow \Gamma_k$ be the projection. The map p and the induced map $p_*: C_*(\pi) \rightarrow C_*(\Gamma_k)$ are both obviously Mod -equivariant.

As mentioned in the introduction, we have a homomorphism called the k th Morita homomorphism. To reiterate, Morita's homomorphism

$$\tilde{\tau}_k: \mathcal{I}(k) \rightarrow H_3(\Gamma_k)$$

is defined by:

$$[\phi] \mapsto [p_* D_{[\phi]}]$$

Here the chain $D_{[\phi]} \in C_3(\pi)$ bounds $[\phi] \cdot C - C$, where $C \in C_2(\pi)$ is a chain bounding the element $-\ell \in B_1(\pi)$ and projecting to a representative the fundamental class of $\pi/\langle \bar{\ell} \rangle$.

It makes sense to talk about the homology class of $p_* D_{[\phi]}$ when $[\phi] \in \mathcal{I}(k)$; because $\mathcal{I}(k)$ acts trivially on Γ_k we have:

$$\partial p_* D_{[\phi]} = p_* \partial D_{[\phi]} = p_* [\phi] \cdot C - p_* C = [\phi] \cdot p_* C - p_* C = 0$$

For proofs that this definition does not depend on the choices involved and that it is a homomorphism, we refer the reader to Morita's paper [16, Theorem 3.1].

4.2 Equivalence of the homomorphisms

In this subsection we show:

Theorem 4.1 For each $k \geq 2$,

$$\tilde{\tau}_k = f \circ \bar{\epsilon}_k$$

where $f: H_3(X_k) \rightarrow H_3(\Gamma_k)$ is the canonical isomorphism.

In order to prove this, we find a chain equivalence between $C_*(X_k)$ and $C_*(\Gamma_k)$ and show that for each $[\phi] \in \mathcal{I}(k)$, this equivalence maps a representative of $\bar{\epsilon}_k([\phi])$ in $C_3(X_k)$ to a representative of $\tilde{\tau}_k([\phi])$ in $C_3(\Gamma_k)$.

We recall that a singular n -simplex $\Delta \in C_n(X)$ on a space X is a map from a fixed n -simplex (considered as a space) to X . We denote the vertices of our fixed n -simplex as v_0, \dots, v_n , so that the vertices of the n -simplex Δ are the points $\Delta(v_0), \dots, \Delta(v_n) \in X$.

Definition 4.1 Suppose the group G is the fundamental group of an aspherical space X , and $\tilde{X} \rightarrow X$ the universal cover of X . Suppose $A \subset \tilde{X}$ is a fundamental domain for the G action on \tilde{X} .

Let $c_A: \tilde{X} \rightarrow G$ to be the map sending \tilde{x} to the unique $g \in G$ with $\tilde{x} \in g \cdot A$. Let $\tilde{f}_A: C_*(\tilde{X}) \rightarrow F_*(G)$ be the G -complex map given by

$$\tilde{f}_A(\Delta) = (c_A(\Delta(v_0)), \dots, c_A(\Delta(v_n))),$$

for Δ an n -simplex in \tilde{X} . Let $f_A: C_*(X) \rightarrow C_*(G)$ be the chain complex map given by

$$f_A(C) = [\tilde{f}_A(\tilde{C})],$$

for $C \in C_*(X)$, where \tilde{C} is a lift of C to \tilde{X} .

Lemma 4.1 *The maps \tilde{f}_A and f_A are well-defined. Furthermore, both maps are homotopy equivalences.*

Proof We leave it to the reader to check \tilde{f}_A is a G -complex map and that f_A is a chain complex map. The map f_A does not depend on the choice of \tilde{C} because the G -action is lost in passing to $C_*(G)$.

The set of n -simplices $\Delta \in C_*(\tilde{X})$ such that $\Delta(v_0) \in A$ form a free $\mathbb{Z}G$ -basis of $C_n(\tilde{X})$. Since \tilde{X} is contractible, the G -complex $C_*(\tilde{X})$ is a free resolution of \mathbb{Z} . Since $C_*(\tilde{X})$ and $F_*(G)$ are both free resolutions of \mathbb{Z} and \tilde{f}_A preserves their respective augmentation maps $C_0(\tilde{X}) \rightarrow \mathbb{Z}$ and $C_0(G) \rightarrow \mathbb{Z}$, it follows from standard theory (see Brown [2, chapter I.7]) that \tilde{f}_A is a homotopy equivalence of G -complexes. It immediately follows that f_A is a homotopy equivalence of chain complexes. \square

Lemma 4.2 *Given the setup of Definition 4.1, suppose that $\tilde{x} \in A$ is a basepoint for \tilde{X} . Let $x \in X$ be the image of \tilde{x} under the covering projection. Suppose Δ_t for $t \in [0, 1]$ is a homotopy of a simplex of $C_*(X)$ in X , subject to the condition that*

$$\Delta_t(v_i) = x$$

for each vertex v_i , for $0 \leq t \leq 1$. Then $f_A(\Delta_t)$ is constant with respect to t .

Proof For any $i \in \{1, \dots, n\}$, we have an element $\gamma_i \in \pi_1(X, x) = G$ equal to the class of the edge of Δ_t from v_0 to v_i , considered as a loop in X . Since Δ_t and $\Delta_{t'}$ are homotopic relative to the vertices for all t and t' , these elements γ_i do not depend on t . For each t , let $\tilde{\Delta}_t$ be the lift of Δ_t to \tilde{X} with the $\tilde{\Delta}_t(v_0) = \tilde{x}$. Then $\tilde{\Delta}_t(v_i) = \gamma_i \cdot \tilde{x}$ and $c_A(\tilde{\Delta}_t(v_i)) = \gamma_i$ for all t . The lemma follows from the definition of f_A . \square

Lemma 4.3 *Suppose $i \in S_k$. Let $\tilde{\Sigma}$ be the universal cover of Σ . It is possible to find a fundamental domain A for the action of π on $\tilde{\Sigma}$, and a fundamental domain B for the action of Γ_k on G_k , such that the following diagram commutes:*

$$\begin{array}{ccc} C_*(\Sigma) & \xrightarrow{f_A} & C_*(\pi) \\ \downarrow i_* & & \downarrow p_* \\ C_*(X_k) & \xrightarrow{f_B} & C_*(\Gamma_k) \end{array}$$

Proof Let $\tilde{\gamma}: \tilde{\Sigma} \rightarrow G_k$ be a lift of i . If $\gamma \in \pi$ and $\tilde{x} \in \tilde{\Sigma}$, we have:

$$\tilde{\gamma}(\gamma \cdot \tilde{x}) = p(\gamma)\tilde{\gamma}(\tilde{x})$$

This follows from the definition of S_k . Since $\tilde{\tau}$ observes this rule, it descends to a map $\hat{\tau}: \tilde{\Sigma}/\ker p \rightarrow G_k$. Let $B \subset G_k$ be a fundamental domain for the action of Γ_k on G_k . Then $\hat{\tau}^{-1}(B)$ is a fundamental domain for the action of Γ_k on $\tilde{\Sigma}/\ker p$. Let $A' \subset \tilde{\Sigma}$ be a fundamental domain for the action of $\ker p$ on $\tilde{\Sigma}$. Set $A = A' \cap \tilde{\tau}^{-1}(B)$.

We claim that A is a fundamental domain for the action of π on $\tilde{\Sigma}$. If $\tilde{x} \in \tilde{\Sigma}$, then there is a unique $\gamma \in \Gamma_k$ with $\gamma\tilde{\tau}(\tilde{x}) \in B$. If $\tilde{\gamma} \in \pi$ is a lift of γ , then there is a unique $\alpha \in \ker p$ with $\alpha\tilde{\gamma}\tilde{x} \in A$. It follows that $\alpha\tilde{\gamma}$ is the unique element of π sending \tilde{x} into A . Since \tilde{x} was arbitrary, this means that A is a fundamental domain.

Now we show that the diagram commutes. Pick $\tilde{x} \in \tilde{\Sigma}$. Let $\alpha = c_A(\tilde{x})$. Then $\tilde{\tau}(\tilde{x}) \in \tilde{\tau}(\alpha \cdot A)$. We deduce that $\tilde{\tau}(\tilde{x}) \in p(\alpha)B$, so $c_B(\tilde{\tau}(\tilde{x})) = p(\alpha)$. We have shown:

$$c_B \circ \tilde{\tau} = p \circ c_A$$

It follows from [Definition 4.1](#) that the diagram commutes. □

At this point, we fix a representative $\ell \in Z_1(\partial\Sigma) \subset Z_1(\Sigma)$ of the boundary loop in Σ and a class $C_\Sigma \in C_2(\Sigma)$ with $\partial C_\Sigma = \ell$. We demand that the simplices that are summands of C_Σ have all of their vertices equal to the basepoint $*$ of Σ . The symbols $C_\Sigma \times [0, 1]$ and $\partial C_\Sigma \times [0, 1]$ have the same (abusive) meanings that they were given in the preamble to [Definition 3.3](#). We also fix a map $i_{\partial\Sigma}: \partial\Sigma \rightarrow X_k$, which determines the space S_k , and we fix a map $i \in S_k$ as in [Definition 3.3](#).

Lemma 4.4 *Suppose $[\phi] \in \mathcal{I}(k)$ and F is the chain*

$$F = (F_\phi)_*(C_\Sigma \times [0, 1]) \in C_3(X_k)$$

where $F_\phi: \Sigma \times [0, 1] \rightarrow X_k$ is a piecewise-smooth homotopy from i to $\phi \cdot i$ through maps in S_k as in [Definition 3.3](#). Suppose B is a fundamental domain for the action of Γ_k on X_k . Then $f_B(F)$ is a boundary.

Proof By definition, the chain C_Σ is a sum of simplices whose vertices are all equal to the basepoint $*$ of Σ . Since the homotopy F_ϕ is a homotopy relative to the basepoint $*$, the homotopy $F_{\phi,t}(\Delta)$ satisfies the conditions of [Lemma 4.2](#), where Δ varies over the summands of C_Σ . This means that $f_B(F_{\phi,t}(C_\Sigma))$ is constant with respect to t . So since $f_B(F)$ is the image of a constant homotopy, it is a boundary. □

Lemma 4.5 *Let A be a fundamental domain for the action of π on $\tilde{\Sigma}$. If ϕ is in $\text{Diff}(\Sigma, \partial\Sigma)$ and $[\phi]$ is its image in Mod , we have:*

$$f_A(\phi \cdot C_\Sigma) = [\phi] \cdot f_A(C_\Sigma)$$

Proof Again, we use the fact that the summands of C_Σ have all vertices equal to the basepoint $*$ of Σ . We restrict our attention to a single simplex Δ of C_Σ . We note that, as in the proof of [Lemma 4.2](#), the image of this simplex under f_A is the co-invariance class of the n -tuple of elements of π specified by the edges of Δ (considered as loops) from v_0 to v_i for $i = 0, \dots, n$. Since the action of ϕ on loops in Σ defines the action of $[\phi]$ on π , this proves the lemma. \square

Proof of Theorem 4.1 Let A and B be fundamental domains for the actions of π on $\widetilde{\Sigma}$ and Γ_k on X_k respectively, such that the conclusions of [Lemma 4.3](#) hold. Let $\phi \in \text{Diff}(\Sigma, \partial\Sigma)$ be an element such that $[\phi] \in \mathcal{I}(k)$. Let $D_\phi \in C_3(\Sigma)$ be some chain satisfying

$$\partial D_\phi = \phi_*^{-1} C_\Sigma - C_\Sigma + (\text{proj}_1)_*(\partial C_\Sigma \times [0, 1])$$

where $\text{proj}_1: \partial\Sigma \times [0, 1] \rightarrow \partial\Sigma$ is the first coordinate projection. Let $F_\phi: \Sigma \times [0, 1] \rightarrow X_k$ be a piecewise-smooth homotopy from i to $\phi \cdot i$ through S_k , and let

$$C_\phi = (F_\phi)_*(C_\Sigma \times [0, 1]) - i_* D_\phi$$

so that $\bar{\epsilon}_k([\phi]) = [C_\phi]$.

By [Lemma 4.3](#), we have:

$$\begin{aligned} f_B(C_\phi - (F_\phi)_*(C_\Sigma \times [0, 1])) &= -f_B(i_* D_\phi) \\ &= -p_* f_A(D_\phi) \end{aligned}$$

[Lemma 4.5](#) allows us to compute $\partial f_A(D_\phi)$:

$$\begin{aligned} \partial f_A(D_\phi) &= f_A(\partial D_\phi) \\ &= f_A(\phi \cdot C_\Sigma - C_\Sigma) \\ &= [\phi] \cdot f_A(C_\Sigma) - f_A(C_\Sigma) \end{aligned}$$

Then $f_B(C_\phi - (F_\phi)_*(C_\Sigma \times [0, 1]))$ is $p_*(-[\phi] \cdot f_A(C_\Sigma) + f_A(C_\Sigma))$ where $-f_A(C_\Sigma)$ bounds the reverse boundary loop element $-f_A(\ell)$. This fits the definition of $\tilde{\tau}_k$, so we have shown:

$$\tilde{\tau}_k([\phi]) = [f_B(C_\phi - (F_\phi)_*(C_\Sigma \times [0, 1]))]$$

Then by [Lemma 4.4](#), we are finished:

$$\tilde{\tau}_k([\phi]) = [f_B(C_\phi)] \quad \square$$

Final step in the proof of Main Theorem A By [Lemma 3.9](#), we know that the map $v_* \circ f_{B*}^{-1}: H_3(\Gamma_k) \hookrightarrow C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$ is indeed an injection. By [Lemma 3.8](#), we know that $\epsilon_k|_{\mathcal{I}(k)} = v_* \circ \bar{\epsilon}_k$ and by [Theorem 4.1](#), we know that $f_{B*}^{-1} \circ \tilde{\tau}_k = \bar{\epsilon}_k$. This proves the theorem. \square

5 Extending the higher Johnson homomorphisms

Recall from the introduction that $\mathcal{L}_{k+1} = \pi^{(k-1)}/\pi^{(k)}$. We have:

$$(5-1) \quad 1 \rightarrow \mathcal{L}_{k+1} \rightarrow \Gamma_{k+1} \rightarrow \Gamma_k \rightarrow 1$$

By taking Mal'cev completions (see [Theorem 2.3](#)) we have an exact sequence of Lie groups:

$$1 \rightarrow \mathcal{L}_{k+1} \otimes \mathbb{R} \rightarrow G_{k+1} \rightarrow G_k \rightarrow 1$$

Taking \mathfrak{l}_{k+1} to be the Lie algebra of $\mathcal{L}_{k+1} \otimes \mathbb{R}$, we get the following exact sequence of Lie algebras:

$$(5-2) \quad 0 \rightarrow \mathfrak{l}_{k+1} \rightarrow \mathfrak{g}_{k+1} \rightarrow \mathfrak{g}_k \rightarrow 0$$

The Hochschild–Serre spectral sequence for the homology of this exact sequence has a differential:

$$\begin{aligned} d^2: H_3(\mathfrak{g}_k) &\rightarrow H_1(\mathfrak{g}_k, H_1(\mathfrak{l}_{k+1})) \cong \mathfrak{g}_2 \otimes \mathfrak{l}_{k+1} \\ &\cong (\mathfrak{g}_{k+1} \wedge \mathfrak{l}_{k+1}) / (\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1}) \end{aligned}$$

Here $\mathfrak{g}_{k+1}^{(1)}$ is the commutator subalgebra of \mathfrak{g}_{k+1} , and $\mathfrak{g}_2 = H \otimes \mathbb{R}$ is the Lie algebra of $G_2 = H \otimes \mathbb{R}$. In short, one gets d^2 of a class $[c]$ in $H_3(\mathfrak{g}_k)$ by picking a representative c in $Z_3(\mathfrak{g}_k)$, lifting c to \tilde{c} in $C_3(\mathfrak{g}_{k+1})$, and taking the boundary of the lift. This boundary is in $\mathfrak{g}_{k+1} \wedge \mathfrak{l}_{k+1}$ and is well-defined up to elements of $\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1}$.

We start by extending this differential to a map on $C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$.

Definition 5.1 *The extended differential*

$$\tilde{d}^2: C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k) \rightarrow C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$$

is defined by

$$\tilde{d}^2([c]) = \partial\tilde{c} + \mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1}$$

where $[c] \in C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$, the chain $c \in C_3(\mathfrak{g}_k)$ is a representative of $[c]$, and the chain $\tilde{c} \in C_3(\mathfrak{g}_{k+1})$ is a lift of c .

Claim 5.1 *The extended differential \tilde{d}^2 is well defined and extends the differential $d^2: H_3(\mathfrak{g}_k) \rightarrow (\mathfrak{g}_{k+1} \wedge \mathfrak{l}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$.*

Proof The fact that \tilde{d}^2 extends d^2 is immediate from the definitions.

To see that \tilde{d}^2 is well-defined, we first show that it does not depend on the choice of lift. Suppose $c \in C_3(\mathfrak{g}_k)$ and that \tilde{c} and \tilde{c}' are lifts of c to $C_3(\mathfrak{g}_{k+1})$. Then $\tilde{c} - \tilde{c}'$ is

in the $\ker(C_3(\mathfrak{g}_{k+1}) \rightarrow C_3(\mathfrak{g}_k))$. This kernel is easily seen to be $\mathfrak{g}_{k+1} \wedge \mathfrak{g}_{k+1} \wedge \mathfrak{l}_{k+1}$. The formula for the boundary (Equation (2–1)) and the fact that \mathfrak{l}_{k+1} is central in \mathfrak{g}_{k+1} indicate that image of this kernel under the boundary is in $\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1}$. Therefore the choice of lift is irrelevant.

To show that the choice of representative of the class $[c] \in C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$ is irrelevant, we show that the group of boundaries maps to zero. Take $b \in B_3(\mathfrak{g}_k)$. Pick a class $c \in C_4(\mathfrak{g}_k)$ with $\partial c = b$. Pick a lift $\tilde{c} \in C_4(\mathfrak{g}_{k+1})$. Then $\tilde{b} = \partial \tilde{c} \in C_3(\mathfrak{g}_k)$ is a lift of b . Immediately, we get that $\partial \tilde{b} = \partial \partial \tilde{c} = 0$, which proves the claim. \square

The subspace $\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_k$ is clearly invariant under the action of Mod on $C_2(\mathfrak{g}_k)$. This gives us an induced action of Mod on $C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$.

Lemma 5.1 *The extended differential \tilde{d}^2 is Mod -equivariant.*

Proof This is immediate from the definition. \square

Let H^* denote $\text{Hom}(H, \mathbb{Z})$, the dual to H . Let $D: H \rightarrow H^*$ denote the symplectic duality map (this is the same as the Poincaré duality if we identify H^* with $H^1(\Sigma)$). We have a map

$$D \otimes \text{id}: H \otimes \mathcal{L}_k \rightarrow H^* \otimes \mathcal{L}_k \cong \text{Hom}(H, \mathcal{L}_k).$$

We also use d^2 to denote the differential

$$d^2: H_3(\Gamma_k) \rightarrow H_1(\Gamma_k, H_1(\mathcal{L}_{k+1})) \cong H \otimes \mathcal{L}_{k+1}$$

of the spectral sequence of Equation (5–1).

The following is a theorem of Morita [16, Theorem 3.1].

Theorem 5.1 *We have:*

$$\tau_k = (D \otimes \text{id}) \circ d^2 \circ \tilde{\tau}_k$$

Let T_{k+1} be the fiber of $[e]$ with respect to the map $X_{k+1} \rightarrow X_k$. This space is easily seen to be a torus with fundamental group \mathcal{L}_{k+1} . According to the theory of Section 2.1, we have a map $v: C_*(T_{k+1}; \mathbb{R}) \rightarrow C_*(\mathfrak{l}_k)$ inducing an isomorphism on homology.

Lemma 5.2 *There are homotopy equivalences*

$$f: C_*(\Gamma_k) \rightarrow C_*(X_k)$$

and

$$f': C_*(\mathcal{L}_{k+1}) \rightarrow C_*(T_{k+1})$$

such that the following diagram commutes:

$$\begin{array}{ccccc} H_3(\Gamma_k) & \xrightarrow{d^2} & H_1(\Gamma_k, H_1(\mathcal{L}_{k+1})) & \xrightarrow{\cong} & H_1(\Gamma_k) \otimes H_1(\mathcal{L}_{k+1}) \\ \downarrow (v \circ f)_* & & & & \downarrow (v \circ f)_* \otimes (v \circ f')_* \\ H_3(\mathfrak{g}_k) & \xrightarrow{d^2} & H_1(\mathfrak{g}_k, H_1(\mathfrak{l}_{k+1})) & \xrightarrow{\cong} & H_1(\mathfrak{g}_k) \otimes H_1(\mathfrak{l}_{k+1}), \end{array}$$

where the maps d^2 are the differentials from the Hochschild–Serre spectral sequences for the exact sequences in Equation (5–1) and Equation (5–2), respectively.

Proof We note that since the extensions in Equation (5–1) and Equation (5–2) are central, we can make the following identifications:

$$H_1(\mathfrak{g}_k; H_1(\mathfrak{l}_{k+1})) = H_1(\mathfrak{g}_k) \otimes H_1(\mathfrak{l}_{k+1})$$

and

$$H_1(\Gamma_k; H_1(\mathcal{L}_{k+1})) = H_1(\Gamma_k) \otimes H_1(\mathcal{L}_{k+1})$$

Consider the double complex:

$$C_{pq} = F_p(\Gamma_k) \otimes_{\Gamma_k} (F_q(\Gamma_{k+1})_{\mathcal{L}_{k+1}})$$

Here subscripts denote co-invariants and the boundary maps are induced from the usual ones for the bar resolutions $F_*(\Gamma_k)$ and $F_*(\Gamma_{k+1})$. This double complex gives rise to the Hochschild–Serre spectral sequence for Equation (5–1). See Brown [2, chapter VII.6] for proof.

Let $U\mathfrak{g}_k$ be the universal enveloping algebra of \mathfrak{g}_k . We consider a second double complex:

$$D_{pq} = (U\mathfrak{g}_k \otimes \Lambda^p \mathfrak{g}_k) \otimes_{U\mathfrak{g}_k} ((U\mathfrak{g}_{k+1} \otimes \Lambda^q \mathfrak{g}_{k+1})_{\mathfrak{l}_{k+1}})$$

Again, subscripts denote co-invariants. The boundary maps here are induced from the usual ones for the Chevalley–Eilenberg complexes $U\mathfrak{g}_k \otimes \Lambda^p \mathfrak{g}_k$ and $U\mathfrak{g}_{k+1} \otimes \Lambda^q \mathfrak{g}_{k+1}$. We note that the co-invariants of the Chevalley–Eilenberg complex are the standard Lie algebra chain complex with the usual boundary. See Weibel [23, chapter 7.7] for definitions. This double complex gives rise to the Hochschild–Serre spectral sequence for Equation (5–2). See Weibel [23, chapter 7.5] for proof.

Now let $h_k: C_*(\Gamma_k) \rightarrow C_*(X_k)$ be a homotopy equivalence (these exist by standard theory; see the proof to Lemma 4.1). We also demand that $h_{k+1}|_{C_*(\mathcal{L}_{k+1})}$ has its image in $C_*(T_{k+1})$. We define a map

$$f_{pq}: C_{pq} \rightarrow D_{pq}$$

by its action on a \mathbb{Z} -basis for C_{pq} :

$$f_{pq}: [(\gamma_0, \dots, \gamma_p) \otimes (\gamma'_0, \dots, \gamma'_q)] \\ \mapsto [(1 \otimes v(h_k([\gamma_0, \dots, \gamma_p])) \otimes (1 \otimes v(h_{k+1}([\gamma'_0, \dots, \gamma'_q])))$$

Here the square brackets denote the equivalence class represented by that element after taking co-invariants. The map f_{**} is well-defined because we pass to co-invariants before plugging into h_k and h_{k+1} . We note that f_{**} is a map of double complexes because h_n and v are chain maps and because the Chevalley–Eilenberg complex boundary map induces the usual Lie algebra chain boundary map. This means that f_{**} induces a map of spectral sequences and therefore:

$$d^2 \circ f_{30}^2 = f_{11}^2 \circ d^2: H_3(\Gamma_k) \rightarrow H_1(\mathfrak{g}_k) \otimes H_1(\mathfrak{l}_{k+1})$$

One can easily show that

$$(v \circ h_k)_* = f_{30}^2: H_3(\Gamma_k) \rightarrow H_3(\mathfrak{g}_k)$$

and that:

$$(v \circ h_k)_* \otimes (v \circ h_{k+1}|_{C_*(\mathcal{L}_{k+1})})_* = f_{11}^2: H_1(\Gamma_k) \otimes H_1(\mathcal{L}_{k+1}) \rightarrow H_1(\mathfrak{g}_k) \otimes H_1(\mathfrak{l}_{k+1})$$

This proves the lemma. □

Proof of Main Theorem B First of all, we know from [Lemma 5.1](#) that the map $\tilde{d}^2 \circ \epsilon_k$ is in fact a crossed homomorphism, since it is the composition of a crossed homomorphism and an equivariant map.

Let the map $\text{Hom}(H, \mathcal{L}_{k+1}) \hookrightarrow C_2(\mathfrak{g}_{k+1})/(\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$ be the following long composition of the maps. First is the symplectic duality isomorphism:

$$D^{-1} \otimes \text{id}: \text{Hom}(H, \mathcal{L}_{k+1}) \xrightarrow{\cong} H \otimes \mathcal{L}_{k+1}$$

Second is this isomorphism to a product of homology groups:

$$H \otimes \mathcal{L}_{k+1} \xrightarrow{\cong} H_1(X_k) \otimes H_1(T_{k+1})$$

Then we have this injection, from [Lemma 3.9](#):

$$H_1(X_k) \otimes H_1(T_{k+1}) \hookrightarrow H_1(X_k; \mathbb{R}) \otimes H_1(T_{k+1}; \mathbb{R})$$

This is followed by the dual Nomizu map on each part, which is an isomorphism by [Corollary 2.1](#):

$$v_* \otimes v_*: H_1(X_k; \mathbb{R}) \otimes H_1(T_{k+1}; \mathbb{R}) \xrightarrow{\cong} H_1(\mathfrak{g}_k) \otimes H_1(\mathfrak{l}_{k+1})$$

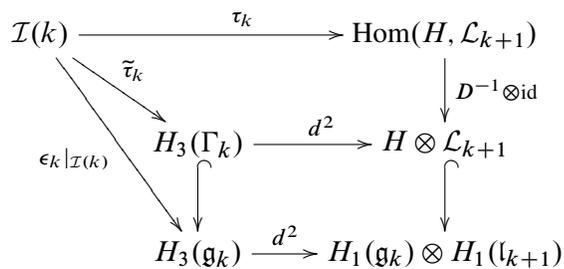


Figure 1: The diagram relating $d^2 \circ \epsilon_k$ and τ_k .

This is then isomorphic to the following vector space:

$$H_1(\mathfrak{g}_k) \otimes H_1(\mathfrak{l}_{k+1}) \xrightarrow{\cong} (\mathfrak{g}_{k+1} \wedge \mathfrak{l}_{k+1}) / (\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$$

This vector space then injects in our desired target:

$$(\mathfrak{g}_{k+1} \wedge \mathfrak{l}_{k+1}) / (\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1}) \hookrightarrow C_2(\mathfrak{g}_{k+1}) / (\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$$

Now consider the diagram in Figure 1. The top quadrilateral commutes by Theorem 5.1, the side triangle commutes by Main Theorem A and the lower right square commutes by Lemma 5.2. By Lemma 3.8 we know that ϵ_k takes values in $H_3(\Gamma_k)$. Since we can see that $\tilde{d}^2 \circ \epsilon_k$ restricts to $d^2 \circ \epsilon_k$ on $\mathcal{I}(k)$, the above commuting diagram proves the theorem. □

For the proof of Main Corollary A, we need the following result of A Heap.

Theorem 5.2 (Heap [8, Corollary 23]) *The kernel of $\tilde{\tau}_k$ is $\mathcal{I}(2k - 1)$.*

Proof of Main Corollary A Since $\mathcal{I}(k)$ acts trivially on Γ_k and the other actions under consideration are induced from this action, we see that $\mathcal{I}(k)$ is in the kernel of the actions of Mod on $C_3(\mathfrak{g}_k) / B_3(\mathfrak{g}_k)$ and $C_2(\mathfrak{g}_{k+1}) / (\mathfrak{g}_{k+1}^{(1)} \wedge \mathfrak{l}_{k+1})$. This means that the semi-direct products in the statement of the corollary are well-defined. The map

$$Mod \hookrightarrow (Mod / \mathcal{I}(k)) \times (C_3(\mathfrak{g}_k) / B_3(\mathfrak{g}_k))$$

by
$$\alpha \mapsto (\alpha \mathcal{I}(k), \epsilon_k(\alpha))$$

is a homomorphism because the actions in the definitions of ϵ_k and the semi-direct product match. The kernel of this map is the kernel of $\tilde{\tau}_k$: if α maps to the trivial element, it must be in $\mathcal{I}(k)$, on which ϵ_k is equal to $\tilde{\tau}_k$. Since the kernel of $\tilde{\tau}_k$ is $\mathcal{I}(2k - 1)$, the first map in the statement of the theorem is the induced map and is a well-defined injection. The proof that the second map is a well-defined injection is similar. \square

6 Final remarks

Many questions arise concerning the best possible range for a crossed homomorphism extending a Johnson or Morita homomorphism to the mapping class group. In the cases that have been computed explicitly, by Morita in [17], and by Perron in [21], there are crossed homomorphisms extending $\tau_{g,1}(2)$ and $\tau_{g,1}(3)$, with ranges in the finite rank abelian groups $\frac{1}{2} \text{Hom}(H, \mathcal{L}_2)$ and $\frac{1}{8} \text{Hom}(H, \mathcal{L}_3)$, respectively. Although it is conceivable that the methods in this paper could be modified to give similar results for $\tilde{\tau}_k$ for arbitrary k , there are two difficulties to be overcome:

- the use of integration in these methods, and the choice of $i: \Sigma \rightarrow X_k$ would have to be carefully managed to ensure well-controlled, rational coefficients, and
- one can easily prove from the definitions that $\partial \circ \epsilon_k$ is a refinement of $\tilde{d}^2 \circ \epsilon_{k-1}$. This indicates that no choice of i will make ϵ_k take values in $H_3(\mathfrak{g}_k)$ for $k > 2$. In order to restrict the range of ϵ_k from $C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$ to $H_3(\mathfrak{g}_k)$, we will need to correct ϵ_k by a cocycle that projects to ϵ_{k-1} but restricts to the zero map on $\mathcal{I}(k)$. It is not at all clear whether such a cocycle should exist for $k > 3$.

There is an advantage to the use of integration in these methods; namely, that it leads to connections between algebraic invariants such as the Morita homomorphism and differential geometric invariants such as flux. In particular, we have used these methods to prove a pair of theorems connecting the extended flux homomorphism to ϵ_2 on a surface with an area form. We hope to present these results in a future paper. Of course, there may be further connections between such differential invariants and ϵ_k for higher k .

We note that the methods of this paper also work for the mapping class group of a closed surface $\Sigma_{g,*}$ with a basepoint. The nilmanifolds analogous to the X_k in this case are the higher Albanese manifolds of a Riemann surface, and each naturally has the structure of a \mathbb{C} -manifold. This is due to R Hain and S Zucker; see Hain [3, Definition 1.4] and [3, Remarks 1.5], or Hain and Zucker [7, Section 5]. There may be a version of ϵ_k over a closed surface that takes advantage of this additional structure. In the case $k = 2$, Hain gives a construction in [4] that seems to be related.

Finally, one can interpret a cocycle on the mapping class group as a topological cocycle in an appropriate local coefficient system on the moduli space of surfaces. R Penner and S Morita do this for Morita's extension of τ_2 in [19], but their methods do not seem to give a good interpretation for the cocycles presented in this paper. It seems natural to ask how one would develop the objects of this paper from such a perspective.

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