Pullbacks of generalized universal coverings

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It is known that there is a wide class of path-connected topological spaces X, which are not semilocally simply-connected but have a *generalized* universal covering, that is, a surjective map $p\colon \widetilde{X} \to X$ which is characterized by the usual unique lifting criterion and the fact that $\widetilde{X}$ is path-connected, locally path-connected and simply-connected.

For a path-connected topological space Y and a map $f\colon Y \to X$, we form the pullback $f^*p\colon f^*\widetilde{X} \to Y$ of such a generalized universal covering $p\colon \widetilde{X} \to X$ and consider the following question: given a path-component $\widetilde{Y}$ of $f^*\widetilde{X}$, when exactly is $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ a generalized universal covering? We show that the classical criterion, of $f_\#\colon \pi_1(Y) \to \pi_1(X)$ being injective, is too coarse a notion to be sufficient in this context and present its appropriate (necessary and sufficient) refinement.

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1 Introduction and preliminaries

We call a continuous function $p\colon \widetilde{X} \to X$, from a path-connected, locally path-connected and simply-connected topological space $\widetilde{X}$ onto a topological space X, a *generalized universal covering of* X if for every path-connected and locally path-connected topological space Z, for every continuous function $g\colon (Z,z) \to (X,x)$ with $g_\#(\pi_1(Z,z)) = 1$, and for every $\widetilde{x}$ in $\widetilde{X}$ with $p(\widetilde{x}) = x$, there exists a *unique* continuous lift $h\colon (Z,z) \to (\widetilde{X},\widetilde{x})$ with $p \circ h = g$.

A generalized universal covering of X, if it exists, is uniquely determined by these properties. Its group of covering transformations $\mathrm{Aut}(\widetilde{X} \overset{p}{\to} X)$ is isomorphic to $\pi_1(X,x_0)$ and it acts freely and transitively on every fiber $p^{-1}(\{x\})$ with $x \in X$.

The main result of Fischer and Zastrow [3] is that for a wide class of path-connected spaces X, which are not necessarily semilocally simply-connected and not necessarily locally path-connected, the generalized universal covering exists and can be built by the following standard construction: Fix a base point $x_0 \in X$ and let $\mathcal{P}(X,x_0)$ denote the set of all continuous paths $\alpha\colon [0,1] \to X$ such that $\alpha(0) = x_0$. On $\mathcal{P}(X,x_0)$, consider

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the equivalence relation given by $\alpha \sim \beta$ if and only if $\alpha(1) = \beta(1)$ and α is homotopic to β within X, relative to their common endpoints. Let $[\alpha]$ denote the equivalence class of α and let $\widetilde{X}$ denote the set of all such equivalence classes. Define $p\colon \widetilde{X} \to X$ by $p([\alpha]) = \alpha(1)$. For each $[\alpha] \in \widetilde{X}$ and each open subset U of X containing $\alpha(1)$, let $B([\alpha], U)$ denote the set of all $[\beta] \in \widetilde{X}$ for which there exists a continuous map $\gamma\colon [0,1] \to U$ such that $\gamma(0) = \alpha(1)$, $\gamma(1) = \beta(1)$ and $[\beta] = [\alpha \cdot \gamma]$; where $\alpha \cdot \gamma$ denotes the usual concatenation of the paths α and γ. Notice that $B([\alpha], X) = \widetilde{X}$ for all $[\alpha] \in \widetilde{X}$ and that if $[\beta] \in B([\alpha], U)$, then $B([\beta], U) = B([\alpha], U)$. Moreover, if $U \subseteq V$, then $B([\alpha], U) \subseteq B([\alpha], V)$. It follows that the collection of all such sets $B([\alpha], U)$ forms a basis for a topology on $\widetilde{X}$, which one employs.

The lift h of g is given by $h(w) = [\alpha \cdot (g \circ \tau)]$ where $\widetilde{x} = [\alpha]$ and $\tau\colon [0,1] \to Z$ is any path from $\tau(0) = z$ to $\tau(1) = w$.

The unique path lifting property of $p\colon \widetilde{X} \to X$ makes it necessary for X to be *homotopically Hausdorff*: for every $x \in X$, the only element of $\pi_1(X, x)$ which can be represented by arbitrarily small loops is the trivial element.

If X happens to be locally path-connected, then $p\colon \widetilde{X} \to X$ is open so that $\widetilde{X}/G$ is homeomorphic to X, where $G = \mathrm{Aut}(\widetilde{X} \overset{p}{\to} X)$. (In case X is locally path-connected and first countable, then the fact that a generalized universal covering of X must be an open map can already be deduced from the path lifting property.)

If X is locally path-connected and semilocally simply-connected, then the generalized universal covering agrees with the classical universal covering. However, while a generalized universal covering is, in particular, a Serre fibration with unique path lifting, it distinguishes itself from a classical covering most notably in that it need not be a Hurewicz fibration.

Spaces which allow for a generalized universal covering, constructed in this manner, include all path-connected 1-dimensional continua, all path-connected planar sets and certain trees of manifolds, including certain Coxeter group boundaries.

We refer the reader to Fischer and Zastrow [3] for more information on generalized universal coverings.

General assumptions Let (X, x_0) be a path-connected topological space such that $p\colon \widetilde{X} \to X$, as constructed above, is a generalized universal covering and let $f\colon Y \to X$ be a continuous map from a path-connected topological space Y.

Consider the *pullback diagram*

$$\begin{array}{ccc} f^*\widetilde{X} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{X} \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\ f\ } & X \end{array}$$

where $f^*\widetilde{X} = \{(y, \widetilde{x}) \in Y \times \widetilde{X} \mid f(y) = p(\widetilde{x})\} \subseteq Y \times \widetilde{X}$, $f^*p\colon f^*\widetilde{X} \to Y$ is given by $f^*p(y, \widetilde{x}) = y$ and $\widetilde{f}\colon f^*\widetilde{X} \to \widetilde{X}$ is given by $\widetilde{f}(y, \widetilde{x}) = \widetilde{x}$. (Recall that $f^*\widetilde{X}$ is uniquely characterized by its universal property: given any space Z and maps $g\colon Z \to Y$ and $h\colon Z \to \widetilde{X}$ such that $f \circ g = p \circ h$, there is a unique map $q\colon Z \to f^*\widetilde{X}$ such that $(f^*p) \circ q = g$ and $\widetilde{f} \circ q = h$.)

The pullback has the following easily verified but important classical property: *If $p\colon \widetilde{X} \to X$ is a classical universal covering of X and if $\widetilde{Y}$ is a path-component of $f^*\widetilde{X}$, then $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ is a classical covering, where $\widetilde{Y}$ is simply-connected if and only if $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective; rendering the classical universal covering of a locally path-connected Y as a so-called "fibered product"* (see Spanier [4]).

We thank Professor Kazuhiro Kawamura for the following inspiring question, whose answer, given in Theorem 4.3 below, enables the appropriate use of pullback constructions in applications of the generalized theory, such as that found in Fischer [2].

Question Given the general assumptions stated above and a path-component $\widetilde{Y}$ of $f^*\widetilde{X}$, when exactly is $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ a generalized universal covering?

Of course, the following two facts are always true.

Lemma 1.1 *Any two path-components of $f^*\widetilde{X}$ are homeomorphic and induce equivalent maps $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$.*

Proof Let $\widetilde{Y}_1$ and $\widetilde{Y}_2$ be two path-components of $f^*\widetilde{X}$. Fix any $(y_i, \widetilde{x}_i) \in \widetilde{Y}_i$. Choose a path $\alpha\colon [0,1] \to Y$ from $\alpha(0) = y_1$ to $\alpha(1) = y_2$. Let $g\colon ([0,1], 0) \to (\widetilde{X}, \widetilde{x}_1)$ be the lift of $f \circ \alpha\colon ([0,1], 0) \to (X, f(y_1))$ with $p \circ g = f \circ \alpha$. Put $\widetilde{z} = g(1)$. Then $(\alpha, g)\colon [0,1] \to f^*\widetilde{X}$ is a path from $(y_1, \widetilde{x}_1)$ to $(y_2, \widetilde{z})$ and $p(\widetilde{z}) = f(y_2) = p(\widetilde{x}_2)$. Since $p\colon \widetilde{X} \to X$ is a generalized universal covering, there is a homeomorphism $h\colon \widetilde{X} \to \widetilde{X}$ such that $p = p \circ h$ and $h(\widetilde{z}) = \widetilde{x}_2$, inducing a homeomorphism $(\mathrm{id}, h)|_{f^*\widetilde{X}}\colon f^*\widetilde{X} \to f^*\widetilde{X}$. Since $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are path-components of $f^*\widetilde{X}$, this yields the desired homeomorphism $(\mathrm{id}, h)|_{\widetilde{Y}_1}\colon \widetilde{Y}_1 \to \widetilde{Y}_2$ with $f^*p = f^*p \circ (\mathrm{id}, h)$. □

Remark 1.2 The number of path-components of $f^*\widetilde{X}$ is equal to the index of $f_\#(\pi_1(Y))$ in $\pi_1(X)$, as a standard calculation shows: first note that, for a fixed $(y,\tilde{x}) \in f^*\widetilde{X}$, an explicit isomorphism Ψ between $\mathrm{Aut}(\widetilde{X} \stackrel{p}{\to} X)$ and $\pi_1(X, f(y))$ is given by the formula $\Psi(h) = [p \circ g]$, where g is any path in $\widetilde{X}$ from $\tilde{x}$ to $h(\tilde{x})$. By the above, this group acts transitively on the collection of path-components of $f^*\widetilde{X}$, where the stabilizer of the path-component containing $(y,\tilde{x})$ corresponds to $f_\#(\pi_1(Y,y))$.

Lemma 1.3 *The path-components of $f^*\widetilde{X}$ are simply-connected if and only if $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective.*

Proof Fix any $(y,\tilde{x}) \in f^*\widetilde{X}$ and let $\widetilde{Y}$ be the path-component of $f^*\widetilde{X}$ containing $(y,\tilde{x})$. Put $x = p(\tilde{x})$. First suppose that $\widetilde{Y}$ is simply-connected. If $[\alpha] \in \pi_1(Y,y)$ is such that $f_\#([\alpha]) = 1 \in \pi_1(X,x)$, then the lift $g\colon ([0,1],0) \to (\widetilde{X},\tilde{x})$ of $f\circ\alpha\colon ([0,1],0) \to (X,x)$ is a loop. This yields a loop $h = (\alpha, g)\colon [0,1] \to \widetilde{Y} \subseteq Y \times \widetilde{X}$ which projects to α. Since $\widetilde{Y}$ is simply-connected, $[\alpha] = 1 \in \pi_1(Y,y)$.

Now suppose $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective and let $h\colon ([0,1],0) \to (\widetilde{Y},(y,\tilde{x}))$ be any loop. Then $\tilde{f}_\#([h]) \in \pi_1(\widetilde{X},\tilde{x}) = \{1\}$, so that $f_\#([(f^*p)\circ h]) = [f\circ(f^*p)\circ h] = [p \circ \tilde{f} \circ h] = p_\# \circ \tilde{f}_\#[h] = 1 \in \pi_1(X,x)$. Hence, $[(f^*p)\circ h] = 1 \in \pi_1(Y,y)$. Any nullhomotopy for $(f^*p)\circ h$ in Y, fixing the endpoints, maps via f to a nullhomotopy for $f\circ(f^*p)\circ h$ in X, from where it can be lifted to a nullhomotopy for $\tilde{f}\circ h$ in $\widetilde{X}$. Combining these nullhomotopies for $(f^*p)\circ h$ and $\tilde{f}\circ h$, yields one for h in $\widetilde{Y}$. □

In order to distill the essence from the above question, we now consider an example in which the induced map $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ differs from the generalized universal covering of Y, although $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective.

As we shall see, local path-connectivity of $\widetilde{Y}$ is the sticking point and is far from guaranteed. Outside the classical context of locally nice spaces, the map $f\colon Y \to X$ will have to satisfy a more rigid condition, the prototypical failure of which is exhibited by the following example.

2 An example

Let Y be the space obtained from joining two copies of the Hawaiian Earring with an arc between their distinguished points. Specifically, let $Y \subseteq \mathbb{R}^2$ be given by $Y = \mathbb{H} \cup A \cup \mathbb{H}'$, where $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y-1-\frac{1}{n})^2 = (\frac{1}{n})^2, n \in \mathbb{N}\}$, $A = \{0\} \times [-1,1]$, and $\mathbb{H}' = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y+1+\frac{1}{n})^2 = (\frac{1}{n})^2, n \in \mathbb{N}\}$. Let X be the quotient space obtained from Y by identifying the arc A to a point and let $f\colon Y \to X$ denote the quotient map, so that X is the one-point union of $\mathbb{H}$ and $\mathbb{H}'$. Put $y_0 = (0,0) \in Y$ and let $x_0 = f(y_0)$.

We will show that $p\colon \widetilde{X} \to X$ is a generalized universal covering, that the induced homomorphism $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective, and that $\widetilde{Y}$ is not locally path-connected. Consequently, $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ is not a generalized universal covering. Indeed, while the generalized universal covering space of Y exists and is contractible, it will be shown that $\widetilde{Y}$ is not contractible.

Recall that for every 1–dimensional continuum Z, the natural homomorphism $\pi_1(Z,*) \to \check{\pi}_1(Z,*)$ to the first Čech homotopy group is injective by Eda and Kawamura [1]. It therefore follows from Fischer and Zastrow [3] that $p\colon \widetilde{X} \to X$ is a generalized universal covering and that both Y and X have generalized universal covering spaces which are $\mathbb{R}$–trees and hence contractible.

The injectivity of $f_\#\colon \pi_1(Y) \to \pi_1(X)$ can be deduced, for example, from that of $f_* \circ \varphi$ in the following commutative diagram:

$$\begin{array}{ccc} \pi_1(Y,y_0) & \xrightarrow{f_\#} & \pi_1(X,x_0) \\ \varphi\downarrow & & \downarrow \\ \check{\pi}_1(Y,y_0) & \xrightarrow{f_*} & \check{\pi}_1(X,x_0) \end{array}$$

To this end, let $[\alpha] \in \pi_1(Y,y_0)$ be such that $f_*(\varphi([\alpha])) = 1 \in \check{\pi}_1(X,x_0)$. Then $\varphi([\alpha]) = 1 \in \check{\pi}_1(Y,y_0)$, because identifying A to a point induces an isomorphism of fundamental groups on every level of the canonical inverse sequences of approximating polyhedra for Y and X. Because φ is injective, we have $[\alpha] = 1 \in \pi_1(Y,y_0)$, as desired.

Note that $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is not surjective, because those elements of $\pi_1(X,x_0)$ which are represented by continuous loops that non-trivially alternate infinitely often between the two copies of the Hawaiian Earring in X are not in the image of $f_\#$. Therefore, by Remark 1.2, $f^*\widetilde{X}$ is not path-connected. Let $\widetilde{x}_0 \in \widetilde{X}$ be the equivalence class of the constant path at x_0 and let $\widetilde{Y}$ be the path-component of $f^*\widetilde{X}$ containing the point $\widetilde{y}_0 = (y_0, \widetilde{x}_0)$.

We claim that $\widetilde{Y}$ is not locally path-connected. Assume otherwise and choose open subsets $V \subseteq Y$ and $U \subseteq X$ with $y_0 \in V$ and $x_0 \in U$, such that every point of $\widetilde{Y} \cap (V \times B(\widetilde{x}_0, U))$ is joined to $\widetilde{y}_0$ by a path in $\widetilde{Y} \cap ((\{0\} \times (-1,1)) \times \widetilde{X})$. Choose any non-trivial loop $\alpha\colon ([0,1],\{0,1\}) \to (Y,\{y_0\})$ with $f \circ \alpha([0,1]) \subseteq U$. Put $\beta = f \circ \alpha$, $\widetilde{x}_1 = [\beta]$ and $\widetilde{y}_1 = (y_0, \widetilde{x}_1)$. Then $\widetilde{x}_0 \neq \widetilde{x}_1 \in B(\widetilde{x}_0, U)$. Let $\widetilde{\beta}\colon ([0,1],0) \to (\widetilde{X}, \widetilde{x}_0)$ denote the lift of $\beta\colon ([0,1],0) \to (X,x_0)$ with $p \circ \widetilde{\beta} = \beta$. Then $(\alpha, \widetilde{\beta})$ is a path in

$f^*\widetilde{X}$ from $\widetilde{y}_0$ to $\widetilde{y}_1$. Hence $\widetilde{y}_1 \in \widetilde{Y} \cap (V \times B(\widetilde{x}_0, U))$. By choice of V and U, there is a path $g\colon [0,1] \to f^*\widetilde{X}$ with $f^*p \circ g([0,1]) \subseteq A$ from $g(0) = \widetilde{y}_0$ to $g(1) = \widetilde{y}_1$. Then $p \circ (\widetilde{f} \circ g) = f \circ (f^*p) \circ g$ is the constant path at x_0. So, by the unique path lifting property of $p\colon \widetilde{X} \to X$, we have that $\widetilde{f} \circ g$ is the constant path at $\widetilde{x}_0$. Hence $\widetilde{x}_1 = \widetilde{x}_0$, contrary to the above, and our claim follows.

Note that if we replace Y by $\mathbb{H} \cup A$ in the above discussion, then $f^*\widetilde{X}$ becomes path-connected, remains non-locally path-connected, but is contractible.

In contrast, we now verify that $\widetilde{Y}$ is not contractible when $Y = \mathbb{H} \cup A \cup \mathbb{H}'$. First observe that because $\widetilde{Y}$ is simply-connected by Lemma 1.3, we can associate to each $\widetilde{y} \in \widetilde{Y}$ with $(f^*p)(\widetilde{y}) = y_0$ a well-defined word-length in the free product $\pi_1(Y, y_0) \approx \pi_1(\mathbb{H}) * \pi_1(\mathbb{H}')$ by choosing any path $g\colon [0,1] \to \widetilde{Y}$ from $g(0) = \widetilde{y}_0$ to $g(1) = \widetilde{y}$ and calculating the word-length of $[(f^*p) \circ g] \in \pi_1(Y, y_0)$. Now suppose, to the contrary, that there is a homotopy $H\colon \widetilde{Y} \times [0,1] \to \widetilde{Y}$ such that $H(\widetilde{y}, 1) = \widetilde{y}$ and $H(\widetilde{y}, 0) = \widetilde{y}_0$ for all $\widetilde{y} \in \widetilde{Y}$. By compactness of $[0,1]$, there are open subsets $V \subseteq Y$ and $U \subseteq X$ with $y_0 \in V$ and $x_0 \in U$ such that $||(f^*p) \circ H(\widetilde{y}_0, t) - (f^*p) \circ H(\widetilde{y}, t)|| < \frac{1}{2}$ for all $\widetilde{y} \in \widetilde{Y} \cap (V \times B(\widetilde{x}_0, U))$ and all $t \in [0,1]$. Since $g(t) = H(\widetilde{y}, t)$ can be used to calculate the word-length for $\widetilde{y}$ when $(f^*p)(\widetilde{y}) = y_0$ and since the length of the arc A equals 2, the above inequality allows us to choose a positive integer m such that for every $\widetilde{y} \in \widetilde{Y} \cap (V \times B(\widetilde{x}_0, U))$ with $(f^*p)(\widetilde{y}) = y_0$, the word-length associated to $\widetilde{y}$ is less than or equal to m. On the other hand, by alternating between $\mathbb{H}$ and $\mathbb{H}'$, we may choose a loop $\alpha\colon ([0,1], \{0,1\}) \to (Y, \{y_0\})$ with $f \circ \alpha([0,1]) \subseteq U$ such that the word-length of $[\alpha] \in \pi_1(Y, y_0)$ is equal to $2m$. As above, let $\beta = f \circ \alpha$, $\widetilde{\beta}\colon ([0,1], 0) \to (\widetilde{X}, \widetilde{x}_0)$ be the lift of β, $\widetilde{x}_1 = [\beta]$ and $\widetilde{y}_1 = (y_0, \widetilde{x}_1)$. Then $g = (\alpha, \widetilde{\beta})$ is a path in $f^*\widetilde{X}$ from $\widetilde{y}_0$ to $\widetilde{y}_1$ so that $\widetilde{y}_1 \in \widetilde{Y} \cap (V \times B(\widetilde{x}_0, U))$, $(f^*p)(\widetilde{y}_1) = y_0$ and the word-length associated to $\widetilde{y}_1$ is equal to $2m$; contradicting the choice of m.

Remark 2.1 The following is a variation of the above example with the same properties but in which $f\colon Y \to X$ is inclusion, so that one can take $f^*\widetilde{X} = p^{-1}(Y)$, $f^*p = p|_{p^{-1}(Y)}$ and $\widetilde{f}$ to be inclusion. Embed $f\colon Y = \mathbb{H} \cup A \cup \mathbb{H}' \hookrightarrow X = \mathbb{H} \times [0,1]$ such that $f(\mathbb{H}) \subseteq \mathbb{H} \times \{0\}$, $f(\mathbb{H}') \subseteq \mathbb{H} \times \{1\}$, $f(A) = \{(0,1)\} \times [0,1]$ and such that $f(\mathbb{H})$ and $f(\mathbb{H}')$ occupy alternating cylinders of X.

3 Gradual π_1–injectivity

We now refine the notion of π_1–injectivity appropriately. For $y \in V \subseteq W \subseteq Y$ and $f(V) \subseteq U \subseteq X$, consider the following commutative diagram of homotopy groups and sets, whose exact rows are induced by inclusions and restrictions, and whose vertical arrows are induced by the map $f\colon (Y,V,y)\to(X,U,f(y))$ and inclusion $i\colon (Y,V,y)\hookrightarrow(Y,W,y)$.

$$
\begin{array}{ccccccc}
\pi_1(W,y) & \longrightarrow & \pi_1(Y,y) & \longrightarrow & \pi_1(Y,W,y) & \longrightarrow & \pi_0(W,y)\\
\uparrow & & \uparrow & & \uparrow{\scriptstyle i_\#} & & \uparrow\\
\pi_1(V,y) & \longrightarrow & \pi_1(Y,y) & \longrightarrow & \pi_1(Y,V,y) & \longrightarrow & \pi_0(V,y)\\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f_\#} & & \downarrow\\
\pi_1(U,f(y)) & \longrightarrow & \pi_1(X,f(y)) & \longrightarrow & \pi_1(X,U,f(y)) & \longrightarrow & \pi_0(U,f(y))
\end{array}
\tag{3–1}
$$

Definition We call the map $f\colon Y\to X$ *gradually π_1–injective* if for every $y\in Y$ and every open subset W of Y with $y\in W$ there exist open subsets V and U of Y and X, respectively, with $y\in V\subseteq W$ and $f(V)\subseteq U$, such that the kernel of $f_\#\colon \pi_1(Y,V,y)\to\pi_1(X,U,f(y))$ in diagram (3–1) is contained in the kernel of $i_\#\colon \pi_1(Y,V,y)\to\pi_1(Y,W,y)$.

Remark 3.1 Let $j\colon V\hookrightarrow Y$ and $k\colon U\hookrightarrow X$ denote inclusions. If V is path-connected and if it happens that

$$f_\#^{-1}(k_\#(\pi_1(U,f(y)))\subseteq j_\#(\pi_1(V,y)),$$

in the lower left square of diagram (3–1), then a quick diagram chase (not involving the top row) implies that the kernel of $f_\#\colon \pi_1(Y,V,y)\to\pi_1(X,U,f(y))$ is trivial.

Remark 3.2 If $f_\#\colon \pi_1(Y)\to\pi_1(X)$ is injective, then Y clearly inherits the property of being homotopically Hausdorff from X.

The following two observations endorse our definition.

Lemma 3.3 *Suppose X is semilocally simply-connected and Y is locally path-connected. If $f_\#\colon \pi_1(Y)\to\pi_1(X)$ is injective, then Y is homotopically Hausdorff and $f\colon Y\to X$ is gradually π_1–injective.*

Proof Let $y\in Y$ and an open subset $W\subseteq Y$ with $y\in W$ be given. Choose an open subset $U\subseteq X$ such that $f(y)\in U$ and such that the inclusion induced homomorphism $k_\#\colon \pi_1(U,f(y))\to\pi_1(X,f(y))$ is trivial. Then choose an open path-connected subset $V\subseteq Y$ with $y\in V\subseteq W$ and $f(V)\subseteq U$. Now apply Remark 3.1 and Remark 3.2. □

Lemma 3.4 *If Y is homotopically Hausdorff and if $f\colon Y\to X$ is gradually π_1–injective, then $f_\#\colon \pi_1(Y)\to\pi_1(X)$ is injective.*

Proof Let a be an element of the kernel of $f_\#\colon \pi_1(Y,y)\to\pi_1(X,f(y))$. Let W, V and U be as in the definition for gradually π_1–injective. A diagram chase reveals that a is in the image of $\pi_1(W,y)\to\pi_1(Y,y)$. Since W can be chosen arbitrarily small and since Y is homotopically Hausdorff, a is trivial. □

4 The main result

The chief difficulty in determining whether $f^*p|_{\widetilde{Y}}\colon \widetilde{Y}\to Y$ is a generalized universal covering lies in checking the local path-connectivity of $\widetilde{Y}$. The next two lemmas characterize this property in terms of gradual π_1–injectivity of $f\colon Y\to X$.

Lemma 4.1 *If $f\colon Y\to X$ is gradually π_1–injective, then the path-components of $f^*\widetilde{X}$ are locally path-connected.*

Proof Let $\widetilde{Y}$ be a path-component of $f^*\widetilde{X}$ and let $(y,\tilde{x})\in\widetilde{Y}$. Say, $\tilde{x}=[\alpha]\in\widetilde{X}$. Let $\widetilde{N}$ be an open subset of $\widetilde{Y}$ with $(y,\tilde{x})\in\widetilde{N}$. Then $(y,\tilde{x})\in\widetilde{Y}\cap(V\times B([\alpha],U))\subseteq\widetilde{N}$ for some open subsets V and U of Y and X, respectively. In particular, $y\in V$ and $f(y)=p(\tilde{x})=\alpha(1)\in U$. Choose an open subset $W\subseteq V$ with $y\in W$ such that $f(W)\subseteq U$. Since $f\colon Y\to X$ is gradually π_1-injective, there are open subsets $V'\subseteq Y$ and $U'\subseteq X$ with $y\in V'\subseteq W$ and $f(V')\subseteq U'$ such that the kernel of $f_\#\colon \pi_1(Y,V',y)\to\pi_1(X,U',f(y))$ lies in the kernel of $i_\#\colon \pi_1(Y,V',y)\to\pi_1(Y,W,y)$. Replacing U' with $U\cap U'$, if necessary, we may assume without loss of generality that $U'\subseteq U$. Then $(y,\tilde{x})\in\widetilde{Y}\cap(V'\times B([\alpha],U'))\subseteq\widetilde{Y}\cap(V\times B([\alpha],U))$.

We will show that every point of $\widetilde{Y}\cap(V'\times B([\alpha],U'))$ is joined to $(y,\tilde{x})$ by a path in $\widetilde{Y}\cap(V\times B([\alpha],U))$. To this end, let $(w,\tilde{z})\in\widetilde{Y}\cap(V'\times B([\alpha],U'))$ be given. Say $\tilde{z}=[\beta]\in\widetilde{X}$. Then $f(w)=p(\tilde{z})=\beta(1)$. Since $\widetilde{Y}$ is path-connected, there is a path $g\colon[0,1]\to\widetilde{Y}$ from $g(0)=(y,\tilde{x})$ to $g(1)=(w,\tilde{z})$. Then $\tilde{f}\circ g\colon[0,1]\to\widetilde{X}$ is a path from $\tilde{f}\circ g(0)=\tilde{x}=[\alpha]$ to $\tilde{f}\circ g(1)=\tilde{z}=[\beta]$ and $(f^*p)\circ g\colon[0,1]\to Y$ is a path from $(f^*p)\circ g(0)=y$ to $(f^*p)\circ g(1)=w$. Put $\gamma=p\circ\tilde{f}\circ g=f\circ(f^*p)\circ g\colon[0,1]\to X$. Then $\gamma(0)=p([\alpha])=\alpha(1)=f(y)$ and $\gamma(1)=f\circ(f^*p)\circ g(1)=f(w)$. Since $p\colon\widetilde{X}\to X$ has the unique path lifting property, $[\beta]=\tilde{f}\circ g(1)=[\alpha\cdot\gamma]$. Since $[\beta]\in B([\alpha],U')$, there is a path $\delta\colon[0,1]\to U'$ such that $[\beta]=[\alpha\cdot\delta]$. Since $[\alpha\cdot\gamma]=[\beta]=[\alpha\cdot\delta]\in\widetilde{X}$, we see that γ and δ are homotopic in X, relative to their common endpoints. This places $[(f^*p)\circ g]$ into the kernel of $f_\#\colon\pi_1(Y,V',y)\to\pi_1(X,U',f(y))$ and hence into the kernel of $i_\#\colon\pi_1(Y,V',y)\to\pi_1(Y,W,y)$. Therefore, there is a path $\xi\colon[0,1]\to W$ which is homotopic to $(f^*p)\circ g$ in Y, relative to their common endpoints. Now let $h\colon([0,1],0)\to(\widetilde{X},\tilde{x})$ be the lift of $f\circ\xi\colon([0,1],0)\to(U,f(y))$. Then $h(1)=[\alpha\cdot(f\circ\xi)]=[\alpha\cdot(f\circ(f^*p)\circ g)]=[\alpha\cdot\gamma]=[\beta]=\tilde{z}$. Therefore,

$(\xi, h)\colon [0,1] \to (V \times B([\alpha], U))$ is a path from $(y, \widetilde{x})$ to $(w, \widetilde{z})$ which lies in $\widetilde{Y}$, because $f \circ \xi = p \circ h$. □

Lemma 4.2 *If the path-components of $f^*\widetilde{X}$ are simply-connected and locally path-connected, then $f\colon Y \to X$ is gradually π_1–injective.*

Proof Let $y \in Y$ and an open subset $W \subseteq Y$ with $y \in W$ be given. Choose any path $\alpha\colon [0,1] \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = f(y)$. Put $\widetilde{x} = [\alpha]$. Then $f(y) = \alpha(1) = p(\widetilde{x})$ so that $(y, \widetilde{x}) \in f^*\widetilde{X}$. Let $\widetilde{Y}$ be the path-component of $f^*\widetilde{X}$ which contains $(y, \widetilde{x})$. Since $\widetilde{Y} \cap (W \times B([\alpha], X))$ is an open subset of $\widetilde{Y}$ containing $(y, \widetilde{x})$, and since $\widetilde{Y}$ is assumed to be locally path-connected, there are open subsets $V \subseteq Y$ and $U \subseteq X$ with $y \in V \subseteq W$ and $f(V) \subseteq U$ such that every point of $\widetilde{Y} \cap (V \times B([\alpha], U))$ is joined to $(y, \widetilde{x})$ by a path in $\widetilde{Y} \cap (W \times B([\alpha], X))$.

Let $[\gamma]$ be an element of the kernel of $f_\#\colon \pi_1(Y, V, y) \to \pi_1(X, U, f(y))$ and put $w = \gamma(1) \in V$. Then there is a path $\beta\colon [0,1] \to U$ such that β is homotopic to $f \circ \gamma$ in X, relative to their common endpoints $\beta(0) = f(y)$ and $\beta(1) = f(w)$. Put $\widetilde{z} = [\alpha \cdot \beta]$. Then $(w, \widetilde{z}) \in V \times B([\alpha], U)$. Let $g\colon ([0,1], 0) \to (\widetilde{X}, \widetilde{x})$ be the lift of $f \circ \gamma\colon ([0,1], 0) \to (X, f(y))$. Then $(\gamma, g)\colon [0,1] \to f^*\widetilde{X}$ is a path from $(y, \widetilde{x})$ to $(w, [\alpha \cdot (f \circ \gamma)]) = (w, [\alpha \cdot \beta]) = (w, \widetilde{z})$. Therefore, $(w, \widetilde{z}) \in \widetilde{Y} \cap (V \times B([\alpha], U))$. By choice of V and U, there is a path $h\colon [0,1] \to \widetilde{Y} \cap (W \times B([\alpha], X))$ from $h(0) = (y, \widetilde{x})$ to $h(1) = (w, \widetilde{z})$. Put $\delta = (f^*p) \circ h$. Then $\delta\colon [0,1] \to W$ is a path from $\delta(0) = y$ to $\delta(1) = w$. Since $p \circ (\widetilde{f} \circ h) = f \circ (f^*p) \circ h = f \circ \delta$ and $\widetilde{f} \circ h(0) = \widetilde{x} = [\alpha]$, it follows from the unique path lifting property of $p\colon \widetilde{X} \to X$ that $\widetilde{f} \circ h(1) = [\alpha \cdot (f \circ \delta)]$. On the other hand, $\widetilde{f} \circ h(1) = \widetilde{z} = [\alpha \cdot \beta]$. Hence $f \circ \delta$ is homotopic to β and therefore also homotopic to $f \circ \gamma$ in X, relative to their common endpoints. By Lemma 1.3, $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective so that we can conclude that δ is homotopic to γ in Y, relative to their common endpoints. Since δ lies in W, this places $[\gamma]$ into the kernel of $i_\#\colon \pi_1(Y, V, y) \to \pi_1(Y, W, y)$, as desired. □

Here is the main result.

Theorem 4.3 *Let $\widetilde{Y}$ be a path-component of $f^*\widetilde{X}$. Then $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ is a generalized universal covering if and only if Y is homotopically Hausdorff and $f\colon Y \to X$ is gradually π_1–injective.*

Proof First assume that $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ is a generalized universal covering. Since $\widetilde{Y}$ is simply-connected, $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective by Lemma 1.1 and Lemma 1.3. Then Y is homotopically Hausdorff by Remark 3.2. Since $\widetilde{Y}$ is also locally path-connected, $f\colon Y \to X$ is gradually π_1–injective by Lemma 4.2.

Now assume that Y is homotopically Hausdorff and that $f\colon Y \to X$ is gradually π_1–injective. Then $f_\#\colon \pi_1(Y) \to \pi_1(X)$ is injective by Lemma 3.4. Hence, $\widetilde{Y}$ is simply-connected by Lemma 1.3 and locally path-connected by Lemma 4.1. Surjectivity of the map $f^*p|_{\widetilde{Y}}\colon \widetilde{Y} \to Y$ follows from the surjectivity of $p\colon \widetilde{X} \to X$ and Lemma 1.1. Finally, the required lifting criterion follows easily: Let Z be a path-connected and locally path-connected topological space and let $g\colon (Z,z) \to (Y,y)$ be a continuous function with $g_\#(\pi_1(Z,z)) = 1$. Let $\widetilde{y}$ in $\widetilde{Y}$ be such that $f^*p(\widetilde{y}) = y$. Put $\widetilde{x} = \widetilde{f}(\widetilde{y})$ and $x = f(y)$. Then $\widetilde{y} = (y, \widetilde{x})$. Since $f \circ g\colon (Z,z) \to (X,x)$ is such that $(f \circ g)_\#(\pi_1(Z,z)) = 1$, there is a unique lift $q\colon (Z,z) \to (\widetilde{X}, \widetilde{x})$ with $p \circ q = f \circ g$. Then $h = (g,q)\colon (Z,z) \to (\widetilde{Y}, \widetilde{y})$ is a lift with $(f^*p) \circ h = g$. Observe that any lift $h'\colon (Z,z) \to (\widetilde{Y}, \widetilde{y})$ with $(f^*p) \circ h' = g$ yields a lift $\widetilde{f} \circ h'\colon (Z,z) \to (\widetilde{X}, \widetilde{x})$ with $p \circ \widetilde{f} \circ h' = f \circ (f^*p) \circ h' = f \circ g$. By uniqueness of q, we obtain $\widetilde{f} \circ h' = q$. Thus, $h' = ((f^*p) \circ h', \widetilde{f} \circ h') = (g,q) = h$. □

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