Dendroidal sets

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We introduce the concept of a dendroidal set. This is a generalization of the notion of a simplicial set, specially suited to the study of (coloured) operads in the context of homotopy theory. We define a category of trees, which extends the category Δ used in simplicial sets, whose presheaf category is the category of dendroidal sets. We show that there is a closed monoidal structure on dendroidal sets which is closely related to the Boardman–Vogt tensor product of (coloured) operads. Furthermore, we show that each (coloured) operad in a suitable model category has a coherent homotopy nerve which is a dendroidal set, extending another construction of Boardman and Vogt. We also define a notion of an inner Kan dendroidal set, which is closely related to simplicial Kan complexes. Finally, we briefly indicate the theory of dendroidal objects in more general monoidal categories, and outline several of the applications and further theory of dendroidal sets.

55P48, 55U10, 55U40; 18D50, 18D10, 18G30

1 Introduction

There is an intimate relation between simplicial sets and categories (and, more generally, between simplicial objects and enriched categories), which plays a fundamental role in many parts of homotopy theory. The goal of this paper is to introduce an extension of the category of simplicial sets, suitable for studying operads. We call the objects of this larger category "dendroidal sets", and denote the inclusion functor by

$$i_!\colon (\text{simplicial sets}) \to (\text{dendroidal sets}).$$

The pair of adjoint functors

$$\tau\colon (\text{simplicial sets}) \rightleftarrows (\text{categories}) :N$$

where N denotes the nerve and τ its left adjoint, will be seen to extend to a pair

$$\tau_d\colon (\text{dendroidal sets}) \rightleftarrows (\text{operads}) :N_d$$

having similar properties.

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Many other properties and constructions of simplicial sets also extend to dendroidal sets. In particular, we will show that the cartesian closed monoidal structure on simplicial sets extends to a (non-cartesian!) closed monoidal structure on dendroidal sets. Here “extends” means that there is a natural isomorphism

$$i_!(X \times Y) \cong i_!(X) \otimes i_!(Y)$$

for any two simplicial sets X and Y. This tensor product of dendroidal sets is closely related to the Boardman–Vogt tensor product of operads. In fact, the latter can be defined in terms of the former by the isomorphism

$$\mathcal{P} \otimes_{BV} \mathcal{Q} \cong \tau_d(N_d\mathcal{P} \otimes N_d\mathcal{Q})$$

for any two operads $\mathcal{P}$ and $\mathcal{Q}$.

We will also define a notion of inner (or weak) Kan complex for dendroidal sets, extending the simplicial one in the sense that for any simplicial set X, one has that X is an inner Kan complex if, and only if, $i_!(X)$ is. The nerve of an operad always satisfies this dendroidal inner Kan condition, just like the nerve of a category satisfies the simplicial inner Kan condition. Moreover, this inner Kan condition has various basic properties related to the monoidal structure on dendroidal sets, the most significant one being that, under some conditions on a dendroidal set X, $\underline{Hom}(X, K)$ is an inner Kan complex whenever K is. The analogous property for simplicial sets was recently proved by Joyal, and forms one of the basic steps in the proof of the existence of the closed model structure on simplicial sets in which the inner Kan complexes are exactly the fibrant objects. Joyal calls these inner Kan complexes quasi-categories, and one might call a dendroidal set a *quasi-operad* if it satisfies our dendroidal version of the inner Kan condition. We expect that there is a closed model structure on dendroidal sets in which the quasi-operads are the fibrant objects. Dendroidal sets also seem to be useful in the theory of homotopy-$\mathcal{P}$–algebras for an operad $\mathcal{P}$ and weak maps between such algebras. As an illustration of this point, we will give an inductive definition of weak higher categories and weak functors between these, based on the theory of inner Kan complexes.

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2 Operads

In this paper, *operad* means *coloured symmetric operad*. (In the literature such operads are also referred to as symmetric multi-categories, see Leinster [20].) We briefly recall the basic definitions, and refer to Berger and Moerdijk [3] for a more extensive discussion. An operad $\mathcal{P}$ is given by a set of colours C, and for each $n \geq 0$ and each sequence of colours $c_1, \dots, c_n, c$ a set $\mathcal{P}(c_1, \dots, c_n; c)$ (to be thought of as operations taking n inputs of colours $c_1, \dots, c_n$ respectively to an output of colour c). Moreover, there are structure maps for units and composition. If we write $I = \{*\}$ for the one-point set, there is for each colour c a unit map

$$u\colon I \to \mathcal{P}(c; c)$$

taking $*$ to 1_c. The composition operations are maps

$$\mathcal{P}(c_1, \dots, c_n; c) \times \mathcal{P}(d_1^1, \dots, d_{k_1}^1; c_1) \times \cdots \times \mathcal{P}(d_1^n, \dots, d_{k_n}^n; c_n) \to \mathcal{P}(d_1^1, \dots, d_{k_n}^n; c)$$

which we denote $p, q_1, \dots, q_n \mapsto p(q_1, \dots, q_n)$. These operations should satisfy the usual associativity and unitary conditions. Furthermore, for each $\sigma \in \Sigma_n$ and colours $c_1, \dots, c_n, c \in C$ there is a map $\sigma^*\colon \mathcal{P}(c_1, \dots, c_n; c) \to \mathcal{P}(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c)$. These maps define a right action of Σ_n in the sense that $(\sigma\tau)^* = \tau^*\sigma^*$, and the composition operations should be equivariant in some natural sense. The definition can equivalently be cast in terms of the units and the "$\circ_i$–operations"

$$\mathcal{P}(c_1, \dots, c_n; c) \times \mathcal{P}(d_1, \dots, d_k; c_i) \xrightarrow{\circ_i} \mathcal{P}(c_1, \dots, c_{i-1}, d_1, \dots, d_k, c_{i+1}, \dots, c_n; c).$$

A coloured operad $\mathcal{P}$ with set C of colours will also be referred to as an operad *coloured by* C, or an operad *over* C. For such an operad $\mathcal{P}$, a $\mathcal{P}$–algebra consists of a family of sets $\{A_c\}_{c \in C}$ together with arrows $\mathcal{P}(c_1, \dots, c_n; c) \times A_{c_1} \times \cdots \times A_{c_n} \to A_c$, satisfying the usual compatibility conditions for unit, compositions, and symmetry.

The same definitions of operad and algebra still make sense if we replace *Set* by an arbitrary cocomplete symmetric monoidal category $\mathcal{E}$. In particular, the strong monoidal functor *Set* $\to \mathcal{E}$, which sends a set S to the S–fold coproduct of copies of the unit I of $\mathcal{E}$, maps every operad $\mathcal{P}$ over C in *Set* to an operad in $\mathcal{E}$, which we denote by $\mathcal{P}_{\mathcal{E}}$, or sometimes again by $\mathcal{P}$.

If $\mathcal{P}$ is an operad over C and $f\colon D \to C$ is a map of sets, then there is an evident induced operad $f^*(\mathcal{P})$ over D, given by

$$f^*(\mathcal{P})(d_1, \dots, d_n; d) = \mathcal{P}(fd_1, \dots, fd_n; fd).$$

If $\mathcal{P}$ and $\mathcal{Q}$ are operads, a *map* $\mathcal{Q} \xrightarrow{f} \mathcal{P}$ is given by a map of sets $f\colon D \to C$, and for each $d_1, \ldots, d_n, d$ a map

$$f_{d_1,\ldots,d_n,d}\colon \mathcal{Q}(d_1, \ldots, d_n; d) \to \mathcal{P}(f(d_1), \ldots, f(d_n); f(d))$$

which commutes with all the operations and the Σ_n–actions. If $D = C$ and $f\colon D \to C$ is the identity, we will call f a map of operads *over* C. For a fixed symmetric monoidal category $\mathcal{E}$, we denote by $\mathit{Operad}(\mathcal{E})$ the category of all coloured operads in $\mathcal{E}$. When $\mathcal{E} = \mathit{Set}$ we will simply write *Operad* instead of $\mathit{Operad}(\mathit{Set})$.

Example 2.1 Let $\mathcal{E}$ be a symmetric monoidal category. Then $\mathcal{E}$ gives rise to a coloured operad $\underline{\mathcal{E}}$, whose colours are the objects of $\mathcal{E}$. For a sequence $X_1, \ldots, X_n, X$ of such objects, $\underline{\mathcal{E}}(X_1, \ldots, X_n; X)$ is the set of arrows $X_1 \otimes \cdots \otimes X_n \to X$ in $\mathcal{E}$. If $\mathcal{E}$ is a symmetric *closed* monoidal category, then $\mathcal{E}$ may be viewed as an operad $\underline{\underline{\mathcal{E}}}$ in $\mathcal{E}$, with the objects of $\mathcal{E}$ as colours again, and with $\underline{\underline{\mathcal{E}}}(X_1, \ldots, X_n; X)$ the internal *Hom*–object $\underline{\mathit{Hom}}_{\mathcal{E}}(X_1 \otimes \cdots \otimes X_n, X)$.

Note that, in general, the objects of $\mathcal{E}$ form a proper class and not a set. However, in this paper, we will largely ignore such set-theoretic issues, and interpret "small" or "set" in terms of a suitable universe. In this context, let us point out that for any *set* S of objects of $\mathcal{E}$, there are operads $\underline{\mathcal{E}}_S$ and $\underline{\underline{\mathcal{E}}}_S$ obtained by restricting $\underline{\mathcal{E}}$ and $\underline{\underline{\mathcal{E}}}$ to the colours in S (If $i\colon S \to \mathit{Objects}(\mathcal{E})$ is the inclusion, then $\underline{\mathcal{E}}_S = i^*(\underline{\mathcal{E}})$, etc). In general, we will often identify a monoidal category with the corresponding operad, and simply write $\mathcal{E}$ for $\underline{\mathcal{E}}$ or $\underline{\underline{\mathcal{E}}}$.

Example 2.2 Any category $\mathcal{C}$ can be considered as an operad $\mathcal{P}_{\mathcal{C}}$ in the following way. The colours of $\mathcal{P}_{\mathcal{C}}$ are the objects of $\mathcal{C}$, and for any sequence of colours $c_1, \ldots, c_n, c$ we set

$$\mathcal{P}_{\mathcal{C}}(c_1, \ldots, c_n; c) = \begin{cases} \mathcal{C}(c_1, c), & \text{if } n = 1 \\ \phi, & \text{if } n \neq 1 \end{cases}$$

the compositions and units are as in $\mathcal{C}$ and the symmetric actions are all trivial. In this way we obtain a functor $j_!\colon \mathit{Cat} \to \mathit{Operad}$ from the category *Cat* of small categories to the category of operads. This functor has an evident right adjoint $j^*\colon \mathit{Operad} \to \mathit{Cat}$, sending an operad $\mathcal{P}$ to the category given by the colours and unary operations of $\mathcal{P}$. In exactly the same way, any $\mathcal{E}$–enriched category can be seen as an operad in $\mathcal{E}$ and we thus obtain adjoint functors $\mathit{Cat}(\mathcal{E}) \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathit{Operad}(\mathcal{E})$.

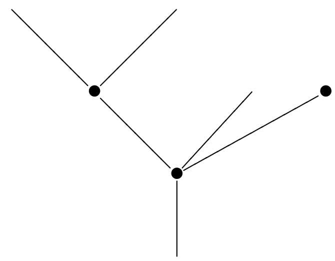
Remark 2.3 There is also the notion of a non-symmetric (also called planar) operad. A planar operad is exactly the same structure as an operad except that there are no symmetric actions involved. The resulting category of planar operads with their obvious notion of maps is denoted by $\mathit{Operad}_\pi(\mathcal{E})$. There is an evident forgetful functor $\mathit{Operad}(\mathcal{E}) \to \mathit{Operad}(\mathcal{E})_\pi$ which maps an operad to the same operad with the symmetric actions forgotten. This functor has a left adjoint *Symm*: $\mathit{Operad}(\mathcal{E})_\pi \to \mathit{Operad}(\mathcal{E})$, which we call the symmetrization functor. This functor is useful in the construction of operads, since sometimes it is easier to directly describe the non-symmetric operad whose algebras are the desired structures in a given context.

Example 2.4 Let S be a set. We describe now a planar operad $\mathcal{B}_S$ whose algebras are categories having S as set of objects. The set of colours of $\mathcal{B}_S$ is $S \times S$, and for any sequence of colours of the form $(s_1, s_2), (s_2, s_3), \dots, (s_{n-1}, s_n)$ there is exactly one operation in $\mathcal{B}_S((s_1, s_2), \dots, (s_{n-1}, s_n); (s_1, s_n))$. There are no other operations except those just given, which then completely determine the operadic structure. We thus have a planar operad in *Set* whose symmetrization we denote by $\mathcal{A}_S$. For any cocomplete monoidal category $\mathcal{E}$ we obtain an operad in $\mathcal{E}$ (still) denoted $\mathcal{A}_S$ which is the image of the original $\mathcal{A}_S$ under the functor $\mathit{Operad}(\mathit{Set}) \to \mathit{Operad}(\mathcal{E})$ described above. It is easy to verify that an $\mathcal{A}_S$–algebra in $\mathcal{E}$ is the same as an $\mathcal{E}$–enriched category having S as set of objects. For the special case where S is a one-point set, $\mathcal{A}_S$ is the familiar operad *Ass*.

We refer the reader to Berger and Moerdijk [3] for more examples of coloured operads.

3 A category of trees

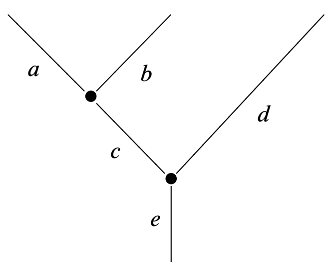
The trees we will consider are finite, non-empty (non-planar) trees with a designated root. As is common in the theory of operads (see Getzler and Kapranov [13], Ginzburg and Kapranov [14] and Markl, Shnider and Stashef [23]) we allow some edges to have a vertex only on one side. These edges are called *outer* (or external) edges, while those having vertices on both sides are called *inner* (or internal) edges. By a designated root we mean a choice of one of the outer edges. The root defines an up-down direction in the tree (towards the root) and thus each vertex has a number of incoming edges (the number is the *valence* of the vertex) and one edge going out of it. We also allow vertices of valence 0. For example, the tree



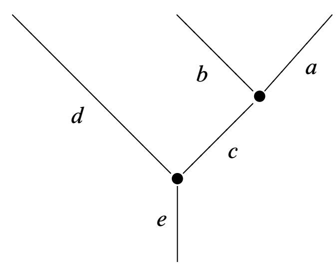
has three vertices, of valence 2, 3, and 0, and three input edges. A tree with no vertices

whose input edge (e say) coincides with its output edge will be denoted by η_e, or simply by η.

When we draw a tree we will always put the root at the bottom. One drawback of drawing a tree on the plane is that it immediately becomes a planar tree; we thus have many different 'pictures' for the same tree. For instance the two trees

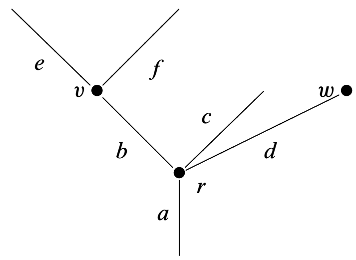


and



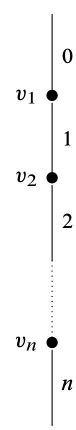
are different planar representations of the same tree.

Any tree T can be viewed as generating an operad $\Omega(T)$, whose colours are the edges of the tree, while the vertices of the tree are the generators of the operations. More explicitly, if we choose a planar representation of T then each vertex v with input edges $e_1, \ldots, e_n$ and output edge e defines an operation $v \in \Omega(T)(e_1, \ldots, e_n; e)$. The other operations are the unit operations and the operations obtained by compositions and by permutations, so as to obtain an operad in which every *Hom* set has at most one object. For example, in the same tree T pictured above, let us name the edges and vertices $a, b, \ldots, f$ and r, v, w.



Then $v \in \Omega(T)(e, f; b)$, $w \in \Omega(T)(\,; d)$ and $r \in \Omega(b, c, d; a))$ are the generators, while the other operations are the units $1_a, 1_b, 1_c \cdots 1_f$, the operations obtained by compositions $r \circ_1 v \in \Omega(T)(e, f, c, d; a)$, $r \circ_3 w \in \Omega(T)(b, c; a)$ and $r(v, 1_c, w) = (r \circ_1 v) \circ_4 w = (r \circ_3 w) \circ_1 v \in \Omega(T)(e, f, c; a)$, and permutations of these. This is a complete description of the operad $\Omega(T)$.

Viewing trees as coloured operads as above enables us to define the category Ω, whose objects are trees, and whose arrows $T \to T'$ are operad maps $\Omega(T) \to \Omega(T')$. The category Ω extends the simplicial category Δ. Indeed, any $n \geq 0$ defines a linear tree

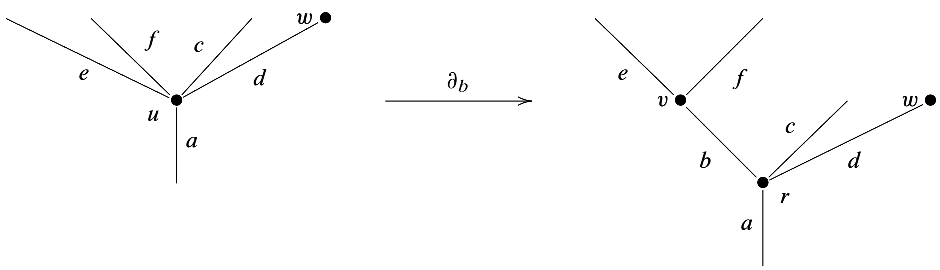


on $n+1$ edges and n vertices $v_1, \ldots, v_n$. We denote this tree by $[n]$. Any order preserving map $\{0, \ldots, n\} \to \{0, \ldots, m\}$ defines an arrow $[n] \to [m]$ in the category Ω. In this way, we obtain an embedding

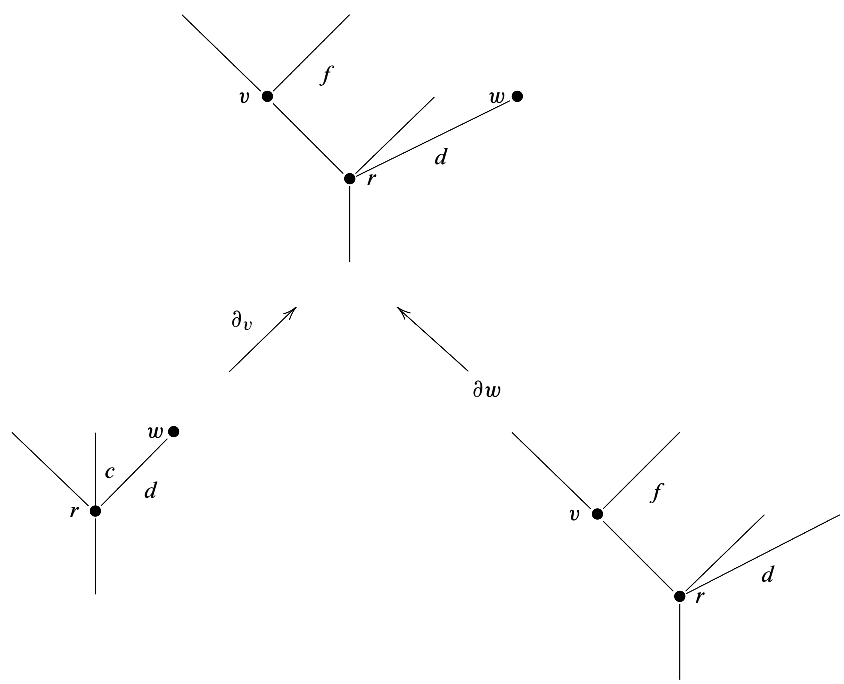
$$\Delta \xrightarrow{i} \Omega$$

This embedding is fully faithful. Moreover, it describes Δ as a sieve (or ideal) in Ω, in the sense that for any arrow $S \to T$ in Ω, if T is linear then so is S.

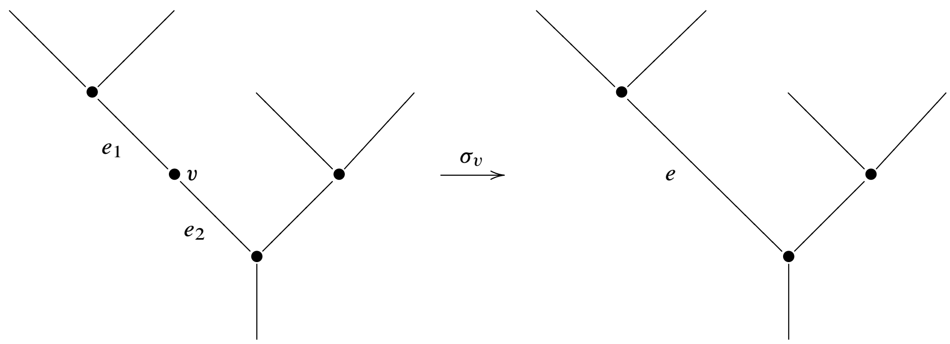
With a tree T one can associate certain maps in Ω as follows. If b is an inner edge in T, let T/b be the tree obtained from T by contracting b. Then there is a natural map $\partial_b\colon T/b \to T$ in Ω, called the *inner face* map associated with b, which locally in the tree looks like this:



Let v be a vertex in T with the property that all but one of the edges incident to v are outer. We call such a vertex an outer cluster. Let T/v be the tree obtained from T by removing the vertex v and all of the outer edges incident to it. Then there is a map $\partial_v : T/v \to T$ in Ω called the *outer face* associated with v. For example, the maps



are two outer faces. We will use the term *face* map to refer to an inner or outer face map. One more type of map is a map that can be associated with a unary vertex v in T as follows. Let T/v be the tree obtained from T by removing the vertex v and merging the two edges incident to it into one edge e. Then there is a map $\sigma_v\colon T \to T/v$ in Ω called the *degeneracy map* associated with v, which sends the vertex v to the identity 1_e, and which can be pictured like this:



The following lemma is the generalization to Ω of the well known fact that in Δ each arrow can be written as a composition of degeneracy maps followed by face maps. We omit the proof.

Lemma 3.1 *Any arrow $f\colon A \to B$ in Ω decomposes as*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\sigma}\downarrow & & \uparrow{\scriptstyle\delta} \\ A' & \xrightarrow{\varphi} & B' \end{array}$$

where $\sigma\colon A \to A'$ is a composition of degeneracy maps, $\varphi\colon A' \to B'$ is an isomorphism, and $\delta\colon B' \to B$ is a composition of face maps.

4 Dendroidal sets

We now define the category *dSet* of dendroidal sets and discuss its relation to the category *sSet* of simplicial sets.

Definition 4.1 A *dendroidal set* is a functor $\Omega^{op} \to Set$. A map between dendroidal sets is a natural transformation. The category of *dendroidal sets* thus defined is denoted *dSet*.

Thus, a dendroidal set X is given by a set X_T for each tree T, and a map $\alpha^*\colon X_T \to X_S$ for each map of trees (arrow in Ω) $\alpha\colon S \to T$; and these maps have to be functorial in α, in the sense that $\mathrm{id}^* = \mathrm{id}$ and $(\alpha\beta)^* = \beta^*\alpha^*$ for $R \xrightarrow{\beta} S \xrightarrow{\alpha} T$ in Ω. A morphism

$Y \xrightarrow{f} X$ of dendroidal sets is given by maps (all denoted) $f\colon Y_T \to X_T$ for each tree T, commuting with the structure maps (that is, $f(\alpha^* y) = \alpha^* f(y)$ for any $y \in Y_T$ and any $\alpha\colon S \to T$). An element of X_T is called a *dendrex* (plural *dendrices*) of *shape* T (This terminology is analogous to simplex, simplices). The dendrices of shape η will be referred to as *vertices*. As for simplicial sets, we call a dendrex $x \in X_T$ *degenerate* if there exists a degeneracy $\sigma\colon T \twoheadrightarrow S$ and a dendrex $y \in X_S$ with $\sigma^*(y) = x$.

Every tree T defines a representable dendroidal set $\Omega[T]$ as follows:

$$\Omega[T]_S = \Omega(S, T)$$

By the Yoneda Lemma each dendrex x of shape T in a dendroidal set X corresponds bijectively to a map $\hat{x}\colon \Omega[T] \to X$ of dendroidal sets. If $\partial_x\colon T \to R$ is a face map associated to an inner edge or an outer cluster x we use the same notation $\partial_x\colon \Omega[T] \to \Omega[R]$ for the induced map of dendroidal sets.

The inclusion functor $i\colon \Delta \to \Omega$ defines an obvious restriction functor

$$i^*\colon dSet \to sSet.$$

This functor has both a left adjoint $i_!$ and a right adjoint i_*, given by left and right Kan extension. The functor $i_!\colon sSet \to dSet$ is "extension by zero",

$$i_!(X)_T = \begin{cases} X_n, & \text{if T is linear with } n \text{ vertices} \\ \phi, & \text{otherwise} \end{cases}$$

(This is clear from the fact that $\Delta \subseteq \Omega$ is a sieve). It follows that $i_!$ is full and faithful, and that $i^* i_!$ is the identity functor on simplicial sets. The pair (i^*, i_*) defines a morphism of toposes $i\colon sSet \to dSet$, which is in fact an *open embedding*.

Example 4.2 If $\mathcal{P}$ is an operad, then the *dendroidal nerve* of $\mathcal{P}$ is the dendroidal set $N_d(\mathcal{P})$ given by

$$N_d(\mathcal{P})_T = Hom_{Operad}(\Omega(T), \mathcal{P}),$$

This construction defines a fully faithful functor

$$N_d\colon Operad \to dSet,$$

which has various nice properties as we will see. As already noted, any monoidal category $\mathcal{E}$ defines an operad $\underline{\mathcal{E}}$. The corresponding dendroidal set $N_d(\underline{\mathcal{E}})$ will simply be written $N_d(\mathcal{E})$ and will be called the *dendroidal nerve* of $\mathcal{E}$. Note that this extends the usual (simplicial) nerve of $\mathcal{E}$, in the sense that

$$i^*(N_d \mathcal{E}) = N(\mathcal{E}).$$

The functor $N_d\colon \mathit{Operad} \to \mathit{dSet}$ has a left adjoint

$$\tau_d\colon \mathit{dSet} \to \mathit{Operad}$$

defined by Kan extension. For a dendroidal set X, we refer to $\tau_d(X)$ as the *operad generated by* X. This functor τ_d extends the functor τ from simplicial sets to categories, left adjoint to $N\colon \mathit{Cat} \to \mathit{sSet}$. In particular, we obtain a diagram of functors

$$\begin{array}{ccc}
\mathit{sSet} & \underset{i^*}{\overset{i_!}{\rightleftarrows}} & \mathit{dSet} \\
{\scriptstyle\tau}\downarrow\uparrow{\scriptstyle N} & & {\scriptstyle\tau_d}\downarrow\uparrow{\scriptstyle N_d} \\
\mathit{Cat} & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & \mathit{Operad}
\end{array}$$

(with left adjoints on the top or on the left), in which the following commutation relations hold up to natural isomorphisms

$$\tau N = \mathrm{id}, \quad \tau_d N_d = \mathrm{id}, \quad i^* i_! = \mathrm{id}, \quad j^* j_! = \mathrm{id}$$

and

$$j_! \tau = \tau_d i_!, \quad N j^* = i^* N_d, \quad i_! N = N_d j_!.$$

The canonical map $\tau i^*(X) \to j^* \tau_d(X)$ is in general not an isomorphism. (For an example, consider the representable dendroidal set $\Omega[T]$ where T is the tree with three edges, one binary and one nullary vertex.)

Remark 4.3 For an arbitrary category $\mathcal{E}$, one can also consider the category $d\mathcal{E}$ of dendroidal objects in $\mathcal{E}$, that is, contra-variant functors from Ω to $\mathcal{E}$. In particular, if one takes for $\mathcal{E}$ the category *Top* of compactly generated topological spaces, one obtains in this way the category *dTop* of dendroidal spaces. Many constructions extend to this more general context. For example, if $\mathcal{P}$ is a topological operad, its dendroidal nerve $N_d(\mathcal{P})$ is naturally a dendroidal space, with the special property that its space $N_d(\mathcal{P})_\eta$ of vertices is discrete. Conversely, from such a dendroidal space X with this property, one can construct a topological operad, $\tau_d(X)$.

4.1 Diagrams of dendroidal sets

If $X\colon \mathbb{S}^{op} \to \mathit{sSet}$ is a diagram of simplicial sets (contravariantly) indexed by a small category $\mathbb{S}$, one can construct a "total" simplicial set $\int_{\mathbb{S}} X$ as follows. An n–simplex of $\int_{\mathbb{S}} X$ is a pair (s, x) where $s = (s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} s_n)$ is an n–simplex in the nerve of $\mathbb{S}$, and x is a function assigning to each map $u\colon [k] \to [n]$ in Δ a k–simplex x_k in $X(s_{u(0)})$, functorial in the following way. If $w = uv\colon [l] \xrightarrow{v} [k] \xrightarrow{u} [n]$, then

$u(0) \leq w(0)$ so there is a composition of α_i's from $s_{u(0)}$ to $s_{w(0)}$ in $\mathbb{S}$, denoted $\alpha_{w,u} = \alpha_{w(0)} \circ \alpha_{w(0)-1} \circ \cdots \circ \alpha_{u(0)+1}$. Then the functorial condition on the x_α's is

$$\alpha^*_{w,u}(x_w) = v^*(x_u).$$

Here $\alpha^*_{w,u}\colon X(s_{w(0)}) \to X(s_{u(0)})$, and this is an identity between l–simplices in $X(s_{u(0)})$. Notice that in the special case where we start with a diagram $\mathbb{C}\colon \mathbb{S}^{op} \to \mathit{Cat}$ of small categories, the diagram $N(\mathbb{C})\colon \mathbb{S}^{op} \to \mathit{sSet}$, obtained by composing with the nerve functor, satisfies the identity

$$\int_{\mathbb{S}} (N(\mathbb{C})) = N(\int_{\mathbb{S}} \mathbb{C}),$$

where $\int_{\mathbb{S}} \mathbb{C}$ on the right is the Grothendieck construction.

We shall now give a similar construction for diagrams of dendroidal sets. This construction will play a role in our definition of weak higher categories in Section 8. For this, we assume that the indexing category $\mathbb{S}$ has finite products. So, let $X\colon \mathbb{S}^{op} \to \mathit{dSet}$ be a diagram of dendroidal sets. We define a dendroidal set $\int_{\mathbb{S}} X$ as follows. For a tree T, an element of $\int_{\mathbb{S}} X_T$ is again a pair (t, x). Here $t \in N_d(\mathbb{S})_T$ is an element of the dendroidal nerve of $\mathbb{S}$ (where $\mathbb{S}$ is viewed as an operad via the cartesian structure). Such an element determines an object $in(t) \in \mathbb{S}$, defined by $in(t) = t(e_1) \times \cdots \times t(e_n)$ where $e_1, \ldots, e_n$ are the input edges of T (in some fixed arbitrary order). Note that for any arrow $u\colon S \to T$ in Ω, the dendrex t determines a map $in(t) \to in(tu)$ in $\mathbb{S}$ (defined by projections, the maps given by t, and the coherence maps in $\mathbb{S}$). Now x is a function which assigns to each such u an element $x_u \in X(in(tu))_S$, functorial in the following way: if $w = u \circ v$ as in $R \xrightarrow{v} S \xrightarrow{u} T$ then there is an induced map $in(tu) \xrightarrow{\alpha_{u,v}} in(tw)$ in $\mathbb{S}$, and we require

$$\alpha^*_{u,v}(x_v) = v^*(x_u)$$

The set $\int_{\mathbb{S}} X_T$ of such pairs (t, x) is contravariant in T, and defines the dendroidal set $\int_{\mathbb{S}} X$.

Note that this construction for dendroidal sets truly extends the one for simplicial sets, in the sense that for a diagram $X\colon \mathbb{S}^{op} \to \mathit{sSet}$ of simplicial sets where $\mathbb{S}$ is cartesian, there is a canonical isomorphism

$$i_! \int_{\mathbb{S}} X = \int_{\mathbb{S}} i_! X.$$

5 The tensor product of dendroidal sets

Like any other category of presheaves of sets, the category *dSet* has a closed cartesian structure. There is, however, another more interesting monoidal structure on the category of dendroidal sets, which we aim to describe in this section. To begin with, we will recall the tensor product for operads from Boardman and Vogt [5, Definition 2.14, page 41].

5.1 The Boardman–Vogt tensor product

Let $\mathcal{P}$ be an operad in *Set* over C and $\mathcal{Q}$ one over D. Their tensor product $\mathcal{P} \otimes_{BV} \mathcal{Q}$ is an operad coloured by the product set $C \times D$. The operations in $\mathcal{P} \otimes_{BV} \mathcal{Q}$ are generated by the following. Any $p \in \mathcal{P}(c_1, \dots, c_n; c)$ and any $d \in D$ define an operation

$$p \otimes d \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d), \dots, (c_n, d); (c, d)).$$

These operations compose in $\mathcal{P} \otimes_{BV} \mathcal{Q}$ in a way to make $p \mapsto p \otimes d$ a map of operads. Similarly, each operation $q \in \mathcal{Q}(d_1, \dots, d_n, d)$ and each $c \in C$ define an operation

$$c \otimes q \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c, d_1), \dots, (c, d_n); (c, d)),$$

and these compose as in $\mathcal{Q}$. Furthermore, the operations from $\mathcal{P}$ and $\mathcal{Q}$ distribute over each other, in the sense that for $p \in \mathcal{P}(c_1, \dots, c_n; c)$ and $q \in \mathcal{Q}(d_1, \dots, d_m; d)$,

$$\sigma^*_{n,m}((p \otimes d)(c_1 \otimes q, \dots, c_n \otimes q)) = (c \otimes q)(p \otimes d_1, \dots, p \otimes d_m)$$

where $\sigma_{n,m} \in \Sigma_{n \cdot m}$ is the permutation described as follows. Consider $\Sigma_{n \cdot m}$ as the set of bijections of the set $\{0, 1, \dots, n \cdot m - 1\}$. Each number in this set can be written uniquely in the form $k \cdot n + j$ where $0 \leq k < m$ and $0 \leq j < n$ as well as in the form $k \cdot m + j$ where $0 \leq k < n$ and $0 \leq j < m$. The permutation $\sigma_{n,m}$ is then defined by $\sigma_{n,m}(k \cdot n + j) = j \cdot m + k$. This tensor product makes the category of operads into a symmetric monoidal category.

This Boardman–Vogt tensor product preserves colimits in each variable separately. In fact, there is a corresponding internal *Hom*, making the category *Operad* into a symmetric *closed* monoidal category. For two operads $\mathcal{P}$ and $\mathcal{Q}$ as above, $\underline{Hom}(\mathcal{P}, \mathcal{Q})$ is the operad whose colours are the maps $\mathcal{P} \to \mathcal{Q}$, and whose operations are suitably defined multi-natural transformations. (Explicitly, for $\alpha_1, \dots, \alpha_n, \beta \colon \mathcal{P} \to \mathcal{Q}$, elements of $\underline{Hom}(\mathcal{P}, \mathcal{Q})(\alpha_1, \dots, \alpha_n; \beta)$ are maps f assigning to each colour $c \in C$ of $\mathcal{P}$ an element $f_c \in \mathcal{Q}(\alpha_1 c, \dots, \alpha_n c; \beta c)$. These f_c should be natural with respect to all operations in $\mathcal{P}$. For example, if $p \in \mathcal{P}(c_1, c_2; c)$ is a binary operation, then $\beta(p)(f_{c_1}, f_{c_2}) \in \mathcal{Q}(\alpha_1 c_1, \dots, \alpha_n c_1, \alpha_1 c_2, \dots, \alpha_n c_2; \beta c)$ is the image under a suitable permutation of $f_c(\alpha_1(p), \dots, \alpha_n(p)) \in \mathcal{Q}(\alpha_1 c_1, \alpha_1 c_2, \dots, \alpha_n c_1, \alpha_n c_2; \beta c)$).

For a symmetric monoidal category $\mathcal{E}$, the Boardman–Vogt tensor product of coloured operads in $\mathcal{E}$ still makes sense for *Hopf* operads $\mathcal{P}$ and $\mathcal{Q}$. For such operads, the categories $Alg_{\mathcal{E}}(\mathcal{P})$ and $Alg_{\mathcal{E}}(\mathcal{Q})$ are again symmetric monoidal, and a $(\mathcal{P} \otimes_{BV} \mathcal{Q})$–algebra in $\mathcal{E}$ is the same thing as a $\mathcal{P}$–algebra in $Alg_{\mathcal{E}}(\mathcal{Q})$, and is also the same thing as a $\mathcal{Q}$–algebra in $Alg_{\mathcal{E}}(\mathcal{P})$.

5.2 The tensor product of dendroidal sets

We now define a tensor product

$$\otimes\colon dSet \times dSet \to dSet$$

which is to preserve colimits in each variable separately. Since each dendroidal set is a colimit of representables, this tensor is completely determined by its effect on representable dendroidal sets $\Omega[S]$ and $\Omega[T]$, which we define as

$$\Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T)),$$

that is, as the dendroidal nerve of the Boardman–Vogt tensor product of the operads $\Omega(S)$ and $\Omega(T)$. It follows by general category theory (see Day [8] and Kelly [17]) that there exists an internal *Hom* for this tensor, defined for two dendroidal sets X and Y and an object T of Ω by

$$\underline{Hom}(X,Y)_T = Hom_{dSet}(\Omega[T] \otimes X, Y)$$

We summarise this discussion in the following proposition:

Proposition 5.1 *There exists a unique (up to natural isomorphism) symmetric closed monoidal structure on dSet, with the property that there is a natural isomorphism* $\Omega[S] \otimes \Omega[T] \cong N_d(\Omega(S) \otimes_{BV} \Omega(T))$ *for any two objects* S, T *of* Ω.

More generally, for suitable symmetric monoidal categories $\mathcal{E}$, there is such a monoidal structure on the category $d\mathcal{E}$ of dendroidal objects. See the Appendix for a discussion of dendroidal objects.

We mention some basic properties of the tensor product on *dSet*, in relation to the tensor product of operads, and to the product of simplicial sets.

Proposition 5.2 *The following properties hold.*

(i) *For any two dendroidal sets* X *and* Y, *there is a natural isomorphism*

$$\tau_d(X \otimes Y) \cong \tau_d(X) \otimes_{BV} \tau_d(Y).$$

(ii) *For any two operads $\mathcal{P}$ and $\mathcal{Q}$, there is a natural isomorphism*

$$\tau_d(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})) \cong \mathcal{P} \otimes_{BV} \mathcal{Q}.$$

Proof It suffices to check (i) for representable X and Y, in which case it follows from the identity $\tau_d N_d \cong \mathrm{id}$. By the same identity, (ii) follows from (i). □

Proposition 5.3 *For any two simplicial sets X and Y, and any dendroidal set D, there are natural isomorphisms*

(i) $i_!(X \times Y) \cong i_!(X) \otimes i_!(Y)$,

(ii) $i^* \underline{Hom}(i_!(X), D) \cong i^*(D)^X$,

(iii) $i^* \underline{Hom}(i_!(X), i_!(Y)) \cong Y^X$.

Proof The isomorphisms of type (ii) and (iii) are deduced from those of type (i), using the fact that $i_!$ is fully faithful. For (i), it suffices again to check this for representable simplicial sets $\Delta[n]$ and $\Delta[m]$. Observe first that, more generally, for any two small categories $\mathbb{C}$ and $\mathbb{B}$,

$$j_!(\mathbb{C} \times \mathbb{B}) \cong j_!(\mathbb{C}) \otimes_{BV} j_!(\mathbb{B}) \tag{1}$$

This holds in particular for the linear orders $[n]$ and $[m]$ viewed as categories, so

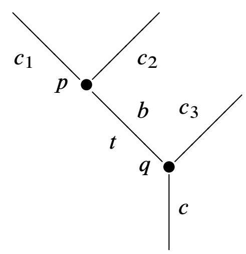
$$\begin{aligned} i_!(\Delta[n] \times \Delta[m]) &\cong i_!(N([n]) \times N([m])) \\ &\cong i_!(N([n] \times [m])) \\ &\cong N_d\, j_!([n] \times [m]) \\ &\cong N_d(j_![n] \otimes_{BV} j_![m]) \quad \text{(by (1))} \\ &\cong N_d(\Omega(n) \otimes_{BV} \Omega(m)) \\ &\cong \Omega[n] \otimes \Omega[m] \\ &\cong i_!(\Delta[n]) \otimes i_!(\Delta[m]). \end{aligned}$$

This shows that (i) holds for representables $\Delta[n]$ and $\Delta[m]$; as said, this completes the proof. □

6 The homotopy coherent nerve

In this section we introduce the homotopy coherent dendroidal nerve of an operad $\mathcal{P}$. This construction plays a crucial role in the definition of homotopy $\mathcal{P}$–algebras and weak higher categories, in Section 8. We begin by recalling the Boardman–Vogt resolution of operads [5] and its generalization (see Berger and Moerdijk [4]).

Let $\mathcal{P} = (C, P)$ be an operad in the category of compactly generated topological spaces, and let $H = [0, 1]$ be the unit interval. One can construct a (cofibrant) resolution $W(\mathcal{P}) \to \mathcal{P}$ as follows. $W(\mathcal{P})$ is again an operad coloured by C. The space $W(\mathcal{P})(c_1, \dots, c_n; c)$ is a quotient of a space of labelled planar trees. The edges of such a tree are labelled by elements of C, where in particular the input edges are labelled by the given $c_1, \dots, c_n$ and the output by c. Moreover, the inner edges carry a label $t \in H$ (a "length"), and each vertex v with input edges labelled $b_1, \dots, b_n \in C$ (in the planar order) and output edge labelled $b \in C$, is labelled by an element $p \in \mathcal{P}(b_1, \dots, b_n; b)$. For example,

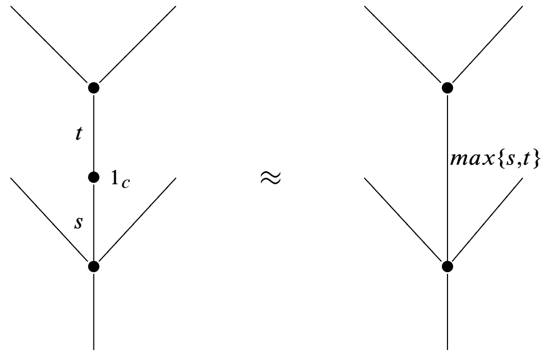


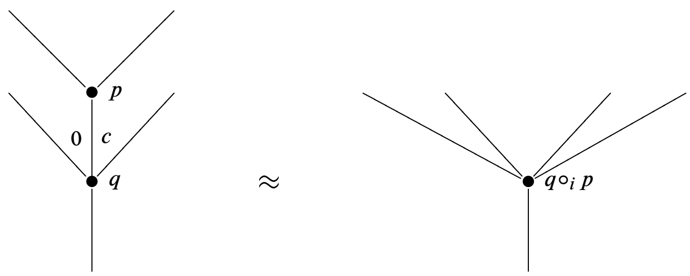
where $p \in \mathcal{P}(c_1, c_2; b)$, $q \in \mathcal{P}(b, c_3; c)$, $t \in [0, 1]$. There is a natural (product) topology on these trees, coming from the topology on $\mathcal{P}$ and that on H. The space $W(\mathcal{P})(c_1, \dots, c_n; c)$ is now the quotient space, obtained (by identifying isomorphic planar trees with the same labelling and) the following two relations (illustrated by the pictures below):

(i) Vertices labelled by an identity can be deleted, taking the maximum of the two adjacent lengths (or forgetting the lengths altogether if one of the adjacent edges is outer).

(ii) Edges of length zero can be contracted, using the $\circ_i$ form of the operad composition of $\mathcal{P}$.

The operad structure of $W(\mathcal{P})$ is given by grafting of trees, giving the newly arising inner edges length 1. The map $W(\mathcal{P}) \to \mathcal{P}$ is given by setting all lengths to zero (that is, forget the lengths and compose in $\mathcal{P}$).





In the article [4] by Berger and Moerdijk it is explained in detail how the above construction can be performed and studied in the more general context of operads in any symmetric monoidal category $(\mathcal{E}, \otimes, I)$, where [0, 1] is replaced by a suitable "interval" H in $\mathcal{E}$. This is an object H equipped with two "points" $0, 1\colon I \rightrightarrows H$, an augmentation $\epsilon\colon H \to I$ satisfying $\epsilon 0 = \mathrm{id} = \epsilon 1$, and a binary operation $\vee\colon H \otimes H \to H$ (playing the role of max) which is associative, and for which 0 is unital and 1 is absorbing ($0 \vee x = x = x \vee 0$ and $1 \vee x = 1 = x \vee 1$). This defines for any operad $\mathcal{P}$ in $\mathcal{E}$ a new operad $W_H(\mathcal{P})$ in $\mathcal{E}$ mapping to $\mathcal{P}$. The algebras for this operad are up-to-homotopy $\mathcal{P}$–algebras.

For example, one can take for $\mathcal{E}$ the category *Cat* of small categories which admits the following model category structure. The weak equivalences are categorical equivalences, the cofibrations are functors that are injective on objects, and the fibrations are those functors having the right lifting property with respect to the functor $0 \to H$, where H is the groupoid $0 \leftrightarrow 1$ with two objects and one isomorphism between them. The groupoid H also plays the role of the interval. We examine this possibility below, when we consider weak n–categories.

Example 6.1 Let $[n]$ be the linear tree, viewed as a (discrete) topological operad. So an $[n]$–algebra consists of a sequence of spaces $X_0, \ldots, X_n$, together with maps

$f_{ji}\colon X_i \to X_j$ for $i \le j$, such that $f_{ii} = \mathrm{id}$ and

$$f_{kj} \circ f_{ji} = f_{ki} \tag{1}$$

if $i \le j \le k$. A $W([n])$–algebra consists of such a sequence of spaces and maps, for which (1) holds only up to specified coherent higher homotopies. Since $W([n])$ is an operad with unary operations only, one can also think of it as a topological category: it has objects $0, 1, \dots, n$, and an arrow $i \to j$ in $W[n]$ is a sequence of "times" $t_{i+1}, \dots, t_{j-1}$ (each $t_k \in [0,1]$). In other words, $W[n](i,j)$ is the cube $[0,1]^{j-i-1}$ for $i+1 \le j$, a point for $i = j$, and the empty set for $i > j$. Composition is given by juxtaposing two such sequences, putting an extra time 1 in the middle: $(t_{i+1}, \dots, t_{j-1})\colon i \to j$ and $(t_{j+1}, \dots, t_{k-1})\colon j \to k$ compose to give

$$(t_{i+1}, \dots, t_{j-1}, 1, t_{j+1}, \dots, t_{k-1}).$$

If $\mathcal{C}$ is a category enriched in *Top* (that is, a topological category with a discrete set of objects), then the sets of continuous functors

$$\mathit{Top}(W[n], \mathcal{C})$$

for varying n define a simplicial set, which is exactly the homotopy coherent nerve of $\mathcal{C}$, described by Vogt [27].

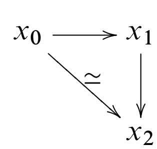
More generally, if $\mathcal{E}$ is a symmetric monoidal category with interval H, one can construct an $\mathcal{E}$–enriched category $W_H[n]$ with

$$W_H[n](i,j) = H^{\otimes j-i-1}$$

and define for each $\mathcal{E}$–enriched category $\mathcal{C}$ its homotopy coherent nerve $hcN(\mathcal{C})$ as the simplicial set given by

$$hcN(\mathcal{C})_n = \mathcal{E}\text{-}\mathit{Cat}(W_H[n], \mathcal{C}),$$

that is, the set of all $\mathcal{E}$–enriched functors from $W_H[n]$ to $\mathcal{C}$. For example, if $\mathcal{E} = \mathit{Cat}$ and $H = 0 \leftrightarrow 1$ as above, then an element of $hcN(\mathcal{C})_2$ is given by a triangle



which composes up to a specified invertible 2–cell in $\mathcal{C}$.

The above generalizes in a completely straightforward way to operads. Suppose $\mathcal{E}$ and H are as above. Each tree T defines an operad $\Omega(T)$ in *Set*, which we can view as an operad in $\mathcal{E}$ (via the functor $Operad \to Operad(\mathcal{E})$). Applying the generalized Boardman–Vogt construction yields an operad $W_H(T)$ in $\mathcal{E}$. This construction produces a functor $\Omega \to Operad(\mathcal{E})$, which induces an adjunction

$$Operad(\mathcal{E}) \underset{|\cdot|_H}{\overset{hcN_d}{\rightleftarrows}} dSet$$

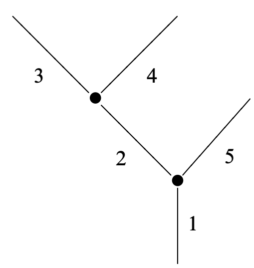
by Kan extension. For an operad $\mathcal{Q}$ in $\mathcal{E}$ the dendroidal set $hcN_d(\mathcal{Q})$ is called the *homotopy coherent dendroidal nerve* of $\mathcal{Q}$, and is given explicitly by

$$hcN_d(\mathcal{Q})_T = Hom_{Operad(\mathcal{E})}(W_H(T), \mathcal{Q}).$$

Remark 6.2 The functor $|-|_H$ is closely related to the W–construction for operads. In fact, if $\mathcal{P}$ is an operad in *Set*, then the Boardman–Vogt resolution $W_H(\mathcal{P}_\mathcal{E})$, of $\mathcal{P}$ viewed as an operad in $\mathcal{E}$, is isomorphic to the operad $|N_d(\mathcal{P})|_H$, as follows by direct inspection of the explicit construction of $W_H(\mathcal{P}_\mathcal{E})$ in the article by Berger and Moerdijk [4]. In particular, for an operad $\mathcal{P}$ in *Set* and an operad $\mathcal{Q}$ in $\mathcal{E}$, there is a natural bijective correspondence

$$Hom_{Operad(\mathcal{E})}(W_H(\mathcal{P}_\mathcal{E}), \mathcal{Q}) = Hom_{dSet}(N_d(\mathcal{P}), hcN_d(\mathcal{Q})).$$

Remark 6.3 Consider the special case where $\mathcal{E}$ is the category *Top* of compactly generated spaces, and H is the unit interval. If $\mathcal{P}$ is a topological operad and T is a tree (an object of Ω), then the set $hcN_d(\mathcal{P})_T$ of maps of topological operads $W_H(\Omega(T)) \to \mathcal{P}$ has a natural topology, as a topological sum of generalized mapping fibrations. For example, for the tree T with edges numbered $1, \ldots, 5$,



$hcN_d(\mathcal{P})_T$ is the sum, over all 5–tuples $c_1, \ldots, c_5$ of colours of $\mathcal{P}$, of mapping fibrations of the maps

$$\mathcal{P}(c_3, c_4; c_2) \times \mathcal{P}(c_2, c_5; c_1) \to \mathcal{P}(c_3, c_4, c_5; c_1).$$

Let $(dTop)^\delta$ be the category of dendroidal spaces with discrete set of vertices. Then $hcN_d(\mathcal{P})$ with this topology defines a functor $hcN_d(-)\colon Operad(Top) \to (dTop)^\delta$. This functor again has a left adjoint $|-|_H$, which relates to the Boardman–Vogt resolution of topological operads in the same way as above, by a natural isomorphism

$$W_H(\mathcal{P}) \cong |N_d(\mathcal{P})|_H,$$

where N_d and $|-|_H$ are now viewed as functors between the categories *Operad*(*Top*) and $(dTop)^\delta$.

7 The inner Kan condition for dendroidal sets

Let us begin by recalling some well known facts for simplicial sets (see for example Gabriel and Zisman [12]). Let $\Lambda^k[n] \subseteq \Delta[n]$ be the sub-simplicial set of the standard n–simplex, defined as the union of all the faces of $\Delta[n]$ except the one opposite the k th vertex. A simplicial set X is said to satisfy the Kan condition, or to be a Kan complex, if for any $n \geq 0$ and any k with $0 \leq k \leq n$, any map $\Lambda^k[n] \to X$ can be extended to a map $\Delta[n] \to X$. When this is required for $0 < k < n$ only, X is said to be an *inner Kan complex*, or, to satisfy the *inner Kan condition*. This condition was introduced (under the name "restricted Kan condition") by Boardman and Vogt in [5], while inner Kan complexes are being studied by Joyal [16] under the name *quasi-categories*. Observe that the nerve of a category is always an inner Kan complex. In this section we extend the notion of an inner Kan complex to the context of dendroidal sets.

Consider a tree T. Recall that a *face* of T is a map $S \to T$ which corresponds to either contracting an inner edge in T or pruning an outer cluster in T. Those corresponding to an edge contraction, that is, $\partial_e \colon T/e \to T$ for an inner edge e in T, are called *inner faces*. Let $\Lambda^e[T] \subseteq \Omega[T]$ be the dendroidal subset of the representable dendroidal set $\Omega[T]$, generated by all the faces of T *except* the inner face ∂_e. A dendroidal set X is said to satisfy the *inner Kan condition* if, for any tree T and any inner edge e in T, any map $\Lambda^e[T] \to X$ extends to a map $\Omega[T] \to X$ (that is, to an element in X_T). A dendroidal set satisfying the inner Kan condition is also called an *inner Kan complex.*

We now list some examples and properties of dendroidal inner Kan complexes. Some of the proofs involved are quite technical, and we refer to a companion paper [25] for a detailed exposition of the proofs.

Example 7.1 For any operad $\mathcal{P} \in$ *Operad*, one can easily check that the dendroidal nerve $N_d(\mathcal{P})$ is an inner Kan complex. In fact, any map $\Lambda^e[T] \to N_d(\mathcal{P})$ admits a *unique* extension to a map $\Omega[T] \to N_d(\mathcal{P})$, and this property characterizes those dendroidal sets that are nerves of operads.

More generally we have the following.

Proposition 7.2 *Let $\mathcal{E}$ be a monoidal model category with a chosen interval H. If $\mathcal{P} \in Operad(\mathcal{E})$ is a fibrant operad in the sense that each $\mathcal{P}(c_1, \ldots, c_n; c)$ is fibrant, then the homotopy coherent nerve $hcN_d(\mathcal{P})$ satisfies the inner Kan condition.*

A special case of this for simplicial categories was proved by Cordier and Porter [7].

Remark 7.3 Inner Kan simplicial sets and inner Kan dendroidal sets are related as follows. For a simplicial set X, the dendroidal set $i_!(X)$ is inner Kan if, and only if, X is. It is also true that if Y is a dendroidal set satisfying the inner Kan condition, then the simplicial set $i^*(Y)$ is again an inner Kan complex. The characterization of the nerves of operads as those dendroidal sets having unique fillers, is then the direct analogue of the well known fact that a simplicial set is the nerve of a category if, and only if, it is inner Kan with unique fillers.

The Grothendieck construction introduced above respects the inner Kan condition in the following sense.

Proposition 7.4 *If $X\colon \mathbb{S}^{op} \to dSet$ is a diagram of dendroidal sets, each of which is an inner Kan complex, then the dendroidal set $\int_{\mathbb{S}} X$ is also an inner Kan complex.*

Following Cisinski [6], we call a dendroidal set X *normal* if, for every object T of Ω and for every non-degenerate dendrex $x \in X_T$, the only automorphism of T which fixes x is the identify. For example if X is any simplicial set, then $i_!(X)$ is normal. And if $\mathcal{P}$ is a Σ–free operad (that is, each Σ_n acts freely), then $N_d(\mathcal{P})$ is normal.

Theorem 7.5 *Let K be a dendroidal set satisfying the inner Kan condition and let X be a normal dendroidal set. Then the dendroidal set $\underline{Hom}_{dSet}(X, K)$ satisfies the inner Kan condition.*

The proof is based on a careful analysis of shuffles of trees, together with the fact that normal dendroidal sets admit a nice skeletal filtration. This theorem specializes to simplicial sets. Indeed, if X and K are simplicial sets and K is inner Kan, then so is $i_!(K)$, and hence $\underline{Hom}(i_!(X), i_!(K))$ is a dendroidal inner Kan complex. Applying i^* to it, we see that Proposition 5.3(iii) implies that $\underline{Hom}(X, K)$ is a simplicial inner Kan complex. This simplicial result was already proved by Joyal [16]. Our proof of Theorem 7.5 thus provides in particular a proof of Joyal's result, different from the one given in [16] (and similar to the one in the thesis of Nichols-Barrer [26]).

8 Applications and further developments

In this last, somewhat speculative section, we would like to point out some possible further developments of the theory of dendroidal sets, related to "weak" maps between up-to-homotopy algebras, to enriched and weak higher categories, and to Quillen model categories.

To begin with, let $\mathcal{P}$ be an operad in Set. If $\mathcal{E}$ is a symmetric monoidal model category with a suitable interval H, then $W_H(\mathcal{P})$ is an operad in $\mathcal{E}$ whose algebras are homotopy $\mathcal{P}$–algebras (as mentioned in Section 6 above). The maps of $W_H(\mathcal{P})$–algebras are maps of homotopy $\mathcal{P}$–algebras which strictly commute with all higher homotopies, and this is a notion of map which for many purposes is too restrictive. It is possible to define a weaker notion of map between homotopy $\mathcal{P}$–algebras, but then the question arises to what extent these weak maps form a category.

Boardman and Vogt [5] construct a "quasi-category" of weak maps in the context of topological spaces; in Berger and Moerdijk [4, Theorem 6.9] a kind of Segal category of weak maps is constructed in the context of left proper monoidal model categories; in Hess, Parent and Scott [15] this question is approached via bimodules. The theory of dendroidal sets is relevant here. Indeed, $W_H(\mathcal{P})$–algebras in $\mathcal{E}$ are the same thing as operad maps $W_H(\mathcal{P}) \to \mathcal{E}$, or equivalently, as maps of dendroidal sets $N_d(\mathcal{P}) \to hcN_d(\mathcal{E})$ (see Remark 6.2 above). They thus arise as the vertices of the dendroidal set

$$\underline{Hom}_{dSet}(N_d(\mathcal{P}), hcN_d(\mathcal{E})). \tag{2}$$

Dendrices of shape $i[1]$ (where $i\colon \Delta \to \Omega$) encode a suitable notion of weak map, and such weak maps can be composed (in an up-to-homotopy way) whenever this dendroidal set (2) is an inner Kan complex. This is the case, for example, when $\mathcal{P}$ is Σ–free and every object in $\mathcal{E}$ is fibrant, cf Proposition 7.2.

Notice that, more generally, one might consider (weak) $\mathcal{P}$–algebras with values in any dendroidal set X, as vertices of the dendroidal Hom–set

$$\underline{Hom}_{dSet}(N_d(\mathcal{P}), X).$$

If $\mathcal{P}$ is Σ–free then this dendroidal set is an inner Kan complex whenever X is (Theorem 7.5), in which case maps between $\mathcal{P}$–algebras (again defined as dendrices of shape $i[1]$) can be composed. The case $X = hcN_d(\mathcal{E})$ is the one discussed before. It is also possible to iterate this construction, and consider for another operad $\mathcal{Q}$ the dendroidal set

$$\underline{Hom}_{dSet}(N_d\mathcal{Q}, \underline{Hom}_{dSet}(N_d\mathcal{P}, X))$$

which is of course isomorphic to

$$\underline{Hom}_{dSet}(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q}), X).$$

This dendroidal set admits a map from

$$\underline{Hom}_{dSet}(N_d(\mathcal{P} \otimes_{BV} \mathcal{Q}), X)$$

but is in general not isomorphic to it, unless X is (the dendroidal nerve of) an operad. In particular, for the case $X = hcN_d(\mathcal{E})$, one has a map

$$\underline{Hom}(W_H(\mathcal{P} \otimes_{BV} \mathcal{Q}), \mathcal{E}) \to \underline{Hom}(|N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})|_H, \mathcal{E})$$

which gives different but related notions of iterated weak algebras in $\mathcal{E}$. It would be interesting to compare this to the work of Dunn, Fiedorowicz, and Vogt on the tensor product of operads (see, for example, Dunn [9] and Fiedorowicz [10]) . (In this context, we should point out that, up to now, $\mathcal{P}$ and $\mathcal{Q}$ have been operads in *Set*, but the same applies to topological operads. Indeed, for the category *Top* of compactly generated spaces, the homotopy coherent dendroidal nerve $hcN_d(Top)$ with respect to the usual unit interval is naturally a (large) dendroidal space. If $\mathcal{P}$ is an operad in *Top*, then homotopy $\mathcal{P}$–algebras in *Top* are the vertices of the dendroidal space $\underline{Hom}(N_d(\mathcal{P}), hcN_d(Top))$, etc. We expect that (under suitable cofibrancy conditions on $\mathcal{P}$) this dendroidal space satisfies the inner Kan condition.

We would like to consider the special case of the operad $\mathcal{A}_S$ whose algebras are categories with a given set S as objects (Example 2.4). Note that this operad is Σ–free (like any operad obtained by symmetrization, cf. Remark 2.3). For a fixed dendroidal set X, one can consider the dendroidal set

$$\underline{Hom}(N_d(\mathcal{A}_S), X).$$

By definition, we call its vertices X–enriched categories over S. Its dendrices of shape $i[1]$ provide an interpretation of the notion of "functor" between X–enriched categories over S. By varying S, one obtains a *Set*–indexed diagram of dendroidal sets, which the dendroidal Grothendieck construction (see Section 4.1) assembles into a single dendroidal set

$$\underline{Cat}(X) := \int_{Set} \underline{Hom}(N_d(\mathcal{A}_S), X).$$

By definition, its vertices are categories enriched in X, while its dendrices of shape $i[1]$ are functors between such categories. In this context, it is relevant to observe that by Theorem 7.5 and Proposition 7.4, $\underline{Cat}(X)$ is a dendroidal inner Kan complex whenever X is, so that a composition of functors between X–enriched categories exists. We also note that the construction can be iterated, so as to form the dendroidal inner Kan complex

$$\underline{Cat}^2(X) = \underline{Cat}(\underline{Cat}(X))$$

of X–enriched bicategories, and so on.

Let us consider a few special cases of this construction. First of all, if $\mathcal{E}$ is a symmetric monoidal category, one can construct its dendroidal nerve $N_d(\mathcal{E})$. The dendroidal set $\underline{Cat}(N_d(\mathcal{E}))$ then captures the *usual* notion of $\mathcal{E}$–enriched categories and functors. More precisely, it is isomorphic to the dendroidal nerve of the usual monoidal category $Cat(\mathcal{E})$ of $\mathcal{E}$–enriched categories,

$$\underline{Cat}(N_d(\mathcal{E})) \cong N_d(Cat(\mathcal{E})),$$

where $Cat(\mathcal{E})$ is considered as an operad via the usual tensor product of enriched categories. As a particular case, consider the category $\underline{Cat}$ of small categories with its cartesian monoidal structure. Then the dendroidal set $Cat^n(N_d(\underline{Cat}))$, obtained by iterating the construction n times, is the dendroidal nerve of the category of strict $(n+1)$–categories. It also encodes all higher structure of functors, natural transformations, modifications, and so on.

If $\mathcal{E}$ is a monoidal model category with a suitable interval H, one can consider categories enriched in the homotopy coherent nerve $hcN_d(\mathcal{E})$ (defined in terms of H). For example, if $\mathcal{E}$ is the category of chain complexes over a ring R (with the projective model structure and the usual interval H of normalized chains on the standard 1–simplex), then $\underline{Cat}(hcN_d(\mathcal{E}))$ is a dendroidal inner Kan complex whose vertices are precisely A_∞–categories (see Fukaya [11], Lefèvre-Hasegawa [18] and Lyubashenko [22]). As another example, let $\mathcal{E} = Top$ with the unit interval, and consider for the one-point set $*$ the operad $Ass = \mathcal{A}_*$ and the dendroidal inner Kan complex

$$A_\infty = \underline{Hom}(N_d(Ass), hcN_d(\mathcal{E})).$$

The vertices of this dendroidal set are precisely A_∞–spaces, while dendrices of more general shapes encode operations between A_∞–spaces. Again, the construction can be iterated to form dendroidal inner Kan complexes $A_\infty^{(1)} = A_\infty$ and

$$A_\infty^{(n+1)} = \underline{Hom}(N_d(Ass), A_\infty^{(n)}).$$

It would be interesting to study the relation between $A_\infty^{(n)}$ and n–fold loop spaces in topology (see Dunn [9] and May [24]). In this context, it is important to note that, although $Ass \otimes_{BV} Ass = Comm$, the dendroidal tensor product $N_d(Ass) \otimes N_d(Ass)$ is considerably larger than $N_d(Comm)$, and in fact cannot be the nerve of an operad.

Finally, the category $\underline{Cat}$ of small categories itself is a monoidal model category with interval H as in Section 6 above, and

$$\underline{Hom}(N_d(\mathcal{A}_S), hcN_d(\underline{Cat}))$$

is a dendroidal inner Kan complex capturing the notion of a *bicategory* with S as set of objects (see Bénabou [2]) or rather the notion of an unbiased bicategory (see Leinster [20]). This construction can again be iterated. For example, the dendroidal inner Kan complex $\underline{Hom}(N_d(Ass), \underline{Hom}(N_d(Ass), hcN_d(\underline{Cat})))$ captures braided monoidal categories (and all higher maps between them). The above construction of categories enriched in $\mathcal{E}$ yields, by considering $\mathcal{E} = \underline{Cat}$, an inductive definition of weak n–categories. More precisely, let $WCat_1 = \underline{Cat}$ and for $n > 1$ let

$$WCat_n = \underline{Cat}^{n-1}(hcN_d(\underline{Cat})).$$

For each $n \geq 1$, $WCat_n$ is a dendroidal inner Kan complex. Its vertices are weak n–categories of a special kind. (They have an underlying strict category of 1–cells, and for any two objects x and y, the same is true at level $n-1$ for the dendroidal set $Hom(x, y)$). There are many alternative notions of weak n–categories in the literature (see Leinster [19] for a survey of 10 such definitions and Baez and Dolan [1] for a more general discussion of weak n–categories), and we expect that for any reasonable notion, a weak n–category can be "strictified" to a weak n–category in our sense.

Finally, we would like to say a few words about possible Quillen model structures on dendroidal Sets. Recall from Joyal [16] and Lurie [21] that there is a Quillen model structure on simplicial sets, in which the inner Kan complexes are exactly the fibrant objects. This model structure is related to the "folk" monoidal model structure on $\underline{Cat}$ already mentioned above, in which the weak equivalences are the equivalences of categories and the cofibrations are the functors which are injective on objects. Indeed, a map $X \to Y$ between simplicial sets is a weak equivalence in Joyal's model structure if, and only if, for every simplicial inner Kan complex K, the map $\tau(K^Y) \to \tau(K^X)$ is an equivalence of categories (here $\tau\colon sSet \to Cat$ is the functor discussed in Section 4). The analog of Theorem 7.5 for simplicial sets, which states that K^X and K^Y are again inner Kan complexes, plays an important role in Joyal's model structure.

The folk model structure on $\underline{Cat}$ generalizes without much effort to one on (coloured) operads, in which a map $f\colon \mathcal{Q} \to \mathcal{P}$, from an operad $\mathcal{Q}$ on D to an operad $\mathcal{P}$ on C (as in Section 2) is a weak equivalence if, and only if, $j^*(f)\colon j^*\mathcal{Q} \to j^*\mathcal{P}$ is an equivalence of categories, and moreover f induces a bijection

$$\mathcal{Q}(d_1, \ldots, d_n; d) \to \mathcal{P}(fd_1, \ldots, fd_n; fd)$$

for any sequence $d_1, \dots, d_n, d$ of colours in D. We conjecture that the inner Kan complexes are the fibrant objects in a model structure on dendroidal sets in which a map $X \to Y$ is a weak equivalence if, and only if, for any dendroidal inner Kan complex K, the map

$$\tau_d(\underline{Hom}_{dSet}(Y, K)) \to \tau_d(\underline{Hom}_{dSet}(X, K))$$

is a weak equivalence of operads. Theorem 7.5 should be a substantial step towards a proof of this conjecture.

Appendix A The tensor product of dendroidal objects

Let $\mathcal{E}$ be a symmetric monoidal category. The category of dendroidal objects in $\mathcal{E}$ is the functor category $\mathcal{E}^{\Omega^{op}}$, which we denote by $d\mathcal{E}$. This category has a Boardman–Vogt style tensor product, and a corresponding internal $\underline{Hom}$ whenever $\mathcal{E}$ itself is closed. The construction and its basic properties are explained most easily after recalling some basic facts about "enriched Kan extensions", so we'll do that first. None of the material in Section A.1 is really new, and we refer the reader to Kelly [17] for more background.

A.1 Enriched Kan extensions

We begin by developing a bit of formalism similar to the language of rings and bimodules. Let $\mathcal{E}$ be a symmetric monoidal category, and let $\mathcal{S}$ be any $\mathcal{E}$–enriched category. Suppose $\mathcal{S}$ is *tensored* over $\mathcal{E}$. (This means that one can construct an object $E \otimes S$ in $\mathcal{S}$ for E in $\mathcal{E}$ and S in $\mathcal{S}$, with the property that there is a natural $\underline{Hom}$–tensor correspondence between maps $E \otimes S \to T$ in $\mathcal{S}$ and $E \to \underline{Hom}(S, T)$ in $\mathcal{E}$; see Kelly [17] for a formal definition. For small categories $\mathbb{A}$ and $\mathbb{B}$ (in *Set*), we write

$${}_{\mathbb{A}}\mathcal{E}_{\mathbb{B}} = \mathcal{E}^{\mathbb{B}^{op} \times \mathbb{A}}$$

for the category of functors $\mathbb{B}^{op} \times \mathbb{A} \to \mathcal{E}$. For objects $X \in {}_{\mathbb{A}}\mathcal{E}_{\mathbb{B}}$ and A in $\mathbb{A}$, B in $\mathbb{B}$, we write

$${}_{A}X_{B} = X(B, A) \in \mathcal{E}$$

for the value at (B, A). Also, if $\mathbb{A}$ or $\mathbb{B}$ is the trivial category $\star$ we delete it from the notation. So

$${}_{\mathbb{A}}\mathcal{E}_{\star} = {}_{\mathbb{A}}\mathcal{E} = \mathcal{E}^{\mathbb{A}}, \quad {}_{\star}\mathcal{E}_{\mathbb{B}} = \mathcal{E}_{\mathbb{B}} = \mathcal{E}^{\mathbb{B}^{op}}.$$

Now assume $\mathcal{E}$ has small limits and $\mathcal{S}$ has small colimits. There is a tensor product functor

(3) $$\otimes_{\mathbb{B}} = {}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}} \times {}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}} \to {}_{\mathbb{C}}\mathcal{S}_{\mathbb{A}}$$

defined for E in ${}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}}$ and S in ${}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}}$, by the usual coequalizer

$$_C(E \otimes_{\mathbb{B}} S)_A \longleftarrow \coprod_B ({}_C E_B) \otimes ({}_B S_A) \leftleftarrows \coprod_{B \to B'} ({}_C E_{B'}) \otimes ({}_B S_A)$$

for any two objects $C \in \mathbb{C}$, $A \in \mathbb{A}$. This tensor product has a corresponding internal $\underline{Hom}$,

$$\text{(4)} \qquad \underline{Hom}_{\mathbb{A}}\colon \ {}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}} \times {}_{\mathbb{C}}\mathcal{S}_{\mathbb{A}} \to {}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}},$$

satisfying the usual adjunction property stating a bijective correspondence between maps

$$E \otimes_{\mathbb{B}} S \to T \ \text{ in } {}_{\mathbb{C}}\mathcal{S}_{\mathbb{A}}$$

and maps

$$E \to \underline{Hom}_{\mathbb{A}}(S, T) \ \text{ in } {}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}}.$$

We point out two special cases of this *Hom*–tensor correspondence. First, if F is an element of ${}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}}$, that is, $F\colon \mathbb{B} \to \mathcal{S}_{\mathbb{A}}$, then we obtain adjoint functors

$$f_!\colon \mathcal{E}_{\mathbb{B}} \rightleftarrows \mathcal{S}_{\mathbb{A}} : f^*$$

defined in terms of the previous functors for the special case $\star = \mathbb{C}$, by

$$f_!(E) = E \otimes_{\mathbb{B}} F \quad f^*(S) = \underline{Hom}_{\mathbb{A}}(F, S) = \underline{Hom}(F, S).$$

These functors f^* and $f_!$ are the right and left Kan extensions along F. Secondly, there are "external" tensor and *Hom* functors

$$\text{(5)} \qquad \underline{\otimes}\colon \mathcal{E}_{\mathbb{C}} \times \mathcal{S}_{\mathbb{A}} \to \mathcal{S}_{\mathbb{C}\times\mathbb{A}}$$

$$\text{(6)} \qquad \underline{Hom}_{\mathbb{A}}\colon \mathcal{S}_{\mathbb{A}} \times \mathcal{S}_{\mathbb{C}\times\mathbb{A}} \to \mathcal{E}_{\mathbb{C}}$$

for which there is a natural correspondence between maps

$$X \underline{\otimes} Y \to Z \ \text{ in } \mathcal{S}_{\mathbb{C}\times\mathbb{A}}$$

and

$$X \to \underline{Hom}_{\mathbb{A}}(Y, Z) \ \text{ in } \mathcal{E}_{\mathbb{C}}$$

Indeed, this is the special case where $\mathbb{B} = \star$ while $\mathbb{C}$ is replaces by $\mathbb{C}^{op}$, so that (3) and (4) can be rewritten as $\otimes\colon {}_{\mathbb{C}^{op}}\mathcal{E} \times \mathcal{E}_{\mathbb{A}} \to {}_{\mathbb{C}^{op}}\mathcal{E}_{\mathbb{A}}$ and $\underline{Hom}_{\mathbb{A}}\colon \mathcal{S}_{\mathbb{A}} \times {}_{\mathbb{C}^{op}}\mathcal{S}_{\mathbb{A}} \to {}_{\mathbb{C}^{op}}\mathcal{E}$, defining (5) and (6).

Now consider a functor $F\colon \mathbb{A} \times \mathbb{A} \to \mathcal{S}_{\mathbb{A}}$, that is, $F \in_{\mathbb{A}\times\mathbb{A}} \mathcal{S}_{\mathbb{A}}$. Then by Kan extension we have a functor

$$\mathcal{E}_{\mathbb{A}} \times \mathcal{S}_{\mathbb{A}} \xrightarrow{\underline{\otimes}} \mathcal{S}_{\mathbb{A}\times\mathbb{A}} \xrightarrow{f_!} \mathcal{S}_{\mathbb{A}}$$

which we write as $\otimes^{(F)}$; so

$$E \otimes^{(F)} S = f_!(E \underline{\otimes} S) = (E \underline{\otimes} S) \otimes_{\mathbb{A}\times\mathbb{A}} F.$$

The above discussion also yields a corresponding *Hom*–functor, denoted

$$\underline{Hom}^{(F)}\colon \mathcal{S}_{\mathbb{A}} \times \mathcal{S}_{\mathbb{A}} \to \mathcal{E}_{\mathbb{A}},$$

for which there is a bijective correspondence between maps

$$E \otimes^{(F)} S \to T \quad (\text{in } \mathcal{S}_{\mathbb{A}})$$

and maps

$$E \to \underline{Hom}^{(F)}(S,T) \quad (\text{in } \mathcal{E}_{\mathbb{A}}).$$

Indeed, one can simply define $\underline{Hom}^{(F)}$ in terms of the earlier $\underline{Hom}$ and the right adjoint f^*, as

$$\underline{Hom}^{(F)}(S,T) = \underline{Hom}_{\mathcal{A}}(S, f^*T)$$

A.2 Monoidal closed structure of $d\mathcal{E}$

Let us now consider a complete and cocomplete symmetric closed monoidal category $\mathcal{E}$, and the category $d\mathcal{E}$ of dendroidal object in $\mathcal{E}$. Let $\mathbb{A} = \Omega$, let $\mathcal{S} = \mathcal{E}$, and let $F = BV$ be the Boardman–Vogt tensor product of Hopf operads in $\mathcal{E}$, restricted to operads coming from Ω:

$$BV\colon \Omega \times \Omega \longrightarrow Operad(\mathcal{E}) \xrightarrow{N_d} d\mathcal{E}$$

Then the last construction of Section A.1 yields a functor

$$\otimes^{(BV)}\colon d\mathcal{E} \times d\mathcal{E} \to d\mathcal{E}$$

and a corresponding *Hom*–functor

$$\underline{Hom}^{(BV)}\colon d\mathcal{E} \times d\mathcal{E} \to d\mathcal{E}$$

satisfying the usual properties, and *making $d\mathcal{E}$ into a closed symmetric monoidal category*.

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