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Quantum Teichmüller spaces and Kashaev's $6j$–symbols

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The Kashaev invariants of 3–manifolds are based on $6j$–symbols from the representation theory of the Weyl algebra, a Hopf algebra corresponding to the Borel subalgebra of $U_q(sl(2,\mathbb{C}))$. In this paper, we show that Kashaev's $6j$–symbols are intertwining operators of local representations of quantum Teichmüller spaces. This relates Kashaev's work with the theory of quantum Teichmüller space, which was developed by Chekhov–Fock, Kashaev and continued by Bonahon–Liu.

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1 Introduction

Since the early eighties, the theory of quantum invariants of links and 3–manifolds has grown up rapidly as a very active domain of research. However, it is not yet clear which topological information are carried by these invariants. One of the most important conjectural relations between the topology of a link in a manifold and its quantum invariants has been constructed by Kashaev [13; 14] with his Volume Conjecture, based on a family of complex valued invariants $\{\langle L\rangle_N\}$ of links L in S^3. Later Murakami–Murakami [18] identified Kashaev's invariants as evaluations of colored Jones polynomials at the root of unity $\exp(2\pi i/N)$, which brought the Volume Conjecture to the forefront.

The Volume Conjecture is extended to various aspects, such as Baseilhac–Benedetti [6], Murakami et al [19], Gukov [12]. Also see the references in Costantino [9] for the recent progress on the Volume Conjecture.

In a different area, the theory of the Quantum Teichmüller Space was developed by Chekhov–Fock [11] and, independently, by Kashaev [15]. Bonahon–Liu [7] investigated the finite dimensional representation theory of the (exponential version of the) Quantum Teichmüller Space to construct invariants of surface diffeomorphism; see also Bai [2], Bai–Bonahon–Liu [3]. In computations, these invariants look very similar to those of Baseilhac–Benedetti [6] and Kashaev [14; 15]. Although the objects involved appear very different, the role played by the Pentagon Relation and by the quantum dilogarithm function also hints at a connection between these two points of view. This paper is devoted to making this connection explicit.

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The Kashaev invariants are based on the representation theory of the Weyl algebra $\mathcal{W}$, a Hopf algebra corresponding to the Borel subalgebra of $U_q(sl(2,\mathbb{C}))$. In particular, the Hopf algebra structure is used in a critical way.

If $\mu\colon \mathcal{W}\to \mathrm{End}(V_\mu)$ and $\nu\colon \mathcal{W}\to \mathrm{End}(V_\nu)$ form a *regular* pair of (see Section 2.1 for specific definition) representations of the Weyl algebra $\mathcal{W}$, then all irreducible components of the representation $\mu\otimes\nu$ are isomorphic to a representation $\mu\nu$. This irreducible component $\mu\nu$ has multiplicity N, in the sense that the space $V_{(\mu,\nu)} = \mathrm{Hom}_{\mathcal{W}}(V_{\mu\nu}, V_\mu\otimes V_\nu)$ has dimension N. The elements of $V_{(\mu,\nu)}$ are the *Clebsch–Gordan operators* of μ and ν. There is a natural *canonical map* Ω

$$\Omega = \Omega(\rho,\mu)\colon V_{(\rho,\mu)}\otimes V_{\rho\mu}\to V_\rho\otimes V_\mu$$

defined by $\Omega(f\otimes v) = f(v)$.

For a regular triple (μ,ν,σ) of representations of the Weyl algebra $\mathcal{W}$, all irreducible components of the representation $\mu\otimes\nu\otimes\sigma$ are isomorphic to a representation $\mu\nu\sigma$. with multiplicity N^2. The two different ways of grouping terms in $(\mu\otimes\nu)\otimes\sigma = \mu\otimes(\nu\otimes\sigma)$ lead to two different ways of grouping together the embedding of $V_{\mu\nu\sigma}$ in $V_\mu\otimes V_\nu\otimes V_\sigma$. The key ingredient of the Kashaev invariant of [13; 5] is *Kashaev's $6j$–symbol*

$$R(\mu,\nu,\sigma)\colon V_{(\mu,\nu)}\otimes V_{(\mu\nu,\sigma)}\to V_{(\mu,\nu\sigma)}\otimes V_{(\nu,\sigma)}$$

which describes this correspondence.

More precisely, $R(\mu,\nu,\sigma)$ is defined by the five-term relation

$$\Omega_{23}(\nu,\sigma)\Omega_{13}(\mu,\nu\sigma)R_{12}(\mu,\nu,\sigma) = \Omega_{12}(\mu,\nu)\Omega_{23}(\mu\nu,\sigma).$$

involving linear maps $V_{(\mu,\nu)}\otimes V_{(\mu\nu,\sigma)}\otimes V_{\mu\nu\sigma}\to V_\mu\otimes V_\nu\otimes V_\sigma$, see Section 2.2 for details.

The quantum Teichmüller space $\mathcal{T}_S^q$ of a punctured surface S is based on completely different data. It is a non-commutative deformation of the algebra of rational functions on the classical Teichmüller space. The non-commutativity is measured by a parameter $q = e^{\pi i\hbar}$, and the classical case corresponds to $q=1$.

More precisely, consider an ideal triangulation λ of the surface S, with edges $\lambda_1,\dots,\lambda_n$, and triangles $T_1,\dots,T_m$. The *Chekhov–Fock algebra* $\mathcal{T}_\lambda^q = \mathbb{C}[X_1,X_2,\dots,X_n]_\lambda^q$ of λ is generated by $X_1^{\pm1}, X_2^{\pm1},\dots,X_n^{\pm1}$ associated to the edges of λ, with relations $X_iX_j = q^{2\sigma_{ij}}X_jX_i$, with integers $\sigma_{ij}\in\{0,\pm1,\pm2\}$ determined by the combinatorics of the ideal triangulation. As one moves from one ideal triangulation λ to another λ', Chekhov and Fock [10; 11] make this construction independent of the choice of ideal triangulations by introducing explicit *coordinate change isomorphisms* $\varphi_{\lambda\lambda'}^q\colon \mathcal{T}_{\lambda'}^q\to\mathcal{T}_\lambda^q$.

A *local representation* of the Chekhov–Fock algebra $\mathcal{T}^q_\lambda$ is a certain type of representation $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)$, where $V_\lambda = V_1 \otimes \cdots \otimes V_m$ and the V_j are associated to the triangles T_j; see Section 3.3 for precise definition. A *local representation* ρ of the quantum Teichmüller space $\mathcal{T}^q_S$ consists of the data of a local representation $\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)$ for every ideal triangulation λ, in such a way that $\rho_\lambda \circ \varphi^q_{\lambda\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V_\lambda)$ and $\rho_{\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V_{\lambda'})$ are isomorphic for any pair of ideal triangulations λ, λ'. The *intertwining operator* $L^\rho_{\lambda\lambda'}$ of ρ are the linear maps such that

$$L^\rho_{\lambda\lambda'}\colon V_1 \otimes \cdots \otimes V_m \to V'_1 \otimes \cdots \otimes V'_m$$

that realize the isomorphism $\rho_\lambda \circ \varphi^q_{\lambda\lambda'} \cong \rho_{\lambda'}$, namely such that $L^\rho_{\lambda\lambda'} \circ \rho_\lambda \circ \varphi^q_{\lambda\lambda'}(X') = \rho_{\lambda'}(X') \circ L^\rho_{\lambda\lambda'}$ for every $X' \in \mathcal{T}^q_{\lambda'}$.

The intertwining operators $L^\rho_{\lambda\lambda'}$ are only defined up to scalar multiplication. They are uniquely determined by the case when the surface S is a square and when λ and λ' differ only by the choice of diagonal; see [3].

Our main result is the following theorem.

Theorem 1 *Let V_1, V_2, V'_1 and V'_2 be $\mathcal{W}$–modules for the Weyl algebra $\mathcal{W}$, and $L\colon V_1 \otimes V_2 \to V'_1 \otimes V'_2$ be a linear map (not supposed to be $\mathcal{W}$–linear). Then the following are equivalent.*

1. *There exists a regular pair (μ, ν) of irreducible representations of the Weyl algebra $\mathcal{W}$ and isomorphisms of $\mathcal{W}$–spaces*

$$V_1 \cong V_{(\mu,\nu)}, V_2 \cong V_{\mu\nu}, V'_1 \cong V_\mu, V'_2 \cong V_\nu,$$

   for which L corresponds to a scalar multiple of the canonical map

$$\Omega(\mu,\nu)\colon V_{(\mu,\nu)} \otimes V_{\mu\nu} \to V_\mu \otimes V_\nu.$$

2. *There exists a regular triple (μ, ν, σ) of irreducible representations of the Weyl algebra $\mathcal{W}$ and isomorphisms of $\mathcal{W}$–spaces*

$$V_1 \cong V_{(\mu,\nu)}, V_2 \cong V_{(\mu\nu,\sigma)}, V'_1 \cong V_{(\mu,\nu\sigma)}, V'_2 \cong V_{(\nu,\sigma)},$$

   for which L corresponds to a scalar multiple of Kashaev's $6j$–symbol

$$R(\mu,\nu,\sigma)\colon V(\mu,\nu) \otimes V(\mu\nu,\sigma) \to V(\mu,\nu\sigma) \otimes V(\nu,\sigma).$$

3. *L is an intertwining operator*

$$L^\rho_{\lambda\lambda'}\colon V_1 \otimes V_2 \to V'_1 \otimes V'_2,$$

namely is an isomorphism between $\rho_\lambda \circ \varphi^q_{\lambda\lambda'}$ *and* $\rho_{\lambda'}$. *Here* $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_1 \otimes V_2),\ \rho_{\lambda'} \colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V_1' \otimes V_2')\}$ *is a local representation of the quantum Teichmüller space of the square* S.

A byproduct of Theorem 1 is the fact, already observed by Kashaev [13] and Baseilhac [4], that the Clebsch–Gordan operators and the $6j$–symbols have the same matrix expression in suitable basis, although their approach used the representation of the canonical element of the Heisenberg double of the Weyl algebra.

In future, we will try to identify Kashaev-type invariants with invariants coming out of the quantum Teichmüller theory, and generate these invariants in a more general framework of some kind of TQFT (Topological Quantum Field Theory) structure related to $PSL(2,\mathbb{C})$, as the familar Reshetikhin–Turaev invariants[20] related to $SU(2)$.

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2 Kashaev’s $6j$–symbol

2.1 The Weyl algebra and its representations

The Weyl algebra $\mathcal{W} = \mathcal{W}^q$ is the Hopf algebra over $\mathbb{C}[q, q^{-1}]$ generated by $X^{\pm 1}, Y$ with relation $XY = q^2 YX$. Its comultiplication, counit and antipode are given by

$$\begin{aligned} \Delta(X) &= X \otimes X, & \Delta(Y) &= X^{-1} \otimes Y + Y \otimes 1; \\ \epsilon(X) &= 1, & \epsilon(Y) &= 0; \\ S(X) &= X^{-1}, & S(Y) &= -XY. \end{aligned}$$

The reader will notice that the notation for the Weyl algebra $\mathcal{W}$ is slightly different from that in [5; 13], but it is clearly equivalent and better suited to our purpose.

A *representation* of the Hopf algebra $\mathcal{W}$ is a left $\mathcal{W}$–module, here $\mathcal{W}$ is simply regarded as an algebra. Given a representation $\mu \colon \mathcal{W} \to \mathrm{End}(V_\mu)$ of $\mathcal{W}$, the action of $W \in \mathcal{W}$ on an element $v \in V_\mu$ is denoted by $W_\mu \cdot v = \mu(W)(v)$, or simply $W \cdot v$ if no ambiguity on the representation.

The comultiplication of $\mathcal{W}$ allows one to make the tensor product of representations of $\mathcal{W}$ into a new representation. If $\mu\colon \mathcal{W} \to \mathrm{End}(V_\mu)$ and $\nu\colon \mathcal{W} \to \mathrm{End}(V_\nu)$ are representations of $\mathcal{W}$, then the representation $\mu \otimes \nu$ is defined by

$$W \cdot (v \otimes w) = \Delta(W) \cdot (v \otimes w) = \sum W_{(1)} \cdot v \otimes W_{(2)} \cdot w$$

for any $W \in \mathcal{W}$, $v \in V_\mu$ and $w \in V_\nu$. Here we use Sweedler's sigma notation

$$\Delta(W) = \sum W_{(1)} \otimes W_{(2)}.$$

For general properties of representation theory of Hopf algebras, we refer readers to Chari–Pressley [8] or Montgomery [17].

Any linear space V can be given a *trivial* representation of $\mathcal{W}$ by

$$W \cdot v = \epsilon(W)v \quad \forall W \in \mathcal{W}, v \in V.$$

It is of interest to consider non-trivial finite dimensional representations of $\mathcal{W}$. They exist only when q^2 is a root of unity. We consequently assume that q^2 be a primitive Nth root of unity for some positive integer N.

A representation $\mu\colon \mathcal{W} \to \mathrm{End}(V_\mu)$ over a finite dimensional complex vector space V_μ is *cyclic* if the operators $\mu(X)$ and $\mu(Y)$ are invertible. A sequence of cyclic representations $(\mu_1, \dots, \mu_n)$ is *regular* if for any $1 \leqslant i < n$ and $1 \leqslant j \leqslant n-i$ the representations $\mu_i \otimes \mu_{i+1} \otimes \cdots \otimes \mu_{i+j}$ is cyclic.

The following is completely elementary.

Proposition 2 *Let q^2 be a primitive Nth root of unity.*

1. *Every cyclic representation of the Weyl algebra $\mathcal{W}$ is completely reducible, and its irreducible factors are cyclic.*
2. *Up to isomorphism, an irreducible cyclic representation $\mu\colon \mathcal{W} \to \mathrm{End}(V_\mu)$ of the Weyl algebra is completely determined by the complex numbers $x_\mu, y_\mu \in \mathbb{C}^*$ such that*
$$X_\mu^N = x_\mu \cdot \mathrm{Id}_{V_\mu}, \quad Y_\mu^N = y_\mu \cdot \mathrm{Id}_{V_\mu}.$$
3. *More precisely, the irreducible cyclic representation μ is isomorphic to a representation $\mu'\colon \mathcal{W} \to \mathrm{End}(\mathbb{C}^N)$ defined by*
$$X_{\mu'} = x_\mu^{1/N} A, \quad Y_{\mu'} = y_\mu^{1/N} B$$
where $x_\mu^{1/N}$ and $y_\mu^{1/N}$ are arbitrary Nth roots of x_μ and y_μ, and where $A, B \in \mathrm{End}(\mathbb{C}^N)$ are the unitary matrices defined by
$$Ae_k = q^{2k} e_k, \quad Be_k = e_{k+1} \text{ for } 0 \le k \le N-1$$

on the standard basis $\{e_0,\dots,e_{N-1}\}$ *of* $\mathbb{C}^N$*, with indices considered modulo* N*.*

2.2 The Clebsch–Gordan operators and $6j$–symbols

Suppose that two irreducible representations μ and ν form a regular pair, and are classified by parameters (x_μ, y_μ) and (x_ν, y_ν) as in Proposition 2. From the calculation of

$$\Delta(X^N) = X^N \otimes X^N, \quad \Delta(Y^N) = X^{-N} \otimes Y^N + Y^N \otimes 1,$$

we see that all the irreducible components of $\mu \otimes \nu$ are isomorphic to the representation $\mu\nu$ classified by the parameters

$$x_{\mu\nu} = x_\mu x_\nu, \quad y_{\mu\nu} = x_\mu^{-1} y_\nu + y_\mu.$$

Thus we have the following lemma.

Lemma 3 *If* (μ,ν) *is a regular pair of irreducible representations of the Weyl algebra* $\mathcal{W}$*, then the representation* $\mu \otimes \nu\colon \mathcal{W} \to \mathrm{End}(V_\mu \otimes V_\nu)$ *splits as a direct sum of* N *representations isomorphic to the representation* $\mu\nu$*.*

As a consequence, the space $V_{(\mu,\nu)} = \mathrm{Hom}_{\mathcal{W}}(V_{\mu\nu}, V_\mu \otimes V_\nu)$ *of* $\mathcal{W}$*–equivariant linear maps* $V_{\mu\nu} \to V_\mu \otimes V_\nu$ *is a vector space of dimension* N*.*

The elements of $V_{(\mu,\nu)}$ are called the *Clebsch–Gordan operators*. The linear map $\Omega = \Omega(\mu,\nu)\colon V_{(\mu,\nu)} \otimes V_{\mu\nu} \to V_\mu \otimes V_\nu$ given by

$$\Omega(f \otimes v) = f(v), \quad \forall f \in V_{(\mu,\nu)}, v \in V_{\mu\nu}$$

is the *canonical map*.

The Kashaev $6j$–symbol is associated to a regular sequence (μ,ν,σ) of three irreducible representations of the Weyl algebra $\mathcal{W}$. In this situation, the diagram

$$\begin{tikzcd} & V_{\mu\nu} \otimes V_\sigma \arrow[r] & V_\mu \otimes V_\nu \otimes V_\sigma \arrow[d, equal] \\ V_{\mu\nu\sigma} \arrow[ur] \arrow[dr] & & \\ & V_\mu \otimes V_{\nu\sigma} \arrow[r] & V_\mu \otimes V_\nu \otimes V_\sigma \end{tikzcd}$$

provides natural isomorphisms between $\mathrm{Hom}_{\mathcal{W}}(V_{\mu\nu\sigma}, V_\mu \otimes V_\nu \otimes V_\sigma)$ and both $V_{(\mu,\nu)} \otimes V_{(\mu\nu,\sigma)}$ and $V_{(\mu,\nu\sigma)} \otimes V_{(\nu,\sigma)}$. The *Kashaev's* $6j$*–symbol* of (μ,ν,σ) is defined as the resulting isomorphism

$$R(\mu,\nu,\sigma)\colon V_{(\mu,\nu)} \otimes V_{(\mu\nu,\sigma)} \to V_{(\mu,\nu\sigma)} \otimes V_{(\nu,\sigma)}.$$

In other words, $R(\mu,\nu,\sigma)$ is a linear operator defined by equation

$$\Omega_{23}(\nu,\sigma)\Omega_{13}(\mu,\nu\sigma)R_{12}(\mu,\nu,\sigma)=\Omega_{12}(\mu,\nu)\Omega_{23}(\mu\nu,\sigma),$$

which is better illustrated as the commutativity of pentagon diagram

$$\begin{tikzcd}
& V_\mu\otimes V_{(\nu,\sigma)}\otimes V_{\nu\sigma} \arrow[dl, "\Omega_{23}(\nu,\sigma)"'] & V_{(\mu,\nu\sigma)}\otimes V_{(\nu,\sigma)}\otimes V_{\mu\nu\sigma} \arrow[l, "\Omega_{13}(\mu,\nu\sigma)"'] \\
V_\mu\otimes V_\nu\otimes V_\sigma & & \\
& V_{(\mu,\nu)}\otimes V_{\mu\nu}\otimes V_\sigma \arrow[ul, "\Omega_{12}(\mu,\nu)"] & V_{(\mu,\nu)}\otimes V_{(\mu\nu,\sigma)}\otimes V_{\mu\nu\sigma} \arrow[l, "\Omega_{23}(\mu\nu,\sigma)"] \arrow[uu, "R_{12}(\mu,\nu,\sigma)"']
\end{tikzcd}$$

Here we use the standard notation that, if an operator F acts on the i th and j th components $(1\le i<j\le 3)$ of the tensor product of spaces $U\otimes V\otimes W$, then F_{ij} denotes the operator of $U\otimes V\otimes W$ that acts by F on the i th and j th components, and by the identity on the remaining component of the tensor product.

3 The quantum Teichmüller space

3.1 The triangle algebra and its representations

The *triangle algebra* $\mathcal{T}$ is the algebra over $\mathbb{C}$ generated by $X^{\pm1},Y^{\pm1},Z^{\pm1}$ with relations

$$XY=q^2YX,\quad YZ=q^2ZY,\quad ZX=q^2XZ.$$

The center of the triangle algebra $\mathcal{T}$ is generated by the *principal central element* $H=q^{-1}XYZ$ and the three other elements X^N,Y^N and Z^N.

The finite dimensional representation theory of the quantum Teichmüller space works best when $q^N=(-1)^{N+1}$, which is slightly more restrictive than the condition that q^2 is a primitive N th root of unity [3; 7]. Consequently we henceforth assume that q is a primitive N th root of $(-1)^{N+1}$.

Let $\{e_0,\dots,e_{N-1}\}$ be the standard basis of $\mathbb{C}^N$, with indices modulo N. Let $A,B,C\in\mathrm{End}(\mathbb{C}^N)$ be the unitary matrices given by

$$Ae_k=q^{2k}e_k,\ Be_k=e_{k+1},\ Ce_k=q^{1-2k}e_{k-1}.$$

Note that $AB=q^2BA,\ BC=q^2CB,\ CA=q^2AC$ and

$$A^N=B^N=C^N=q^{-1}ABC=\mathrm{Id}_{\mathbb{C}^N}.$$

The following analogue of Proposition 2 is elementary.

Proposition 4 *Let q^2 be a primitive N th root of unity.*

1. *Up to isomorphism, an irreducible representation $\rho\colon \mathcal{T} \to \mathrm{End}(V_\rho)$ of the triangle algebra $\mathcal{T}$ is completely determined by the complex numbers $x_\rho, y_\rho, z_\rho, h_\rho \in \mathbb{C}^*$ with relation $h_\rho^N = x_\rho y_\rho z_\rho$, such that*

   $$X_\rho^N = x_\rho \cdot \mathrm{Id}_{V_\rho}, \quad Y_\rho^N = y_\rho \cdot \mathrm{Id}_{V_\rho}, \quad Z_\rho^N = z_\rho \cdot \mathrm{Id}_{V_\rho}, \quad H_\rho = h_\rho \cdot \mathrm{Id}_{V_\rho}.$$

   In addition, any number $x, y, z, h \in \mathbb{C}^*$ with $h^N = xyz$ can be realized by such a representation.

2. *More precisely, the irreducible representation ρ is isomorphic to a representation $\rho'\colon \mathcal{T} \to \mathrm{End}(\mathbb{C}^N)$ defined by*

   $$X_{\rho'} = x_\rho^{1/N} A, \quad Y_{\rho'} = y_\rho^{1/N} B, \quad Z_{\rho'} = z_\rho^{1/N} C, \quad H_{\rho'} = h_\rho \cdot \mathrm{Id}_{\mathbb{C}^N}$$

   where $x_\rho^{1/N}$, $y_\rho^{1/N}$ and $z_\rho^{1/N}$ are arbitrary N th roots of x_ρ, y_ρ and z_ρ satisfying $x_\rho^{1/N} \cdot y_\rho^{1/N} \cdot z_\rho^{1/N} = h_\rho$, and where $A, B, C \in \mathrm{End}(\mathbb{C}^N)$ are the unitary matrices defined as above.

The number h_ρ is called the *central charge* of the representation ρ.

3.2 The quantum Teichmüller space

Let S be an oriented surface of genus g with $p \geq 1$ punctures, obtained by removing p points $\{v_1, v_2, \dots, v_p\}$ from the closed oriented surface $\overline{S}$ of genus g. An *ideal triangulation* λ of S consists of finitely many disjoint simple arcs $\lambda_1, \lambda_2, \dots, \lambda_n$ going from puncture to puncture and decomposing S into triangles. Any ideal triangulation has precisely $n = -3\chi(S) = 6g + 3p - 6$ arcs, where $\chi(S)$ is the Euler characteristic of the punctured surface. Let $\Lambda(S)$ denote the set of isotopy classes of all ideal triangulations.

Following the terminology of [7; 16], the *Chekhov–Fock algebra* associated to the ideal triangulation λ is the algebra $\mathcal{T}_\lambda^q = \mathbb{C}[X_1, X_2, \dots, X_n]_\lambda^q$ over $\mathbb{C}$ generated by $X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}$ respectively associated to the edges of λ with relations $X_i X_j = q^{2\sigma_{ij}} X_j X_i$, where $\sigma_{ij} \in \{0, \pm 1, \pm 2\}$ are integers determined by the combinatorics of the ideal triangulation. This algebra has a well-defined fraction division algebra $\mathbb{C}(X_1, X_2, \dots, X_n)_\lambda^q$, consisting of all formal rational fractions in variables X_i that skew-commute according to the relations $X_i X_j = q^{2\sigma_{ij}} X_j X_i$.

As one moves from one ideal triangulation λ to another λ', Chekhov and Fock [10; 11] (as developed in Liu [16]) introduce explicit *coordinate change isomorphisms* $\varphi^q_{\lambda\lambda'}\colon \mathbb{C}(X'_1, X'_2, \dots, X'_n)^q_{\lambda'} \to \mathbb{C}(X_1, X_2, \dots, X_n)^q_{\lambda}$. These are algebra isomorphisms which satisfy the natural property that $\varphi^q_{\lambda\lambda''} = \varphi^q_{\lambda\lambda'} \circ \varphi^q_{\lambda'\lambda''}$ for any ideal triangulations $\lambda, \lambda', \lambda'' \in \Lambda(S)$. These algebra isomorphisms are essentially unique by Bai [1], once we require them to satisfy a certain number of natural conditions.

The *quantum Teichmüller space* of S is the algebra defined in a triangulation independent way as

$$\begin{aligned}\mathcal{T}^q_S &= \Big(\bigsqcup_{\lambda \in \Lambda(S)} \mathbb{C}(X_1, X_2, \dots, X_n)^q_\lambda \Big) / \sim \\ &= \{(\lambda, X) : \lambda \in \Lambda(S), \quad X \in \mathbb{C}(X_1, X_2, \dots, X_n)^q_\lambda\} / \sim\end{aligned}$$

where the equivalence relation $\sim$ is defined by

$$(\lambda, X) \sim (\lambda', X') \Leftrightarrow X = \varphi^q_{\lambda\lambda'}(X').$$

3.3 Local representations of the quantum Teichmüller space

Every ideal triangulation λ of S has exactly $m = -2\chi(S)$ triangles. Each triangle T_j determines a triangle algebra $\mathcal{T}_j$, with generators associated to the three sides of T_j. The Chekhov–Fock algebra has a natural embedding ι_λ into the tensor product of triangle algebras

$$\iota_\lambda\colon \mathcal{T}^q_\lambda \to \bigotimes_{j=1}^m \mathcal{T}_j,$$

see [2; 3]. To describe ι_λ, consider the generator $X_i \in \mathcal{T}^q_\lambda$ corresponding to the edge λ_i of λ. There are two possible cases:

(1) The edge λ_i separates two triangles T_j and $T_{j'}$. Define $\iota_\lambda(X_i) = X_{ij} \otimes X_{ij'}$ where X_{ij} ($X_{ij'}$ resp.) is the generator of $\mathcal{T}_j$ ($\mathcal{T}_{j'}$ resp.) associated to the edge λ_i.

(2) The two sides of λ_i belong to the same triangle T_j. Let X_{ij_1} and X_{ij_2} be the generators of $\mathcal{T}_j$ associated to those two sides, indexed such that $X_{ij_1}X_{ij_2} = q^2 X_{ij_2}X_{ij_1}$, then define $\iota_\lambda(X_i) = q^{-1}X_{ij_1}X_{ij_2} = qX_{ij_2}X_{ij_1}$.

By convention, when describing an element $Z_1 \otimes \cdots \otimes Z_m$ of $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_m$, we omit in the tensor product those Z_j that are equal to the identity element 1 of $\mathcal{T}_j$.

Suppose that we are given an irreducible representation $\rho_j\colon \mathcal{T}_j \to \mathrm{End}(V_j)$ for every triangle T_j. The tensor product $\otimes_{j=1}^m \rho_j$ restricts to a representation $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)$

where $V_\lambda = V_1 \otimes \cdots \otimes V_m$. By definition, a *local representation* of $\mathcal{T}_\lambda^q$ is any representation obtained in this way.

A *local representation* of the quantum Teichmüller space $\mathcal{T}_S^q$ consists of the data of a local representation $\rho_\lambda\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)$ of the Chekhov–Fock algebra for every ideal triangulation λ, in such a way that $\rho_\lambda \circ \varphi_{\lambda\lambda'}^q\colon \mathcal{T}_{\lambda'}^q \to \mathrm{End}(V_\lambda)$ and $\rho_{\lambda'}\colon \mathcal{T}_{\lambda'}^q \to \mathrm{End}(V_{\lambda'})$ are isomorphic for any pair of ideal triangulations λ, λ'.

Let $\rho = \left\{\rho_\lambda\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\right\}_{\lambda\in\Lambda(S)}$ be a local representation of $\mathcal{T}_S^q$. For every pair of ideal triangulations λ, λ', an *intertwining operator* $L_{\lambda\lambda'}^\rho$ for the local representation ρ is defined as the linear map

$$L_{\lambda\lambda'}^\rho\colon V_1 \otimes \cdots \otimes V_m \to V_1' \otimes \cdots \otimes V_m'$$

such that $L_{\lambda\lambda'}^\rho \circ \rho_\lambda \circ \varphi_{\lambda\lambda'}^q(X') = \rho_{\lambda'}(X') \circ L_{\lambda\lambda'}^\rho$ for every $X' \in \mathcal{T}_{\lambda'}^q$. This $L_{\lambda\lambda'}^\rho$ is actually what is denoted by $L_{\lambda\lambda'}^{\rho\rho}$ in [3].

In [3, Theorem 20] it is shown that the family of intertwining operators $L_{\lambda\lambda'}^\rho$ is uniquely determined, provided that we impose certain natural conditions on them. In particular, the intertwining operators can be explicitly obtained, for all surfaces S, from the intertwining operator associated to the case where S is a square and λ, λ' are ideal triangulations of S differing from each other by different choices of a diagonal of that square.

Here, a *square* is the surface obtained from a closed disk by removing 4 punctures from its boundary.

3.4 The quantum Teichmüller space of the square

We consequently analyze the case of the square in more details.

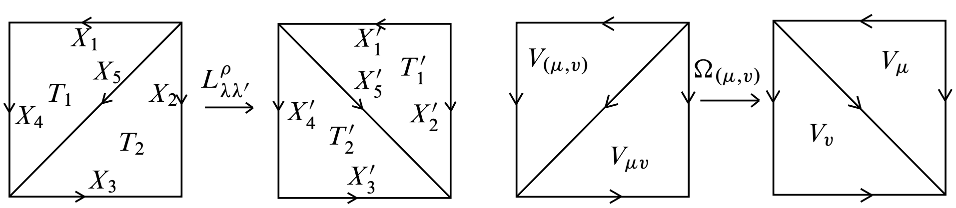

Figure 1: The intertwining operator and the canonical map

Let λ and λ' be the two ideal triangulations of the square S represented in Figure 1, with the indexing of edges and triangles indicated there. Let $\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_1 \otimes V_2)$ and $\rho_{\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V'_1 \otimes V'_2)$ be local representations of the Chekhov–Fock algebra of λ and λ'.

The *principal central element* of the Chekhov–Fock algebra $\mathcal{T}^q_\lambda$ is

$$H = q^{-\sum_{i<j}\sigma_{ij}} X_1 X_2 \dots X_5 = X_1 X_2 \dots X_5.$$

Similarly, the principal central element of Chekhov–Fock algebra $\mathcal{T}^q_{\lambda'}$ is

$$H' = q^{-\sum_{i<j}\sigma_{ij}} X'_1 X'_2 \dots X'_5 = q^{-2} X'_1 X'_2 \dots X'_5.$$

Compare with the principal central element of the triangle algebra in Section 3.1.

The following is proved in [3], although the proof is elementary.

Proposition 5 *Up to isomorphism, the local representation $\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_1 \otimes V_2)$ is uniquely determined by the numbers $x_1, x_2, x_3, x_4, h_{\rho_\lambda} \in \mathbb{C}^*$ such that $\rho_\lambda(X_i^N) = x_i \cdot \mathrm{Id}$ $(i = 1, 2, 3, 4)$ and $\rho_\lambda(H) = h_{\rho_\lambda} \cdot \mathrm{Id}$. In addition, any five tuple of numbers $x_1, x_2, x_3, x_4, h \in \mathbb{C}^*$ are associated to a local representation in this way.*

The number h_{ρ_λ} is called the *central charge* of the local representation ρ_λ. Note that $\rho_\lambda(X_5^N) = x_5 \cdot \mathrm{Id}$ where $x_5 = h_{\rho_\lambda}^N/(x_1 x_2 x_3 x_4)$.

The same result holds for λ', so that $\rho_{\lambda'}$ is classified by numbers $x'_1, x'_2, x'_3, x'_4, h_{\rho_{\lambda'}} \in \mathbb{C}^*$.

Proposition 6 *$\rho_\lambda \circ \varphi^q_{\lambda\lambda'}$ is isomorphic to $\rho_{\lambda'}$ if and only if*

$$\begin{cases} x'_1 &= (1 + x_5)x_1 \\ x'_2 &= (1 + x_5^{-1})^{-1} x_2 \\ x'_3 &= (1 + x_5)x_3 \\ x'_4 &= (1 + x_5^{-1})^{-1} x_4 \\ h_{\rho_{\lambda'}} &= h_{\rho_\lambda} \end{cases}$$

where $x_5 = h_{\rho_\lambda}^N/(x_1 x_2 x_3 x_4)$.

Proof This is an immediate consequence of Theorem 12 of [3], and of the specific form of the coordinate change isomorphism $\varphi^q_{\lambda\lambda'}$ for the square. □

We should point out that the role of x_i and x'_i are symmetric, in the sense that the above formula is equivalent to the following one

$$\begin{cases} x_1 &= (1+{x'}_5^{-1})^{-1}x'_1 \\ x_2 &= (1+x'_5)x'_2 \\ x_3 &= (1+{x'}_5^{-1})^{-1}x'_3 \\ x_4 &= (1+x'_5)x'_4 \\ h_{\rho_\lambda} &= h_{\rho_{\lambda'}} \end{cases}$$

where $x'_5 = h^N_{\rho_{\lambda'}}/(x'_1x'_2x'_3x'_4) = x_5^{-1}$.

Lemma 7 *Suppose that $\rho_\lambda \circ \varphi^q_{\lambda\lambda'}$ is isomorphic to $\rho_{\lambda'}$ by an intertwining operator $L^q_{\lambda\lambda'}\colon V_1 \otimes V_2 \to V'_1 \otimes V'_2$. Then there exist unique local representations $\rho'_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_1 \otimes V_2)$ and $\rho'_{\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V'_1 \otimes V'_2)$ with central charges equal to 1 and such that*

$$\rho'_\lambda(X_i) = \rho_\lambda(X_i), \quad \rho'_{\lambda'}(X'_i) = \rho_{\lambda'}(X'_i)$$

for $i = 1, 2, 3, 4$.

In addition, the same $L^q_{\lambda\lambda'}$ defines an isomorphism between $\rho'_\lambda \circ \varphi^q_{\lambda\lambda'}$ and $\rho'_{\lambda'}$.

Proof Define ρ'_λ by the property that $\rho'_\lambda(X_i)$ is equal to $\rho_\lambda(X_i)$ if $1 \le i \le 4$, and $\rho'_\lambda(X_5) = \rho_\lambda(X_5)/h$. Similarly, define $\rho'_{\lambda'}$. □

As a consequence, we can restrict our attention to local representations with central charge 1.

4 The Weyl algebra and the triangle algebra

As an algebra, the Weyl algebra $\mathcal{W}$ is clearly very similar to the triangle algebra $\mathcal{T}$. Let us make this correspondence explicit.

Let T be a triangle, with an additional structure of an arrow on each edge arranged so that the following holds: the three vertices of the triangle can be ordered so that the ordering goes counterclockwise, and each edge is oriented from the lower vertex to the higher vertex. Figure 1 offers many examples.

With this additional data, any irreducible regular representation $\mu\colon \mathcal{W} \to \mathrm{End}(V_\mu)$ of the Weyl algebra gives a unique representation $\mathcal{T}_T \to \mathrm{End}(V_\mu)$, which we will also denote by μ, of the triangle algebra $\mathcal{T}_T$ associated to the triangle T. Indeed, the generators X, Y, Z of the triangle algebra $\mathcal{T}_T$ can be arranged such that X is associated to the edge going from the lowest vertex to the middle one, while Y is associated to the edge going from lowest vertex to the highest one.

In this setting, define the representation $\mu\colon \mathcal{T}_T \to \mathrm{End}(V_\mu)$ (induced from the representation $\mu\colon \mathcal{W} \to \mathrm{End}(V_\mu)$) by

$$X_\mu = X_\mu, \quad Y_\mu = Y_\mu, \quad Z_\mu = qY_\mu^{-1}X_\mu^{-1}.$$

Note that the central charge of μ is 1, as $H_\mu = (q^{-1}XYZ)_\mu = \mathrm{Id}_{V_\mu}$. Every irreducible representation of $\mathcal{T}_T$ with central charge 1 is obtained in the way.

5 An action of the Weyl algebra on the Clebsch–Gordan operators

Assume that $\mu\colon \mathcal{W} \to \mathrm{End}(V_\mu)$, $\nu\colon \mathcal{W} \to \mathrm{End}(V_\nu)$ form a regular pair of irreducible representations of the Weyl algebra. Let $V_{(\mu,\nu)} = \mathrm{Hom}_{\mathcal{W}}(V_{\mu\nu}, V_\mu \otimes V_\nu)$ be the corresponding space of Clebsch–Gordan operators.

Consider the square S, with the two triangulations λ, λ' and the edge orientations indicated in Figure 1. Associate to each triangle of λ, λ' vector spaces V_μ, V_ν, $V_{\mu\nu}$ and $V_{(\mu,\nu)}$ as in Figure 1.

Note that the spaces V_μ, V_ν, $V_{\mu\nu}$ come with an action of the Weyl algebra $\mathcal{W}$, and therefore, using the arrows indicated, with an action of the triangle algebra of the corresponding triangle.

If, in addition, the space of Clebsch–Gordan operators $V_{(\mu,\nu)}$ is also endowed with an action of the Weyl algebra $\mathcal{W}$, this will define two local representations $\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_{(\mu,\nu)} \otimes V_{\mu\nu})$ and $\rho'_\lambda\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V_\mu \otimes V_\nu)$ of corresponding Chekhov–Fock algebras.

Proposition 8 *Assume that $\mu\colon \mathcal{W} \to \mathrm{End}(V_\mu)$ and $\nu\colon \mathcal{W} \to \mathrm{End}(V_\nu)$ form a regular pair of irreducible representations of the Weyl algebra. Then there exists a unique action the Weyl algebra on the space of Clebsch–Gordan operators $V_{(\mu,\nu)}$, such that the canonical map $\Omega(\mu,\nu)\colon V_{(\mu,\nu)} \otimes V_{\mu\nu} \to V_\mu \otimes V_\nu$ is an intertwining operator for the local representation ρ of the quantum Teichmüller space of the square, which consists of $\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_{(\mu,\nu)} \otimes V_{\mu\nu})$ and $\rho'_\lambda\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V_\mu \otimes V_\nu)$.*

Proof Let us denote the possible action of the Weyl algebra on the Clebsch–Gordan operators by $(\mu,\nu)\colon \mathcal{W} \to \mathrm{End}(V_{(\mu,\nu)})$. If the canonical map $\Omega = \Omega(\mu,\nu)$ is an intertwining operator for the local representation ρ, then by definition $\Omega \circ \rho_\lambda \circ \varphi^q_{\lambda\lambda'} = \rho'_\lambda \circ \Omega$, or equivalently, $\Omega \circ \rho_\lambda = \rho'_\lambda \circ (\varphi^q_{\lambda\lambda'})^{-1} \circ \Omega$.

A typical set of generators of $\mathcal{T}_\lambda^q$ consists of $X_1, \ldots, X_5$ as described in Section 3.4. This set corresponds to $\{X \otimes 1, 1 \otimes Y, 1 \otimes Z, Z \otimes 1, Y \otimes X\}$ in the algebra tensor $\mathcal{T}_{T_1} \otimes \mathcal{T}_{T_2}$ via the natural embedding of Section 3.3, with the arrows as in Figure 1.

In particular, for any $f \in V_{(\mu,\nu)}$ and $v \in V_{\mu\nu}$, the equation

$$\Omega \circ \rho_\lambda(X \otimes 1)(f \otimes v) = \rho'_\lambda \circ (\varphi^q_{\lambda\lambda'})^{-1}(X \otimes 1) \circ \Omega(f \otimes v)$$

gives that

$$(X_{(\mu,\nu)} \cdot f)(v) = (X_\mu^{-1} \otimes \mathrm{Id}_{V_\nu} + Y_\mu \otimes Y_\nu^{-1})^{-1} \cdot f(v),$$

and the equation

$$\Omega \circ \rho_\lambda(Y \otimes X)(f \otimes v) = \rho'_\lambda \circ (\varphi^q_{\lambda\lambda'})^{-1}(Y \otimes X) \circ \Omega(f \otimes v)$$

gives that

$$(Y_{(\mu,\nu)} \cdot f)(X_{\mu\nu} \cdot v) = (Z_\mu^{-1} \otimes Y_\nu^{-1}) \cdot f(v).$$

If we require that $Y_{(\mu,\nu)} \cdot f$ is still in $V_{(\mu,\nu)}$, namely is $\mathcal{W}$–equivariant, then it forces that

$$(Y_{(\mu,\nu)} \cdot f)(v) = (X_\mu^{-1} \otimes X_\nu^{-1}) \circ (Z_\mu^{-1} \otimes Y_\nu^{-1}) \cdot f(v) = (q^{-1} Y_\mu \otimes X_\nu^{-1} Y_\nu^{-1}) \cdot f(v).$$

Hence the possible action (μ, ν) is completely determined by the formulas

$$\begin{aligned}(X_{(\mu,\nu)} \cdot f)(v) &= (X_\mu^{-1} \otimes \mathrm{Id}_{V_\nu} + Y_\mu \otimes Y_\nu^{-1})^{-1} \cdot f(v),\\ (Y_{(\mu,\nu)} \cdot f)(v) &= (q^{-1} Y_\mu \otimes X_\nu^{-1} Y_\nu^{-1}) \cdot f(v).\end{aligned}$$

The condition that (μ, ν) forms a regular pair of irreducible representations of the Weyl algebra is critical, which guarantees that both the operators $X_\mu^{-1} \otimes \mathrm{Id}_{V_\nu} + Y_\mu \otimes Y_\nu^{-1}$ and $q^{-1} Y_\mu \otimes X_\nu^{-1} Y_\nu^{-1}$ are invertible in $\mathrm{End}(V_\mu \otimes V_\nu)$.

We need to show that the above formula really defines an action of the Weyl algebra on the space of Clebsch–Gordan operators $V_{(\mu,\nu)}$, namely that for any $f \in V_{(\mu,\nu)} = \mathrm{Hom}_{\mathcal{W}}(V_{\mu\nu}, V_\mu \otimes V_\nu)$, these operators $X_{(\mu,\nu)} \cdot f$ and $Y_{(\mu,\nu)} \cdot f$ are still in $V_{(\mu,\nu)}$, or, are $\mathcal{W}$–equivariant.

The comultiplication Δ of the Weyl algebra $\mathcal{W}$ has the property that $\Delta(X) = X \otimes X$ and $\Delta(Y) = X^{-1} \otimes Y + Y \otimes 1$, see Section 2.1. It is straightforward to check that both the elements $X^{-1} \otimes 1 + Y \otimes Y^{-1}$ and $q^{-1} Y \otimes X^{-1} Y^{-1}$ commute with either $\Delta(X)$ or $\Delta(Y)$ in $\mathcal{W} \otimes \mathcal{W}$, therefore commute with $\Delta(W)$ for any $W \in \mathcal{W}$.

Now $X_{(\mu,\nu)} \cdot f \in V_{(\mu,\nu)}$ is immediate from the following calculation

$$\begin{aligned}(X_{(\mu,\nu)} \cdot f)(a_{\mu\nu} \cdot v) &= (X_\mu^{-1} \otimes \mathrm{Id} + Y_\mu \otimes Y_\nu^{-1})^{-1} \cdot f(a_{\mu\nu} \cdot v) \\ &= (X_\mu^{-1} \otimes \mathrm{Id} + Y_\mu \otimes Y_\nu^{-1})^{-1} \circ \Delta(a) \cdot f(v) \\ &= \Delta(a) \circ (X_\mu^{-1} \otimes \mathrm{Id} + Y_\mu \otimes Y_\nu^{-1})^{-1} \cdot f(v) \\ &= \Delta(a) \circ (X_{(\mu,\nu)} \cdot f)(v).\end{aligned}$$

The same argument proves that $Y_{(\mu,\nu)} \cdot f \in V_{(\mu,\nu)}$.

By establishing the representation of the Weyl algebra on the space of Clebsch–Gordan operators, we have shown that

$$\Omega \circ \rho_\lambda = \rho'_\lambda \circ (\varphi^q_{\lambda\lambda'})^{-1} \circ \Omega$$

holds for $X \otimes 1$ and $Y \otimes X$. The reader can easily check that the equation holds for other generators of the Chekhov–Fock algebra $\mathcal{T}^q_\lambda$.

This completes the proof the proposition. □

Calculation shows that

$$X^N_{(\mu,\nu)} = (x_\mu^{-1} + y_\mu y_\nu^{-1})^{-1} \cdot \mathrm{Id}, \quad Y^N_{(\mu,\nu)} = y_\mu x_\nu^{-1} y_\nu^{-1} \cdot \mathrm{Id}.$$

That means, the representation (μ,ν) of the Weyl algebra is classified by the parameter $(x_{(\mu,\nu)} = (x_\mu^{-1} + y_\mu y_\nu^{-1})^{-1}, y_{(\mu,\nu)} = y_\mu x_\nu^{-1} y_\nu^{-1})$ in the sense of Proposition 2. Once again, we note that the number $x_\mu^{-1} + y_\mu y_\nu^{-1} \neq 0$ due to the fact that μ and ν forms a regular pair.

6 The canonical maps, Kashaev's $6j$ –symbols and intertwining operators

Theorem 9 *Let V_1, V_2, V_1' and V_2' be $\mathcal{W}$–modules for the Weyl algebra $\mathcal{W}$, and $L\colon V_1 \otimes V_2 \to V_1' \otimes V_2'$ be a linear map (not supposed to be $\mathcal{W}$–linear). Then the following are equivalent:*

1. *There exists a regular pair (μ,ν) of irreducible representations of the Weyl algebra $\mathcal{W}$ and isomorphisms of $\mathcal{W}$–spaces*

$$V_1 \cong V_{(\mu,\nu)}, V_2 \cong V_{\mu\nu}, V_1' \cong V_\mu, V_2' \cong V_\nu,$$

for which L corresponds to a scalar multiple of the canonical map

$$\Omega(\mu,\nu)\colon V_{(\mu,\nu)} \otimes V_{\mu\nu} \to V_\mu \otimes V_\nu.$$

2. *There exists a regular triple (μ,ν,σ) of irreducible representations of the Weyl algebra $\mathcal{W}$ and isomorphisms of $\mathcal{W}$–spaces*

$$V_1 \cong V_{(\mu,\nu)}, V_2 \cong V_{(\mu\nu,\sigma)}, V_1' \cong V_{(\mu,\nu\sigma)}, V_2' \cong V_{(\nu,\sigma)},$$

for which L corresponds to a scalar multiple of Kashaev's $6j$–symbol

$$R(\mu,\nu,\sigma)\colon V(\mu,\nu)\otimes V(\mu\nu,\sigma)\to V(\mu,\nu\sigma)\otimes V(\nu,\sigma).$$

3. *L is an intertwining operator*

$$L^{\rho}_{\lambda\lambda'}\colon V_1\otimes V_2\to V_1'\otimes V_2',$$

namely is an isomorphism between $\rho_\lambda\circ\varphi^q_{\lambda\lambda'}$ and $\rho_{\lambda'}$. Here $\rho=\{\rho_\lambda\colon \mathcal{T}^q_\lambda\to \mathrm{End}(V_1\otimes V_2),\ \rho_{\lambda'}\colon \mathcal{T}^q_{\lambda'}\to\mathrm{End}(V_1'\otimes V_2')\}$ is a local representation of the quantum Teichmüller space of the square S.

Proof of $1\Rightarrow 3$ This is a consequence of Proposition 8. □

Proof of $3\Rightarrow 1$ Let us pick $V_\mu = V_1'$, $V_\nu = V_2'$.

First, we need to show that there exist isomorphisms of $\mathcal{W}$–spaces between $V_{\mu\nu}$ and V_2, also between $V_{(\mu,\nu)}$ and V_1. Let us consider the relationship between the corresponding parameters.

Notice that in every representation of the triangle algebra which is induced from an irreducible regular representation of the Weyl algebra, the product of the three parameters x,y,z is equal to 1, see Section 4. Thus from $V_\nu = V_1'$, $V_\nu = V_2'$, the following holds

$$x_\mu = x_1',\quad y_\mu = x_2',\quad x_\nu = x_4',\quad y_\nu = x'^{-1}_3 x'^{-1}_4.$$

Now combining with the relation

$$x_1 = (1+x'^{-1}_5)^{-1}x_1',\quad x_2=(1+x_5')x_2',\quad x_3=(1+x'^{-1}_5)^{-1}x_3',\quad x_4=(1+x_5')x_4'$$

from Section 3.4 (where $x_5' = 1/(x_1'x_2'x_3'x_4')$), the relation

$$x_{(\mu,\nu)} = (x_\mu^{-1}+y_\mu y_\nu^{-1})^{-1}, y_{(\mu,\nu)} = y_\mu x_\nu^{-1}y_\nu^{-1}$$

from Section 5 and the relation

$$x_{\mu\nu} = x_\mu x_\nu,\quad y_{\mu\nu} = x_\mu^{-1}y_\nu + y_\mu$$

from Section 2.2, we immediately get

$$x_{(\mu,\nu)} = x_1,\quad y_{(\mu,\nu)} = x_1^{-1}x_4^{-1},\quad x_{\mu\nu} = x_2^{-1}x_3^{-1},\quad y_{\mu\nu} = x_2.$$

By Proposition 2, this is exactly the relationship among the parameters which gives the desirable isomorphisms of $\mathcal{W}$–spaces.

Then, we need to show that L corresponds to the canonical map $\Omega(\mu,\nu)$, up to scalar multiplication.

Proposition 8 tells us that $\Omega(\mu,\nu)$ is an intertwining operator for the local representation ρ of the quantum Teichmüller space of the square, which consists of $\rho_\lambda\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_{(\mu,\nu)} \otimes V_{\mu\nu})$ and $\rho'_\lambda\colon \mathcal{T}_{\lambda'}^q \to \mathrm{End}(V_\mu \otimes V_\nu)$. But any irreducible representation of the quantum Teichmüller space of the square is of dimension N^2 [7], hence both $\rho_\lambda \circ \varphi_{\lambda\lambda'}^q$ and ρ'_λ are irreducible. Then Schur's lemma implies that the intertwining operator is unique, up to scalar multiplication. This finishes our proof. □

Proof of $2 \Rightarrow 3$ Assume that (μ,ν,σ) forms a regular triple of irreducible representations of the Weyl algebra $\mathcal{W}$. We interpret the defining equation of Kashaev's $6j$–symbol

$$R(\mu,\nu,\sigma)\colon V(\mu,\nu) \otimes V(\mu\nu,\sigma) \to V(\mu,\nu\sigma) \otimes V(\nu,\sigma)$$

as the commutativity of the Figure 2.

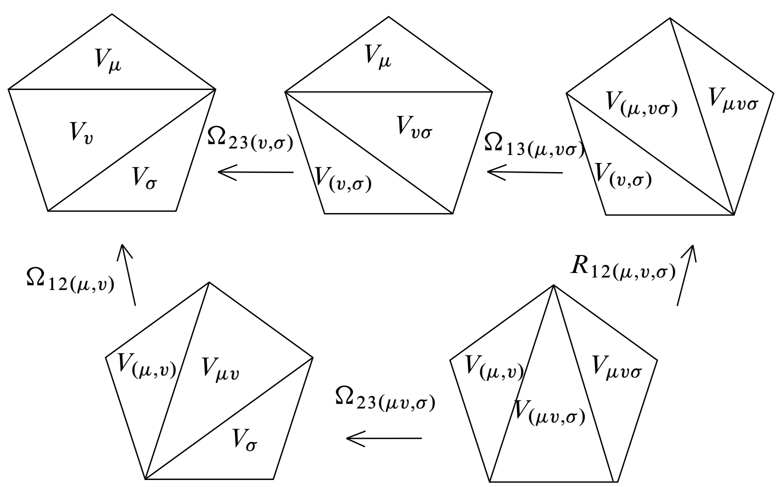

Figure 2: $6j$–symbol as intertwining operator

By the result of $1 \Rightarrow 3$, every Ω–map is an intertwining operator between the local representations of the Chekhov–Fock algebras associated to the five distinct ideal triangulations of the pentagon in Figure 2.

Note that, by the property of the coordinate change isomorphisms $\varphi^q_{\lambda'\lambda} = \left(\varphi^q_{\lambda\lambda'}\right)^{-1}$, the inverse of an intertwining operator is still an intertwining operator, since $L^{\rho}_{\lambda\lambda'} \circ \rho_\lambda \circ \varphi^q_{\lambda\lambda'} = \rho_{\lambda'} \circ L^{\rho}_{\lambda\lambda'}$ implies $\left(L^{\rho}_{\lambda\lambda'}\right)^{-1} \circ \rho_{\lambda'} \circ \varphi^q_{\lambda'\lambda} = \rho_\lambda \circ \left(L^{\rho}_{\lambda\lambda'}\right)^{-1}$.

Then

$$R(\mu,\nu,\sigma) \otimes \mathrm{Id}_{V_{\mu\nu\sigma}} = \Omega^{-1}_{13}(\mu,\nu\sigma) \circ \Omega^{-1}_{23}(\nu,\sigma) \circ \Omega_{12}(\mu,\nu) \circ \Omega_{23}(\mu\nu,\sigma)$$

is the composition of the intertwining operators, so again is an intertwining operator between the local representations of corresponding Chekhov–Fock algebras of the pentagon. Consequently, Kashaev's $6j$–symbol $R(\mu,\nu,\sigma)$ is an intertwining operator for a local representation of the quantum Teichmüller space of the square. □

Proof of $3 \Rightarrow 2$ Let $\rho = \{\rho_\lambda, \rho_{\lambda'}\}$ be a local representation of the quantum Teichmüller space of the square S. Assume that the local representation $\rho_{\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V'_1 \otimes V'_2)$ is classified by the number $x'_1, x'_2, x'_3, x'_4 \in \mathbb{C}^*$ ($h_{\rho_{\lambda'}} = 1$) in the sense of Proposition 5. If there exists a regular triple (μ,ν,σ) of irreducible representations of the Weyl algebra $\mathcal{W}$, such that the representation $(\mu,\nu\sigma) \otimes (\nu,\sigma)$ of the Weyl algebra induces the local representation $\rho_{\lambda'}$ of the Chekhov–Fock algebra, then by Proposition 6 the parameters necessarily have relation

$$\begin{cases} x'_1 &= x_{(\mu,\nu\sigma)} &= \left(x_\mu^{-1} + y_\mu (x_\nu^{-1} y_\sigma + y_\nu)^{-1}\right)^{-1} \\ x'_2 &= y_{(\mu,\nu\sigma)} &= y_\mu \cdot x_\nu^{-1} x_\sigma^{-1} \cdot (x_\nu^{-1} y_\sigma + y_\nu)^{-1} \\ x'_3 &= x_{(\nu,\sigma)} &= \left(x_\nu^{-1} + y_\nu y_\sigma^{-1}\right)^{-1} \\ x'_4 &= z_{(\nu,\sigma)} &= (x_\nu^{-1} + y_\nu y_\sigma^{-1}) \cdot y_\nu^{-1} x_\sigma y_\sigma \end{cases}$$

The second equality of each row is derived from the relations from Section 5 and Section 2.2, see the proof of $3 \Rightarrow 1$.

This is easily solved as

$$\begin{cases} x_\mu = x'^{-1}_1 - x'_2 x'_3 x'_4 (x'^{-1}_3 y_\nu^{-1} y_\sigma - 1)^{-1}, & y_\mu = x'_2 x'_4 (x'^{-1}_3 y_\nu^{-1} - y_\sigma^{-1})^{-1} \\ x_\nu = (x'^{-1}_3 - y_\nu y_\sigma^{-1})^{-1}, & y_\nu = y_\nu \\ x_\sigma = x'_3 x'_4 y_\nu y_\sigma^{-1}, & y_\sigma = y_\sigma \end{cases}$$

for every y_ν, y_σ such that

$$y_\nu \cdot y_\sigma \cdot (y_\sigma y_\nu^{-1} - x'_3) \cdot (y_\sigma y_\nu^{-1} - (x'_1 x'_2 x'_3 x'_4 + 1) x'_3) \neq 0.$$

Any such solution to these equations defines regular irreducible representations μ,ν,σ of the Weyl algebra and $\mathcal{W}$–isomorphisms

$$V_1 \cong V_{(\mu,\nu)}, V_2 \cong V_{(\mu\nu,\sigma)}, V'_1 \cong V_{(\mu,\nu\sigma)}, V'_2 \cong V_{(\nu,\sigma)}.$$

By the proof of $2 \Rightarrow 3$, we know that Kashaev's $6j$–symbol $R(\mu,\nu,\sigma)$: $V(\mu,\nu) \otimes V(\mu\nu,\sigma) \to V(\mu,\nu\sigma) \otimes V(\nu,\sigma)$ is an intertwining operator, and we already see that, in the case of the square, the intertwining operator is unique up to scalar multiplication. It follows that this $6j$–symbol $R(\mu,\nu,\sigma)$ corresponds to a scalar multiple of L. $\square$

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