The homotopy infinite symmetric product represents stable homotopy

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We modify the definition of the infinite symmetric product of a based space X by applying the homotopy colimit instead of the colimit. This gives a topological monoid $SP_h(X)$ and using formal properties of homotopy colimits, we prove that its group completion represents the stable homotopy of X. In this way we get a streamlined approach to the Barratt–Priddy–Quillen theorem.

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1 Introduction

A classical theorem of Dold and Thom [4] states, that if X is a based connected CW–complex, then the infinite symmetric product $SP(X)$ represents the reduced integral homology groups of X in the sense that these groups may be identified with the homotopy groups of $SP(X)$. One way to formulate the definition of $SP(X)$ is as follows. Let ω be the set of natural numbers $1, 2, \dots$, and let M be the monoid of injective self maps of ω under composition. Given a based CW–complex X, let X^∞ be the colimit of the sequence of maps

$$* \to X \to X^2 \to \cdots \to X^n \to X^{n+1} \to \cdots ,$$

defined by including X^n in X^{n+1} as the subspace of points whose last coordinate equals the base point. We define a left M–action on X^∞ by letting an injective map α act on an element $\mathbf{x} = (x_i)$ by $\alpha \cdot \mathbf{x} = \mathbf{y}$, where

$$y_j = \begin{cases} x_i, & \text{if } \alpha(i) = j \\ *, & \text{if } j \notin \alpha(\omega). \end{cases} \tag{1–1}$$

By definition, $SP(X)$ is the associated orbit space X^∞_M. Let now X^∞_{hM} be the corresponding homotopy orbit space, that is, the homotopy colimit of the M–diagram X^∞ obtained by viewing M as a category with a single object in the usual way. We shall relate this to the space

$$Q(X) = \mathrm{hocolim}_n\, \Omega^n(\Sigma^n X)$$

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that represents the stable homotopy groups of X.

Theorem 1.2 *If X is connected, then there is a natural chain of homotopy equivalences $X^{\infty}_{hM} \simeq Q(X)$.*

From this point of view, the canonical map $X^{\infty}_{hM} \to X^{\infty}_{M}$ represents the Hurewicz homomorphism if X is connected. For general X, we may view $Q(X)$ as a group completion of X^{∞}_{hM}. In order to make this precise, we reformulate the result as follows. Let $\mathcal{I}$ be the category whose objects are the finite sets $\mathbf{n} = \{1, \dots, n\}$ and whose morphisms are the injective maps between such sets. The empty set $\mathbf{0}$ is an initial object. The concatenation $\mathbf{m} \sqcup \mathbf{n}$ defined by letting $\mathbf{m}$ correspond to the first m elements and $\mathbf{n}$ to the last n elements of $\{1, \dots, m+n\}$ gives $\mathcal{I}$ the structure of a symmetric monoidal category. The symmetric structure is given by the shuffle permutations $\mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$. A based space X gives rise to an $\mathcal{I}$–space $X^{\bullet}\colon \mathbf{n} \mapsto X^{n}$ by letting an injective map $\alpha\colon \mathbf{m} \to \mathbf{n}$ act on an element $\mathbf{x} \in X^{m}$ as in (1–1), replacing $\alpha(\omega)$ by $\alpha(\mathbf{m})$. The categorical colimit of this diagram may again be identified with $SP(X)$. We write $SP_h(X)$ for the corresponding homotopy colimit,

$$SP_h(X) = \operatorname{hocolim}_{\mathcal{I}} X^{\bullet}.$$

Based on an argument by Jeff Smith, we prove in Proposition 3.7 that the spaces X^{∞}_{hM} and $SP_h(X)$ are homotopy equivalent. The latter has the advantage that it inherits the structure of a topological monoid from the monoidal structure of $\mathcal{I}$, hence we may define its group completion by

$$SP^{\wedge}_h(X) = \Omega B(SP_h(X)).$$

If X is connected, then so is $SP_h(X)$ and the canonical map

$$SP_h(X) \to SP^{\wedge}_h(X)$$

is therefore an equivalence in this case. Theorem 1.2 then follows from the following more general result.

Theorem 1.3 *If X is a based CW–complex, then there is a natural chain of homotopy equivalences $SP^{\wedge}_h(X) \simeq Q(X)$.*

More generally, if X is a well-based space (see below), then, since the functors in the theorem preserve weak homotopy equivalences, there results a natural chain of weak homotopy equivalences relating $SP^{\wedge}_h(X)$ and $Q(X)$.

Theorem 1.3 is of course closely related to the theorem of Barratt–Priddy–Quillen relating $Q(X)$ to the action of the symmetric groups on the spaces X^{n}. In the case of a space of the form X_+, that is, a space with a disjoint base point, Segal [13] defines a symmetric monoidal topological category $\Sigma(X)$ such that the group completion of $B\Sigma(X)$ is equivalent to $Q(X_+)$. More generally, Barratt and Eccles [1] construct a model of $Q(X)$ for any CW–complex X. Using the by now standard properties of homotopy colimits, we present in this paper a new streamlined approach to these results.

1.1 Organization of the paper

In Section 2 we consider general $\mathcal{I}$–spaces, that is, $\mathcal{I}$–diagrams of spaces. There is a symmetric monoidal product $X \boxtimes Y$ of $\mathcal{I}$–spaces and the main result in this section is Proposition 2.5, which states that under suitable cofibrancy conditions, this is equivalent to the homotopy invariant version $X \boxtimes_h Y$. In Section 3 we specialize to $SP(X)$ and make the observation that if X and Y are based spaces, then the $\mathcal{I}$–space $X^\bullet \boxtimes Y^\bullet$ is isomorphic to $(X \vee Y)^\bullet$. Using this, the proof of Theorem 1.3 is based on standard properties of homotopy colimits together with the fact that if X is $(n-1)$–connected, then the inclusion of $X \vee X$ in $X \times X$ is $(2n-1)$–connected. (A map is k–connected if its homotopy fibers are $(k-1)$–connected). In Section 4 we compare $SP_h(X)$ to the constructions by Segal and Barrett–Eccles mentioned above.

1.2 Notation and conventions

We shall work in the category $\mathcal{U}$ of compactly generated weak Hausdorff spaces. By an equivalence in $\mathcal{U}$ we mean a homotopy equivalence and by a cofibration we understand a map having the homotopy extension property in the usual sense (see, for example, Steenrod [15, Section 7]). (The use of homotopy equivalences instead of weak homotopy equivalences is not essential; we could have worked with weak homotopy equivalences throughout the paper). We let $\mathcal{T}$ be the analogous category of based spaces and say that an object of $\mathcal{T}$ is well-based if the inclusion of the base point is a cofibration. Alternatively, one may interpret $\mathcal{U}$ and $\mathcal{T}$ as the categories of unbased and based simplicial sets throughout the paper. In this setting some of the arguments simplify since a cofibration is then simply an injective map.

1.3 Homotopy colimits

Given a small category $\mathcal{C}$ and a functor $X\colon \mathcal{C} \to \mathcal{U}$, the homotopy colimit $\operatorname{hocolim}_{\mathcal{C}} X$ is the realization of the simplicial space

$$[k] \longmapsto \coprod_{\left\{C_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_k} C_k\right\}} X(C_k)$$

as defined by Bousfield and Kan [3, Section XII.5.1]. In the case of a diagram of based spaces one modifies this definition by replacing the disjoint union by the wedge product. Given a based space X, let UX denote the unbased space obtained by forgetting the base point and if X is a $\mathcal{C}$–diagram of based spaces, let UX be the associated unbased diagram. Assuming that X is a diagram of well-based spaces, we then have a cofibration sequence

$$B\mathcal{C} \to \operatorname{hocolim}_{\mathcal{C}} UX \to U \operatorname{hocolim}_{\mathcal{C}} X.$$

If furthermore $B\mathcal{C}$ is contractible, it follows that the second map is an equivalence. In particular, this is the case if $\mathcal{C}$ has an initial object $\varnothing$. Notice that in this case, the difference between a based and an unbased diagram amounts to a choice of base point in $X(\varnothing)$ and that the categorical colimit of the unbased diagram UX may be identified with $U\operatorname{colim}_{\mathcal{C}} X$.

1.4 Homotopy Kan extensions

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between small categories $\mathcal{A}$ and $\mathcal{B}$. Given a functor $X\colon \mathcal{A} \to \mathcal{U}$, the homotopy Kan extension is the functor $L_h^F X\colon \mathcal{B} \to \mathcal{U}$ defined by

$$L_h^F X(b) = \operatorname{hocolim}_{F/b} X \circ \pi_b.$$

The homotopy colimit is over the comma category F/b whose objects are pairs (a, β) in which a is an object in $\mathcal{A}$ and $\beta\colon F(a) \to b$ a morphism in $\mathcal{B}$. A morphism $(a, \beta) \to (a', \beta')$ is a morphism $\alpha\colon a \to a'$ in $\mathcal{A}$ such that $\beta = \beta' \circ F(\alpha)$. The functor $\pi_b\colon F/b \to \mathcal{A}$ is defined by $(a, \beta) \mapsto a$. We recall that the categorical Kan extension is defined using the categorical colimit instead of the homotopy colimit, see Mac Lane [8, Section X]. It will often be convenient to omit the functor π_b from the notation when writing such homotopy colimits. The result in the following lemma may be viewed as a statement about the composition of two derived functors, see Hirschhorn [6]. We include a direct proof here for completeness. Arguments of this sort goes back to Quillen [11].

Lemma 1.4 *Given functors $F\colon \mathcal{A} \to \mathcal{B}$ and $X\colon \mathcal{A} \to \mathcal{U}$ as above, there is a canonical equivalence*

$$\pi\colon \operatorname{hocolim}_{\mathcal{B}} L_h^F X \xrightarrow{\sim} \operatorname{hocolim}_{\mathcal{A}} X.$$

Proof The functors π_b define a map of $\mathcal{B}$–diagrams from $L_h^F X$ to the constant $\mathcal{B}$–diagram $\operatorname{hocolim}_{\mathcal{A}} X$ and π is the induced map. Thus, π is induced by a map of bisimplicial spaces

$$\coprod_{\{b_0\leftarrow\cdots\leftarrow b_i\}} \coprod_{\substack{\{a_0\leftarrow\cdots\leftarrow a_j\} \\ \{b_i\leftarrow F(a_0)\}}} X(a_j) \longmapsto \coprod_{\{a_0\leftarrow\cdots\leftarrow a_j\}} X(a_j),$$

where we view the target as a bisimplicial space which is constant in the i–direction. If we fix j and form the realization of the resulting simplicial spaces, the domain decomposes into a disjoint union

$$\coprod_{\{a_0\leftarrow\cdots\leftarrow a_j\}} B(F(a_0)/\mathcal{B}) \times X(a_j).$$

The category $F(a_0)/\mathcal{B}$ has an initial object and its classifying space is therefore contractible. Consequently, the bisimplicial map is an equivalence for each fixed j and since these are good simplicial spaces in the sense of Segal [13], it is itself an equivalence. □

2 The monoidal structure on $\mathcal{I}$–spaces

We define an *$\mathcal{I}$–space* to be a functor $X\colon \mathcal{I} \to \mathcal{U}$ and let $\mathcal{I}\mathcal{U}$ be the category of $\mathcal{I}$–spaces in which a morphism is a natural transformation. Given $\mathcal{I}$–spaces X and Y, let $X \times Y$ be the $\mathcal{I}^2$–diagram defined by

$$(\mathbf{n}_1, \mathbf{n}_2) \mapsto X(\mathbf{n}_1) \times Y(\mathbf{n}_2).$$

Using the monoidal structure $\sqcup\colon \mathcal{I}\times\mathcal{I} \to \mathcal{I}$, we internalize this by letting $X \boxtimes Y$ be the $\mathcal{I}$–space defined by the Kan extension

$$X \boxtimes Y(\mathbf{n}) = \operatorname{colim}_{\sqcup/\mathbf{n}} X \times Y.$$

In this way, $\mathcal{I}\mathcal{U}$ inherits the structure of a symmetric monoidal category from that of $\mathcal{I}$. The unit is the constant diagram $\mathcal{I}(\mathbf{0}, \bullet)$. We refer to Mandell, May, Schwede and Shipley [9] for a general discussion of induced symmetric monoidal structures on diagram categories. By the universal property of the Kan extension, the data defining a monoid in $\mathcal{I}\mathcal{U}$ amounts to an associative natural transformation

$$X(\mathbf{m}) \times X(\mathbf{n}) \to X(\mathbf{m} \sqcup \mathbf{n})$$

of $\mathcal{I}^2$–diagrams, together with a multiplicative unit $1_X \in X(\mathbf{0})$. Given a monoid X in $\mathcal{I}\mathcal{U}$, the categorical colimit $X_{\mathcal{I}}$ has the structure of a topological monoid with multiplication

$$X_{\mathcal{I}} \times X_{\mathcal{I}} = \operatorname{colim}_{\mathcal{I}\times\mathcal{I}} X(\mathbf{n}_1) \times X(\mathbf{n}_2) \to \operatorname{colim}_{\mathcal{I}\times\mathcal{I}} X(\mathbf{n}_1 \sqcup \mathbf{n}_2) \to X_{\mathcal{I}}.$$

If X is commutative, then $X_{\mathcal{I}}$ is a commutative topological monoid. Similarly, the homotopy colimit $X_{h\mathcal{I}}$ inherits the structure of a topological monoid, and in this case commutativity of X implies that the group completion of $X_{h\mathcal{I}}$ is an infinite loop space, see [12, Section 5]. In the case of the constant diagram $\mathcal{I}(\mathbf{0}, \bullet)$, this gives the contractible topological monoid $B\mathcal{I}$. Notice, that since the latter is not commutative, $X_{h\mathcal{I}}$ cannot be strictly commutative either.

Example 2.1 Let X be a based space and $X^{\bullet}$ the $\mathcal{I}$–space introduced in Section 1. The identity maps $X^m \times X^n \to X^{m+n}$ make this a commutative monoid in $\mathcal{IU}$ and the induced monoid structures on the colimit and the homotopy colimit are the monoid structures on $SP(X)$ and $SP_h(X)$ considered in Section 1.

Example 2.2 A symmetric spectrum E gives rise to an $\mathcal{I}$–space $\mathbf{n} \mapsto \Omega^n(E_n)$ (see, for example, [12]). If E is a ring spectrum, then this inherits a monoid structure which is commutative if E is. These $\mathcal{I}$–spaces are used in Bökstedt's definition of topological Hochschild homology and play an important role in the theory of symmetric spectra, see Shipley [14].

There also is a homotopy invariant version of the $\boxtimes$–product, where we consider the $\mathcal{I}$–space $X \boxtimes_h Y$ defined by the homotopy Kan extension

$$X \boxtimes_h Y(\mathbf{n}) = \operatorname{hocolim}_{\sqcup/\mathbf{n}} X \times Y.$$

The canonical projection from the homotopy colimit to the colimit defines a map of $\mathcal{I}$–spaces $X \boxtimes_h Y \to X \boxtimes Y$. We shall now introduce a criterion on X and Y which ensures that this map is a level-wise equivalence. Consider a commutative diagram in $\mathcal{I}$ of the form

$$\begin{tikzcd} \mathbf{m} \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"] & \mathbf{n}_1 \arrow[d, "\beta_1"] \\ \mathbf{n}_2 \arrow[r, "\beta_2"] & \mathbf{n}, \end{tikzcd} \tag{2–3}$$

and let γ denote the composite $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$. We say that an $\mathcal{I}$–space X is *flat* if for any diagram of the form (2–3), such that the intersection of the images of β_1 and β_2 equals the image of γ, the induced map

$$X(n_1) \cup_{X(m)} X(n_2) \to X(n)$$

is an cofibration. By Lillig's union theorem [7], this is equivalent to the requirement that (i) any morphism α: $\mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces an cofibration $X(m) \to X(n)$, and (ii) that the intersection of the images of $X(n_1)$ and $X(n_2)$ in $X(n)$ equals the image of $X(m)$.

Example 2.4 If X is well-based, then the $\mathcal{I}$–space $X^\bullet$ is flat.

Proposition 2.5 *If X and Y are flat $\mathcal{I}$–spaces, then the canonical map $X \boxtimes_h Y \to X \boxtimes Y$ is a level-wise equivalence.*

Proof Let $\mathcal{A}(n)$ be the full subcategory of $\sqcup/\mathbf{n}$ whose objects $\alpha\colon \mathbf{n}_1 \sqcup \mathbf{n}_2 \to \mathbf{n}$ are such that the restrictions to $\mathbf{n}_1$ and $\mathbf{n}_2$ are order preserving. Since this is a skeleton subcategory in the sense of Mac Lane [8, Section IV.4], it suffices to show that the canonical map

$$\operatorname{hocolim}_{\mathcal{A}(n)} X(n_1) \times Y(n_2) \to \operatorname{colim}_{\mathcal{A}(n)} X(n_1) \times Y(n_2)$$

is an equivalence for each n. Notice, that $\mathcal{A}(n)$ may be identified with the partially ordered set of pairs (U, V) of disjoint subsets of $\mathbf{n}$, so that we may write the diagram in the form

$$(U, V) \mapsto X(U) \times Y(V)$$

The categories $\mathcal{A}(n)$ are very small in the sense of Dwyer and Spaliński [5, Section 10.13]. Thus, using the Strøm model category structure on $\mathcal{U}$ [16] it follows from the discussion in [5, Section 10] that it is sufficient to show that the canonical map

$$\operatorname{colim}_{(U,V)\subsetneq(U_0,V_0)} X(U) \times Y(V) \to X(U_0) \times Y(V_0)$$

is a cofibration for each fixed object (U_0, V_0). Since cofibrations are preserved under products and are closed inclusions, we may view each of the spaces $X(U) \times Y(V)$ as a closed subspace of $X(U_0) \times Y(V_0)$. By the assumptions on X and Y we then have the equality

$$X(U) \times Y(V) \cap X(U') \times Y(V') = X(U \cap U') \times Y(V \cap V')$$

for each pair of objects (U, V) and (U', V'). Thus, it follows from the pasting lemma for maps defined on a union of closed subspaces that the colimit in question may be identified with the union of these subspaces. The conclusion now follows from an inductive argument using Lillig's union theorem for cofibrations [7]. $\square$

In particular, we conclude from Lemma 1.4 that $(X \boxtimes Y)_{h\mathcal{I}}$ is equivalent to $X_{h\mathcal{I}} \times Y_{h\mathcal{I}}$ if X and Y are flat.

3 The homotopy infinite symmetric product

Given a based space X, let $X^\bullet$ be the $\mathcal{I}$–space $\mathbf{n} \mapsto X^n$ introduced in Section 1 and let $SP_h(X)$ be the corresponding unbased homotopy colimit. It is sometimes more convenient to view $X^\bullet$ as a diagram of based spaces and we write $SP_h^*(X)$ for the corresponding based homotopy colimit. Recall from Section 1.3, that if X is well-based, then $SP_h(X)$ and $SP_h^*(X)$ are equivalent since $\mathcal{I}$ has an initial object. However, $SP_h^*(X)$ does not have a strictly associative multiplication, since homotopy colimits only commute with products in the unbased setting. On the other hand, $SP_h^*(X)$ has the advantage that there is a natural based map

$$X \wedge SP_h^*(Y) \to SP_h^*(X \wedge Y)$$

obtained by including the X–coordinate of a point in $X \wedge Y^n$ diagonally in $(X \wedge Y)^n$.

Lemma 3.1 *Given based spaces X and Y, there is an isomorphism of $\mathcal{I}$–spaces $X^\bullet \boxtimes Y^\bullet \simeq (X \vee Y)^\bullet$.*

Proof As in the proof of Proposition 2.5, we identify $X^\bullet \boxtimes Y^\bullet(\mathbf{n})$ with the colimit of the diagram $X^U \times Y^V$, where U and V runs over all pairs of disjoint subsets of $\mathbf{n}$. Given $\mathbf{x} \in X^U$ and $\mathbf{y} \in Y^V$, let $\mathbf{z}$ be the element in $(X \vee Y)^n$ defined by

$$z_i = x_i \text{ if } i \in U, \quad z_i = y_i \text{ if } i \in V, \quad z_i = * \text{ if } i \notin U \cup V.$$

This defines a homeomorphism from $X^U \times Y^V$ to a closed subspace $Z(U,V)$ of $(X \vee Y)^n$. Since

$$Z(U_1, V_1) \cap Z(U_2, V_2) = Z(U_1 \cap U_2, V_1 \cap V_2),$$

these maps assemble to give the required homeomorphism. □

Using the above lemma, we shall apply the analysis of the $\boxtimes$–product in Section 2 to the study of $SP_h(X)$.

Lemma 3.2 *The natural map*

$$SP_h(X \vee Y) \to SP_h(X) \times SP_h(Y)$$

is an equivalence for all based CW–complexes X and Y.

Proof Since $X^\bullet$ and $Y^\bullet$ are flat, we have by Proposition 2.5 and Lemma 3.1 a level-wise equivalence

$$X^\bullet \boxtimes_h Y^\bullet \xrightarrow{\sim} X^\bullet \boxtimes Y^\bullet \xrightarrow{\sim} (X \vee Y)^\bullet. \tag{3–3}$$

The left hand side is a homotopy Kan extension along the functor $\mathcal{I}\times\mathcal{I}\to\mathcal{I}$ and by the formal property of homotopy Kan extensions recalled in Lemma 1.4, we thus get a canonical equivalence

$$\operatorname{hocolim}_{\mathcal{I}} X^\bullet \boxtimes_h Y^\bullet \xrightarrow{\sim} \operatorname{hocolim}_{\mathcal{I}\times\mathcal{I}} X^\bullet \times Y^\bullet. \tag{3–4}$$

Consider then the commutative diagram

$$\begin{tikzcd}
\operatorname{hocolim}_{\mathcal{I}}(X\vee Y)^\bullet \arrow[r] & \operatorname{hocolim}_{\mathcal{I}} X^\bullet \times \operatorname{hocolim}_{\mathcal{I}} Y^\bullet \\
\operatorname{hocolim}_{\mathcal{I}} X^\bullet \boxtimes_h Y^\bullet \arrow[r] \arrow[u, "\simeq"'] & \operatorname{hocolim}_{\mathcal{I}}(X^\bullet \boxtimes_h *) \times \operatorname{hocolim}_{\mathcal{I}}(* \boxtimes_h Y^\bullet), \arrow[u, "\simeq"']
\end{tikzcd}$$

where the vertical maps are equivalences by Proposition 2.5. Applying Lemma 1.4 to both sides of the bottom map and using that $B\mathcal{I}$ is contractible, we see that the latter is an equivalence as well. The same then holds for the upper horizontal map as claimed. $\square$

Consider now the based map $X\to SP_h^*(X)$ obtained by including X in the 0–skeleton of the simplicial space defining the homotopy colimit.

Lemma 3.5 *If X is a $(n-1)$–connected CW–complex, then $X\to SP_h^*(X)$ is $(2n-1)$–connected.*

Proof Consider the based $\mathcal{I}$–space $\mathbf{n}\mapsto \vee_{i=1}^n X$ and notice that since the inclusion $\vee_{i=1}^n X\to X^n$ is at least $(2n-1)$–connected, the same holds for the induced map of based homotopy colimits

$$\big(\operatorname{hocolim}_{\mathcal{I}} \mathbf{n}\big)_+ \wedge X \simeq \operatorname{hocolim}_{\mathcal{I}} \bigvee_{i=1}^n X \to \operatorname{hocolim}_{\mathcal{I}} X^n.$$

The first equivalence is the homeomorphism obtained by identifying $\vee_{i=1}^n X$ with $\mathbf{n}_+\wedge X$ in the obvious way. It follows from the definition that $\operatorname{hocolim}_{\mathcal{I}}\mathbf{n}$ is homeomorphic to the classifying space of the category $\mathbf{1}/\mathcal{I}$, hence contractible since the latter has an initial object. This concludes the proof. $\square$

Proof of Theorem 1.3 Let $S^1_\bullet$ be the standard simplicial circle with k non-base point simplices in degree k. Applying Lemma 3.2 for each k, we get a sequence of equivalences

$$SP_h(S^1_k\wedge X)\xrightarrow{\sim} SP_h(X)^k = B_k SP_h(X)$$

that gives a degree-wise equivalence of simplicial spaces, hence an equivalence of the topological realizations. This fits into the following commutative diagram in the homotopy category

$$\begin{array}{ccccc} S^1 \wedge SP_h^*(X) & \xleftarrow{\sim} & S^1 \wedge SP_h(X) & = & S^1 \wedge SP_h(X) \\ \downarrow & & \downarrow & & \downarrow \\ SP_h^*(S^1 \wedge X) & \xleftarrow{\sim} & SP_h(S^1 \wedge X) & \xrightarrow{\sim} & BSP_h(X), \end{array}$$

where we use the contractibility of $B\mathcal{I}$ to fill in the vertical map in the middle. Consider now the following diagram

$$\Omega SP_h^*(S^1 \wedge X) \xrightarrow{\sim} \mathrm{hocolim}_n\, \Omega^n SP_h^*(S^n \wedge X) \xleftarrow{\sim} Q(X).$$

We claim that the maps are equivalences as indicated. For the left hand map this follows by applying the above argument to $S^n \wedge X$, using that the adjoint of the vertical map on the right hand side, that is, the group completion map, is an equivalence for connected X. As for the right hand map, it follows from Lemma 3.5 that it is a weak homotopy equivalence, hence an equivalence since these spaces have the homotopy types of CW–complexes by Milnor [10]. □

Remark 3.6 The above argument may be reformulated in terms of Segal's Γ–spaces [13]. Indeed, it follows from Lemma 3.2 that the functor that to a finite based set S associates the space $SP_h^*(X \wedge S)$ is a Γ–space as defined in that paper (this is what is called a special Γ–space in Bousfield and Friedlander [2]). By [13, Proposition 1.4] the associated spectrum is then an Ω–spectrum in positive degrees and by Lemma 3.5, this spectrum is equivalent to the suspension spectrum of X.

Recall the definition of X_{hM}^{∞} from Section 1. Applying an argument by Jeff Smith, we shall relate this to $SP_h(X)$.

Proposition 3.7 *The spaces X_{hM}^{∞} and $SP_h(X)$ are equivalent for any based CW–complex X.*

Proof Consider more generally an $\mathcal{I}$–space X. Let $\mathcal{I}_\omega$ be the category whose objects are the finite sets $\mathbf{n}$ in $\mathcal{I}$ together with the set ω. A morphisms is an injective map between such sets. Let $L_h^i X$ be the homotopy Kan extension of X along the inclusion $i\colon \mathcal{I} \to \mathcal{I}_\omega$. Then M acts on $L_h^i X(\omega)$ from the left and it follows from Shipley [14, Proposition 2.2.9] that the associated homotopy orbit space is equivalent to $\mathrm{hocolim}_{\mathcal{I}} X$.

Let T be the subcategory of $\mathcal{I}$ containing only the standard subset inclusions $\mathbf{m} \to \mathbf{n}$ for $m \le n$, and consider the commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim}_T X & \longrightarrow & \operatorname{hocolim}_{i/\omega} X \\ \downarrow & & \downarrow \\ \operatorname{colim}_T X & \longrightarrow & \operatorname{colim}_{i/\omega} X. \end{array}$$

By definition, $L_h^i X(\omega)$ is the homotopy colimit in the upper right hand corner of the diagram. The horizontal maps are induced by the functor t that takes an object $\mathbf{n}$ in T to the standard inclusion of $\mathbf{n}$ in ω. This functor is final in the sense that the categories u/t are contractible for each object u in i/ω, hence the upper horizontal map is an equivalence by the dual version of [3, Theorem XI.9.2] and the lower horizontal map is a homeomorphism by [8, Theorem IX.3.1]. Suppose now that the maps in the $\mathcal{I}$–diagram X are cofibrations. Then the left hand vertical map is an equivalence and therefore the same holds for the vertical map on the right hand side. Notice also that the latter is M–equivariant. Applying this to the $\mathcal{I}$–space $X^\bullet$ associated to a based CW–complex, we get an equivalence $L_h^i X^\bullet(\omega) \xrightarrow{\sim} X^\infty$. Letting M act on X^∞ as in Section 1, this is M–equivariant, hence the induced map of homotopy orbit spaces is also an equivalence. □

4 The Barratt–Priddy–Quillen Theorem

We begin by reformulating the definition of $SP_h(X)$. Given a based space X, let $\mathcal{I}(X)$ be the topological category whose objects have the form $(\mathbf{m}, \mathbf{x})$ for $\mathbf{x} \in X^m$, and in which a morphism $(\mathbf{m}, \mathbf{x}) \to (\mathbf{n}, \mathbf{y})$ is a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ such that $\alpha_* \mathbf{x} = \mathbf{y}$. It follows from the definition of the homotopy colimit, that $SP_h(X)$ may be identified with the classifying space $B\mathcal{I}(X)$. From this point of view, the monoid structure on $SP_h(X)$ is induced by the symmetric strict monoidal structure of $\mathcal{I}(X)$ inherited from $\mathcal{I}$.

4.1 Segal's construction

In Segal's formulation of the Barratt–Priddy–Quillen Theorem, one associates to an *unbased* space X the topological category $\Sigma(X)$ whose objects have the form $(\mathbf{m}, \mathbf{x})$ for $\mathbf{x} \in X^m$, and in which a morphism $(\mathbf{m}, \mathbf{x}) \to (\mathbf{n}, \mathbf{y})$ is an *isomorphism* $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ such that $\alpha_* \mathbf{x} = \mathbf{y}$; thus $\mathbf{m} = \mathbf{n}$. We wish to compare this category to $\mathcal{I}(X_+)$, where X_+ is the based space obtained by adjoining a disjoint base point. Given an object $(\mathbf{n},\mathbf{x})$ of the latter, consider the diagram

$$\begin{array}{ccc} \bar{\mathbf{n}} & \xrightarrow{\rho_{\mathbf{x}}} & \mathbf{n} \\ \big\downarrow{\scriptstyle \bar{\mathbf{x}}} & & \big\downarrow{\scriptstyle \mathbf{x}} \\ X & \longrightarrow & X_+, \end{array} \tag{4–1}$$

where $\rho_{\mathbf{x}}$ is the unique order preserving morphism in $\mathcal{I}$ whose image equals $\mathbf{x}^{-1}(X)$. Given a morphism $(\mathbf{m},\mathbf{x}) \to (\mathbf{n},\mathbf{y})$ in $\mathcal{I}(X_+)$ represented by a morphism α in $\mathcal{I}$, let $\pi(\alpha)$ be the morphism in $\Sigma(X)$ determined by the commutative diagram

$$\begin{array}{ccc} (\bar{\mathbf{m}},\bar{\mathbf{x}}) & \xrightarrow{\pi(\alpha)} & (\bar{\mathbf{n}},\bar{\mathbf{y}}) \\ \big\downarrow{\scriptstyle \rho_{\mathbf{x}}} & & \big\downarrow{\scriptstyle \rho_{\mathbf{y}}} \\ (\mathbf{m},\mathbf{x}) & \xrightarrow{\alpha} & (\mathbf{n},\mathbf{y}), \end{array}$$

where $\bar{\mathbf{m}} = \bar{\mathbf{n}}$. In this way we get a functor

$$\pi\colon \mathcal{I}(X_+) \to \Sigma(X), \quad (\mathbf{n},\mathbf{x}) \mapsto (\bar{\mathbf{n}},\bar{\mathbf{x}}) \tag{4–2}$$

that is natural with respect to the unbased space X. Furthermore, if we give $\Sigma(X)$ the symmetric strict monoidal structure induced by concatenation of permutations, then this functor is symmetric monoidal.

Proposition 4.3 *The functor π is an equivalence of symmetric monoidal categories and gives rise to an equivalence*

$$SP^{\wedge}_h(X_+) \xrightarrow{\sim} \Omega B(B\Sigma(X))$$

for any (unbased) CW–complex X.

Proof If we view $\Sigma(X)$ as a subcategory of $\mathcal{I}(X_+)$ via the inclusion of X in X_+, then the functor π provides a retraction. Furthermore, we may view the map $\rho_{\mathbf{x}}$ in Diagram (4–1) as a natural transformation relating the other composition to the identity on $\mathcal{I}(X_+)$. It follows that $B\Sigma(X)$ is a deformation retract of $B\mathcal{I}(X_+)$ and that the retraction

$$B\pi\colon B\mathcal{I}(X_+) \xrightarrow{\sim} B\Sigma(X)$$

is an equivalence. Applying the functor ΩB to this homomorphism, we get the required equivalence. $\square$

4.2 The Barratt–Eccles construction

Let Σ_n denote the symmetric group on the set $\mathbf{n}$. The correspondence $\mathbf{n} \mapsto \Sigma_n$ defines a contravariant $\mathcal{I}$–diagram $\Sigma_\bullet$ of sets as follows. Given an injective map $\alpha\colon \mathbf{m} \to \mathbf{n}$ and a permutation $\sigma \in \Sigma_n$, there is a unique factorization of the form

$$\sigma \circ \alpha = \sigma_*(\alpha) \circ \alpha^*(\sigma)$$

such that $\alpha^*(\sigma) \in \Sigma_m$ and $\sigma_*(\alpha)\colon \mathbf{m} \to \mathbf{n}$ is order preserving. The induced map $\alpha^*\colon \Sigma_n \to \Sigma_m$ is then defined by $\sigma \mapsto \alpha^*(\sigma)$. Let $\widetilde{\Sigma}_n$ be the translation category associated to Σ_n. This has as its objects the elements of Σ_n and a morphism $\rho\colon \sigma \to \tau$ is an element $\rho \in \Sigma_n$ such that $\rho \circ \sigma = \tau$. Notice, that since Σ_n is a group, a morphism is determined by its domain and target. It follows that in order to define a functor $\mathcal{C} \to \widetilde{\Sigma}_n$ from a category $\mathcal{C}$, one needs only specify the behaviour on objects. In particular, the $\mathcal{I}$–diagram $\Sigma_\bullet$ extends uniquely to an $\mathcal{I}$–diagram of categories $\mathbf{n} \mapsto \widetilde{\Sigma}_n$, hence to an $\mathcal{I}$–diagram of spaces $B\widetilde{\Sigma}_\bullet$. Given a based space X, the Barrett–Eccles construction [1] is the coend

$$\Gamma^+(X) = B\widetilde{\Sigma}_\bullet \otimes_{\mathcal{I}} X^\bullet,$$

that is, the quotient space

$$\left(\coprod_{n=0}^{\infty} B\widetilde{\Sigma}_n \times X^n\right) \Big/ (\alpha^*(\boldsymbol{\sigma}), \mathbf{x}) \sim (\boldsymbol{\sigma}, \alpha_*(\mathbf{x})),$$

where α runs through the morphisms in $\mathcal{I}$. We refer to Mac Lane [8, Section XI.6] for a general discussion of coends. Using the functors $\widetilde{\Sigma}_m \times \widetilde{\Sigma}_n \to \widetilde{\Sigma}_{m+n}$ given by concatenation of permutations, we define a monoid structure on $\Gamma^+(X)$ by

$$[\sigma, \mathbf{x}] \cdot [\tau, \mathbf{y}] = [\sigma \sqcup \tau, (\mathbf{x}, \mathbf{y})].$$

Notice, that for a based space of the form X_+, we have that

$$\Gamma^+(X_+) = \coprod_{n=0}^{\infty} B\widetilde{\Sigma}_n \times_{\Sigma_n} X^n \cong B\Sigma(X). \tag{4–4}$$

Proposition 4.5 *There is an equivalence of topological monoids*

$$\pi\colon SP_h(X) \xrightarrow{\sim} \Gamma^+(X)$$

for any based CW–complex X.

It follows of course that $SP_h^\wedge(X)$ is equivalent to $\Omega B(\Gamma^+(X))$. The latter is equivalent to the algebraic group completion $\Gamma(X)$ introduced by Barratt and Eccles in the simplicial setting (see Barratt and Eccles [1]). The proof of Proposition 4.5 requires yet another description of $SP_h(X)$. Consider the contravariant functor $B(\bullet/\mathcal{I})$ that to an object $\mathbf{n}$ in $\mathcal{I}$ associates the classifying space of the category $\mathbf{n}/\mathcal{I}$ of objects in $\mathcal{I}$ under $\mathbf{n}$. Then it follows from the definition of the homotopy colimit that $SP_h(X)$ is homeomorphic to the coend $B(\bullet/\mathcal{I}) \otimes_{\mathcal{I}} X^{\bullet}$. For each object $\mathbf{n}$ in $\mathcal{I}$ we now define a functor $\mathbf{n}/\mathcal{I} \to \widetilde{\Sigma}_n$ by mapping an object $\alpha\colon \mathbf{n} \to \mathbf{m}$ of the domain category to the permutation $\alpha^*(1_m)$ in Σ_n. Letting $\mathbf{n}$ vary, this defines a natural transformation of $\mathcal{I}^o$–diagrams of categories, hence we get a map of $\mathcal{I}^o$–diagrams

$$\pi\colon B(\bullet/\mathcal{I}) \to B\widetilde{\Sigma}_{\bullet}.$$

The map π in Proposition 4.5 is the induced map

$$\pi\colon SP_h(X) = B(\bullet/\mathcal{I}) \otimes_{\mathcal{I}} X^{\bullet} \to B\widetilde{\Sigma}_{\bullet} \otimes_{\mathcal{I}} X^{\bullet} = \Gamma^+(X).$$

One checks that for a space of the form X_+, this map agrees with that induced by the functor π in Section 4.1 if we identify the right hand sides as in (4–4).

Proof of Proposition 4.5 Consider first the case, where X is a based discrete set. Then we may view X as a space with a disjoint base point and the result follows from the proof of Proposition 4.3. Consider then the case where X is the realization of a based simplicial set $X_{\bullet}$. We then have a commutative diagram

$$\begin{array}{ccc} SP_h(X) & \longrightarrow & \Gamma^+(X) \\ \| & & \| \\ |SP_h(X_{\bullet})| & \longrightarrow & |\Gamma^+(X_{\bullet})|, \end{array}$$

where the vertical maps are homeomorphisms and the bottom map is the realization of a degree-wise equivalence, hence itself an equivalence. Thus, the result also holds for X of this form. The general case now follows, since both functors are homotopy functors and any CW–complex is equivalent to the realization of a simplicial set, for example the total singular complex. □

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