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Non-finiteness results for Nil-groups

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Generalizing an idea of Farrell we prove that for a ring Λ and a ring automorphism α of finite order the groups $\mathrm{Nil}_0(\Lambda;\alpha)$ and all of its p–primary subgroups are either trivial or not finitely generated as an abelian group. We also prove that if β and γ are ring automorphisms such that $\beta\circ\gamma$ is of finite order then $\mathrm{Nil}_0(\Lambda;\Lambda_\beta,\Lambda_\gamma)$ and all of its p–primary subgroups are either trivial or not finitely generated as an abelian group. These Nil-groups include the Nil-groups appearing in the decomposition of K_i of virtually cyclic groups for $i\leq 1$.

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1 Introduction

Let Λ be a unital ring and α a ring automorphism. Farrell defined in his PhD thesis [2] twisted Nil-groups, $\mathrm{Nil}_i(\Lambda;\alpha)$ for $i\in\mathbb{N}$. We denote the twisted polynomial ring by $\Lambda_\alpha[t]$. The Nil-group $\mathrm{Nil}_i(\Lambda;\alpha)$ is the kernel of the map $\epsilon\colon K_{i+1}(\Lambda_\alpha[t])\to K_{i+1}(\Lambda)$ which is induced by the augmentation map. Farrell–Hsiang [4] and Grayson [6] generalized the *fundamental lemma of algebraic K–theory* to twisted Laurent polynomial rings. They proved the exactness of the following sequence, relating the K–theory of the twisted Laurent polynomial ring $\Lambda_\alpha[t,t^{-1}]$ to the K–theory of Λ:

$$\cdots\longrightarrow K_{i+1}(\Lambda)\xrightarrow{1-\alpha_*}K_{i+1}(\Lambda)\longrightarrow$$
$$K_{i+1}(\Lambda_\alpha[t,t^{-1}])/\big(\mathrm{Nil}_i(\Lambda,\alpha)\oplus\mathrm{Nil}_i(\Lambda,\alpha^{-1})\big)\longrightarrow K_i(\Lambda)\longrightarrow\cdots.$$

Let A, B and C be rings and let $\alpha\colon C\to A$ and $\beta\colon C\to B$ be inclusions which are *pure* and *free* (for a definition of pure and free see Waldhausen [10]). Let R be the push-out of the diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \\ B & & \end{array}$$

in the category of rings. Waldhausen proved that there is a Mayer-Vietoris sequence for algebraic K–theory, which is exact up to Nil-groups $\mathrm{Nil}_i(C;A',B')$, where A' is

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defined to be the C–bimodule such that $A = \alpha(C) \oplus A'$ and B' is defined similarly [10; 11]. More precisely he proved that the following sequence is exact:

$$\cdots \longrightarrow K_{i+1}(C) \longrightarrow K_{i+1}(A) \oplus K_{i+1}(B) \longrightarrow$$
$$K_{i+1}(R)/\operatorname{Nil}_i\left(C; A', B'\right) \longrightarrow K_i(C) \longrightarrow \cdots .$$

In the article at hand we use an idea which goes back to Farrell [3] to prove that Nil-groups and its p–primary subgroups have the mysterious property of being either trivial or not finitely generated as an abelian group. For a ring automorphism $\alpha\colon \Lambda \to \Lambda$ we denote the Λ–bimodule Λ with Λ–action from the left via the identity and from the right via α by Λ_α. For an abelian group G and a prime p, define

$$G_p = \{x \in G : p^n x = 0 \text{ for some } n \geq 0\}.$$

G_p is called the *p–primary subgroup* of G.

Theorem 1.1 *Let Λ be a ring, p a prime and α a ring automorphism of finite order. The groups $\operatorname{Nil}_0(\Lambda;\alpha)$ and $\operatorname{Nil}_0(\Lambda;\alpha)_p$ are either trivial or not finitely generated as an abelian group. If β and γ are ring automorphisms such that $\beta \circ \gamma$ is of finite order then $\operatorname{Nil}_0(\Lambda;\Lambda_\beta,\Lambda_\gamma)$ and $\operatorname{Nil}_0(\Lambda;\Lambda_\beta,\Lambda_\gamma)_p$ are either trivial or not finitely generated as an abelian group.*

The non-finiteness of $\operatorname{Nil}_0(\Lambda;\alpha)$ for $\alpha = \mathrm{id}$ was already known [3] and the non-finiteness of $\operatorname{Nil}_0(\mathbb{Z}G;\alpha)$ for a finite group G was independently proven by Ramos [9].

For topology the K–theory of group rings is of special importance and the Farrell-Jones conjecture, which is known to be true for a large class of groups, predicts that the building blocks of the K–theory of a group ring is the K–theory of virtually cyclic groups. There are two types of infinite virtually cyclic groups:

(i) the semidirect product $G \rtimes \mathbb{Z}$ of a finite group G and the infinite cyclic group;

(ii) the amalgamated product $G_1 *_H G_2$ of two finite groups G_1 and G_2 over a subgroup H such that $[G_1 : H] = 2 = [G_2 : H]$;

If one decomposes the K–theory of infinite virtually cyclic groups the Nil-groups of finite groups appear. Since for a finite group all automorphisms are of finite order we obtain the following corollary about Nil-groups of finite groups.

Corollary 1.2 *Let R be a ring, G a finite group, p a prime and α and β group automorphisms. The groups $\operatorname{Nil}_i(RG;\alpha)$, $\operatorname{Nil}_i(RG;\alpha)_p$, $\operatorname{Nil}_i(RG;RG_\alpha,RG_\beta)$ and $\operatorname{Nil}_i(RG;RG_\alpha,RG_\beta)_p$ are either trivial or not finitely generated as an abelian group for $i \leq 0$.*

For $R = \mathbb{Z}$ the considered Nil-groups are known to vanish for $i \leq -2$ (see Farrell and Jones [5]) and are known to be n–torsion for an arbitrary group of finite order n (see Kuku and Tang [8]).

2 Non-finiteness results for Nil-groups

In the following Λ will always be a unital ring and α a ring automorphism of finite order n, that is, $\alpha^n = \mathrm{id}$.

For $m \in \mathbb{N}$ we have canonical inclusion maps

$$\sigma_m \colon \Lambda_{\alpha^{nm+1}}[t^{nm+1}] \to \Lambda_\alpha[t].$$

Those maps induce transfer and induction maps

$$\begin{aligned} \sigma_*^m \colon K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]) &\to K_1(\Lambda_\alpha[t]) \\ \sigma_m^* \colon K_1(\Lambda_\alpha[t]) &\to K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]). \end{aligned}$$

Since $\mathrm{Nil}_0(\Lambda;\alpha) = \mathrm{Nil}_0(\Lambda;\alpha^{nm+1})$ we have an embedding

$$\iota' \colon \mathrm{Nil}_0(\Lambda;\alpha) \hookrightarrow K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]).$$

The proof of the non-finiteness result is based on the following diagram:

$$\begin{array}{ccc} \mathrm{Nil}_0(\Lambda;\alpha) & \xrightarrow{\iota'} & K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]) \\ & & \downarrow \sigma_*^m \\ \mathrm{Nil}_0(\Lambda;\alpha) & \xrightarrow{\iota} & K_1(\Lambda_\alpha[t]) \\ & & \downarrow \sigma_m^* \\ & & K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]). \end{array}$$

The idea is to choose, for a finitely generated Nil-group, m such that $\sigma_m^*\sigma_*^m\iota'$ is a monomorphism (Lemma 2.1) and trivial (Proposition 2.3). Thus every finitely generated Nil-group is trivial.

Lemma 2.1 *Let G be a finitely generated subgroup of $K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}])$. For every $K \in \mathbb{N}$ there is an $m \geq K$ such that $\sigma_m^*\sigma_*^m$ is a monomorphism on G.*

Proof Let T be the exponent of the torsion subgroup of G and let F be the rank of a maximal torsion free subgroup. Choose $\ell \in \mathbb{N}$ such that $\ell \cdot T \geq K$. For $x \in$

$K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}])$ we have

$$\sigma^*_{\ell\cdot T}\sigma_*^{\ell\cdot T}(x)=\sum_{i=0}^{n\cdot\ell\cdot T}\alpha_*^i(x)=x+\ell\cdot T\sum_{i=1}^{n}\alpha_*^i(x)$$

where $\alpha_*\colon K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}])\to K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}])$ is the map which is induced by the ring automorphism on $\Lambda_{\alpha^{nm+1}}[t^{nm+1}]$ which sends an element $\sum r_i t^i$ to the element $\sum\alpha(r_i)t^i$. The automorphism α_* restricts to an automorphism of $A:=\cup_{i=1}^n\alpha^i(\mathbb{Z}^F)$ where $\mathbb{Z}^F$ is a maximal torsion free subgroup of G. The map $\alpha_*|_A$ is conjugate to a diagonal matrix, that is,

$$g\alpha_*|_A g^{-1}=\begin{pmatrix}\zeta_1 & & \\ & \ddots & \\ & & \zeta_r\end{pmatrix}$$

where $g\in\mathrm{GL}_r(\mathbb{C})$ and $\zeta_1,\dots,\zeta_r\in\mathbb{C}$ are nth roots of unity. We can find $k\in\mathbb{N}$ such that $\sum_{i=1}^{k\cdot\ell\cdot T}\zeta_j^i\neq -1$ for all $j\in\{1,\dots,r\}$. One verifies easily that $\sigma^*_{k\cdot\ell\cdot T}\sigma_*^{k\cdot\ell\cdot T}(x)$ is a monomorphism. □

Lemma 2.2 *The image of ι' is mapped into the image of ι by every σ_*^m.*

Proof The result follows since the diagram

$$\begin{array}{ccc} K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]) & \xrightarrow{\epsilon} & K_1(\Lambda) \\ \downarrow{\scriptstyle\sigma_*^m} & & \downarrow{\scriptstyle\mathrm{id}} \\ K_1(\Lambda_\alpha[t]) & \xrightarrow{\epsilon} & K_1(\Lambda)\end{array}$$

commutes. We denote the maps which are induced by the augmentation map by ϵ. □

Proposition 2.3 *For every $x\in\mathrm{Nil}_0(\Lambda;\alpha)$, there exists an integer $K(x)$ such that $\sigma_m^*(x)=0$ for all integers $m\geq K(x)$.*

For the proof of Proposition 2.3 we need the following lemma. We denote by $\mathrm{GL}_n(\Lambda)$ the group of invertible $n\times n$ matrices, by $\mathrm{GL}(\Lambda)$ the colimit over $\mathrm{GL}_n(\Lambda)$ and by $\mathrm{E}(\Lambda_\alpha[t])$ the subgroup of $\mathrm{GL}(\Lambda_\alpha[t])$ generated by all elementary matrices. For a matrix N we denote by $\alpha(N)$ the matrix obtained for N by applying α to each component.

Lemma 2.4 *Every matrix $B\in\mathrm{GL}(\Lambda_\alpha[t])$ can be reduced, modulo $\mathrm{GL}(\Lambda)$ and $\mathrm{E}(\Lambda_\alpha[t])$, to a matrix of the from $1+Nt$, where*

$$\prod_{j=0}^{M}\alpha^{-j}(N)=0$$

for some $M \in \mathbb{N}$.

Proof We have

$$B = B_0 + B_1 t + \dots + B_n t^n$$

with $B_i \in \mathrm{Mat}_m(\Lambda)$. In $\mathrm{GL}(\Lambda_\alpha[t])$ we have

$$B = \begin{pmatrix} B & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Modulo $\mathrm{E}(\Lambda_\alpha[t])$ we have:

$$\begin{pmatrix} B & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} B & B_n t^n \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} B - B_n t^n & B_n t^n \\ -t & \mathrm{id} \end{pmatrix}.$$

This implies by induction that

$$B = \widetilde{B}_0 + \widetilde{B}_1 t.$$

Since $B \in \mathrm{GL}_k(\Lambda_\alpha[t])$ there exists B^{-1} with

$$B^{-1} = C_0 + C_1 t + \dots + C_m t^m.$$

where $C_i \in \mathrm{Mat}_k(\Lambda)$. We have

$$1 = BB^{-1} = B_0 C_0 + B_1 t C_0 + \dots + B_1 t C_m t^m.$$

Thus $B_0 C_0 = 1$ and therefore $B = 1 + Nt$ module $\mathrm{GL}(\Lambda)$. Let $L = L_0 + L_1 t + \dots + L_m t^m$ be the inverse of $(1 + Nt)$. We have

$$\begin{aligned} 1 &= (1 + Nt)(L_0 + L_1 t + \dots + L_m t^m) \\ &= L_0 + NtL_0 + L_1 t + NtL_1 t + \dots + L_m t^m + NtL_m t^m \\ &= L_0 + \sum_{i=0}^{m-1} (N\alpha^{-1}(L_i) + L_{i+1}) t^{i+1} + N\alpha^{-1}(L_m) t^{m+1} \end{aligned}$$

This implies the following identities:

$$\begin{aligned} L_0 &= 1 \\ N\alpha^{-1}(L_0) + L_1 &= 0 \\ &\vdots \\ N\alpha^{-1}(L_i) + L_{i+1} &= 0 \\ &\vdots \\ N\alpha^{-1}(L_{m-1}) + L_m &= 0 \\ N\alpha^{-1}(L_m) &= 0. \end{aligned}$$

Thus

$$\prod_{j=0}^{m-1} \alpha^{-j}(N) = 0.$$

□

Proof of Proposition 2.3 By Lemma 2.4 we have

$$x = 1 + Nt$$

with

$$\prod_{i=0}^{M} \alpha^{-i}(N) = 0.$$

The element $\sigma_*^m(x)$ is represented by the matrix

$$\begin{pmatrix} \mathrm{id} & & & & \alpha^{-nm}(N)t^{nm+1} \\ N & \mathrm{id} & & & \\ & \alpha^{-1}(N) & \ddots & & \\ & & \ddots & \mathrm{id} & \\ & & & \alpha^{-nm+1}(N) & \mathrm{id} \end{pmatrix}.$$

Thus $\sigma_*^m(x)$ is also represented by the following matrix:

$$\begin{pmatrix} \mathrm{id}+(-1)^{nm}\left(\prod_{i=0}^{nm}\alpha^{-i}(N)\right)t^{nm+1} & \cdots & (-1)^{nm-j}\left(\prod_{i=j}^{nm}\alpha^{-i}(N)\right)t^{nm+1} & \cdots & \alpha^{-nm}(N)t^{nm+1} \\ & \mathrm{id} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \mathrm{id} \end{pmatrix}$$

This implies that for m such that $n \cdot m \geq M$ we have $\sigma_*^m(x) = 0$. □

Theorem 2.5 *Let Λ be a ring, p a prime and α a ring automorphism of finite order. The groups $\mathrm{Nil}_0(\Lambda;\alpha)$ and $\mathrm{Nil}_0(\Lambda;\alpha)_p$ are either trivial or not finitely generated as an abelian group.*

Proof Assume $\mathrm{Nil}_0(\Lambda;\alpha)$ to be a finitely generated abelian group. By Lemma 2.2 and Proposition 2.3 we can find K such that $\sigma_m^*\sigma_*^m\iota'(x) = 0$ for all $x \in \mathrm{Nil}(\Lambda;\alpha)$ and $m \geq K$. By Lemma 2.1 we can find an $m \geq K$ such that $\sigma_m^*\sigma_*^m\iota'$ is a monomorphism. Thus $\mathrm{Nil}_0(\Lambda;\alpha)$ is the trivial group. The proof for $\mathrm{Nil}(\Lambda;\alpha)_p$ goes in exactly the same way. □

Corollary 2.6 *Let Λ be a ring, p be a prime and α and β be ring automorphisms such that $\alpha \circ \beta$ is of finite order. The groups $\mathrm{Nil}_0(\Lambda; \Lambda_\alpha, \Lambda_\beta)$ and $\mathrm{Nil}_0(\Lambda; \Lambda_\alpha, \Lambda_\beta)_p$ are either trivial or not finitely generated as an abelian group.*

Proof It is a result of Kuku and Tang [8] that $\mathrm{Nil}_0(\Lambda; \Lambda_\alpha, \Lambda_\beta)$ can also be described as a Nil-group of type $\mathrm{Nil}_0(\Lambda \times \Lambda; \gamma)$ where γ is the ring automorphism defined by

$$\gamma\colon (a,b) \mapsto \big(\beta(b), \alpha(a)\big). \qquad \square$$

Corollary 2.7 *Let R be a ring, G a finite group, p a prime and α and β group automorphisms. The groups $\mathrm{Nil}_i(RG;\alpha)$, $\mathrm{Nil}_i(RG;\alpha)_p$, $\mathrm{Nil}_i(RG; RG_\alpha, RG_\beta)$ and $\mathrm{Nil}_i(RG; RG_\alpha, RG_\beta)_p$ are either trivial or not finitely generated as an abelian group for $i \leq 0$.*

Proof Using the suspension ring construction as explained by Bartels and Lück [1] and by the author [7], one gets that the considered Nil-groups are covered by Theorem 2.5 and Corollary 2.6. $\square$

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