Non-finiteness results for Nil-groups

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Generalizing an idea of Farrell we prove that for a ring \( \Lambda \) and a ring automorphism \( \alpha \) of finite order the groups \( \text{Nil}_i(\Lambda; \alpha) \) and all of its \( p \)-primary subgroups are either trivial or not finitely generated as an abelian group. We also prove that if \( \beta \) and \( \gamma \) are ring automorphisms such that \( \beta \circ \gamma \) is of finite order then \( \text{Nil}_i(\Lambda; \Lambda_\beta, \Lambda_\gamma) \) and all of its \( p \)-primary subgroups are either trivial or not finitely generated as an abelian group. These Nil-groups include the Nil-groups appearing in the decomposition of \( K_i \) of virtually cyclic groups for \( i \leq 1 \).

1 Introduction

Let \( \Lambda \) be a unital ring and \( \alpha \) a ring automorphism. Farrell defined in his PhD thesis [2] twisted Nil-groups, \( \text{Nil}_i(\Lambda; \alpha) \) for \( i \in \mathbb{N} \). We denote the twisted polynomial ring by \( \Lambda_\alpha[t] \). The Nil-group \( \text{Nil}_i(\Lambda; \alpha) \) is the kernel of the map \( \epsilon: K_{i+1}(\Lambda_\alpha[t]) \to K_i(\Lambda) \) which is induced by the augmentation map. Farrell–Hsiang [4] and Grayson [6] generalized the fundamental lemma of algebraic \( K \)-theory to twisted Laurent polynomial rings. They proved the exactness of the following sequence, relating the \( K \)-theory of the twisted Laurent polynomial ring \( \Lambda_\alpha[t, t^{-1}] \) to the \( K \)-theory of \( \Lambda \):

\[
\cdots \to K_{i+1}(\Lambda) \xrightarrow{1-\alpha} K_{i+1}(\Lambda) \xrightarrow{\epsilon} K_{i+1}(\Lambda_\alpha[t]) / \left( \text{Nil}_i(\Lambda, \alpha) \oplus \text{Nil}_i(\Lambda, \alpha^{-1}) \right) \xrightarrow{\alpha^{-1}} K_i(\Lambda) \xrightarrow{\beta} K_i(\Lambda_\alpha[t]) \xrightarrow{\epsilon} K_{i+1}(\Lambda) \to \cdots.
\]

Let \( A, B \) and \( C \) be rings and let \( \alpha: C \to A \) and \( \beta: C \to B \) be inclusions which are pure and free (for a definition of pure and free see Waldhausen [10]). Let \( R \) be the push-out of the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & A \\
\downarrow{\beta} & & \\
B & & \\
\end{array}
\]

in the category of rings. Waldhausen proved that there is a Mayer-Vietoris sequence for algebraic \( K \)-theory, which is exact up to Nil-groups \( \text{Nil}_i(C; A', B') \), where \( A' \) is...
defined to be the \( C \)-bimodule such that \( A = \alpha(C) \oplus A' \) and \( B' \) is defined similarly \([10; 11]\). More precisely he proved that the following sequence is exact:

\[
\cdots \rightarrow K_{i+1}(C) \rightarrow K_{i+1}(A) \oplus K_{i+1}(B) \rightarrow K_{i+1}(R) / \text{Nil}_i(C; A', B') \rightarrow K_i(C) \rightarrow \cdots.
\]

In the article at hand we use an idea which goes back to Farrell \([3]\) to prove that Nil-groups and its \( p \)-primary subgroups have the mysterious property of being either trivial or not finitely generated as an abelian group. For a ring automorphism \( \alpha: \Lambda \rightarrow \Lambda \) we denote the \( \Lambda \)-bimodule \( \Lambda \) with \( \Lambda \)-action from the left via the identity and from the right via \( \alpha \) by \( \Lambda_\alpha \). For an abelian group \( G \) and a prime \( p \), define

\[
G_p = \{ x \in G : p^n x = 0 \text{ for some } n \geq 0 \}.
\]

\( G_p \) is called the \( p \)-primary subgroup of \( G \).

**Theorem 1.1** Let \( \Lambda \) be a ring, \( p \) a prime and \( \alpha \) a ring automorphism of finite order. The groups \( \text{Nil}_0(\Lambda; \alpha) \) and \( \text{Nil}_0(\Lambda; \alpha)_p \) are either trivial or not finitely generated as an abelian group. If \( \beta \) and \( \gamma \) are ring automorphisms such that \( \beta \circ \gamma \) is of finite order then \( \text{Nil}_0(\Lambda; \Lambda_\beta, \Lambda_\gamma) \) and \( \text{Nil}_0(\Lambda; \Lambda_\beta, \Lambda_\gamma)_p \) are either trivial or not finitely generated as an abelian group.

The non-finiteness of \( \text{Nil}_0(\Lambda; \alpha) \) for \( \alpha = \text{id} \) was already known \([3]\) and the non-finiteness of \( \text{Nil}_0(\mathbb{Z}G; \alpha) \) for a finite group \( G \) was independently proven by Ramos \([9]\).

For topology the \( K \)-theory of group rings is of special importance and the Farrell-Jones conjecture, which is known to be true for a large class of groups, predicts that the building blocks of the \( K \)-theory of a group ring is the \( K \)-theory of virtually cyclic groups. There are two types of infinite virtually cyclic groups:

(i) the semidirect product \( G \rtimes \mathbb{Z} \) of a finite group \( G \) and the infinite cyclic group;

(ii) the amalgamated product \( G_1 *_H G_2 \) of two finite groups \( G_1 \) and \( G_2 \) over a subgroup \( H \) such that \([G_1 : H] = 2 = [G_2 : H]\);

If one decomposes the \( K \)-theory of infinite virtually cyclic groups the Nil-groups of finite groups appear. Since for a finite group all automorphisms are of finite order we obtain the following corollary about Nil-groups of finite groups.

**Corollary 1.2** Let \( R \) be a ring, \( G \) a finite group, \( p \) a prime and \( \alpha \) and \( \beta \) group automorphisms. The groups \( \text{Nil}_i(RG; \alpha) \), \( \text{Nil}_i(RG; \alpha)_p \), \( \text{Nil}_i(RG; RG_\alpha, RG_\beta) \) and \( \text{Nil}_i(RG; RG_\alpha, RG_\beta)_p \) are either trivial or not finitely generated as an abelian group for \( i \leq 0 \).
For $R = \mathbb{Z}$ the considered Nil-groups are known to vanish for $i \leq -2$ (see Farrell and Jones [5]) and are known to be $n$–torsion for an arbitrary group of finite order $n$ (see Kuku and Tang [8]).

2 Non-finiteness results for Nil-groups

In the following $\Lambda$ will always be a unital ring and $\alpha$ a ring automorphism of finite order $n$, that is, $\alpha^n = \text{id}.$

For $m \in \mathbb{N}$ we have canonical inclusion maps
\[ \sigma_m : \Lambda_{\alpha^{nm+1}}[t^{nm+1}] \to \Lambda_{\alpha}[t]. \]

Those maps induce transfer and induction maps
\[ \sigma_*^m : K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]) \to K_1(\Lambda_{\alpha}[t]) \]
\[ \sigma_*^m : K_1(\Lambda_{\alpha}[t]) \to K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]). \]

Since $\text{Nil}_0(\Lambda; \alpha) = \text{Nil}_0(\Lambda; \alpha^{nm+1})$ we have an embedding
\[ \iota' : \text{Nil}_0(\Lambda; \alpha) \hookrightarrow K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]). \]

The proof of the non-finiteness result is based on the following diagram:
\[
\begin{array}{c}
\text{Nil}_0(\Lambda; \alpha) \\ \downarrow \iota'
\end{array}
\quad \xymatrix{
K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]) \ar[d]^{\sigma_*^m} \\
\text{Nil}_0(\Lambda; \alpha) \\ \downarrow \iota
\end{array}
\quad \xymatrix{
K_1(\Lambda_{\alpha}[t]) \ar[d]^{\sigma_*^m} \\
K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}]).
\end{array}
\]

The idea is to choose, for a finitely generated Nil-group, $m$ such that $\sigma_*^m \sigma_*^m \iota'$ is a monomorphism (Lemma 2.1) and trivial (Proposition 2.3). Thus every finitely generated Nil-group is trivial.

**Lemma 2.1** Let $G$ be a finitely generated subgroup of $K_1(\Lambda_{\alpha^{nm+1}}[t^{nm+1}])$. For every $K \in \mathbb{N}$ there is an $m \geq K$ such that $\sigma_*^m \sigma_*^m$ is a monomorphism on $G$.

**Proof** Let $T$ be the exponent of the torsion subgroup of $G$ and let $F$ be the rank of a maximal torsion free subgroup. Choose $\ell \in \mathbb{N}$ such that $\ell \cdot T \geq K$. For $x \in$
\( K_1(\Lambda_{\alpha^{nm+1}[t^{nm+1}]}) \) we have
\[
\sigma_{\ell,T}^* \alpha_{\ell,T}^* (x) = \sum_{i=0}^{n} \alpha_i^*(x) = x + \ell \cdot T \sum_{i=1}^{n} \alpha_i^*(x)
\]
where \( \alpha_*: K_1(\Lambda_{\alpha^{nm+1}[t^{nm+1}]}) \rightarrow K_1(\Lambda_{\alpha^{nm+1}[t^{nm+1}]}) \) is the map which is induced by the ring automorphism on \( \Lambda_{\alpha^{nm+1}[t^{nm+1}]} \) which sends an element \( \sum r_i t^i \) to the element \( \sum \alpha(r_i) t^i \). The automorphism \( \alpha_* \) restricts to an automorphism of \( A := \bigcup_{i=1}^{n} \alpha^i(\mathbb{Z}^F) \) where \( \mathbb{Z}^F \) is a maximal torsion free subgroup of \( G \). The map \( \alpha_*|_A \) is conjugate to a diagonal matrix, that is,
\[
g \alpha_*|_A g^{-1} = \begin{pmatrix} \zeta_1 & & \\ & \ddots & \\ & & \zeta_r \end{pmatrix}
\]
where \( g \in \text{GL}_r(\mathbb{C}) \) and \( \zeta_1, \ldots, \zeta_r \in \mathbb{C} \) are \( n \)th roots of unity. We can find \( k \in \mathbb{N} \) such that \( \sum_{i=1}^{k} \zeta_i \neq -1 \) for all \( j \in \{ 1, \ldots, r \} \). One verifies easily that \( \sigma_{k\ell,T}^* \alpha_{k\ell,T}^* (x) \) is a monomorphism.

**Lemma 2.2** The image of \( \ell' \) is mapped into the image of \( \ell \) by every \( \sigma_m^* \).

**Proof** The result follows since the diagram

\[
\begin{array}{ccc}
K_1(\Lambda_{\alpha^{nm+1}[t^{nm+1}]}) & \xrightarrow{\epsilon} & K_1(\Lambda) \\
\downarrow \sigma_m^* & & \downarrow \text{id} \\
K_1(\Lambda_{\alpha[t]}) & \xrightarrow{\epsilon} & K_1(\Lambda)
\end{array}
\]

commutes. We denote the maps which are induced by the augmentation map by \( \epsilon \).

**Proposition 2.3** For every \( x \in \text{Nil}_0(\Lambda; \alpha) \), there exists an integer \( K(x) \) such that \( \sigma_m^*(x) = 0 \) for all integers \( m \geq K(x) \).

For the proof of **Proposition 2.3** we need the following lemma. We denote by \( \text{GL}_n(\Lambda) \) the group of invertible \( n \times n \) matrices, by \( \text{GL}(\Lambda) \) the colimit over \( \text{GL}_n(\Lambda) \) and by \( \text{E}(\Lambda_{\alpha[t]}) \) the subgroup of \( \text{GL}(\Lambda_{\alpha[t]}) \) generated by all elementary matrices. For a matrix \( N \) we denote by \( \alpha(N) \) the matrix obtained for \( N \) by applying \( \alpha \) to each component.

**Lemma 2.4** Every matrix \( B \in \text{GL}(\Lambda_{\alpha[t]}) \) can be reduced, modulo \( \text{GL}(\Lambda) \) and \( \text{E}(\Lambda_{\alpha[t]}) \), to a matrix of the from \( t^M \), where
\[
\prod_{j=0}^{M} \alpha^{-j}(N) = 0
\]
for some $M \in \mathbb{N}$.

**Proof** We have

$$B = B_0 + B_1 t + \ldots + B_n t^n$$

with $B_i \in \text{Mat}_m(\Lambda)$. In $\text{GL}(\Lambda[t])$ we have

$$B = \begin{pmatrix} B & 0 \\ 0 & \text{id} \end{pmatrix}.$$ 

Modulo $E(\Lambda[t])$ we have:

$$\begin{pmatrix} B & 0 \\ 0 & \text{id} \end{pmatrix} = \begin{pmatrix} B & B_n t^n \\ 0 & \text{id} \end{pmatrix} = \begin{pmatrix} B - B_n t^n & B_n t^n \\ -t & \text{id} \end{pmatrix}.$$ 

This implies by induction that

$$B = \widetilde{B}_0 + \widetilde{B}_1 t.$$ 

Since $B \in \text{GL}_k(\Lambda[t])$ there exists $B^{-1}$ with

$$B^{-1} = C_0 + C_1 t + \ldots + C_m t^m.$$ 

where $C_i \in \text{Mat}_k(\Lambda)$. We have

$$1 = BB^{-1} = B_0 C_0 + B_1 t C_0 + \ldots + B_1 t C_m t^m.$$ 

Thus $B_0 C_0 = 1$ and therefore $B = 1 + N t$ modulo $\text{GL}(\Lambda)$. Let $L = L_0 + L_1 t + \ldots + L_m t^m$ be the inverse of $(1 + N t)$. We have

$$1 = (1 + N t)(L_0 + L_1 t + \ldots + L_m t^m)$$

$$= L_0 + N t L_0 + L_1 t + N t L_1 t + \ldots + L_m t^m + N t L_m t^m$$

$$= L_0 + \sum_{i=0}^{m-1} (N \alpha^{-1}(L_i) + L_{i+1}) t^{i+1} + N \alpha^{-1}(L_m) t^{m+1}.$$ 

This implies the following identities:

$$L_0 = 1$$

$$N \alpha^{-1}(L_0) + L_1 = 0$$

$$\vdots$$

$$N \alpha^{-1}(L_{i+1}) = 0$$

$$\vdots$$

$$N \alpha^{-1}(L_{m-1}) + L_m = 0$$

$$N \alpha^{-1}(L_m) = 0.$$

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Thus
\[
\prod_{j=0}^{m-1} \alpha^{-j}(N) = 0.
\]

\[\square\]

**Proof of Proposition 2.3** By Lemma 2.4 we have
\[
x = 1 + Nt
\]
with
\[
\prod_{i=0}^{M} \alpha^{-i}(N) = 0.
\]

The element \(\sigma^m(x)\) is represented by the matrix
\[
\begin{pmatrix}
\text{id} & \alpha^{-nm}(N)t^{nm+1} \\
N & \text{id} \\
\alpha^{-1}(N) & \ddots \\
& \ddots & \text{id} \\
& & \alpha^{-nm+1}(N) & \text{id}
\end{pmatrix}.
\]

Thus \(\sigma^m(x)\) is also represented by the following matrix:
\[
\begin{pmatrix}
\text{id} & (-1)^{nm}(\prod_{i=0}^{nm} \alpha^{-i}(N))t^{nm+1} & \cdots & (-1)^{nm-j}(\prod_{i=j}^{nm} \alpha^{-i}(N))t^{nm+1} & \cdots & \alpha^{-nm}(N)t^{nm+1} \\
\text{id} & & \ddots & & & \\
& \ddots & & \ddots & & \\
& & \ddots & & \ddots & \\
& & & \ddots & \ddots & \text{id}
\end{pmatrix}
\]

This implies that for \(m\) such that \(n \cdot m \geq M\) we have \(\sigma^m(x) = 0\).

\[\square\]

**Theorem 2.5** Let \(\Lambda\) be a ring, \(p\) a prime and \(\alpha\) a ring automorphism of finite order. The groups \(\text{Nil}_0(\Lambda; \alpha)\) and \(\text{Nil}_0(\Lambda; \alpha)_p\) are either trivial or not finitely generated as an abelian group.

**Proof** Assume \(\text{Nil}_0(\Lambda; \alpha)\) to be a finitely generated abelian group. By Lemma 2.2 and Proposition 2.3 we can find \(K\) such that \(\sigma^m\sigma^m(x) = 0\) for all \(x \in \text{Nil}(\Lambda; \alpha)\) and \(m \geq K\). By Lemma 2.1 we can find an \(m \geq K\) such that \(\sigma^m\sigma^m\) is a monomorphism. Thus \(\text{Nil}_0(\Lambda; \alpha)\) is the trivial group. The proof for \(\text{Nil}(\Lambda; \alpha)_p\) goes in exactly the same way.

\[\square\]
Corollary 2.6  Let $\Lambda$ be a ring, $p$ be a prime and $\alpha$ and $\beta$ be ring automorphisms such that $\alpha \circ \beta$ is of finite order. The groups $\Nil_0(\Lambda; \Lambda_{\alpha}, \Lambda_{\beta})$ and $\Nil_0(\Lambda; \Lambda_{\alpha}, \Lambda_{\beta})_p$ are either trivial or not finitely generated as an abelian group.

Proof  It is a result of Kuku and Tang [8] that $\Nil_0(\Lambda; \Lambda_{\alpha}, \Lambda_{\beta})$ can also be described as a Nil-group of type $\Nil_0(\Lambda \times \Lambda; \gamma)$ where $\gamma$ is the ring automorphism defined by $\gamma: (a, b) \mapsto (\beta(b), \alpha(a))$. 

Corollary 2.7  Let $R$ be a ring, $G$ a finite group, $p$ a prime and $\alpha$ and $\beta$ group automorphisms. The groups $\Nil_i(RG; \alpha), \Nil_i(RG; \alpha)_p, \Nil_i(RG; RG_{\alpha}, RG_{\beta})$ and $\Nil_i(RG; RG_{\alpha}, RG_{\beta})_p$ are either trivial or not finitely generated as an abelian group for $i \leq 0$.

Proof  Using the suspension ring construction as explained by Bartels and Lück [1] and by the author [7], one gets that the considered Nil-groups are covered by Theorem 2.5 and Corollary 2.6.

References


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