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The foam and the matrix factorization sl_3 link homologies are equivalent

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We prove that the universal rational sl_3 link homologies which were constructed by Khovanov in [3] and the authors in [7], using foams, and by Khovanov and Rozansky in [4], using matrix factorizations, are naturally isomorphic as projective functors from the category of links and link cobordisms to the category of bigraded vector spaces.

57M27; 57M25, 81R50, 18G60

1 Introduction

In [3] Khovanov constructed a bigraded integer link homology categorifying the sl_3 link polynomial. His construction used singular cobordisms called foams. Working in a category of foams modulo certain relations, the authors in [7] generalized Khovanov's theory and constructed the universal integer sl_3–link homology (see also Morrison and Nieh [8] for a slightly different approach). In [4] Khovanov and Rozansky (KR) constructed a rational bigraded theory that categorified the sl_n link polynomial for all $n > 0$. They conjectured that their theory is isomorphic to the one in [3] for $n = 3$, after tensoring the latter with $\mathbb{Q}$. Their construction uses matrix factorizations and can be generalized to give the universal rational link homology for all $n > 0$ (see Gornik [1], Rasmussen [9] and Wu [10]). In this paper we prove that the universal rational KR link homology for $n = 3$ is equivalent to the foam link homology in [7] tensored with $\mathbb{Q}$.

One of the main difficulties one encounters when trying to relate both theories mentioned above is that the foam approach uses ordinary *webs*, which are ordinary oriented trivalent graphs, subject to the condition that at each vertex all edges have the same orientation, inward or outward, whereas the KR theory uses *KR–webs*, which are trivalent graphs containing two types of edges: oriented simple edges and unoriented thick edges. In Khovanov and Rozansky's setup in [4] there is a unique way to associate a matrix factorization to each KR–web. In general there are several KR–webs that one can associate to an ordinary web, so there is no obvious choice of a KR matrix factorization

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to associate to a web. However, we show that the KR–matrix factorizations for all these KR–webs are homotopy equivalent and that between two of them there is a canonical choice of homotopy equivalence in a certain sense. This allows us to associate an equivalence class of KR–matrix factorizations to each ordinary web. After that it is relatively straightforward to show the equivalence between the foam and the KR sl_3 link homologies.

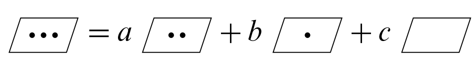
In Section 2 we review the category $\mathbf{Foam}_{/\ell}$ and the main results of [7]. In Section 3 we recall some basic facts about matrix factorizations and define the universal KR homology for $n = 3$. Section 4 is the core of the paper. In this section we show how to associate equivalence classes of matrix factorizations to ordinary webs and use them to construct a link homology that is equivalent to Khovanov and Rozansky's. In Section 5 we establish the equivalence between the foam sl_3 link homology and the one presented in Section 4.

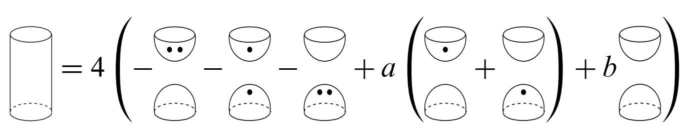
We assume familiarity with the papers [4] and [7].

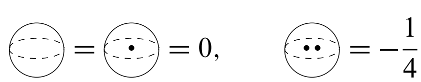
2 The category $\mathbf{Foam}_{/\ell}$ revisited

This section contains a brief review of the universal rational sl_3 link homology using foams as constructed by the authors [7] following Khovanov's ideas in [3]. Here we simply state the basics and the modifications that are necessary to relate it to Khovanov and Rozansky's universal sl_3 link homology using matrix factorizations. We refer to [7] for details.

The category $\mathbf{Foam}_{/\ell}$ has webs as objects and $\mathbb{Q}[a,b,c]$–linear combinations of foams as morphisms divided by the set of relations $\ell = \{$ (3D), (CN), (S), (Θ) $\}$ and the *closure relation*, which are explained below. Note that we are using a different normalization of the coefficients[1] in our relations compared to [7]. These are necessary to establish the connection with the KR link homology later on.

(3D) 

(CN) 

(S) 

Let $\theta(\alpha, \beta, \delta)$ denote the theta foam in Figure 1, where α, β and δ are the number of

[1] We thank S Morrison for spotting a mistake in the coefficients in a previous version of this paper.

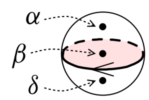


Figure 1: A theta-foam

dots on each facet. For $\alpha, \beta, \delta \leq 2$ we have

$$(\Theta) \qquad \theta(\alpha,\beta,\delta) = \begin{cases} \frac{1}{8} & (\alpha,\beta,\delta) = (1,2,0) \text{ or a cyclic permutation} \\ -\frac{1}{8} & (\alpha,\beta,\delta) = (2,1,0) \text{ or a cyclic permutation} \\ 0 & \text{else} \end{cases}$$

The *closure relation* says that any $\mathbb{Q}[a,b,c]$–linear combination of foams, all of which have the same boundary, is equal to zero if and only if any common way of closing these foams yields a $\mathbb{Q}[a,b,c]$–linear combination of closed foams whose evaluation is zero.

The category $\mathbf{Foam}_{/\ell}$ is additive and graded. The q–grading in $\mathbb{Q}[a,b,c]$ is defined as

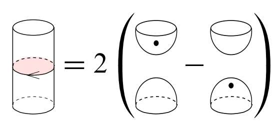
$$q(1) = 0, \quad q(a) = 2, \quad q(b) = 4, \quad q(c) = 6$$

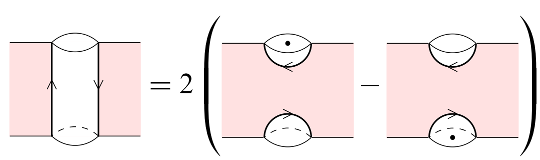
and the degree of a foam f with $|\bullet|$ dots is given by

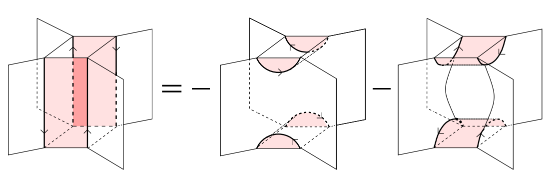
$$q(f) = -2\chi(f) + \chi(\partial f) + 2|\bullet|,$$

where χ denotes the Euler characteristic.

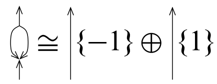
Using the relations ℓ one can prove the identities (RD), (DR) and (CN), and Lemma 2.1 below (for detailed proofs see [7]). We note that Morrison and Nieh [8] have proved that the relations (DR) and (SqR) are equivalent to the closure relation in the category of webs and foams modulo ℓ.

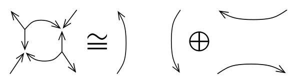
(RD) 

(DR) 

(SqR) 

Lemma 2.1 (Khovanov–Kuperberg relations [3; 6]) *We have the following decompositions in* $\mathbf{Foam}_{/\ell}$*:*

(Digon Removal) 

(Square Removal) 

where $\{j\}$ *denotes a positive shift in the* q*–grading by* j*.*

The construction of the topological complex from a link diagram is well known by now and uses the elementary foams in Figure 2, which we call the 'zip' and the 'unzip', to build the differential. We follow the conventions in [7] and read foams from bottom to top when interpreted as morphisms.

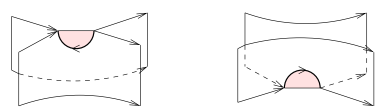


Figure 2: Elementary foams

The tautological functor C from $\mathbf{Foam}_{/\ell}$ to the category $\mathbf{Mod_{gr}}$ of graded $\mathbb{Q}[a,b,c]$–modules maps a closed web Γ to $C(\Gamma)=\mathrm{Hom}_{\mathbf{Foam}_{/\ell}}(\varnothing,\Gamma)$ and, for a foam f between two closed webs Γ and Γ', the $\mathbb{Q}[a,b,c]$–linear map $C(f)$ from $C(\Gamma)$ to $C(\Gamma')$ is the one given by composition. The $\mathbb{Q}[a,b,c]$–module $C(\Gamma)$ is graded and the degree of $C(f)$ is equal to $q(f)$.

Denote by **Link** the category of oriented links in S^3 and ambient isotopy classes of oriented link cobordisms properly embedded in $S^3\times[0,1]$ and by $\mathbf{Mod_{bg}}$ the category of bigraded $\mathbb{Q}[a,b,c]$–modules. The functor C extends to the category $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$ of chain complexes in $\mathbf{Foam}_{/\ell}$ and the composite with the homology functor defines a projective functor $U_{a,b,c}\colon \mathbf{Link}\to\mathbf{Mod_{bg}}$.

The theory described above is equivalent to the one in [7] after tensoring the latter with $\mathbb{Q}$.

3 Deformations of Khovanov–Rozansky sl_3 link homology

3.1 Review of matrix factorizations

This subsection contains a brief review of matrix factorizations and the properties that will be used throughout this paper. We assume familiarity with [4]. All the matrix factorizations in this paper are $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}$–graded. Let R be a polynomial ring over $\mathbb{Q}$. We take the degree of each polynomial to be twice its total degree. This way R is $\mathbb{Z}$–graded. Let W be a homogeneous element of R of degree $2m$. A matrix factorization of W over R is a $\mathbb{Z}/2\mathbb{Z}$–graded free R–module $M = M_0 \oplus M_1$ with R–homomorphisms of degree m

$$M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0$$

such that $d_1 d_0 = W\,\mathrm{Id}_{M_0}$ and $d_0 d_1 = W\,\mathrm{Id}_{M_1}$. The $\mathbb{Z}$–grading of R induces a $\mathbb{Z}$–grading on M. The shift functor $\{k\}$ acts on M as

$$M\{k\} = M_0\{k\} \xrightarrow{d_0} M_1\{k\} \xrightarrow{d_1} M_0\{k\}.$$

A homomorphism $f\colon M \to M'$ of matrix factorizations of W is a pair of maps of the same degree $f_i\colon M_i \to M_i'$ $(i=0,1)$ such that the diagram

$$\begin{tikzcd}
M_0 \arrow[r, "d_0"] \arrow[d, "f_0"'] & M_1 \arrow[r, "d_1"] \arrow[d, "f_1"'] & M_0 \arrow[d, "f_0"'] \\
M_0' \arrow[r, "d_0'"] & M_1' \arrow[r, "d_1'"] & M_0'
\end{tikzcd}$$

commutes. It is an isomorphism of matrix factorizations if f_0 and f_1 are isomorphisms of the underlying modules. Denote the set of homomorphisms of matrix factorizations from M to M' by

$$\mathrm{Hom}_{\mathbf{MF}}(M, M').$$

It has an R–module structure with the action of R given by $r(f_0, f_1) = (rf_0, rf_1)$ for $r \in R$. Matrix factorizations over R with homogeneous potential W and homomorphisms of matrix factorizations form a graded additive category, which we denote by $\mathbf{MF}_R(W)$. If $W=0$ we simply write $\mathbf{MF}_R$.

The free R–module $\mathrm{Hom}_R(M, M')$ of graded R–module homomorphisms from M to M' is a 2–complex

$$\mathrm{Hom}^0_R(M, M') \xrightarrow{D} \mathrm{Hom}^1_R(M, M') \xrightarrow{D} \mathrm{Hom}^0_R(M, M')$$

where

$$\mathrm{Hom}^0_R(M, M') = \mathrm{Hom}_R(M_0, M_0') \oplus \mathrm{Hom}_R(M_1, M_1')$$
$$\mathrm{Hom}^1_R(M, M') = \mathrm{Hom}_R(M_0, M_1') \oplus \mathrm{Hom}_R(M_1, M_0')$$

and for f in $\mathrm{Hom}^i_R(M,M')$ the differential acts as

$$Df = d_{M'}f - (-1)^i f d_M.$$

We define

$$\mathrm{Ext}(M,M') = \mathrm{Ext}^0(M,M') \oplus \mathrm{Ext}^1(M,M') = \mathrm{Ker}\, D/\, \mathrm{Im}\, D,$$

and write $\mathrm{Ext}_{\{m\}}(M,M')$ for the elements of $\mathrm{Ext}(M,M')$ with $\mathbb{Z}$–degree m.

Note that for $f \in \mathrm{Hom}_{\mathbf{MF}}(M,M')$ we have $Df = 0$. We say that two homomorphisms f, g in $\mathrm{Hom}_{\mathbf{MF}}(M,M')$ are homotopic if there is an element h in $\mathrm{Hom}^1_R(M,M')$ such that $f - g = Dh$.

Denote by $\mathrm{Hom}_{\mathbf{HMF}}(M,M')$ the R–module of homotopy classes of homomorphisms of matrix factorizations from M to M' and by $\mathbf{HMF}_R(W)$ the homotopy category of $\mathbf{MF}_R(W)$.

We have

$$\mathrm{Ext}^0(M,M') \cong \mathrm{Hom}_{\mathbf{HMF}}(M,M')$$
$$\mathrm{Ext}^1(M,M') \cong \mathrm{Hom}_{\mathbf{HMF}}(M,M'\langle 1\rangle)$$

We denote by $M\langle 1\rangle$ and $M_\bullet$ the factorizations

$$M_1 \xrightarrow{-d_1} M_0 \xrightarrow{-d_0} M_1$$
$$\text{and} \qquad (M_0)^* \xrightarrow{-(d_1)^*} (M_1)^* \xrightarrow{(d_0)^*} (M_0)^*$$

respectively. Factorization $M\langle 1\rangle$ has potential W while factorization $M_\bullet$ has potential $-W$.

The tensor product $M \otimes_R M_\bullet$ has potential zero and is therefore a 2–complex. Denoting by $\mathrm{H}_{\mathbf{MF}}$ the homology of matrix factorizations with potential zero we have

$$\mathrm{Ext}(M,M') \cong \mathrm{H}_{\mathbf{MF}}(M' \otimes_R M_\bullet)$$

and, if M is a matrix factorization with $W = 0$,

$$\mathrm{Ext}(R,M) \cong \mathrm{H}_{\mathbf{MF}}(M).$$

Koszul Factorizations For a and b homogeneous elements of R, the *elementary Koszul factorization* $\{a,b\}$ over R with potential ab is a factorization of the form

$$R \xrightarrow{a} R\left\{\tfrac{1}{2}(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a)\right\} \xrightarrow{b} R.$$

When we need to emphasize the ring R we write this factorization as $\{a, b\}_R$. It is well known that the tensor product of matrix factorizations M_i with potentials W_i is a matrix factorization with potential $\sum_i W_i$. We restrict to the case where all the W_i are homogeneous of the same degree. Throughout this paper we use tensor products of elementary Koszul factorizations $\{a_j, b_j\}$ to build bigger matrix factorizations, which we write in the form of a *Koszul matrix* as

$$\begin{Bmatrix} a_1 \ , & b_1 \\ \vdots & \vdots \\ a_k \ , & b_k \end{Bmatrix}$$

We denote by $\{\mathbf{a}, \mathbf{b}\}$ the Koszul matrix with columns $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$.

Note that the action of the shift $\langle 1 \rangle$ on $\{\mathbf{a}, \mathbf{b}\}$ is equivalent to switching terms in one line of $\{\mathbf{a}, \mathbf{b}\}$:

$$\{\mathbf{a}, \mathbf{b}\}\langle 1 \rangle = \begin{Bmatrix} \vdots & \vdots \\ a_{i-1} \ , & b_{i-1} \\ -b_i \ , & -a_i \\ a_{i+1} \ , & b_{i+1} \\ \vdots & \vdots \end{Bmatrix} \left\{\tfrac{1}{2}(\deg_{\mathbb{Z}} b_i - \deg_{\mathbb{Z}} a_i)\right\}.$$

If we choose a different row to switch terms we get a factorization which is isomorphic to this one.

We also have that

$$\{\mathbf{a}, \mathbf{b}\}_{\bullet} \cong \{\mathbf{a}, -\mathbf{b}\}\langle k \rangle \{s_k\},$$

where

$$s_k = \sum_{i=1}^{k} \deg_{\mathbb{Z}} a_i - \frac{k}{2} \deg_{\mathbb{Z}} W.$$

Let $R = \mathbb{Q}[x_1, \dots, x_k]$ and $R' = \mathbb{Q}[x_2, \dots, x_k]$. Suppose that $W = \sum_i a_i b_i \in R'$ and $x_1 - b_i \in R'$, for a certain $1 \le i \le k$. Let $\{\widehat{\mathbf{a}}^i, \widehat{\mathbf{b}}^i\}$ be the matrix factorization obtained from $\{\mathbf{a}, \mathbf{b}\}$ by deleting the ith row and substituting x_1 by $x_1 - b_i$.

Lemma 3.1 (excluding variables) *The matrix factorizations $\{\mathbf{a}, \mathbf{b}\}$ and $\{\widehat{\mathbf{a}}^i, \widehat{\mathbf{b}}^i\}$ are homotopy equivalent.*

In [4] one can find the proof of this lemma and its generalization with several variables.

The following lemma contains three particular cases of [4, Proposition 3] (see also [5]):

Lemma 3.2 (Row operations) *We have the following isomorphisms of matrix factorizations*

$$\begin{Bmatrix} a_i \ , & b_i \\ a_j \ , & b_j \end{Bmatrix} \overset{[i,j]_\lambda}{\cong} \begin{Bmatrix} a_i - \lambda a_j \ , & b_i \\ a_j \ , & b_j + \lambda b_i \end{Bmatrix}, \qquad \begin{Bmatrix} a_i \ , & b_i \\ a_j \ , & b_j \end{Bmatrix} \overset{[i,j]'_\lambda}{\cong} \begin{Bmatrix} a_i + \lambda b_j \ , & b_i \\ a_j - \lambda b_i \ , & b_j \end{Bmatrix}$$

for $\lambda \in R$. If λ is invertible in R, we also have

$$\{a_i \ , \ b_j\} \overset{[i]_\lambda}{\cong} \{\lambda a_i \ , \ \lambda^{-1} b_i\}.$$

Recall that a sequence $(a_1, a_2, \ldots, a_k)$ is called *regular* in R if a_j is not a zero divisor in $R/_{(a_1,a_2,\ldots,a_{j-1})R}$, for $j = 1, \ldots, k$. The proof of the following lemma can be found in [5].

Lemma 3.3 *Let $\mathbf{b} = (b_1, b_2, \ldots, b_k)$, $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ and $\mathbf{a}' = (a'_1, a'_2, \ldots, a'_k)$ be sequences in R. If $\mathbf{b}$ is regular and $\sum_i a_i b_i = \sum_i a'_i b_i$ then the factorizations*

$$\{\mathbf{a} \ , \mathbf{b}\} \ \text{and} \ \{\mathbf{a}' \ , \mathbf{b}\}$$

are isomorphic.

A factorization M with potential W is said to be *contractible* if it is isomorphic to a direct sum of copies of

$$R \xrightarrow{1} R\left\{\tfrac{1}{2}\deg_{\mathbb{Z}} W\right\} \xrightarrow{W} R \quad \text{and} \quad R \xrightarrow{W} R\left\{-\tfrac{1}{2}\deg_{\mathbb{Z}} W\right\} \xrightarrow{1} R.$$

3.2 Khovanov–Rozansky homology

Definition 3.4 A *KR–web* is a trivalent graph with two types of edges, oriented edges and unoriented thick edges, such that each oriented edge has at least one mark. We allow open webs which have oriented edges with only one endpoint glued to the rest of the graph. Every thick edge has exactly two oriented edges entering one endpoint and two leaving the other.

Suppose there are k marks in a KR–web Γ and let $\mathbf{x}$ denote the set $\{x_1, x_2, \ldots, x_k\}$. Denote by R the polynomial ring $\mathbb{Q}[a, b, c, \mathbf{x}]$, where a, b, c are formal parameters. We define a q–grading in R declaring that

$$q(1) = 0, \quad q(x_i) = 2, \ \text{for all } i, \quad q(a) = 2, \quad q(b) = 4, \quad q(c) = 6.$$

The universal rational Khovanov–Rozansky theory for $N = 3$ is related to the polynomial

$$p(x) = x^4 - \frac{4a}{3}x^3 - 2bx^2 - 4cx.$$

$$y \longrightarrow x$$

Figure 3: An oriented arc

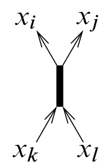


Figure 4: A thick edge

As in [5] we denote by $\widehat{\Gamma}$ the matrix factorization associated to a KR–web Γ.

To an oriented arc with marks x and y, as in Figure 3, we assign the potential

$$W = p(x) - p(y) = x^4 - y^4 - \frac{4a}{3}\big(x^3 - y^3\big) - 2b\big(x^2 - y^2\big) - 4c\big(x - y\big)$$

and the *arc factorization* $\widehat{\uparrow}$, which is given by the Koszul factorization

$$\widehat{\uparrow} = \{\pi_{xy}\ ,\ x - y\} = R \xrightarrow{\pi_{xy}} R\{-2\} \xrightarrow{x-y} R$$

where

$$\pi_{xy} = \frac{x^4 - y^4}{x - y} - \frac{4a}{3}\frac{x^3 - y^3}{x - y} - 2b\frac{x^2 - y^2}{x - y} - 4c.$$

To the thick edge in Figure 4 we associate the potential

$$W = p(x_i) + p(x_j) - p(x_k) - p(x_l)$$

and the *dumbell factorization* $\widehat{\Upsilon}$, which is defined as the tensor product of the factorizations

$$R\{-1\} \xrightarrow{u_{ijkl}} R\{-3\} \xrightarrow{x_i + x_j - x_k - x_l} R\{-1\}$$

and

$$R \xrightarrow{v_{ijkl}} R \xrightarrow{x_i x_j - x_k x_l} R$$

where

$$\begin{aligned} u_{ijkl} &= \frac{(x_i + x_j)^4 - (x_k + x_l)^4}{x_i + x_j - x_k - x_l} - (2b + 4x_i x_j)(x_i + x_j + x_k + x_l) \\ &\quad - \frac{4a}{3}\left(\frac{(x_i + x_j)^3 - (x_k + x_l)^3}{x_i + x_j - x_k - x_l} - 3x_i x_j\right) - 4c, \\ v_{ijkl} &= 2(x_i x_j + x_k x_l) - 4(x_k + x_l)^2 + 4a(x_k + x_l) + 4b. \end{aligned}$$

We can write the dumbell factorization as the Koszul matrix

$$\widehat{\Upsilon} = \begin{Bmatrix} u_{ijkl}\ , & x_i + x_j - x_k - x_l \\ v_{ijkl}\ , & x_i x_j - x_k x_l \end{Bmatrix} \{-1\}.$$

The matrix factorization $\widehat{\Gamma}$ of a general KR–web Γ composed of E arcs and T thick edges is built from the arc and the dumbell factorizations as

$$\widehat{\Gamma} = \bigotimes_{e \in E} \widehat{\uparrow}_e \otimes \bigotimes_{t \in T} \widehat{\Upsilon}_t,$$

which is a matrix factorization with potential $W = \sum \epsilon_i\, p(x_i)$ where i runs over all free ends. By convention $\epsilon_i = 1$ if the corresponding arc is oriented outward and $\epsilon_i = -1$ in the opposite case.

3.2.1 Maps χ_0 and χ_1 Let $\widehat{)(}$ and $\widehat{\Upsilon}$ denote the factorizations in Figure 5.

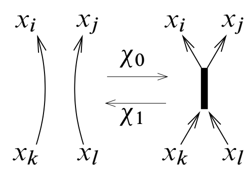


Figure 5: Maps χ_0 and χ_1

The factorization $\widehat{)(}$ is given by

$$\begin{pmatrix} R \\ R\{-4\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R\{-2\} \\ R\{-2\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R \\ R\{-4\} \end{pmatrix}$$

with

$$P_0 = \begin{pmatrix} \pi_{ik} & x_j - x_l \\ \pi_{jl} & -x_i + x_k \end{pmatrix}, \qquad P_1 = \begin{pmatrix} x_i - x_k & x_j - x_l \\ \pi_{jl} & -\pi_{ik} \end{pmatrix}.$$

The factorization $\widehat{\Upsilon}$ is given by

$$\begin{pmatrix} R\{-1\} \\ R\{-3\} \end{pmatrix} \xrightarrow{Q_0} \begin{pmatrix} R\{-3\} \\ R\{-1\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R\{-1\} \\ R\{-3\} \end{pmatrix}$$

with

$$Q_0 = \begin{pmatrix} u_{ijkl} & x_i x_j - x_k x_l \\ v_{ijkl} & -x_i - x_j + x_k + x_l \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} x_i + x_j - x_k - x_l & x_i x_j - x_k x_l \\ v_{ijkl} & -u_{ijkl} \end{pmatrix}.$$

The maps χ_0 and χ_1 can be described by the pairs of matrices:

$$\chi_0 = \left(2 \begin{pmatrix} -x_k + x_j & 0 \\ -\alpha & -1 \end{pmatrix}, 2 \begin{pmatrix} -x_k & x_j \\ 1 & -1 \end{pmatrix} \right)$$

and

$$\chi_1 = \left(\begin{pmatrix} 1 & 0 \\ -\alpha & x_k - x_j \end{pmatrix}, \begin{pmatrix} 1 & x_j \\ 1 & x_k \end{pmatrix} \right)$$

where

$$\alpha = -v_{ijkl} + \frac{u_{ijkl} + x_i v_{ijkl} - \pi_{jl}}{x_i - x_k}.$$

The maps χ_0 and χ_1 have degree 1. A straightforward calculation shows that χ_0 and χ_1 are homomorphisms of matrix factorizations, and that

$$\chi_0 \chi_1 = m(2x_j - 2x_k)\,\mathrm{Id}(\widehat{\Upsilon}) \qquad \chi_1 \chi_0 = m(2x_j - 2x_k)\,\mathrm{Id}(\widehat{)(}),$$

where $m(x_*)$ is multiplication by x_*. Note that we are using a sign convention that is different from the one in [4]. This choice was made to match the signs in the *Digon Removal* relation in [7]. Note also that the map χ_0 is twice its analogue in [4]. This way we obtain a theory that is equivalent to the one in [4] and consistent with our choice of normalization in Section 2.

There is another description of the maps χ_0 and χ_1 when the webs in Figure 5 are closed. In this case both $\widehat{)(}$ and $\widehat{\Upsilon}$ have potential zero. Acting with a row operation on $\widehat{)(}$ we get

$$\widehat{)(} \cong \begin{Bmatrix} \pi_{ik}, & x_i + x_j - x_k - x_l \\ \pi_{jl} - \pi_{ik}, & x_j - x_l \end{Bmatrix}.$$

Excluding the variable x_k from $\widehat{)(}$ and from $\widehat{\Upsilon}$ yields

$$\widehat{)(} \cong \{\pi_{jl} - \pi_{ik},\ x_j - x_l\}_R, \qquad \widehat{\Upsilon} \cong \{v_{ijkl},\ (x_i - x_l)(x_j - x_l)\}_R,$$

with $R = \mathbb{Q}[x_i, x_j, x_k, x_l]/(x_k + x_i + x_j - x_l)$. It is straightforward to check that χ_0 and χ_1 correspond to the maps $(-2(x_i - x_l), -2)$ and $(1, x_i - x_l)$ respectively. This description will be useful in Section 5.

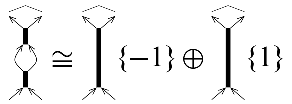
For a link L, we denote by $\mathrm{KR}_{a,b,c}(L)$ the universal rational Khovanov–Rozansky cochain complex, which can be obtained from the data above in the same way as in [4]. Let $\mathrm{HKR}_{a,b,c}(L)$ denote the universal rational Khovanov–Rozansky homology. We have

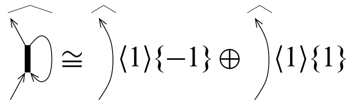
$$\mathrm{HKR}_{a,b,c}(\bigcirc) \cong \left(\mathbb{Q}[x,a,b,c]_{/x^3 - ax^2 - bx - c}\right)\langle 1\rangle\{-2\}.$$

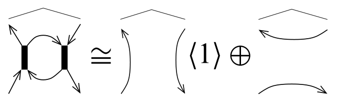
3.3 MOY web moves

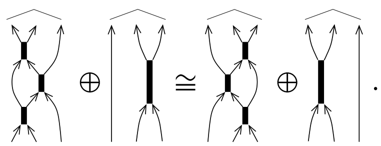
One of the main features of the Khovanov–Rozansky theory is the categorification of the MOY web moves. For $n = 3$ these categorified moves are described by the homotopy equivalences below.

Lemma 3.5 *We have the following direct sum decompositions:*

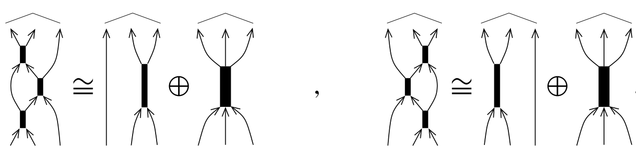
(1) 

(2) 

(3) 

(4) 

The last relation is a consequence of two relations involving a triple edge



The factorization assigned to the triple edge in Figure 6 has potential

$$W = p(x_1) + p(x_2) + p(x_3) - p(x_4) - p(x_5) - p(x_6).$$

Let h be the unique three-variable polynomial such that

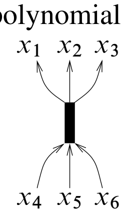


Figure 6: Triple edge factorization

$$h(x+y+z, xy+xz+yz, xyz) = p(x) + p(y) + p(z)$$

and let

$$e_1 = x_1 + x_2 + x_3, \quad e_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad e_3 = x_1x_2x_3,$$
$$s_1 = x_4 + x_5 + x_6, \quad s_2 = x_4x_5 + x_4x_6 + x_5x_6, \quad s_3 = x_4x_5x_6.$$

Define

$$h_1 = \frac{h(e_1,e_2,e_3)-h(s_1,e_2,e_3)}{e_1-s_1}$$
$$h_2 = \frac{h(s_1,e_2,e_3)-h(s_1,s_2,e_3)}{e_2-s_2}$$
$$h_3 = \frac{h(s_1,s_2,e_3)-h(s_1,s_2,s_3)}{e_3-s_3}$$

so that we have $W=h_1(e_1-s_1)+h_2(e_2-s_2)+h_3(e_3-s_3)$. The matrix factorization $\widehat{\Upsilon}$ corresponding to the triple edge is defined by the Koszul matrix

$$\widehat{\Upsilon} = \begin{Bmatrix} h_1 \ , & e_1-s_1 \\ h_2 \ , & e_2-s_2 \\ h_3 \ , & e_3-s_3 \end{Bmatrix}_R \{-3\},$$

where $R=\mathbb{Q}[a,b,c,x_1,\dots,x_6]$. The matrix factorization $\widehat{\Upsilon}$ is the tensor product of the matrix factorizations

$$R \xrightarrow{h_i} R\{2i-4\} \xrightarrow{e_i-s_i} R, \qquad i=1,2,3,$$

shifted down by 3.

3.4 Cobordisms

In this subsection we show which homomorphisms of matrix factorizations we associate to the elementary singular KR–cobordisms. We will need these in Section 5. It is not clear to us whether one can associate a homomorphism of matrix factorizations to arbitrary singular KR–cobordisms.

Recall that the elementary KR–cobordisms are the *zip*, the *unzip* (see Figure 5) and the elementary cobordisms in Figure 7. To the zip and the unzip we associate the maps χ_0 and χ_1, respectively, as defined in Section 3.2.1. For each elementary cobordism in Figure 7 we define a homomorphism of matrix factorizations as below, following Khovanov and Rozansky [4].

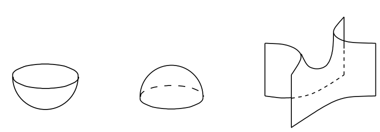


Figure 7: Elementary cobordisms

The *unit* and the *trace* map

$$\iota\colon \mathbb{Q}[a,b,c]\langle 1\rangle \to \widehat{\bigcirc}$$
$$\varepsilon\colon \widehat{\bigcirc} \to \mathbb{Q}[a,b,c]\langle 1\rangle$$

are the homomorphisms of matrix factorizations induced by the maps (denoted by the same symbols)

$$\iota\colon \mathbb{Q}[a,b,c] \to \mathbb{Q}[a,b,c][X]_{/X^3-aX^2-bX-c}\{-2\}, \qquad 1 \mapsto 1$$
$$\varepsilon\colon \mathbb{Q}[a,b,c][X]_{/X^3-aX^2-bX-c}\{-2\} \to \mathbb{Q}[a,b,c], \qquad X^k \mapsto \begin{cases} -\frac{1}{4}, & k=2 \\ 0, & k<2 \end{cases}$$

using the isomorphisms

$$\hat{\varnothing} \cong \mathbb{Q}[a,b,c] \to 0 \to \mathbb{Q}[a,b,c]$$
$$\text{and} \qquad \widehat{\bigcirc} \cong 0 \to \mathbb{Q}[a,b,c][X]_{/X^3-aX^2-bX-c}\{-2\} \to 0.$$

Let $\widehat{)(}$ and $\widehat{\asymp}$ be the factorizations in Figure 8.

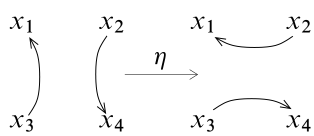


Figure 8: Saddle point homomorphism

The matrix factorization $\widehat{)(}$ is given by

$$\begin{pmatrix} R \\ R\{-4\} \end{pmatrix} \xrightarrow{\begin{pmatrix} \pi_{13} & x_4-x_2 \\ \pi_{24} & x_3-x_1 \end{pmatrix}} \begin{pmatrix} R\{-2\} \\ R\{-2\} \end{pmatrix} \xrightarrow{\begin{pmatrix} x_1-x_3 & x_4-x_2 \\ \pi_{24} & -\pi_{13} \end{pmatrix}} \begin{pmatrix} R \\ R\{-4\} \end{pmatrix}$$

and $\widehat{\asymp}\langle 1\rangle$ is given by

$$\begin{pmatrix} R\{-2\} \\ R\{-2\} \end{pmatrix} \xrightarrow{\begin{pmatrix} x_2-x_1 & x_3-x_4 \\ -\pi_{34} & \pi_{12} \end{pmatrix}} \begin{pmatrix} R \\ R\{-4\} \end{pmatrix} \xrightarrow{\begin{pmatrix} -\pi_{12} & x_3-x_4 \\ -\pi_{34} & x_1-x_2 \end{pmatrix}} \begin{pmatrix} R\{-2\} \\ R\{-2\} \end{pmatrix}$$

To the saddle cobordism between the webs $)($ and $\asymp$ we associate the homomorphism of matrix factorizations $\eta\colon \widehat{)(} \to \widehat{\asymp}\langle 1\rangle$ described by the pair of matrices

$$\eta_0 = \begin{pmatrix} e_{123}+e_{124} & 1 \\ -e_{134}-e_{234} & 1 \end{pmatrix}, \qquad \eta_1 = \begin{pmatrix} -1 & 1 \\ -e_{123}-e_{234} & -e_{134}-e_{123} \end{pmatrix}$$

where

$$\begin{aligned} e_{ijk} &= \frac{(x_k - x_j)p(x_i) + (x_i - x_k)p(x_j) + (x_j - x_i)p(x_k)}{2(x_i - x_j)(x_j - x_k)(x_k - x_i)} \\ &= \frac{1}{2}\big(x_i^2 + x_j^2 + x_k^2 + x_i x_j + x_i x_k + x_j x_k\big) - \frac{2a}{3}\big(x_i + x_j + x_k\big) - b. \end{aligned}$$

The homomorphism η has degree 2. If and belong to open KR–webs the homomorphism η is defined only up to a sign (see [4]). For closed KR–webs we do not know whether there is a sign problem. In Section 5 we will deal with this problem in a slightly different setting.

4 A matrix factorization theory for sl_3 webs

As mentioned in the introduction, the main problem in comparing the foam and the matrix factorization sl_3 link homologies is that one has to deal with two different sorts of webs. In the foam approach, where one uses ordinary trivalent webs, all edges are thin and oriented, whereas for the matrix factorizations Khovanov and Rozansky used KR–webs, which also contain thick unoriented edges. In general there are various KR–webs that one can associate to a given web. Therefore it apparently is not clear which KR matrix factorization to associate to a web. However, in Proposition 4.2 we show that this ambiguity is not problematic. Our proof of this result is rather roundabout and requires a matrix factorization for each vertex. In this way we associate a matrix factorization to each web. For each choice of KR–web associated to a given web the KR–matrix factorization is a quotient of ours, obtained by identifying the vertex variables pairwise according to the thick edges. We then show that for a given web two such quotients are always homotopy equivalent. This is the main ingredient which allows us to establish the equivalence between the foam and the matrix factorization sl_3 link homologies.

Recall that a *web* is an oriented trivalent graph where near each vertex either all edges are oriented into it or away from it. We call the former vertices of $(-)$–*type* and the latter vertices of $(+)$–*type*. To associate matrix factorizations to webs we impose that each edge have at least one mark.

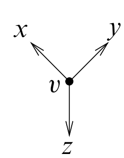


Figure 9: A vertex of $(+)$–type

4.1 The 3–vertex

Consider the 3–vertex of $(+)$-type in Figure 9, with emanating edges marked x, y, z. The polynomial

$$p(x)+p(y)+p(z)=x^4+y^4+z^4-\tfrac{4}{3}a\big(x^3+y^3+z^3\big)-2b\big(x^2+y^2+z^2\big)-4c\big(x+y+z\big)$$

can be written as a polynomial in the elementary symmetric polynomials

$$p_v(x+y+z, xy+xz+yz, xyz)=p_v(e_1,e_2,e_3).$$

Using the methods of Section 3 we can obtain a matrix factorization of p_v, but if we tensor together two of these, then we obtain a matrix factorization which is not homotopy equivalent to the dumbell matrix factorization. This can be seen quite easily, since the new Koszul matrix has 6 rows and only one extra variable. This extra variable can be excluded at the expense of 1 row, but then we get a Koszul matrix with 5 rows, whereas the dumbell Koszul matrix has only 2. To solve this problem we introduce a set of three new variables for each vertex[2]. Introduce the *vertex variables* v_1, v_2, v_3 with $q(v_i)=2i$ and define the *vertex ring*

$$R_v=\mathbb{Q}[a,b,c][x,y,z,v_1,v_2,v_3].$$

We define the potential as

$$W_v=p_v(e_1,e_2,e_3)-p_v(v_1,v_2,v_3).$$

We have

$$\begin{aligned}W_v &= \frac{p_v(e_1,e_2,e_3)-p_v(v_1,e_2,e_3)}{e_1-v_1}\big(e_1-v_1\big)\\ &\quad+\frac{p_v(v_1,e_2,e_3)-p_v(v_1,v_2,e_3)}{e_2-v_2}\big(e_2-v_2\big)\\ &\quad+\frac{p_v(v_1,v_2,e_3)-p_v(v_1,v_2,v_3)}{e_3-v_3}\big(e_3-v_3\big)\\ &= g_1\big(e_1-v_1\big)+g_2\big(e_2-v_2\big)+g_3\big(e_3-v_3\big),\end{aligned}$$

where the polynomials g_i $(i=1,2,3)$ have the explicit form

$$g_1=\frac{e_1^4-v_1^4}{e_1-v_1}-4e_2(e_1+v_1)+4e_3-\frac{4a}{3}\left(\frac{e_1^3-v_1^3}{e_1-v_1}-3e_2\right)-2b(e_1+v_1)-4c \tag{5}$$

$$g_2=2(e_2+v_2)-4v_1^2+4av_1+4b \tag{6}$$

$$g_3=4(v_1-a). \tag{7}$$

[2]M Khovanov had already observed this problem for the undeformed case and suggested to us the introduction of one vertex variable in that case.

We define the 3–vertex factorization $\widehat{\curlyvee}_{v+}$ as the tensor product of the factorizations

$$R_v \xrightarrow{g_i} R_v\{2i-4\} \xrightarrow{e_i - v_i} R_v, \qquad (i = 1, 2, 3)$$

shifted by $-3/2$ in the q–grading and by $1/2$ in the $\mathbb{Z}/2\mathbb{Z}$–grading, which we write in the form of the Koszul matrix

$$\widehat{\curlyvee}_{v+} = \begin{Bmatrix} g_1 \ , \ e_1 - v_1 \\ g_2 \ , \ e_2 - v_2 \\ g_3 \ , \ e_3 - v_3 \end{Bmatrix}_{R_v} \{-3/2\}\langle 1/2 \rangle.$$

If $\curlyvee_v$ is a 3–vertex of $-$–type with incoming edges marked x, y, z we define

$$\widehat{\curlyvee}_{v-} = \begin{Bmatrix} g_1 \ , \ v_1 - e_1 \\ g_2 \ , \ v_2 - e_2 \\ g_3 \ , \ v_3 - e_3 \end{Bmatrix}_{R_v} \{-3/2\}\langle 1/2 \rangle,$$

with g_1, g_2, g_3 as above.

Lemma 4.1 *We have the following homotopy equivalences in* $\mathrm{End}_{\mathbf{MF}}(\widehat{\curlyvee}_{v\pm})$*:*

$$m(x+y+z) \cong m(a), \quad m(xy+xz+yz) \cong m(-b), \quad m(xyz) \cong m(c).$$

Proof For a matrix factorization $\widehat{M}$ over R with potential W the homomorphism

$$R \to \mathrm{End}_{\mathbf{MF}}(\widehat{M}), \quad r \mapsto m(r)$$

factors through the Jacobi algebra of W and up to shifts, the Jacobi algebra of W_v is

$$J_{W_v} \cong \mathbb{Q}[a, b, c, x, y, z]_{/\{x+y+z=a,\ xy+xz+yz=-b,\ xyz=c\}}. \qquad \square$$

4.2 Vertex composition

The elementary webs considered so far can be combined to produce bigger webs. Consider a general web Γ_v composed by E arcs between marks and V vertices. Denote by ∂E the set of free ends of Γ_v and by $(v_{i_1}, v_{i_2}, v_{i_3})$ the vertex variables corresponding to the vertex v_i. We have

$$\widehat{\Gamma}_v = \bigotimes_{e \in E} \widehat{\uparrow}_e \otimes \bigotimes_{v \in V} \widehat{\curlyvee}_v.$$

Factorization $\widehat{\uparrow}_e$ is the arc factorization introduced in Section 3.2. This is a matrix factorization with potential

$$W = \sum_{i \in \partial E(\Gamma)} \epsilon_i p(x_i) + \sum_{v_j \in V(\Gamma)} \epsilon_j^v p_v(v_{j_1}, v_{j_2}, v_{j_3}) = W_{\mathbf{x}} + W_{\mathbf{v}}$$

where $\epsilon_i = 1$ if the corresponding arc is oriented to it or $\epsilon_i = -1$ in the opposite case and $\epsilon_j^v = 1$ if v_j is of positive type and $\epsilon_j^v = -1$ in the opposite case.

From now on we only consider open webs in which the number of free ends oriented inwards equals the number of free ends oriented outwards. This implies that the number of vertices of $(+)$–type equals the number of vertices of $(-)$–type.

Let R and $R_{\mathbf{v}}$ denote the rings $\mathbb{Q}[a,b,c][\mathbf{x}]$ and $R[\mathbf{v}]$ respectively. Given two vertices, v_i and v_j, of opposite type, we can take the quotient by the ideal generated by $v_i - v_j$. The potential becomes independent of v_i and v_j, because they appeared with opposite signs, and we can exclude the common vertex variables corresponding to v_i and v_j as in Lemma 3.1. This is possible because in all our examples the Koszul matrices have linear terms which involve vertex and edge variables. The matrix factorization which we obtain in this way we represent graphically by a virtual edge, as in Figure 10. A

Figure 10: Virtual edges

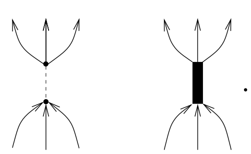
virtual edge can cross other virtual edges and ordinary edges and does not have any mark.

If we pair every positive vertex in Γ_v to a negative one, the above procedure leads to a *complete identification* of the vertices of Γ_v and a corresponding matrix factorization $\zeta(\widehat{\Gamma}_v)$. A different complete identification yields a different matrix factorization $\zeta'(\widehat{\Gamma}_v)$.

Proposition 4.2 *Let Γ_v be a closed web. Then $\zeta(\widehat{\Gamma}_v)$ and $\zeta'(\widehat{\Gamma}_v)$ are isomorphic, up to a shift in the $\mathbb{Z}/2\mathbb{Z}$–grading.*

To prove Proposition 4.2 we need some technical results.

Lemma 4.3 *Consider the web Γ_v and the KR–web Υ below.*



Γ_v Υ

Then $\zeta(\widehat{\Gamma}_v\langle 1\rangle)$ is the factorization of the triple edge $\widehat{\Upsilon}$ of Khovanov–Rozansky for $n = 3$.

Proof Immediate. □

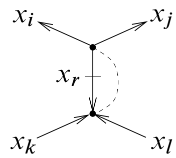
$\Gamma_v =$ 

Figure 11: A double edge

Lemma 4.4 *Let Γ_v be the web in Figure 11. Then $\zeta(\widehat{\Gamma}_v)$ is isomorphic to the factorization assigned to the thick edge of Khovanov–Rozansky for $n = 3$.*

Proof Let $\widehat{\Gamma}_+$ and $\widehat{\Gamma}_-$ be the Koszul factorizations for the upper and lower vertex in Γ_v respectively and $v_i^{\pm}$ denote the corresponding sets of vertex variables. We have

$$\widehat{\Gamma}_+ = \left\{\begin{matrix} g_1^+ , & x_i + x_j + x_r - v_1^+ \\ g_2^+ , & x_i x_j + x_r(x_i + x_j) - v_2^+ \\ g_3^+ , & x_i x_j x_r - v_3^+ \end{matrix}\right\}_{R_{v_+}} \{-3/2\}\langle 1/2\rangle$$

and

$$\widehat{\Gamma}_- = \left\{\begin{matrix} g_1^- , & v_1^- - x_k - x_l - x_r \\ g_2^- , & v_2^- - x_k x_l - x_r(x_k + x_l) \\ g_3^- , & v_3^- - x_k x_l x_r \end{matrix}\right\}_{R_{v_-}} \{-3/2\}\langle 1/2\rangle.$$

The explicit form of the polynomials $g_i^{\pm}$ is given in Equations (5)–(7). Taking the tensor product of $\widehat{\Gamma}_+$ and $\widehat{\Gamma}_-$, identifying vertices v_+ and v_- and excluding the vertex variables yields

$$\zeta(\widehat{\Gamma}_v) = \big(\widehat{\Gamma}_+ \otimes \widehat{\Gamma}_-\big)_{/\mathbf{v}_+ - \mathbf{v}_-} \cong \left\{\begin{matrix} g_1, & x_i + x_j - x_k - x_l \\ g_2, & x_i x_j - x_k x_l + x_r(x_i + x_j - x_k - x_l) \\ g_3, & x_r(x_i x_j - x_k x_l) \end{matrix}\right\}_R \{-3\}\langle 1\rangle,$$

where

$$g_i = g_i^+\big|_{\{v_1^+ = x_k + x_l + x_r,\ v_2^+ = x_k x_l + x_r(x_k + x_l),\ v_3^+ = x_r x_k x_l\}}.$$

This is a factorization over the ring

$$R = \mathbb{Q}[a,b,c][x_i, x_j, x_k, x_l, x_r, \mathbf{v}]_{/I} \cong \mathbb{Q}[a,b,c][x_i, x_j, x_k, x_l, x_r]$$

where I is the ideal generated by

$$\{v_1 - x_r - x_k - x_l,\ v_2 - x_k x_l - x_r(x_k + x_l),\ v_3 - x_k x_l x_r\}.$$

Using $g_3 = 4(x_r + x_k + x_l - a)$ and acting with the shift functor $\langle 1\rangle$ on the third row one can write

$$\zeta(\widehat{\Gamma}_v) \cong \left\{\begin{matrix} g_1\ , & x_i + x_j - x_k - x_l \\ g_2\ , & x_i x_j - x_k x_l + x_r(x_i + x_j - x_k - x_l) \\ -x_r(x_i x_j - x_k x_l)\ , & -4(x_r + x_k + x_l - a) \end{matrix}\right\}_R \{-1\}$$

which is isomorphic, by a row operation, to the factorization

$$\left\{\begin{matrix} g_1 + x_r g_2\ , & x_i + x_j - x_k - x_l \\ g_2\ , & x_i x_j - x_k x_l \\ -x_r(x_i x_j - x_k x_l)\ , & -4(x_r + x_k + x_l - a) \end{matrix}\right\}_R \{-1\}.$$

Excluding the variable x_r from the third row gives

$$\zeta(\widehat{\Gamma}_v) \cong \left\{\begin{matrix} g_1 + (a - x_k - x_l)g_2\ , & x_i + x_j - x_k - x_l \\ g_2\ , & x_i x_j - x_k x_l \end{matrix}\right\}_{R'} \{-1\}$$

where $R' = \mathbb{Q}[a,b,c][x_i,x_j,x_l,x_r]_{/x_r - a + x_k + x_l} \cong \mathbb{Q}[a,b,c][x_i,x_j,x_k,x_l]$.

The claim follows from Lemma 3.3, since both are factorizations over R' with the same potential and the same second column, the terms in which form a regular sequence in R'. As a matter of fact, using a row operation one can write

$$\zeta(\widehat{\Gamma}_v) \cong \left\{\begin{matrix} g_1 + (a - x_k - x_l)g_2 + 2(a - x_k - x_l)(x_i x_j - x_k x_l), & x_i + x_j - x_k - x_l \\ g_2 + 2(a - x_k - x_l)(x_i + x_j - x_k - x_l), & x_i x_j - x_k x_l \end{matrix}\right\} \{-1\}$$

and check that the polynomials in the first column are exactly the polynomials u_{ijkl} and v_{ijkl} corresponding to the factorization assigned to the thick edge in Khovanov–Rozansky theory. □

Lemma 4.5 *Let Γ_v be a closed web and ζ and ζ' two complete identifications that only differ in the region depicted in Figure 12, where T is a part of the diagram whose orientation is not important. Then there is an isomorphism $\zeta(\widehat{\Gamma}_v) \cong \zeta'(\widehat{\Gamma}_v)\langle 1\rangle$.*

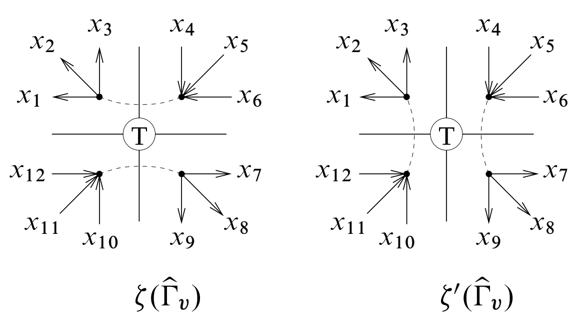


Figure 12: Swapping virtual edges

Proof Denoting by $\widehat{M}$ the tensor product of $\widehat{T}$ with the factorization corresponding to the part of the diagram not depicted in Figure 12 we have

$$\zeta(\widehat{\Gamma}_v) \cong \widehat{M} \otimes \left\{\begin{matrix} g_1\,, & x_1+x_2+x_3-x_4-x_5-x_6 \\ g_2\,, & x_1x_2+(x_1+x_2)x_3-x_5x_6-x_4(x_5+x_6) \\ g_3\,, & x_1x_2x_3-x_4x_5x_6 \\ g'_1\,, & x_7+x_8+x_9-x_{10}-x_{11}-x_{12} \\ g'_2\,, & x_8x_9+x_7(x_8+x_9)-x_{10}x_{11}-(x_{10}+x_{11})x_{12} \\ g'_3\,, & x_7x_8x_9-x_{10}x_{11}x_{12} \end{matrix}\right\}\{-6\}$$

with polynomials g_i and g'_i $(i=1,2,3)$ given by Equations (5)–(7). Similarly

$$\zeta'(\widehat{\Gamma}_v) \cong \widehat{M} \otimes \left\{\begin{matrix} h_1\,, & x_7+x_8+x_9-x_4-x_5-x_6 \\ h_2\,, & x_8x_9+x_7(x_8+x_9)-x_5x_6-x_4(x_5+x_6) \\ h_3\,, & x_7x_8x_9-x_4x_5x_6 \\ h'_1\,, & x_1+x_2+x_3-x_{10}-x_{11}-x_{12} \\ h'_2\,, & x_1x_2+(x_1+x_2)x_3-x_{10}x_{11}-(x_{10}+x_{11})x_{12} \\ h'_3\,, & x_1x_2x_3-x_{10}x_{11}x_{12} \end{matrix}\right\}\{-6\}$$

where the polynomials h_i and h'_i $(i=1,2,3)$ are as above. The factorizations $\zeta(\widehat{\Gamma}_v)$ and $\zeta'(\widehat{\Gamma}_v)$ have potential zero. Using the explicit form

$$g_3 = h_3 = 4(x_4+x_5+x_6-a), \qquad g'_3 = h'_3 = 4(x_{10}+x_{11}+x_{12}-a)$$

we exclude the variables x_4 and x_{12} from the third and sixth rows in $\zeta(\widehat{\Gamma}_v)$ and $\zeta'(\widehat{\Gamma}_v)$. This operation transforms the factorization $\widehat{M}$ into the factorization $\widehat{M}'$, which again is a tensor factor which $\zeta(\widehat{\Gamma}_v)$ and $\zeta'(\widehat{\Gamma}_v)$ have in common. Ignoring common overall shifts we obtain

$$\zeta(\widehat{\Gamma}_v) \cong \widehat{M}' \otimes \left\{\begin{matrix} g_1\,, & x_1+x_2+x_3-a \\ g_2\,, & x_1x_2+(x_1+x_2)x_3-x_5x_6-(x_5+x_6)(a-x_5-x_6) \\ g'_1\,, & x_7+x_8+x_9-a \\ g'_2\,, & x_8x_9+x_7(x_8+x_9)-x_{10}x_{11}-(x_{10}+x_{11})(a-x_{10}-x_{11}) \end{matrix}\right\},$$

$$\zeta'(\widehat{\Gamma}_v) \cong \widehat{M}' \otimes \left\{\begin{matrix} h_1\,, & x_7+x_8+x_9-a \\ h_2\,, & x_8x_9+x_7(x_8+x_9)-x_5x_6-(x_5+x_6)(a-x_5-x_6) \\ h'_1\,, & x_1+x_2+x_3-a \\ h'_2\,, & x_1x_2+(x_1+x_2)x_3-x_{10}x_{11}-(x_{10}+x_{11})(a-x_{10}-x_{11}) \end{matrix}\right\}.$$

Using Equation (5) we see that $g_1 = h'_1$ and $g'_1 = h_1$ and therefore, absorbing in $\widehat{M}'$ the corresponding Koszul factorizations, we can write

$$\zeta(\widehat{\Gamma}_v) \cong \widehat{M}'' \otimes \widehat{K} \quad \text{and} \quad \zeta'(\widehat{\Gamma}_v) \cong \widehat{M}'' \otimes \widehat{K}'$$

where

$$\widehat{K} = \begin{Bmatrix} g_2\,, & x_1x_2+(x_1+x_2)x_3-x_5x_6-(x_5+x_6)(a-x_5-x_6) \\ g_2'\,, & x_8x_9+x_7(x_8+x_9)-x_{10}x_{11}-(x_{10}+x_{11})(a-x_{10}-x_{11}) \end{Bmatrix}$$

$$\widehat{K}' = \begin{Bmatrix} h_2\,, & x_8x_9+x_7(x_8+x_9)-x_5x_6-(x_5+x_6)(a-x_5-x_6) \\ h_2'\,, & x_1x_2+(x_1+x_2)x_3-x_{10}x_{11}-(x_{10}+x_{11})(a-x_{10}-x_{11}) \end{Bmatrix}.$$

To simplify notation define the polynomials $\alpha_{i,j,k}$ and $\beta_{i,j}$ by

$$\alpha_{i,j,k} = x_ix_j + (x_i + x_j)x_k, \qquad \beta_{i,j} = x_ix_j + (x_i + x_j)(a+x_i+x_j).$$

In terms of $\alpha_{i,j,k}$ and $\beta_{i,j}$ we have:

$$\widehat{K} = \begin{Bmatrix} 2(\alpha_{1,2,3} + \beta_{5,6}) + 4b\,, & \alpha_{1,2,3}+\beta_{5,6} \\ 2(\alpha_{7,8,9} + \beta_{10,11}) + 4b\,, & \alpha_{7,8,9}+\beta_{10,11} \end{Bmatrix} \tag{8}$$

$$\widehat{K}' = \begin{Bmatrix} 2(\alpha_{7,8,9} + \beta_{5,6}) + 4b\,, & \alpha_{7,8,9}+\beta_{5,6} \\ 2(\alpha_{1,2,3} + \beta_{10,11}) + 4b\,, & \alpha_{1,2,3}+\beta_{10,11}) \end{Bmatrix} \tag{9}$$

Factorizations $\widehat{K}$ and $\widehat{K}'\langle 1\rangle$ can now be written in matrix form as

$$\widehat{K} = \begin{pmatrix} R \\ R \end{pmatrix} \xrightarrow{P} \begin{pmatrix} R \\ R \end{pmatrix} \xrightarrow{Q} \begin{pmatrix} R \\ R \end{pmatrix} \quad \text{and} \quad \widehat{K}'\langle 1\rangle = \begin{pmatrix} R \\ R \end{pmatrix} \xrightarrow{P'} \begin{pmatrix} R \\ R \end{pmatrix} \xrightarrow{Q'} \begin{pmatrix} R \\ R \end{pmatrix},$$

where

$$P = \begin{pmatrix} 2(\alpha_{1,2,3} + \beta_{5,6}) + 4b & \alpha_{7,8,9} - \beta_{10,11} \\ 2(\alpha_{7,8,9} + \beta_{10,11}) + 4b & -\alpha_{1,2,3} + \beta_{5,6} \end{pmatrix},$$

$$Q = \begin{pmatrix} \alpha_{1,2,3} - \beta_{5,6} & \alpha_{7,8,9} - \beta_{10,11} \\ 2(\alpha_{7,8,9} + \beta_{10,11}) + 4b & -2(\alpha_{1,2,3} + \beta_{5,6}) - 4b \end{pmatrix},$$

$$P' = \begin{pmatrix} -\alpha_{7,8,9} + \beta_{5,6} & -\alpha_{1,2,3} + \beta_{10,11} \\ -2(\alpha_{1,2,3} + \beta_{10,11}) - 4b & 2(\alpha_{7,8,9} + \beta_{5,6}) + 4b \end{pmatrix},$$

$$Q' = \begin{pmatrix} -2(\alpha_{7,8,9} + \beta_{5,6}) - 4b & -\alpha_{1,2,3} + \beta_{10,11} \\ -2(\alpha_{1,2,3} + \beta_{10,11}) - 4b & \alpha_{7,8,9} - \beta_{5,6} \end{pmatrix}.$$

Define a homomorphism $\psi = (f_0, f_1)$ from $\widehat{K}$ to $\widehat{K}'\langle 1\rangle$ by the pair of matrices

$$\left(\begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 \end{pmatrix} \right).$$

It is immediate that ψ is an isomorphism with inverse (f_1, f_0).

It follows that $1_{\widehat{M}''} \otimes \psi$ defines an isomorphism between $\zeta(\widehat{\Gamma}_v)$ and $\zeta'(\widehat{\Gamma}_v)\langle 1\rangle$. $\square$

Although having ψ in this form will be crucial in the proof of Proposition 4.2 an alternative description will be useful in Section 5. Note that we can reduce $\widehat{K}$ and $\widehat{K}'$ in Equations (8) and (9) further by using the row operations $[1,2]_1 \circ [1,2]'_{-2}$. We obtain

$$\widehat{K} \cong \left\{ \begin{matrix} -4(\alpha_{7,8,9} - \beta_{5,6}), & \alpha_{1,2,3} - \beta_{5,6} \\ 2(\alpha_{7,8,9} + \beta_{10,11} + \alpha_{1,2,3} - \beta_{5,6}), & \alpha_{1,2,3} + \alpha_{7,8,9} - \beta_{5,6} - \beta_{10,11} \end{matrix} \right\}$$

$$\text{and} \quad \widehat{K}' \cong \left\{ \begin{matrix} -4(\alpha_{1,2,3} - \beta_{5,6}), & \alpha_{7,8,9} - \beta_{5,6} \\ 2(\alpha_{7,8,9} + \beta_{10,11} + \alpha_{1,2,3} - \beta_{5,6}), & \alpha_{1,2,3} + \alpha_{7,8,9} - \beta_{5,6} - \beta_{10,11} \end{matrix} \right\}.$$

Since the second lines in $\widehat{K}$ and $\widehat{K}'$ are equal we can write

$$\widehat{K} \cong \left\{ -4(\alpha_{7,8,9} - \beta_{5,6}),\ \alpha_{1,2,3} - \beta_{5,6} \right\} \otimes \widehat{K}_2$$

$$\text{and} \quad \widehat{K}' \cong \left\{ -4(\alpha_{1,2,3} - \beta_{5,6}),\ \alpha_{7,8,9} - \beta_{5,6} \right\} \otimes \widehat{K}_2.$$

An isomorphism ψ' between $\widehat{K}$ and $\widehat{K}'\langle 1\rangle$ can now be given as the tensor product between $\big(-m(2),\ -m(1/2)\big)$ and the identity homomorphism of $\widehat{K}_2$.

Corollary 4.6 *The homomorphisms ψ and ψ' are equivalent.*

Proof The first thing to note is that we obtained the homomorphism ψ by first writing the differential (d_0, d_1) in $\widehat{K}'$ as 2×2 matrices and then its shift $\widehat{K}'\langle 1\rangle$ using $(-d_1, -d_0)$, but in the computation of ψ' we switched the terms and changed the signs in the first line of the Koszul matrix corresponding to $\widehat{K}'$. The two factorizations obtained are isomorphic by a non-trivial isomorphism, which is given by

$$T = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Bearing in mind that ψ and ψ' have $\mathbb{Z}/2\mathbb{Z}$–degree 1 and using

$$[1,2]_\lambda = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \right), \qquad [1,2]'_\lambda = \left(\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

it is straightforward to check that the composite homomorphism

$$T[1,2]_1[1,2]'_{-2}\psi[1,2]'_2[1,2]_{-1}$$

is

$$\left(\begin{pmatrix} -2 & 0 \\ 0 & -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 & 0 \\ 0 & -2 \end{pmatrix} \right)$$

which is the tensor product of $\big(-m(2), -m(1/2)\big)$ and the identity homomorphism of $\widehat{K}_2$. □

Proof of Proposition 4.2 We claim that $\zeta'(\widehat{\Gamma}_v) \cong \zeta(\widehat{\Gamma}_v)\langle k\rangle$ with k a nonnegative integer. We transform $\zeta'(\widehat{\Gamma}_v)$ into $\zeta(\widehat{\Gamma}_v)\langle k\rangle$ by repeated application of Lemma 4.5 as follows. Choose a pair of vertices connected by a virtual edge in $\zeta(\widehat{\Gamma}_v)$. Do nothing if the same pair is connected by a virtual edge in $\zeta'(\widehat{\Gamma}_v)$ and use Lemma 4.5 to connect them in the opposite case. Iterating this procedure we can turn $\zeta'(\widehat{\Gamma}_v)$ into $\zeta(\widehat{\Gamma}_v)$ with a shift in the $\mathbb{Z}/2\mathbb{Z}$–grading by $(k \mod 2)$ where k is the number of times we applied Lemma 4.5.

It remains to show that the shift in the $\mathbb{Z}/2\mathbb{Z}$–grading is independent of the choices one makes. To do so we label the vertices of Γ_v of $(+)$– and $(-)$–type by $(v_1^+,\dots,v_k^+)$ and $(v_1^-,\dots,v_k^-)$ respectively. Any complete identification of vertices in Γ_v is completely determined by an ordered set $J_\zeta = (v^-_{\sigma(1)},\dots,v^-_{\sigma(k)})$, with the convention that v_j^+ is connected through a virtual edge to $v^-_{\sigma(j)}$ for $1\le j\le k$. Complete identifications of the vertices in Γ_v are therefore in one-to-one correspondence with the elements of the symmetric group on k letters S_k. Any transformation of $\zeta'(\widehat{\Gamma})$ into $\zeta(\widehat{\Gamma})$ by repeated application of Lemma 4.5 corresponds to a sequence of elementary transpositions whose composite is equal to the quotient of the permutations corresponding to $J_{\zeta'}$ and J_ζ. We conclude that the shift in the $\mathbb{Z}/2\mathbb{Z}$–grading is well-defined, because any decomposition of a given permutation into elementary transpositions has a fixed parity. □

In Section 5 we want to associate an equivalence class of matrix factorizations to each closed web and an equivalence class of homomorphism to each foam. In order to do that consistently, we have to show that the isomorphisms in the proof of Proposition 4.2 are canonical in a certain sense (see Corollary 4.8).

Choose an ordering of the vertices of Γ_v such that v_i^+ is paired with v_i^- for all i and let ζ be the corresponding vertex identification. Use the linear entries in the Koszul matrix of $\zeta(\widehat{\Gamma}_v)$ to exclude one variable corresponding to an edge entering in each vertex of $(-)$–type, as in the proof of Lemma 4.5, so that the resulting Koszul factorization has the form $\zeta(\widehat{\Gamma}_v) = \widehat{K}_{\mathrm{lin}} \otimes \widehat{K}_{\mathrm{quad}}$ where $\widehat{K}_{\mathrm{lin}}$ (resp. $\widehat{K}_{\mathrm{quad}}$) consists of the lines in $\zeta(\widehat{\Gamma}_v)$ having linear (resp. quadratic) terms as its right entries. From the proof of Lemma 4.5 we see that changing a pair of virtual edges leaves $\widehat{K}_{\mathrm{lin}}$ unchanged.

Let σ_i be the element of S_k corresponding to the elementary transposition, which sends the complete identification $(v_1^-,\dots,v_i^-,v_{i+1}^-,\dots,v_k^-)$ to $(v_1^-,\dots,v_{i+1}^-,v_i^-,\dots,v_k^-)$, and let $\Psi_i = 1_{/i}\otimes\psi$ be the corresponding isomorphism of matrix factorizations from the proof of Lemma 4.5. The homomorphism ψ only acts on the ith and $(i+1)$th lines in $\widehat{K}_{\mathrm{quad}}$ and $1_{/i}$ is the identity morphism on the remaining lines. For the composition $\sigma_i\sigma_j$ we have the composite homomorphism $\Psi_i\Psi_j$.

Lemma 4.7 *The assignment $\sigma_i \mapsto \Psi_i$ defines a representation of S_k on $\zeta(\widehat{\Gamma})_0 \oplus \zeta(\widehat{\Gamma})_1$.*

Proof Let $\widehat{K}$ be the Koszul factorization corresponding to the lines i and $i+1$ in $\zeta(\widehat{\Gamma}_v)$ and let $|\,00\,\rangle$, $|\,11\,\rangle$, $|\,01\,\rangle$ and $|\,10\,\rangle$ be the standard basis vectors of $\widehat{K}_0 \oplus \widehat{K}_1$. The homomorphism ψ found in the proof of Lemma 4.5 can be written as only one matrix acting on $\widehat{K}_0 \oplus \widehat{K}_1$:

$$\psi = \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -1 & -1 \\ 1 & -\frac{1}{2} & 0 & 0 \\ -1 & -\frac{1}{2} & 0 & 0 \end{pmatrix}.$$

We have that ψ^2 is the identity matrix and therefore it follows that Ψ_i^2 is the identity homomorphism on $\zeta(\widehat{\Gamma}_v)$. It is also immediate that $\Psi_i\Psi_j = \Psi_j\Psi_i$ for $|i-j|>1$. To complete the proof we need to show that $\Psi_i\Psi_{i+1}\Psi_i = \Psi_{i+1}\Psi_i\Psi_{i+1}$, which we do by explicit computation of the corresponding matrices. Let $\widehat{K}'$ be the Koszul matrix consisting of the three lines i, $i+1$ and $i+2$ in $\widehat{K}_{\mathrm{quad}}$. To show that $\Psi_i\Psi_{i+1}\Psi_i = \Psi_{i+1}\Psi_i\Psi_{i+1}$ is equivalent to showing that ψ satisfies the Yang–Baxter equation

$$(\psi\otimes 1)(1\otimes\psi)(\psi\otimes 1) = (1\otimes\psi)(\psi\otimes 1)(1\otimes\psi), \tag{10}$$

with $1\otimes\psi$ and $\psi\otimes 1$ acting on $\widehat{K}'$. Note that, in general, the tensor product of two homomorphisms of matrix factorizations f and g is defined by

$$(f\otimes g)|\,v\otimes w\,\rangle = (-1)^{|g||v|}|\,fv\otimes gw\,\rangle.$$

Let $|\,000\,\rangle$, $|\,011\,\rangle$, $|\,101\,\rangle$, $|\,110\,\rangle$, $|\,001\,\rangle$, $|\,010\,\rangle$, $|\,100\,\rangle$ and $|\,111\,\rangle$ be the standard basis vectors of $\widehat{K}'_0 \oplus \widehat{K}'_1$. With respect to this basis the homomorphisms $\psi\otimes 1$ and $1\otimes\psi$ have the form of block matrices

$$\psi\otimes 1 = \left(\begin{array}{cccc|cccc} & & & & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ & 0 & & & 1 & 0 & 0 & -\frac{1}{2} \\ & & & & -1 & 0 & 0 & -\frac{1}{2} \\ & & & & 0 & -1 & -1 & 0 \\ \hline 0 & \frac{1}{2} & -\frac{1}{2} & 0 & & & & \\ 1 & 0 & 0 & -\frac{1}{2} & & 0 & & \\ -1 & 0 & 0 & -\frac{1}{2} & & & & \\ 0 & -1 & -1 & 0 & & & & \end{array}\right)$$

and

$$1\otimes\psi=\left(\begin{array}{cccc|cccc} & & & & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ & & & & -1 & -1 & 0 & 0 \\ & 0 & & & 0 & 0 & -1 & \frac{1}{2} \\ & & & & 0 & 0 & 1 & \frac{1}{2} \\ \hline 1 & -\frac{1}{2} & 0 & 0 & & & & \\ -1 & -\frac{1}{2} & 0 & 0 & & 0 & & \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & & & & \\ 0 & 0 & 1 & 1 & & & & \end{array}\right).$$

By a simple exercise in matrix multiplication we find that both sides in Equation (10) are equal and it follows that $\Psi_i\Psi_{i+1}\Psi_i=\Psi_{i+1}\Psi_i\Psi_{i+1}$. □

Corollary 4.8 *The isomorphism $\zeta'(\widehat{\Gamma}_v)\cong\zeta(\widehat{\Gamma}_v)\langle k\rangle$ in Proposition 4.2 is uniquely determined by ζ' and ζ.*

Proof Let σ be the permutation that relates ζ' and ζ. Recall that in the proof of Proposition 4.2 we defined an isomorphism $\Psi_\sigma\colon\zeta'(\widehat{\Gamma}_v)\to\zeta(\widehat{\Gamma}_v)\langle k\rangle$ by writing σ as a product of transpositions. The choice of these transpositions is not unique in general. However, Lemma 4.7 shows that Ψ_σ only depends on σ. □

From now on we write $\widehat{\Gamma}$ for the equivalence class of $\widehat{\Gamma}_v$ under complete vertex identification. Graphically we represent the vertices of $\widehat{\Gamma}$ as in Figure 13. We need to

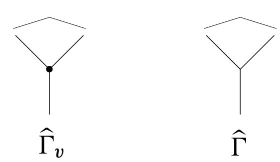


Figure 13: A vertex and its equivalence class under vertex identification

neglect the $\mathbb{Z}/2\mathbb{Z}$ grading, which we do by imposing that $\widehat{\Gamma}$, for any closed web Γ, have only homology in degree zero, applying a shift if necessary.

Let Γ and Λ be arbitrary webs. We have to define the morphisms between $\widehat{\Gamma}$ and $\widehat{\Lambda}$. Let $\zeta(\widehat{\Gamma}_v)$ and $\zeta'(\widehat{\Gamma}_v)$ be representatives of $\widehat{\Gamma}$ and $\zeta(\widehat{\Lambda}_v)$ and $\zeta'(\widehat{\Lambda}_v)$ be representatives of $\widehat{\Lambda}$. Let

$$f\in\mathrm{Hom}_{\mathbf{MF}}\big(\zeta(\widehat{\Gamma}_v),\zeta(\widehat{\Lambda}_v)\big)\quad\text{and}\quad g\in\mathrm{Hom}_{\mathbf{MF}}\big(\zeta'(\widehat{\Gamma}_v),\zeta'(\widehat{\Lambda}_v)\big)$$

be two homomorphisms. We say that f and g are equivalent, denoted by $f \sim g$, if and only if there exists a commuting square

$$\begin{tikzcd}
\zeta(\widehat{\Gamma}_v) \arrow[r, "\cong"] \arrow[d, "f"] & \zeta'(\widehat{\Gamma}_v) \arrow[d, "g"] \\
\zeta(\widehat{\Lambda}_v) \arrow[r, "\cong"] & \zeta'(\widehat{\Lambda}_v)
\end{tikzcd}$$

with the horizontal isomorphisms being of the form as discussed in Proposition 4.2. The composition rule for these equivalence classes of homomorphisms, which relies on the choice of representatives within each class, is well-defined by Corollary 4.8. Note that we can take well-defined linear combinations of equivalence classes of homomorphisms by taking linear combinations of their representatives, as long as the latter have all the same source and the same target. By Corollary 4.8, homotopy equivalences are also well-defined on equivalence classes. We take

$$\mathrm{Hom}(\widehat{\Gamma}, \widehat{\Lambda})$$

to be the set of equivalence classes of homomorphisms of matrix factorizations between $\widehat{\Gamma}$ and $\widehat{\Lambda}$ modulo homotopy equivalence. The additive category that we get this way is denoted by

$$\widehat{\mathbf{Foam}_{/\ell}}.$$

Note that we can define the homology of $\widehat{\Gamma}$, for any closed web Γ. This group is well-defined up to isomorphism and we denote it by $\widehat{\mathrm{H}}(\Gamma)$.

Next we show how to define a link homology using the objects and morphisms in $\widehat{\mathbf{Foam}_{/\ell}}$. For any link L, first take the universal rational Khovanov–Rozansky cochain complex $\mathrm{KR}_{a,b,c}(L)$. The ith cochain group $\mathrm{KR}^i_{a,b,c}(L)$ is given by the direct sum of cohomology groups of the form $\mathrm{H}(\Gamma_v)$, where Γ_v is a total flattening of L. By the remark above it makes sense to consider $\widehat{\mathrm{KR}}^i_{a,b,c}(L)$, for each i. The differential $d^i\colon \mathrm{KR}^i_{a,b,c}(L) \to \mathrm{KR}^{i+1}_{a,b,c}(L)$ induces a map

$$\widehat{d}^i\colon \widehat{\mathrm{KR}}^i_{a,b,c}(L) \to \widehat{\mathrm{KR}}^{i+1}_{a,b,c}(L),$$

for each i. The latter map is well-defined and therefore the homology

$$\widehat{\mathrm{HKR}}^i_{a,b,c}(L)$$

is well-defined, for each i.

Let $u\colon L \to L'$ be a link cobordism. Khovanov and Rozansky [4] constructed a cochain map which induces a homomorphism

$$\mathrm{HKR}_{a,b,c}(u)\colon \mathrm{HKR}_{a,b,c}(L) \to \mathrm{HKR}_{a,b,c}(L').$$

The latter is only defined up to a $\mathbb{Q}$–scalar. The induced map

$$\widehat{\mathrm{HKR}}_{a,b,c}(u)\colon \widehat{\mathrm{HKR}}_{a,b,c}(L) \to \widehat{\mathrm{HKR}}_{a,b,c}(L')$$

is also well-defined up to a $\mathbb{Q}$–scalar. The following result follows immediately:

Lemma 4.9 $\mathrm{HKR}_{a,b,c}$ *and* $\widehat{\mathrm{HKR}}_{a,b,c}$ *are naturally isomorphic as projective functors from* **Link** *to* $\mathbf{Mod_{bg}}$.

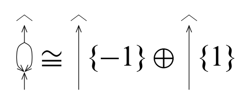
In the next section we will show that $U_{a,b,c}$ and $\widehat{\mathrm{HKR}}_{a,b,c}$ are naturally isomorphic as projective functors.

By Lemma 3.5 we also get the following

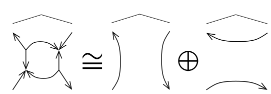
Lemma 4.10 *We have the Khovanov–Kuperberg decompositions in* $\widehat{\mathbf{Foam}}_{/\ell}$ *:*

(Disjoint Union) $$\widehat{\bigcirc\Gamma} \cong \widehat{\bigcirc} \otimes_{\mathbb{Q}[a,b,c]} \widehat{\Gamma}$$

(Digon Removal)



(Square Removal)



Although Lemma 4.10 follows from Lemma 3.5 and Lemma 4.4, an explicit proof will be useful in the sequel.

Proof (Disjoint Union) is a direct consequence of the definitions. To prove (Digon Removal) define the grading-preserving homomorphisms

by Figure 14.

If we choose to create the circle on the other side of the arc in α we obtain a homomorphism homotopic to α and the same holds for β. Define the homomorphisms

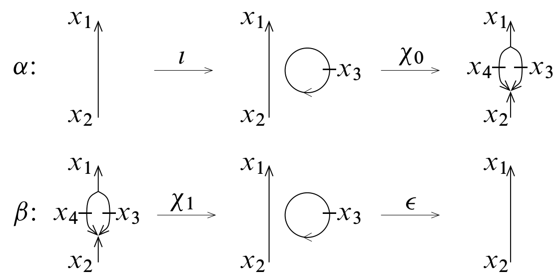
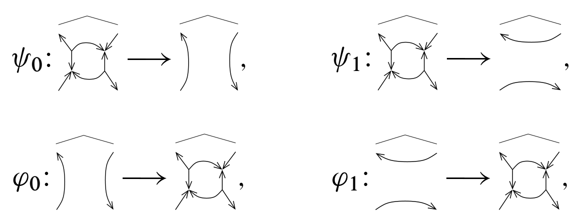


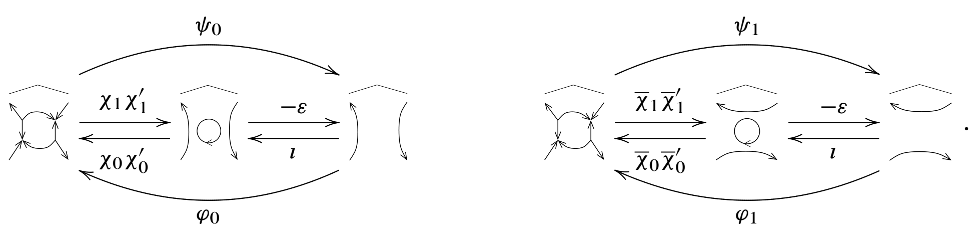
Figure 14: Homomorphisms α and β

by $\alpha_0 = 2\alpha$ and $\alpha_1 = 2\alpha \circ m(-x_2)$. Note that the homomorphism α_1 is homotopic to the homomorphism $2\alpha \circ m(x_1 + x_3 - a)$. Similarly define

by $\beta_0 = -\beta \circ m(x_3)$ and $\beta_1 = -\beta$. A simple calculation shows that $\beta_j \alpha_i = \delta_{ij}\,\mathrm{Id}(\widehat{\uparrow})$. Since the cohomologies of the factorizations and $\widehat{\uparrow}\{-1\} \oplus \widehat{\uparrow}\{1\}$ have the same graded dimension (see [4]) we have that $\alpha_0 + \alpha_1$ and $\beta_0 + \beta_1$ are homotopy inverses of each other and that $\alpha_0\beta_0 + \alpha_1\beta_1$ is homotopic to the identity in End( ). To prove (Square Removal) define grading preserving homomorphisms



by the composed homomorphisms below



We have that $\psi_0\varphi_0 = \mathrm{Id}( )$ and $\psi_1\varphi_1 = \mathrm{Id}( )$. We also have $\psi_1\varphi_0 = \psi_0\varphi_1 = 0$ because Ext(,) $\cong \widehat{\mathrm{HKR}}_{a,b,c}(\bigcirc)\{4\}$ which is zero in q–degree zero and so any homomorphism of degree zero between and is homotopic to the zero homomorphism. Since the cohomologies of and $\oplus$ have the same graded dimension (see

[4]) we have that $\psi_0 + \psi_1$ and $\varphi_0 + \varphi_1$ are homotopy inverses of each other and that $\varphi_0\psi_0 + \varphi_1\psi_1$ is homotopic to the identity in $\mathrm{End}(\widehat{\Box})$. □

5 The equivalence functor

In this section we first define a functor

$$\widehat{\ }\colon \mathbf{Foam}_{/\ell} \to \widehat{\mathbf{Foam}_{/\ell}}.$$

Then we show that this functor is well-defined and an isomorphism of categories. Finally we show that this implies that the link homology functors $U_{a,b,c}$ and $\widehat{\mathrm{HKR}}_{a,b,c}$ from **Link** to $\mathbf{Mod_{bg}}$ are naturally isomorphic.

On objects the functor $\widehat{\ }$ is defined by

$$\Gamma \to \widehat{\Gamma},$$

as explained in the previous section. We now define $\widehat{\ }$ on morphisms. Let $f \in \mathrm{Hom}_{\mathbf{Foam}_{/\ell}}(\Gamma, \Gamma')$. Suffice it to consider the case in which f can be given by one singular cobordism, also denoted f. If f is given by a linear combination of singular cobordisms, one can simply extend the following arguments to all terms. Slice f up between critical points, so that each slice contains one elementary foam, i.e. a zip or unzip, a saddle-point cobordism, or a cap or a cup, glued horizontally to the identity foam on the rest of the source and target webs. For each slice choose compatible vertex identifications on the source and target webs, such that in the region where both webs are isotopic the vertex identifications are the same and in the region where they differ the vertex identifications are such that we can apply the homomorphism of matrix factorizations $\chi_0, \chi_1, \eta, \iota$ or ϵ. This way we get a homomorphism of matrix factorizations for each slice. We can take its $\widehat{\ }$ equivalence class. Composing all these morphisms gives a morphism $\widehat{f}$ between $\widehat{\Gamma}_v$ and $\widehat{\Gamma}'_v$. For its definition we had to choose a representative singular cobordism of the foam f, a way to slice it up and complete vertex identifications for the source and target of each slice. Of course we have to show that $\widehat{f} \in \mathrm{Hom}_{\widehat{\mathbf{Foam}_{/\ell}}}(\widehat{\Gamma}, \widehat{\Gamma}')$ is independent of those choices. But before we do that we have to show that there is no sign problem in the definition of $\widehat{\eta}$ (see the remark at the end of Section 3). Recall that η is the homomorphism of matrix factorizations induced by a saddle-point cobordism.

Lemma 5.1 *The morphism $\widehat{\eta}$ is well defined for closed webs.*

Proof Let Γ and Γ' be two closed webs and $\Sigma\colon \Gamma \to \Gamma'$ a cobordism which is the identity everywhere except for one saddle-point. By a slight abuse of notation, let $\widehat{\eta}$

denote the homomorphism of matrix factorizations which corresponds to Σ. Note that Γ and Γ' have the same number of vertices, which we denote by v. Our proof that $\widehat{\eta}$ is well-defined proceeds by induction on v. If $v = 0$, then the lemma holds, because Γ consists of one circle and Γ' of two circles, or vice-versa. These circles have no marks and are therefore indistinguishable. To each circle we associate the complex $\widehat{\bigcirc}$ and $\widehat{\eta}$ corresponds to the product or the coproduct in $\mathbb{Q}[a,b,c][X]/X^3 - aX^2 - bX - c$. Note that as soon as we mark the two circles, they will become distinguishable, and a minus-sign creeps in when we switch them. However, this minus-sign then cancels against the minus-sign showing up in the homomorphism associated to the saddle-point cobordism.

Let $v > 0$. This part of our proof uses some ideas from the proof of Jeong and Kim [2, Theorem 2.4]. Any web can be seen as lying on a 2–sphere. Let V, E and F denote the number of vertices, edges and faces of a web, where a face is a connected component of the complement of the web in the 2–sphere. Let $F = \sum_i F_i$, where F_i is the number of faces with i edges. Note that we only have faces with an even number of edges. It is easy to see that the following equations hold:

$$3V = 2E \qquad\qquad V - E + F = 2 \qquad\qquad 2E = \sum_i iF_i$$

Therefore, we get

$$6 = 3F - E = 2F_2 + F_4 - F_8 - \cdots,$$

which implies

$$6 \leq 2F_2 + F_4. \tag{11}$$

This lower bound holds for any web, in particular for Γ and Γ'. Note that for $F_2 = 3$ and $F_4 = 0$ we have a theta-web, which is the intersection of a theta-foam and a plane. Since all edges have to have the same orientation at both vertices, there is no way to apply a saddle point cobordism to this web. For all other values of F_2 and F_4 there is always a digon or a square in Γ and Γ' on which $\widehat{\eta}$ acts as the identity, i.e. which does not get changed by the saddle-point in the cobordism to which $\widehat{\eta}$ corresponds. To see how this follows from (11), just note that one saddle point cobordism never involves more than three squares, one digon and two squares, or two digons and one square. Since the MOY-moves in Lemma 4.10 are all given by isomorphisms which correspond to the zip and the unzip and the birth and the death of a circle (see the proof of Lemma 4.10), this shows that there is always a set of MOY-moves which can be applied both to Γ and Γ' whose target webs, say Γ_1 and Γ_1', have less than v vertices, and which commute with $\widehat{\eta}$. Here we denote the homomorphism of matrix factorizations corresponding to the saddle-point cobordism between Γ_1 and Γ_1' by $\widehat{\eta}$

again. By induction, $\widehat{\eta}\colon \widehat{\Gamma}_1 \to \widehat{\Gamma'}_1$ is well-defined. Since the MOY-moves commute with $\widehat{\eta}$, we conclude that $\widehat{\eta}\colon \widehat{\Gamma} \to \widehat{\Gamma}'$ is well-defined. □

Lemma 5.2 *The functor* $\widehat{\ }\colon \mathbf{Foam}_{/\ell} \to \widehat{\mathbf{Foam}_{/\ell}}$ *is well-defined.*

Proof The fact that $\widehat{f}$ does not depend on the vertex identifications follows immediately from Corollary 4.8 and the equivalence relation $\sim$ on the Hom–spaces in $\widehat{\mathbf{Foam}_{/\ell}}$.

Next we prove that $\widehat{f}$ does not depend on the way we have sliced it up. By Lemma 4.10 we know that, for any closed web Γ, the class $\widehat{\Gamma}$ is homotopy equivalent to a direct sum of terms of the form $\widehat{\bigcirc}^k$. Note that $\widehat{\mathrm{Ext}}(\varnothing, \bigcirc)$ is generated by $X^s\iota$, for $0 \leq s \leq 2$, and that all maps in the proof of Lemma 4.10 are induced by cobordisms with a particular slicing. This shows that $\widehat{\mathrm{Ext}}(\varnothing, \Gamma)$ is generated by maps of the form $\widehat{u}$, where u is a cobordism between $\varnothing$ and Γ with a particular slicing. A similar result holds for $\widehat{\mathrm{Ext}}(\Gamma, \varnothing)$. Now let f and f' be given by the same cobordism between Γ and Λ but with different slicings. If $\widehat{f} \neq \widehat{f'}$, then, by the previous arguments, there exist maps $\widehat{u}$ and $\widehat{v}$, where $u\colon \varnothing \to \Gamma$ and $v\colon \Lambda \to \varnothing$ are cobordisms with particular slicings, such that $\widehat{vfu} \neq \widehat{vf'u}$. This reduces the question of independence of slicing to the case of closed cobordisms. Note that we already know that $\widehat{\ }$ is well-defined on the parts that do not involve singular circles, because it is the generalization of a 2d TQFT. It is therefore easy to see that $\widehat{\ }$ respects the relation (CN). Thus we can cut up any closed singular cobordism near the singular circles to obtain a linear combination of closed singular cobordisms isotopic to spheres and theta-foams. The spheres do not have singular circles, so $\widehat{\ }$ is well-defined on them and it is easy to check that it respects the relation (S).

Finally, for theta-foams we do have to check something. There is one basic Morse move that can be applied to one of the discs of a theta-foam, which we show in Figure 15. We have to show that $\widehat{\ }$ is invariant under this Morse move.

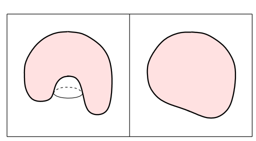


Figure 15: Singular Morse move

In other words, we have to show that the composite homomorphism in Figure 16 is homotopic to the identity. It suffices to do the computation on the homology. First we note that the theta web has homology only in $\mathbb{Z}/2\mathbb{Z}$–degree 0. From the remark at the

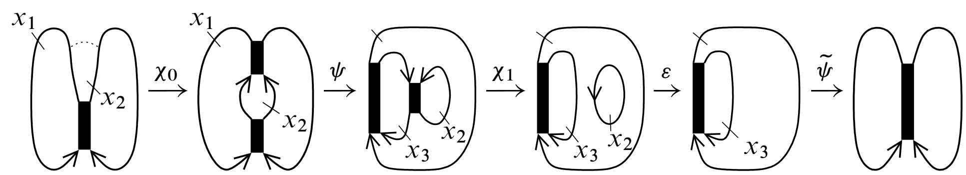


Figure 16: Homomorphism Φ. To avoid cluttering only some marks are shown

end of Section 3.2.1 it follows that χ_0 is equivalent to multiplication by $-2(x_1 - x_2)$ and χ_1 to multiplication by $x_2 - x_3$, where we used the fact that ψ has $\mathbb{Z}/2\mathbb{Z}$–degree 1. From Corollary 4.6 we have that ψ is equivalent to multiplication by -2 and from the definition of vertex identification it is immediate that $\tilde{\psi}$ is the identity. Therefore we have that

$$\Phi = \varepsilon\big(4(x_2 - x_3)(x_1 - x_2)\big) = 1.$$

It is also easy to check that $\widehat{\ }$ respects the relation (Θ).

Note that the arguments above also show that, for an open foam f, we have $\hat{f} = 0$ if $u_1 f u_2 = 0$ for all singular cobordisms $u_1\colon \varnothing \to \Gamma_v$ and $u_2\colon \Gamma'_v \to \varnothing$. This proves that $\widehat{\ }$ is well-defined on foams, which are equivalence classes of singular cobordisms. □

Corollary 5.3 *The functor* $\widehat{\ }\colon \mathbf{Foam}_{/\ell} \to \widehat{\mathbf{Foam}_{/\ell}}$ *is an isomorphism of categories.*

Proof On objects $\widehat{\ }$ is clearly a bijection. On morphisms it is also a bijection by Lemma 4.10 and the proof of Lemma 5.2. □

Theorem 5.4 *The projective functors* $U_{a,b,c}$ *and* $\widehat{\mathrm{HKR}}_{a,b,c}$ *from* **Link** *to* $\mathbf{Mod_{bg}}$ *are naturally isomorphic.*

Proof Let D be a diagram of L, $C_{\mathbf{Foam}_{/\ell}}(D)$ the complex for D constructed with foams in Section 2 and $\widehat{KR}_{a,b,c}(D)$ the complex constructed with equivalence classes of matrix factorizations in Section 4. From Lemma 5.2 and Corollary 5.3 it follows that for all i we have isomorphisms of graded $\mathbb{Q}[a,b,c]$–modules $C^i_{\mathbf{Foam}_{/\ell}}(D) \cong \widehat{KR}^i_{a,b,c}(D)$ where i is the homological degree. By a slight abuse of notation we denote these isomorphisms by $\widehat{\ }$ too. The differentials in $\widehat{KR}_{a,b,c}(D)$ are induced by χ_0 and χ_1, which are exactly the maps that we associated to the zip and the unzip. This shows that $\widehat{\ }$ commutes with the differentials in both complexes and therefore that it defines an isomorphism of complexes.

The naturality of the isomorphism between the two functors follows from Corollary 5.3 and the fact that all elementary link cobordisms are induced by the elementary foams and their respective images with respect to $\widehat{\ }$. □

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