Nielsen type numbers and homotopy minimal periods for maps on 3–solvmanifolds

JONG BUM LEE
XUEZHI ZHAO

For all continuous maps on 3–solvmanifolds, we give explicit formulas for a complete computation of the Nielsen type numbers $NP_n(f)$ and $N\Phi_n(f)$. The most general cases were explored by Heath and Keppelmann [3] and the complementary part is studied in this paper. While studying the homotopy minimal periods of all maps on 3–solvmanifolds, we give a complete description of the sets of homotopy minimal periods of all such maps, including a correction to Jezierski, Kędra and Marzantowicz's results in [5].

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1 Introduction

In Nielsen fixed point theory, there are well known invariants which are the Lefschetz number $L(f)$ and the Nielsen number $N(f)$. It is known that the Nielsen number is much more powerful than the Lefschetz number but computing it is very hard. For periodic points, two Nielsen type numbers $NP_n(f)$ and $N\Phi_n(f)$ were introduced by Jiang, which are lower bounds for the number of periodic points of least period exactly n and the set of periodic points of period n, respectively.

It is obvious that these Nielsen numbers are much more powerful than the Lefschetz number in describing the periodic point sets of self-maps, but the computation of those homotopy invariants is difficult in general. However using fiber techniques on nilmanifolds and some solvmanifolds, Heath and Keppelmann [3] (see also Keppelmann and McCord [8]) succeeded in showing that the Nielsen numbers and the two Nielsen type numbers are related to each other under certain conditions.

One of the natural problems in dynamical systems is the study of the existence of periodic points of least period exactly n. Homotopically, a new concept, namely homotopy minimal periods,

$$\mathrm{HPer}(f) = \bigcap_{g\simeq f} \{m \mid g^m(x) = x, g^q(x) \neq x, q < m \text{ for some } x\}$$

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was introduced by Alsedà, Baldwin, Llibre, Swanson and Szlenk in [1]. Since the homotopy minimal period is preserved under a small perturbation of a self-map f on a manifold X, we can say that the set $\mathrm{HPer}(f)$ of homotopy minimal periods of f describes the rigid part of the dynamics of f. A complete description of the set of homotopy minimal periods of all self-maps was obtained on 3–nilmanifolds by Jezierski and Marzantowicz [6], and Lee and Zhao [10] and on 3–solvmanifolds by Jezierski, Kędra and Marzantowicz [5].

There are six simply connected 3–dimensional unimodular Lie groups: the abelian Lie group $\mathbb{R}^3$, the nilpotent Lie group Nil, the solvable Lie group Sol, the universal covering group $\widetilde{\mathrm{E}}_0(2)$ of the connected component of the Euclidean group, the universal covering group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the special linear group and the special unitary group $\mathrm{SU}(2)$. There are also infinitely many simply connected 3–dimensional non-unimodular solvable Lie groups, see Ha and Lee [2]. It is easy to see that the solvable group $\widetilde{\mathrm{E}}_0(2)$ is not of type (NR) (see the next section for the definition). Since a solvable Lie group which admits a lattice is unimodular, the non-unimodular solvable Lie groups do not concern us. In this paper, we are concerned with closed 3–manifold $\Gamma\backslash\mathrm{Sol}$ which are quotient spaces of Sol by its lattices Γ, and continuous maps between them. Those manifolds will be called 3*–solvmanifolds* with Sol*-geometry* or in short 3*–solvmanifolds*.

Jezierski and Marzantowicz in [6], and the authors in [10] studied homotopy minimal periods for maps on 3–manifolds with Nil-geometry. In [5] (see also Kim, Lee and Yoo [9]), Jezierski, Kędra and Marzantowicz carried out a further study of homotopy minimal periods for maps on 3–solvmanifolds. One of motivations of the present work is to correct Proposition 4.3 of [5], in which the entries of the matrices are claimed to be integers. However this is not true; see Example 3.6. Every continuous map on $\Gamma\backslash\mathrm{Sol}$ which is induced from a homomorphism on Sol must preserve the lattice Γ. This will provide the necessary conditions for those entries; see Remark 3.7.

2 The lattices of Sol

The Lie group Sol is one of the eight geometries that one considers in the study of 3–manifolds, see Scott [11]. One can describe Sol as a semi-direct product $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the map:

$$\varphi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

Its Lie algebra $\mathfrak{sol}$ is given as $\mathfrak{sol} = \mathbb{R}^2 \rtimes_\sigma \mathbb{R}$ where:

$$\sigma(s) = \begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix}$$

The Lie group Sol can be faithfully represented into $\mathrm{Aff}(3)$ by

$$\begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where x, y and t are real numbers, and hence its Lie algebra $\mathfrak{sol}$ is isomorphic to the algebra of matrices:

$$\begin{bmatrix} t & 0 & 0 & a \\ 0 & -t & 0 & b \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we will discuss the existence of a lattice in Sol (cf Thurston [12, Theorem 4.7.13] or Scott [11, page 472]).

Lemma 2.1 *A group Γ is a lattice of $\mathrm{Sol} = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ if and only if it is an extension of the form*

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 1$$

where the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is generated by an element A of $\mathrm{SL}(2,\mathbb{Z})$ with trace greater than 2.

Proof Suppose Γ is a lattice. Since $\mathbb{R}^2$ is the nil-radical of $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$, $\Gamma \cap \mathbb{R}^2$ is a lattice in $\mathbb{R}^2$. Thus Γ is of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where $\mathbb{Z} = \langle t_0 \rangle \subset \mathbb{R}$. We obtain that

$$\begin{aligned}\Gamma &= \left\{ \begin{bmatrix} 1 & 0 & 0 & kx_1 + lx_2 \\ 0 & 1 & 0 & ky_1 + ly_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; k, l \in \mathbb{Z} \right\} \rtimes_\varphi \langle t_0 \rangle \\ &= \left\{ \begin{bmatrix} e^{mt_0} & 0 & 0 & kx_1 + lx_2 \\ 0 & e^{-mt_0} & 0 & ky_1 + ly_2 \\ 0 & 0 & 1 & mt_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; k, l, m \in \mathbb{Z} \right\}\end{aligned}$$

where (x_1, y_1) and (x_2, y_2) are linear independent, and $t_0 \neq 0$. Since Γ is a subgroup of Sol, the elements

$$\begin{bmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{-t_0} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{t_0} & 0 & 0 & e^{t_0} x_i \\ 0 & e^{-t_0} & 0 & e^{-t_0} y_i \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (i = 1, 2)$$

must lie in Γ. Hence, there are integers $\ell_{11}, \ell_{12}, \ell_{21}, \ell_{22}$ such that

$$\begin{bmatrix} e^{t_0} x_i \\ e^{-t_0} y_i \end{bmatrix} = \ell_{1i} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \ell_{2i} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \quad i = 1, 2.$$

This implies that:

$$\begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

By a computation, we have $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is generated by: $A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$ Since A is conjugate to $\begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix}$, the determinate of A is 1, ie, $A \in \mathrm{SL}(2, \mathbb{Z})$.The trace of A is $\ell_{11} + \ell_{22} = e^{t_0} + e^{-t_0}$ and hence is greater than 2.

Let us consider the converse. Given a matrix $A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ with trace $\ell_{11} + \ell_{22} > 2$, A has two distinct irrational eigenvalues $\frac{\ell_{11}+\ell_{22} \pm D}{2}$, where $D = \sqrt{(\ell_{11} + \ell_{22})^2 - 4}$. With two corresponding eigenvectors, we form a real invertible matrix $P = \begin{bmatrix} \frac{\ell_{11}-\ell_{22}+D}{2\ell_{21}} & \frac{\ell_{11}-\ell_{22}-D}{2\ell_{21}} \\ 1 & 1 \end{bmatrix}$. Then we have:

$$P^{-1} A P = \begin{bmatrix} \frac{\ell_{11}+\ell_{22}+D}{2} & 0 \\ 0 & \frac{\ell_{11}+\ell_{22}-D}{2} \end{bmatrix}$$

The semi-product

$$\left\{ \begin{bmatrix} e^{mt_0} & 0 & 0 & k\frac{\ell_{21}}{D} + l\frac{\ell_{22}-\ell_{11}+D}{2D} \\ 0 & e^{-mt_0} & 0 & -k\frac{\ell_{21}}{D} + l\frac{\ell_{11}-\ell_{22}+D}{2D} \\ 0 & 0 & 1 & mt_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \, k, l, m \in \mathbb{Z} \right\} = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$$

of the linear span of the column vectors $\begin{bmatrix} \frac{\ell_{21}}{D} \\ -\frac{\ell_{21}}{D} \end{bmatrix}$ and $\begin{bmatrix} \frac{\ell_{22}-\ell_{11}+D}{2D} \\ \frac{\ell_{11}-\ell_{22}+D}{2D} \end{bmatrix}$ of P^{-1} and $\mathbb{Z} = \langle t_0 \rangle$, where $t_0 = \ln \frac{\ell_{11}+\ell_{22}+\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2}$, forms a lattice of the Sol group $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$. □

Immediately, we have:

Corollary 2.2 *Any lattice* Γ *of* $\mathrm{Sol} = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ *has a presentation*

$$\langle a, b, s \mid [a, b] = 1,\ sas^{-1} = a^{\ell_{11}} b^{\ell_{21}},\ sbs^{-1} = a^{\ell_{12}} b^{\ell_{22}} \rangle,$$

where $A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ *and*

$$a = \begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & y_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad s = \begin{bmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{-t_0} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

satisfying

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1} \begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1}$$

ie, e^{t_0} *and* e^{-t_0} *are two irrational eigenvalues of* A *and* $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1}$ *consists of two eigenvectors of* A.

For an element $A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ generating the action of $\mathbb{Z}$ on $\mathbb{Z}^2$, since its trace is greater than 2, A must have different real eigenvalues: one is greater than 1, the other is less than 1. Such a matrix is said to be a hyperbolic element of $\mathrm{SL}(2, \mathbb{Z})$. We shall write $\Gamma_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ or $\Gamma_\phi = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ for such a Γ which fits in the extension $1 \to \mathbb{Z}^2 \to \Gamma \to \mathbb{Z} \to 1$.

Furthermore, neither ℓ_{12} nor ℓ_{21} vanishes. In fact, if $\ell_{21}\ell_{12} = 0$, the determinant of A would be $\ell_{11}\ell_{22}$. Since $A \in \mathrm{SL}_2(\mathbb{Z})$, $\ell_{11}\ell_{22} = 1$. The trace of A would be $\ell_{11} + \ell_{22} = \pm 2$. This contradicts Lemma 2.1.

Now we will discuss the homomorphisms θ on the lattices of Sol.

Theorem 2.3 *Let*

$$\Gamma = \langle a, b, s \mid [a,b] = 1,\ sas^{-1} = a^{\ell_{11}} b^{\ell_{21}},\ sbs^{-1} = a^{\ell_{12}} b^{\ell_{22}} \rangle$$

be a lattice of Sol. *Then* $\Lambda = \langle a, b\rangle = \mathbb{Z}^2$ *is a fully invariant subgroup of* Γ*, ie, every homomorphism on* Γ *restricts to a homomorphism on* Λ*.*

Proof Consider the commutator subgroup $[\Gamma,\Gamma]$ of Γ. Since every element of Γ is of the form $a^m b^n s^\ell$, we have:

$$\begin{aligned}[\Gamma,\Gamma] &= \langle [a^m b^n s^\ell, a^{m'} b^{n'} s^{\ell'}] \mid m, m', n, n', \ell, \ell' \in \mathbb{Z}\rangle \\ &= \langle a^{m-m'} b^{n-n'} \phi^\ell(a^{m'} b^{n'}) \phi^{\ell'}(a^{-m} b^{-n}) \mid m, m', n, n', \ell, \ell' \in \mathbb{Z}\rangle\end{aligned}$$

where the homomorphism $\phi\colon \Gamma \to \Gamma$ is given by $\phi(a) = a^{\ell_{11}} b^{\ell_{21}}$ and $\phi(b) = a^{\ell_{12}} b^{\ell_{22}}$. Thus $[\Gamma,\Gamma] \subset \Lambda$, and $\Gamma/[\Gamma,\Gamma]$ is generated by $\bar{a} = a[\Gamma,\Gamma], \bar{b} = b[\Gamma,\Gamma], \bar{s} = s[\Gamma,\Gamma]$. Since $\det(I - A) \neq 0$, the rank of $[\Gamma,\Gamma]$ is 2. Thus $\langle \bar{a}, \bar{b}\rangle$ generates the torsion subgroup of $\Gamma/[\Gamma,\Gamma] = \langle \bar{a}, \bar{b}\rangle \oplus \langle \bar{s}\rangle$, and note also that the inverse image of $\langle \bar{a}, \bar{b}\rangle$ under the natural homomorphism $\pi\colon \Gamma \to \Gamma/[\Gamma,\Gamma]$ is the subgroup of Γ generated by a and b. Thus $\Lambda = \pi^{-1}(\langle \bar{a}, \bar{b}\rangle)$. Since $[\Gamma,\Gamma]$ is fully invariant in Γ, and since $\langle \bar{a}, \bar{b}\rangle$ is fully invariant in $\Gamma/[\Gamma,\Gamma]$, we see that Λ is fully invariant in Γ. □

Theorem 2.4 *Let*

$$\Gamma = \langle a, b, s \mid [a,b] = 1,\ sas^{-1} = a^{\ell_{11}} b^{\ell_{21}},\ sbs^{-1} = a^{\ell_{12}} b^{\ell_{22}} \rangle$$

be a lattice of Sol. *Then any homomorphism* θ *on* Γ *is one of the following:*

$$\begin{array}{rl} \text{Type (I)} & \theta(a) = a^u b^{\frac{\ell_{21}}{\ell_{12}} v},\ \theta(b) = a^v b^{u + \frac{\ell_{22}-\ell_{11}}{\ell_{12}} v},\ \theta(s) = a^p b^q s \\ \text{Type (II)} & \theta(a) = a^{-u} b^v,\ \theta(b) = a^{\frac{\ell_{11}-\ell_{22}}{\ell_{21}} u - \frac{\ell_{12}}{\ell_{21}} v} b^u,\ \theta(s) = a^p b^q s^{-1} \\ \text{Type (III)} & \theta(a) = 1,\ \theta(b) = 1,\ \theta(s) = a^p b^q s^m \ \textit{with } m \neq \pm 1 \end{array}$$

Proof Let $\theta\colon \Gamma \to \Gamma$ be a homomorphism on the lattices Γ of Sol. Theorem 2.3 implies that

$$\theta(a) = a^{m_{11}} b^{m_{21}}, \quad \theta(b) = a^{m_{12}} b^{m_{22}}, \quad \theta(s) = a^p b^q s^m$$

where $m_{11}, m_{12}, m_{21}, m_{22}, p, q, m \in \mathbb{Z}$.

Let $\phi\colon \Gamma \to \Gamma$ be the homomorphism given by $\phi(a) = a^{\ell_{11}} b^{\ell_{21}}$ and $\phi(b) = a^{\ell_{12}} b^{\ell_{22}}$ so that ϕ is the conjugation by s. Note that θ must preserve the relations $sas^{-1} = \phi(a)$ and $sbs^{-1} = \phi(b)$. Since

$$\theta(s)\theta(a)\theta(s)^{-1} = (a^p b^q s^m)(a^{m_{11}} b^{m_{21}})(a^p b^q s^m)^{-1} = \phi^m(a^{m_{11}} b^{m_{21}})$$

$$\theta(s)\theta(b)\theta(s)^{-1} = (a^p b^q s^m)(a^{m_{12}} b^{m_{22}})(a^p b^q s^m)^{-1} = \phi^m(a^{m_{12}} b^{m_{22}})$$

we must have $\phi^m(a^{m_{11}} b^{m_{21}}) = \theta(\phi(a))$, $\phi^m(a^{m_{12}} b^{m_{22}}) = \theta(\phi(b))$. These equations are equivalent to:

$$\begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}^m \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

By Lemma 2.1, there is a real invertible matrix

$$P = \begin{bmatrix} \frac{\ell_{11}-\ell_{22}+\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}} & \frac{\ell_{11}-\ell_{22}-\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}} \\ 1 & 1 \end{bmatrix}$$

such that $P^{-1}AP = \begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix}$, where $t_0 = \ln \frac{\ell_{11}+\ell_{22}+\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2}$ is a positive real number. Thus, we have:

$$\left(P \begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix} P^{-1} \right)^m \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \left(P \begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix} P^{-1} \right)$$

Let Q denote $P^{-1} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} P$. It follows that:

$$\begin{bmatrix} e^{mt_0} & 0 \\ 0 & e^{-mt_0} \end{bmatrix} Q = Q \begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix}$$

There are three possibilities:

(1) $m = 1$, $Q = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$;

(2) $m = -1$, $Q = \begin{bmatrix} 0 & \gamma \\ \delta & 0 \end{bmatrix}$;

(3) $m \neq \pm 1$, $Q = 0$. They yield our three types. □

From this Theorem, there is a one-to-one correspondence from the set of homomorphisms on the lattice $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ to the subset of $\mathbb{Z}^5$:

$$\begin{aligned} \Omega \;=\; & \{(u, v, p, q, 1) \in \mathbb{Z}^5 \mid \tfrac{\ell_{21}}{\ell_{12}} v,\ \tfrac{\ell_{22}-\ell_{11}}{\ell_{12}} v \in \mathbb{Z}\} \\ & \bigcup \{(u, v, p, q, -1) \in \mathbb{Z}^5 \mid \tfrac{\ell_{11}-\ell_{22}}{\ell_{21}} u - \tfrac{\ell_{12}}{\ell_{21}} v \in \mathbb{Z}\} \\ & \bigcup \{(0, 0, p, q, m) \in \mathbb{Z}^5 \mid m \neq \pm 1\} \end{aligned}$$

We shall write $\theta_{(u,v,p,q,m)}$ for the homomorphism on $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ corresponding to the integer vector (u, v, p, q, m), which is indicated in above theorem.

By a direct computation, we have:

Proposition 2.5 *Two homomorphisms $\theta_{(u,v,p,q,m)}$ and $\theta_{(u',v',p',q',m')}$ on $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ are conjugate if and only if one of the following holds for some integers k, k', k'':*

(1) $m = m' = 1$, $[u'\ v'] = [u\ v]A^k$, $\begin{bmatrix} p' \\ q' \end{bmatrix} = A^k \begin{bmatrix} p \\ q \end{bmatrix} + (I - A)\begin{bmatrix} k' \\ k'' \end{bmatrix}$

(2) $m = m' = -1$, $\begin{bmatrix} -u' \\ v' \end{bmatrix} = A^k \begin{bmatrix} -u \\ v \end{bmatrix}$, $\begin{bmatrix} p' \\ q' \end{bmatrix} = A^k \begin{bmatrix} p \\ q \end{bmatrix} + (I - A^{-1})\begin{bmatrix} k' \\ k'' \end{bmatrix}$

(3) $m = m' \neq \pm 1$, $u = u' = v = v' = 0$, $\begin{bmatrix} p' \\ q' \end{bmatrix} = A^k \begin{bmatrix} p \\ q \end{bmatrix} + (I - A^m)\begin{bmatrix} k' \\ k'' \end{bmatrix}$

3 Continuous maps on $\Gamma\backslash\mathrm{Sol}$

Let $f\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ be a continuous map. Up to a homotopy, we may assume that f preserves the base point $\epsilon\Gamma$ of $\Gamma\backslash\mathrm{Sol}$, where ϵ is the unit of Sol. Then f induces a homomorphism $f_\pi\colon \pi_1(\Gamma\backslash\mathrm{Sol}, \epsilon\Gamma) \to \pi_1(\Gamma\backslash\mathrm{Sol}, \epsilon\Gamma)$. Note that $\pi_1(\Gamma\backslash\mathrm{Sol}, \epsilon\Gamma)$ is identified with Γ in a classical way. We can extend this homomorphism on Γ in an obvious way to a homomorphism Θ on $\mathrm{Sol} = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$.

Since the Lie group homomorphism $\Theta\colon \mathrm{Sol} \to \mathrm{Sol}$ sends Γ into Γ, it induces a map Θ_Γ on the 3-solvmanifold $\Gamma\backslash\mathrm{Sol}$. Note that a homomorphism on Γ induced by the map Θ_Γ is the same as the homomorphism θ induced by $f\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$. Since a 3–solvmanifold $\Gamma\backslash\mathrm{Sol}$ is aspherical, we see that f is homotopic to Θ_Γ. In all, we have a classification theorem of self maps on $\Gamma\backslash\mathrm{Sol}$.

Theorem 3.1 *Let $\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ be a lattice of Sol, where $A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$. Then any continuous map on $\Gamma\backslash\mathrm{Sol}$ is homotopic to a map $\Theta_\Gamma\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ induced from a homomorphism $\Theta\colon \mathrm{Sol} \to \mathrm{Sol}$ with the restriction $\theta_{(u,v,p,q,m)}\colon \Gamma \to \Gamma$ in Theorem 2.4.*

Moreover, two homomorphisms on Sol induce homotopic maps on $\Gamma\backslash\mathrm{Sol}$ if and only if their restrictions on Γ satisfy one of relations in Proposition 2.5.

Since the invariants that we are going to deal with are all homotopy invariants, we will assume in what follows that *every continuous map on $\Gamma\backslash\mathrm{Sol}$ is induced by a homomorphism Θ on Sol*.

Definition 3.2 (Keppelmann and McCord [8]) Let $f\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ be a continuous map inducing a homomorphism $\theta\colon \Gamma \to \Gamma$. Then there is a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\hat{\theta}"] & \Gamma \arrow[r] \arrow[d, "\theta"] & \Gamma/\Lambda \arrow[r] \arrow[d, "\bar{\theta}"] & 1 \\
1 \arrow[r] & \Lambda \arrow[r] & \Gamma \arrow[r] & \Gamma/\Lambda \arrow[r] & 1
\end{tikzcd}$$

The induced homomorphisms $\bar{\theta}\colon \mathbb{Z} \to \mathbb{Z}$ and $\hat{\theta}\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is called a *linearization* of f, denoted $F = \begin{bmatrix} \hat{\theta} & 0 \\ 0 & \bar{\theta} \end{bmatrix}$.

By Theorem 2.4, we obtain:

Proposition 3.3 *Let $f\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ be a continuous map, where $\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with $A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$. Then any linearization matrix is an integer matrix of one of the following*

$$\begin{bmatrix} u & v & 0 \\ \frac{\ell_{21}}{\ell_{12}}v & u+\frac{\ell_{22}-\ell_{11}}{\ell_{12}}v & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -u & \frac{\ell_{11}-\ell_{22}}{\ell_{21}}u-\frac{\ell_{12}}{\ell_{21}}v & 0 \\ v & u & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix} \quad (m \neq \pm 1)$$

which are respectively conjugate to:

$$\text{(I)} \quad \begin{bmatrix} u-\frac{\ell_{11}-\ell_{22}-\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{12}}v & 0 & 0 \\ 0 & u-\frac{\ell_{11}-\ell_{22}+\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{12}}v & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{(II)} \quad \begin{bmatrix} 0 & u+\frac{\ell_{11}-\ell_{22}-\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}}v & 0 \\ u+\frac{\ell_{11}-\ell_{22}+\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}}v & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\text{(III)} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix} \quad (m \neq \pm 1)$$

Proof The three integer matrices are induced from Theorem 2.4. After a conjugation $T^{-1}(\cdot)T$ by the following matrix:

$$T=\begin{bmatrix}\frac{\ell_{11}-\ell_{22}+\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}} & \frac{\ell_{11}-\ell_{22}-\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}} & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

we arrive at our conclusion. $\square$

Note that homomorphisms of distinct types are not conjugate to each other. We shall say a map on $\Gamma\backslash\mathrm{Sol}$ is of type (I), (II) or (III) according to its homomorphism on Γ.

Corollary 3.4 *Each linearization matrix of a continuous map on $\Gamma\backslash\mathrm{Sol}$ is conjugate to one of the following*

$$(*)\qquad \text{(I)}\ \begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&1\end{bmatrix},\quad \text{(II)}\ \begin{bmatrix}0&\gamma&0\\ \delta&0&0\\0&0&-1\end{bmatrix},\quad \text{(III)}\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&\xi\end{bmatrix}\quad (\xi\neq\pm1),$$

where $\alpha+\beta,\ \alpha\beta,\ \gamma\delta,\ \xi\in\mathbb{Z}$. Moreover, $\alpha=0$ if and only if $\beta=0$, and $\gamma=0$ if and only if $\delta=0$.

Proof Since the trace $\ell_{11}+\ell_{12}$ of $\begin{bmatrix}\ell_{11}&\ell_{12}\\ \ell_{21}&\ell_{22}\end{bmatrix}$ is an integer greater than 2, $\sqrt{(\ell_{11}+\ell_{22})^2-4}$ must be an irrational number.

In type (I), if $\alpha=u-\frac{\ell_{11}-\ell_{22}-\sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{12}}v=0$, then v must be zero and hence $u=0$ because $u,v,\ell_{11},\ell_{12},\ell_{22}=0$ are all integers. It follows that $\beta=\alpha=0$. The converse is the same.

The proof of the relation between γ and δ in type (II) is similar. $\square$

In fact, $\Gamma\backslash\mathrm{Sol}$ is a torus bundle over the circle. It is known that any map on $\Gamma\backslash\mathrm{Sol}$ is homotopic to a fibre map with respect to the above bundle structure. Then $\overline{\theta}$ is the degree of the induced map on the base space S^1. By Proposition 2.5 and Theorem 3.1, we obtain:

Proposition 3.5 *Let f and f' be two continuous maps on $\Gamma\backslash\mathrm{Sol}$ induced by homomorphisms θ and θ' on Sol, with linearization matrices $\begin{bmatrix}\widehat{\theta}&0\\0&\overline{\theta}\end{bmatrix}$ and $\begin{bmatrix}\widehat{\theta}'&0\\0&\overline{\theta}'\end{bmatrix}$, where $\Gamma=\mathbb{Z}^2\rtimes_A\mathbb{Z}$. If f and f' are homotopic, then $\widehat{\theta}=A^k\widehat{\theta}'$ and $\overline{\theta}=\overline{\theta}'$.*

By this result, $\det\theta$ and $\overline{\theta}$ are both homotopy invariants. A map is of type (I), (II) or (III) according to $\overline{\theta}$ is 1, -1 or the others. It should be noticed that the $\widehat{\theta}$ is not a homotopy invariant. The following is an explicit example of maps on $\Gamma\backslash\mathrm{Sol}$ and their linearizations.

Example 3.6 Let $\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ be a lattice of Sol, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is an element of $\mathrm{SL}(2,\mathbb{Z})$ with trace 3 and eigenvalues $\frac{3\pm\sqrt{5}}{2}$. We take P to be $\begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$. Then:

$$P^{-1}AP = \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} e^{\ln\frac{3+\sqrt{5}}{2}} & 0 \\ 0 & e^{-\ln\frac{3+\sqrt{5}}{2}} \end{bmatrix}$$

Three generators of Γ, as elements in Sol, are:

$$a = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ 0 & 1 & 0 & \frac{\sqrt{5}+1}{2\sqrt{5}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$s = \begin{bmatrix} e^{\ln\frac{3+\sqrt{5}}{2}} & 0 & 0 & 0 \\ 0 & e^{-\ln\frac{3+\sqrt{5}}{2}} & 0 & 0 \\ 0 & 0 & 1 & \ln\frac{3+\sqrt{5}}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The linearization matrix F of any map on $\Gamma\backslash\mathrm{Sol}$ induced by a homomorphism $\Theta\colon \mathrm{Sol}\to\mathrm{Sol}$ is one of the following

$$\begin{bmatrix} u & v & 0 \\ v & u-v & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -u & -v+u & 0 \\ v & u & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix} \quad (m \neq \pm 1),$$

where u, v, m are integers. The corresponding matrices $\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}^{-1} F \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$ are respectively:

$$\begin{bmatrix} u-\frac{1-\sqrt{5}}{2}v & 0 & 0 \\ 0 & u-\frac{1+\sqrt{5}}{2}v & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & u+\frac{1-\sqrt{5}}{2}v & 0 \\ u+\frac{1+\sqrt{5}}{2}v & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix}$$

Remark 3.7 The entries α, β, γ, δ in the linearization F of type (I), (II) (see the list $(*)$) need not be integers. The above example shows in particular that the only conditions for which we can impose are:

$$\alpha+\beta,\ \alpha\beta,\ \gamma\delta,\ \zeta\in\mathbb{Z}$$

Therefore the conditions for the entries in [5, Proposition 4.3] should be changed as above. Notice that a linearization F in the form of $(*)$ with all entries integers is possible only when $v=0$ if F is of type (I) or (II).

The above example also shows that all homomorphisms on Sol need not preserve the lattice $\Gamma_A=\mathbb{Z}^2\rtimes_A\mathbb{Z}$ and hence need not induce continuous maps on $\Gamma_A\backslash\mathrm{Sol}$.

4 Lefschetz numbers and Nielsen type numbers

A connected solvable Lie group G is called of *type* (NR) (for "no roots") if the eigenvalues of $\mathrm{Ad}(x)\colon \mathfrak{G}\to\mathfrak{G}$ are always either equal to 1 or else they are not roots of unity. Solvable Lie groups of type (NR) were considered first in Keppelmann and McCord [8]. Since our solvmanifold $\Gamma\backslash\mathrm{Sol}$ is of type (NR), the following is a direct consequence of the main result, Theorem 3.1, of [8].

Proposition 4.1 *Let $f\colon \Gamma\backslash\mathrm{Sol}\to\Gamma\backslash\mathrm{Sol}$ be any continuous map on the solvmanifold $\Gamma\backslash\mathrm{Sol}$ with linearization matrix $F=\begin{bmatrix}\hat\theta & 0\\ 0 & \bar\theta\end{bmatrix}$. Then for all positive integers n:*

$$L(f^n)=\begin{cases}0 & \text{if } f \text{ is of type (I)}\\ (1-(-1)^n)(1+(\det\hat\theta)^n) & \text{if } f \text{ is of type (II)}\\ 1-\bar\theta^n & \text{if } f \text{ is of type (III)}\end{cases}$$

and $N(f^n)=|L(f^n)|$.

Proof First, we consider the case $n=1$. Since $\Gamma\backslash\mathrm{Sol}$ is of type (R) and hence of type (NR), by [8, Theorem 3.1], we have $L(f)=\det(I-F)$, and $N(f)=|L(f)|$. It is sufficient to consider the Lefschetz number.

If f is of type (I), then $\bar\theta=1$. It follows that $N(f)=L(f)=0$. If f is of type (II), then $\bar\theta=-1$. By Proposition 3.3, there is an invertible matrix P such that $P^{-1}\hat\theta P=\begin{bmatrix}0&\gamma\\ \delta&0\end{bmatrix}$. Hence,

$$L(f)=\det\left(I-\begin{bmatrix}\hat\theta & 0\\ 0 & -1\end{bmatrix}\right)$$

$$
\begin{aligned}
&= \det\left(I - \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \widehat{\theta} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det\left(I - \begin{bmatrix} 0 & \gamma & 0 \\ \delta & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right) \\
&= 2(1-\gamma\delta) = 2(1+\det\widehat{\theta}).
\end{aligned}
$$

If f is of type (III), we have $\widehat{\theta} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and hence:

$$
L(f) = \det\left(I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{\theta} \end{bmatrix}\right) = 1 - \overline{\theta}
$$

For general n, f^n has a linearization matrix $\begin{bmatrix} \widehat{\theta}^n & 0 \\ 0 & \overline{\theta}^n \end{bmatrix}$. The arguments are the same. It should be pointed out that f^n is of type (I) if f is of type (II) and n is even. □

Recall from Proposition 3.5 that the numbers $\det\widehat{\theta}$ and $\overline{\theta}$ are independent of the choice of linearizations of f.

Theorem 4.2 *Let $f\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ be a continuous map with a linearization matrix $F = \begin{bmatrix} \widehat{\theta} & 0 \\ 0 & \overline{\theta} \end{bmatrix}$. We have*

(1) *If f is of type (I), then:*

$$
NP_n(f) = N\Phi_n(f) = 0
$$

(2) *If f is of type (II), then:*

$$
\begin{aligned}
NP_n(f) &= \begin{cases} 2\sum_{m\mid n}\mu(m)|1+(\det\widehat{\theta})^m| & n \text{ is odd,} \\ 0 & n \text{ is even,} \end{cases} \\
N\Phi_n(f) &= 2|1+(\det\widehat{\theta})^{n_0}|, \quad n = 2^r n_0, \quad n_0 \text{ is odd.}
\end{aligned}
$$

(3) *If f is of type (III), then:*

$$
NP_n(f) = \sum_{m\mid n}\mu(m)|1-\overline{\theta}^m|, \quad N\Phi_n(f) = |1-\overline{\theta}^n|
$$

Proof In case (1), the Nielsen number $N(f^m)$ is 0 for all positive integer m. The map f has no essential periodic orbit classes of any period. It follows that $NP_n(f) = N\Phi_n(f) = 0$.

Since $\Gamma\backslash\mathrm{Sol}$ is a solvmanifold of type (NR), by Heath and Keppelmann [3, Theorems 1.2], we have $N\Phi_m(f) = N(f^m)$ and $NP_m(f) = \sum_{q|m} \mu(q) N(f^{\frac{m}{q}})$ for all $m|n$ provided $N(f^n) \neq 0$. This proves our case (2) for odd n and case (3), by Proposition 4.1.

In case (2) for even n, by Proposition 4.1, we have $N(f^n) = 0$. It follows that $NP_n(f) = 0$. Let $n = 2^r n_0$ for odd n_0. By Proposition 4.1 again, f^q has no essential fixed point class for every even factor q of n. Thus, the set of essential fixed point classes of f^q with $q \mid n$ is the same as the set of essential fixed point classes of f^q with $q \mid n_0$. Thus, $N\Phi_n(f) = N\Phi_{n_0}(f)$, which is just $N(f^{n_0}) = 2|1 + (\det \widehat{\theta})^{n_0}|$. $\square$

5 Homotopy minimal periods $\mathrm{HPer}(f)$

In this section, we shall present the homotopy minimal periods for all maps on 3–solvmanifolds.

Theorem 5.1 *Let $f\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ be a continuous map with a linearization matrix $F = \begin{bmatrix} \widehat{\theta} & 0 \\ 0 & \overline{\theta} \end{bmatrix}$. Then:*

$$\mathrm{HPer}(f) = \begin{cases} \{1\} & \text{if } \overline{\theta} = 0 \\ \varnothing & \text{if } \overline{\theta} = 1 \\ \varnothing & \text{if } \overline{\theta} = -1, \det \widehat{\theta} = -1 \\ \{1\} & \text{if } \overline{\theta} = -1, \det \widehat{\theta} = 0, 1 \\ \mathbb{N} - 2\mathbb{N} & \text{if } \overline{\theta} = -1, |\det \widehat{\theta}| > 1 \\ \mathbb{N} - \{2\} & \text{if } \overline{\theta} = -2 \\ \mathbb{N} & \text{otherwise} \end{cases}$$

Proof Since $NP_n(f)$ is a lower bound for the number of periodic points with least period n, by definition, we have $\mathrm{HPer}(f) \supset \{n \mid NP_n(f) \neq 0\}$. By Jezierski [4], we obtain $\mathrm{HPer}(f) = \{n \mid NP_n(f) \neq 0\}$. Note that $N(f^n) = |\det(I - F^n)|$ for any n. Using the same argument as in Jiang and Llibre [7, Theorem 3.4], we have:

$$\begin{aligned} &\{n \mid NP_n(f) \neq 0\} \\ &= \{n \mid N(f^n) \neq 0, N(f^n) \neq N(f^{\frac{n}{q}}) \text{ for all prime } q \mid n, q \leq n\} \end{aligned}$$

Table 1

$\mathrm{HPer}(f)$	Linearization matrix F is conjugate to
$\varnothing$	$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(\alpha+\beta,\ \alpha\beta \in \mathbb{Z})$ or $\begin{bmatrix} 0 & \gamma & 0 \\ \gamma^{-1} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $(\gamma \neq 0)$
$\{1\}$	$\begin{bmatrix} 0 & -\gamma & 0 \\ \gamma^{-1} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $(\gamma \neq 0)$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
$\mathbb{N}-\{2\}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$
$\mathbb{N}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \zeta \end{bmatrix}$ $(\zeta \neq -2,-1,0,1)$
$\mathbb{N}-2\mathbb{N}$	$\begin{bmatrix} 0 & \gamma & 0 \\ \delta & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $(\gamma\delta \neq -1,0,1,\ \gamma\delta \in \mathbb{Z})$

Hence:

$$(**)\qquad \mathrm{HPer}(f) = \{n \mid N(f^n) \neq 0, \\ N(f^n) \neq N(f^{\frac{n}{q}}) \text{ for all prime } q \mid n, q \leq n\}$$

(1) If $\overline{\theta} = 0$, then $N(f^n) = 1$ for all n by Proposition 4.1. Hence $\mathrm{HPer}(f) = \{1\}$.

(2) If $\overline{\theta} = 1$, then f is of type (I). We have $N(f^n) = 0$ for all n. This implies that $\mathrm{HPer}(f) = \varnothing$.

(3) If $\overline{\theta} = -1$, then f is of type (II). By Proposition 4.1, $N(f^n) = 0$ for even n. It follows that $\mathrm{HPer}(f)$ does not contain any even number, ie $\mathrm{HPer}(f) \subset \mathbb{N}-2\mathbb{N}$. Let us consider its subcases:

 (3.1) If $\det\widehat{\theta} = -1$, by Proposition 4.1, $N(f^n) = 0$ for all n. Thus, $\mathrm{HPer}(f) = \varnothing$.

 (3.2) If $\det\widehat{\theta} = 0$, by Proposition 4.1, $N(f^n) = 2$ for all odd n. Since $N(f) \neq 0$, we have $1 \in \mathrm{HPer}(f)$. By $(**)$, we have $n \notin \mathrm{HPer}(f)$ for all odd n with $n > 1$. Thus, $\mathrm{HPer}(f) = \{1\}$.

 (3.3) If $\det\widehat{\theta} = 1$, by Proposition 4.1, $N(f^n) = 4$ for all odd n. Since $N(f) \neq 0$, we have $1 \in \mathrm{HPer}(f)$. By $(**)$, we have $n \notin \mathrm{HPer}(f)$ for all odd n with $n > 1$. Thus, $\mathrm{HPer}(f) = \{1\}$.

Table 2

Type	$\Theta(\cdot)$	Linearization matrix	HPer$(\cdot)$
(I)	$\begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\varnothing$
(II)	$\begin{bmatrix} e^{-t} & 0 & 0 & \frac{3-\sqrt{5}}{2}y \\ 0 & e^t & 0 & \frac{3+\sqrt{5}}{2}x \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\varnothing$
(II)	$\begin{bmatrix} e^{-t} & 0 & 0 & \frac{1-\sqrt{5}}{2}y \\ 0 & e^t & 0 & \frac{1+\sqrt{5}}{2}x \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\{1\}$
(II)	$\begin{bmatrix} e^{-t} & 0 & 0 & (1-\sqrt{5})y \\ 0 & e^t & 0 & (1+\sqrt{5})x \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\mathbb{N}-2\mathbb{N}$
(III)	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\{1\}$
(III)	$\begin{bmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 1 & -2t \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$	$\mathbb{N}-\{2\}$
(III)	$\begin{bmatrix} e^{3t} & 0 & 0 & 0 \\ 0 & e^{-3t} & 0 & 0 \\ 0 & 0 & 1 & 3t \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	$\mathbb{N}$

(4) If $\overline{\theta} = -2$, by Proposition 4.1, we have $N(f^n) = |1-\overline{\theta}^n|$. Especially, $N(f) = N(f^2) = 3$. By $(**)$, $1 \in \mathrm{HPer}(f)$ but $2 \notin \mathrm{HPer}(f)$. Note that $N(f^{n+1}) > N(f^n)$ for all $n > 2$. By $(**)$, $\mathrm{HPer}(f) = \mathbb{N}-\{2\}$.

(5) In the final case, $\overline{\theta} \neq -2, -1, 0, 1$. We still have $N(f^n) = |1-\overline{\theta}^n|$. Notice that $N(f^{n+1}) > N(f^n)$ for all n. By $(**)$, $\mathrm{HPer}(f) = \mathbb{N}$. □

In Table 1 we tabulate this result according to the Linearization matrix.

In fact, for each subset $S \subset \mathbb{N}$ appearing as $\mathrm{HPer}(f)$ and each form of linearization F listed above, there exists a self-map $f\colon \Gamma\backslash\mathrm{Sol} \to \Gamma\backslash\mathrm{Sol}$ such that $\mathrm{HPer}(f) = S$. Examples are given next.

Example 5.2 Fix the lattice $\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ of Sol, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. We list the continuous maps on $\Gamma\backslash\mathrm{Sol}$ by the explicit lifting homomorphism $\Theta\colon \mathrm{Sol} \to \mathrm{Sol}$, which are determined by $\Theta(\cdot) = \Theta\left(\begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$

Hence for each subset $S \subset \mathbb{N}$ appearing as $\mathrm{HPer}(f)$ and each form of linearization F listed in Table 2 can be realized.

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Department of Mathematics, Sogang University, Seoul 121–742, KOREA

Department of Mathematics, Institute of Mathematics and Interdisciplinary Science
Capital Normal University, Beijing 100037, CHINA

jlee@sogang.ac.kr, zhaoxve@mail.cnu.edu.cn

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