The cobordism class of the multiple points of immersions

GÁBOR BRAUN

Using generating functions, we derive a multiple point formula for every generic immersion $f\colon M^m \looparrowright N^n$ between even dimensional oriented manifolds. This produces explicit formulas for the signature and Pontrjagin numbers of the multiple point manifolds. The formulas take a particular simple form in many special cases, for example, when f is nullhomotopic, we recover Szűcs's formulas in [3]. They also include Hirzebruch's virtual signature formula [1, 9.3(4')].

57R20, 57R42; 57R75, 16W60

1 Introduction

Let $f\colon M^m \looparrowright N^n$ be a generic immersion between oriented compact even dimensional smooth manifolds.

Aims Let $M^{\times k}$ be the cartesian product of k copies of M and $f^{\times k}\colon M^{\times k} \to N^{\times k}$ be the map induced by f between the products. Let

$$\Delta \coloneqq \{(x, x, \dots, x) : x \in N\} \subseteq N^{\times k}$$

be the (narrow) diagonal in $N^{\times k}$. Then the k–tuples of the multiple points of f form the manifold

$$\widetilde{\Delta}_k(f) \coloneqq \{(x_1, \dots, x_k) \in (f^{\times k})^{-1}(\Delta) : x_i \neq x_j \text{ if } i \neq j\}. \tag{1–1}$$

Note that by permuting the coordinates, the symmetric group S_k acts on $\widetilde{\Delta}_k(f)$. Factoring out with the action leads to the *manifold of k–tuple points* of f:

$$\Delta_k(f) \coloneqq \widetilde{\Delta}_k(f)/S_k.$$

If the action of S_k on $\widetilde{\Delta}_k(f)$ preserves orientation (ie $n-m$ is even or $k=1$), then the factor $\Delta_k(f)$ is naturally oriented.

We are going to express the signature and the characteristic numbers of the manifold of k–tuple points of f in terms of cohomological invariants of M, N and f (formulas (2–4), (2–6) and (2–10)).

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Special cases Our formulas are particularly simple in many special cases as shown in Subsection 2.3. For example, if N is a sphere, we recover Szűcs's formulas (here (2–20) and (2–21)) from [3, Theorems 4 and 5].

Our formula also generalizes Hirzebruch's virtual signature formula, which we recall now.

Let $f\colon M \to N$ be a union of embeddings of codimension 2 manifolds $M_1, \dots, M_k$ into N. Recall that every codimension 2 embedded manifold is the set of zeros of a transversal section of a 2 dimensional vector bundle. Let V_i be such a bundle for M_i, and let us denote the Euler class of this bundle by e_i. The manifold $\Delta_k(f)$ of k–tuple points of f is exactly the intersection of the submanifolds $M_1, \dots, M_k$. Hirzebruch's virtual signature formula from [1, 9.3(4')] states that its signature is

$$\sigma(\Delta_k(f)) = \left\langle L(N) \prod_{i=1}^{k} e_i L(V_i)^{-1}, [N] \right\rangle, \tag{1–2}$$

where $L(N)$ is the Hirzebruch class of the tangent bundle of N and, similarly, $L(V_i)$ is the Hirzebruch class of V_i.

Hirzebruch uses "index" for "signature" and also uses a different notation. We can get back Hirzebruch's original formula by replacing in the above formula σ with τ, the number k with r, the manifold $\Delta_k(f)$ with V^{n-2r}, the manifold N with M, the expression $\langle -, [N] \rangle$ with $\varkappa^n[-]$, the class $L(N)$ with $\sum_{i=0}^{\infty} L_i(p_1(M^n), \dots, p_i(M^n))$, and $e_i L(V_i)^{-1}$ with $\tanh v_i$. All but the last replacement are just changes in notation. To justify the last replacement, note that the dual class v_i of M_i is the Euler class e_i of V_i and hence

$$e_i L(V_i)^{-1} = e_i \frac{\tanh e_i}{e_i} = \tanh e_i = \tanh v_i. \tag{1–3}$$

Our Corollary 2.3 generalizes Hirzebruch's formula to generic immersions with even codimension by introducing the cohomology class $B_k(f)$ in (2–12) such that for any immersion $f\colon M \looparrowright N$, we get:

$$\sigma(\Delta_k(f)) = \frac{1}{k!} \langle L(N) B_k(f), [N] \rangle. \tag{1–4}$$

When M is the disjoint union of manifolds M_i, each of which is embedded by f, then $B_k(f)$ factors into a product of some classes of the M_i (see Theorem 2.4). If all M_i has codimension 2, then this formula reduces to Hirzebruch's (1–2) (see the discussion after Theorem 2.4).

Equation (1–4) has a version in (2–10) using the cohomology of M instead of the cohomology of N. Szűcs obtained this version in the special case when N is a Euclidean space in [3, Theorem 4], which we reproduce as (2–20).

The main idea Using the definition in (1–1), let $i_k\colon \widetilde{\Delta}_k(f) \to M^{\times k}$ denote the inclusion and $j_k\colon \widetilde{\Delta}_k(f) \to M$ the projection to the first coordinate. What we essentially do is deduce an explicit formula for $j_{k!}i_k^*$ (Theorem 2.2). The formulas for the characteristic numbers are applications of this formula.

How do we compute $j_{k!}i_k^*$? First, in Section 4, we apply Ronga's Clean Intersection Theorem [2, Proposition 2.2] (repeated here as Theorem 4.2) to obtain the recursion (4–1) for this map. Then in Section 5 we interpret this recursion using generating functions, which produces an easy way to solve it.

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2 Main results

2.1 Notation

We fix our notation for the rest of the paper. Let $f\colon M^m \looparrowright N^n$ be a generic immersion of compact oriented smooth manifolds with even codimension. The dimension of the components of the manifolds need not be the same. Let m be the dimension function of M which maps every component of M to its dimension, and, similarly, let n be the dimension function of N. Now, f having even codimension means that for every $x \in M$ the number $n(f(x)) - m(x)$ is even.

Cohomology classes We fix notation for some cohomology classes. Let ν be the normal bundle of f and let $e = e(\nu)$ be the Euler class of ν. Let $L(\xi)$ be the Hirzebruch class of the bundle ξ. For a manifold X, we write $L(X)$ for $L(TX)$, the Hirzebruch class of the tangent bundle of X. Similarly, we define $P(\xi)$ to be the total Pontrjagin class of ξ, and let $P(X) \coloneqq P(TX)$ be the total Pontrjagin class of the manifold X. We use similar notation for Chern classes with c instead of P.

We need further notation from [3]. Let $J = (j_1, j_2, \dots, j_l)$ be a sequence of non-negative integers. For a total cohomology class a, let a_i be its i dimensional part, and let $a_J \coloneqq a_{j_1} a_{j_2} \cdots a_{j_l}$. For example, if X is a compact manifold, then $p_J[X] \coloneqq$

$\langle P(X)_J, [X]\rangle$ is the Pontrjagin number of X corresponding to J (assuming that $\sum j_i$ is the dimension of X).

We write cup products either as ordinary products or with the symbol $\cup$ for the operation. We use $\times$ for cross products.

Equivalence relations Equivalence relations naturally arise in our treatment; see Section 4 or Szűcs [3], which is the starting point of our investigation.

Let $\mathrm{Eq}(k)$ be the set of all equivalence relations on $\{1,\dots,k\}$. We will think of an equivalence relation α as the set of its equivalence classes, thus $\Theta \in \alpha$ will denote that Θ is an equivalence class of α. Every $\Theta \in \alpha$ is a subset of $\{1,\dots,k\}$ and so Θ is ordered by the usual ordering on integers. Moreover, there is an ordering among the equivalence classes themselves: $\Theta_1 < \Theta_2$ if and only if the smallest element of Θ_1 is smaller than the smallest element of Θ_2. Hence every equivalence relation α is the ordered set of its equivalence classes.

Let $\alpha[i]$ be the equivalence class of α containing i. In particular, $\alpha[1]$ is the smallest equivalence class of α.

We will let $0 = 0(k)$ denote the trivial equivalence relation, under which different elements are not equivalent. Let $1 = 1(k)$ denote the universal equivalence relation, under which all elements are equivalent.

Whenever we write $\prod_{\Theta\in\alpha}$ we assume that the terms of the product appear in the order determined by the ordering of α. Similar remark applies to $\prod_{i\in\Theta}$ and other products with ordered index set.

We will denote by $|X|$ the number of elements of the set X. For example, $|\alpha|$ is the number of equivalence classes of α.

Maps We list the maps between topological spaces we will use in our formulas. These are variants of the diagonal map and the graph of $f^{\times k}$. Below k is a positive integer and α is an equivalence relation on $\{1,\dots,k\}$. Moreover, x_i denotes the i–th coordinate of x. For an element x of $M^{\times|\alpha|}$ and $\Theta \in \alpha$, let x_Θ denote the Θ–coordinate of x. We define x_Θ similarly for $M \times N^{\times(|\alpha|-1)}$, where M is the $\alpha[1]$–coordinate and the other coordinates are identified with the other classes of α.

$$\begin{aligned}
\Delta^{1(k)} = \Delta_k\colon M \to M^{\times k}, &\qquad \Delta_k(x)_i \coloneqq x \\
\Delta^{\alpha}\colon M^{\times|\alpha|} \to M^{\times k}, &\qquad \Delta^{\alpha}(x)_i \coloneqq x_{\alpha[i]} \\
\Gamma^{1(k)} = \Gamma_k\colon M \to M \times N^{\times(k-1)}, &\qquad \Gamma_k(x)_i \coloneqq \begin{cases} x & \text{if } i = 1 \\ f(x) & \text{if } i > 1 \end{cases} \\
\Gamma^{\alpha}\colon M \times N^{\times(|\alpha|-1)} \to M \times N^{\times(k-1)}, &\qquad \Gamma^{\alpha}(x)_i \coloneqq \begin{cases} f(x_{\alpha[i]}) & \text{if } \alpha[i] = \alpha[1] \\ & \text{and } i > 1 \\ x_{\alpha[i]} & \text{otherwise} \end{cases}
\end{aligned}$$

Note that if $M = N$ and f is the identity, then $\Gamma^{\alpha} = \Delta^{\alpha}$. Occasionally, we will use Δ_k for manifolds other than M. The context will always make this clear.

We need two maps from the multiple point manifold $\widetilde{\Delta}_k(f)$ defined in (1–1): the canonical inclusion $i_k\colon \widetilde{\Delta}_k(f) \to M^{\times k}$ and the projection $j_k\colon \widetilde{\Delta}_k(f) \to M$ to the first coordinate of $M^{\times k}$. When we want to include f in the notation, we write $i_k^{(f)}$ and $j_k^{(f)}$.

Abbreviation To make formulas more readable, we introduce a shorthand notation for a frequent constant:

$$A_k \coloneqq (-1)^{k-1}(k-1)! \tag{2–1}$$

2.2 The general formula

Now we state our main results, which will be proved in later sections.

Let $f\colon M^m \looparrowright N^n$ be a generic immersion of oriented compact manifolds with even codimension. We start with the general formula for signature and characteristic numbers:

Theorem 2.1 *The signature and the Pontrjagin numbers of $\Delta_k(f)$ are*

$$\sigma(\Delta_k(f)) = \frac{1}{k!}\Big\langle j_{k!} i_k^* \Big(L(M) \times \bigtimes_{i=2}^{k} L(\nu)^{-1} \Big), [M] \Big\rangle \tag{2–2}$$

$$= \frac{1}{k!}\Big\langle L(N) f_! j_{k!} i_k^* \Big(\bigtimes_{i=1}^{k} L(\nu)^{-1} \Big), [N] \Big\rangle, \tag{2–3}$$

$$p_J[\Delta_k(f)] = \frac{1}{k!}\Big\langle j_{k!} i_k^* \Big(P(M) \times \bigtimes_{i=2}^{k} P(\nu)^{-1} \Big)_J, [M] \Big\rangle. \tag{2–4}$$

If M and N are almost complex, then we have a similar formula for Chern numbers:

$$c_J[\Delta_k(f)] = \frac{1}{k!}\Big\langle j_{k!} i_k^* \Big(C(M) \times \bigtimes_{i=2}^{k} C(\nu)^{-1}\Big)_J, [M]\Big\rangle. \tag{2–5}$$

Formulas (2–4) and (2–5) can also be written in the form analogous to (2–3).

To make these formulas explicit, we have to compute $j_{k!} i_k^*$ (or $f_! j_{k!} i_k^*$). Recall that $e = e(\nu)$ is the Euler class of the normal bundle of f.

Theorem 2.2 *For every cohomology class $x \in H^*(M^{\times k})$*

$$j_{k!}(i_k^*(x)) = \sum_{\alpha \in \mathrm{Eq}(k)} \Gamma_{|\alpha|}^* (1 \times f^{\times(|\alpha|-1)})_! \Big(\Big(\bigtimes_{\Theta \in \alpha} A_{|\Theta|} e^{|\Theta|-1}\Big) \cdot \Delta^{\alpha *}(x)\Big), \tag{2–6}$$

$$f_!(j_{k!}(i_k^*(x))) = \sum_{\alpha \in \mathrm{Eq}(k)} \Delta_{|\alpha|}^* f_!^{\times|\alpha|} \Big(\Big(\bigtimes_{\Theta \in \alpha} A_{|\Theta|} e^{|\Theta|-1}\Big) \cdot \Delta^{\alpha *}(x)\Big). \tag{2–7}$$

In particular, if $x_1, \dots, x_k \in H^{2*}(M)$ (ie the x_i have even dimension), then

$$\begin{aligned} j_{k!} i_k^*(x_1 \times \cdots \times x_k) = \sum_{\alpha \in \mathrm{Eq}(k)} & \Big(A_{|\alpha[1]|} e^{|\alpha[1]|-1} \prod_{i \in \alpha[1]} x_i\Big) \\ & \cdot \prod_{\substack{\Theta \in \alpha \\ \Theta > \alpha[1]}} \Big(A_{|\Theta|} f^* f_! \Big(e^{|\Theta|-1} \prod_{i \in \Theta} x_i\Big)\Big), \end{aligned} \tag{2–8}$$

$$f_! j_{k!} i_k^*(x_1 \times \cdots \times x_k) = \sum_{\alpha \in \mathrm{Eq}(k)} \prod_{\Theta \in \alpha} \Big(A_{|\Theta|} f_! \Big(e^{|\Theta|-1} \prod_{i \in \Theta} x_i\Big)\Big). \tag{2–9}$$

The two theorems together provide explicit formulas for the characteristic numbers and signature. We state only the signature formula:

Corollary 2.3 *The signature of the k–fold intersection manifold of f is*

$$\sigma(\Delta_k(f)) = \sum_{l+\sum_{i=1}^{k-1} il_i=k} \frac{(-1)^{k-1-\sum_{i=1}^{k-1} l_i}}{k\prod_{i=1}^{k-1} i^{l_i}\cdot l_i!} \cdot\Big\langle L(M)e^{l-1}L(\nu)^{1-l}\prod_{i=1}^{k-1}\Big(f^*f_!\Big(e^{i-1}L(\nu)^{-i}\Big)\Big)^{l_i},[M]\Big\rangle \tag{2–10}$$
$$= \sum_{\sum_{i=1}^{k} il_i=k} \frac{(-1)^{k-\sum_{i=1}^{k} l_i}}{\prod_{i=1}^{k} i^{l_i}\cdot l_i!}\Big\langle L(N)\prod_{i=1}^{k}\Big(f_!\Big(e^{i-1}L(\nu)^{-i}\Big)\Big)^{l_i},[N]\Big\rangle$$

where the indices l_i run through the non-negative integers and the index l runs through the positive integers.

So the general signature formula is similar to Hirzebruch's formula (1–2):

$$\sigma(\Delta_k(f)) = \frac{1}{k!}\langle L(N)B_k(f),[N]\rangle, \tag{2–11}$$

where $B_k(f)$ generalizes the product in (1–2):

$$B_k(f) := f_! j_{k!} i_k^*\Big(\bigtimes_{i=1}^{k} L(\nu)^{-1}\Big) = \sum_{\sum_{i=1}^{k} il_i=k} \frac{k!(-1)^{k-\sum_{i=1}^{k} l_i}}{\prod_{i=1}^{k} i^{l_i}\cdot l_i!}\prod_{i=1}^{k}\Big(f_!\Big(e^{i-1}L(\nu)^{-i}\Big)\Big)^{l_i}. \tag{2–12}$$

We now examine how this formula reduces to Hirzebruch's (1–2), ie the case when M has several components.

Theorem 2.4 *Let M be the disjoint union of manifolds $M_1,\dots,M_l$. Let f_i denote the restriction of f to M_i. Then we can compute $B_k(f)$ as*

$$B_k(f) = \sum_{k_1+\dots+k_l=k} \frac{k!}{k_1!\cdots k_l!}\prod_{i=1}^{l} B_{k_i}(f_i), \tag{2–13}$$

where the k_i run through the non-negative integers, and $B_0(f_i) := 0$ by definition.

In particular, if the M_i are embedded manifolds, then

$$B_l(f) = \prod_{i=1}^{l} B_1(f_i) = \prod_{i=1}^{l} f_{i!}(L(\nu_i)^{-1}) \tag{2–14}$$

where ν_i is the normal bundle of f_i.

Let V_i be a 2 dimensional vector bundle over N. Let M_i be the set of zeros of a transversal section of V_i. Let f_i be the inclusion of M_i into N. Then the normal bundle of f_i is the restriction of V_i to M_i, which means $\nu_i = f_i^*(V_i)$. Thus we have

$$f_{i!}(L(\nu_i)^{-1}) = f_{i!}(f_i^*(L(V_i)^{-1})) = e_i L(V_i)^{-1} \tag{2–15}$$

where e_i is the Euler class of V_i. Finally, Hirzebruch's formula (1–2) is obtained by combining (2–11), (2–14) and (2–15).

Theorem 2.2 will be proved in Section 4 and Section 5. The other results will be proved in Section 6.

2.3 Special cases

We present some special cases when the formulas above reduce to a product. We leave the proofs to Section 6.

e, $L(\nu)$ comes from N If the cohomology classes e and $L(\nu)$ are in the image of f^*, then the formulas simplify:

$$j_{k!} i_k^* (f^{\times k})^*(y) = \Delta_k^* (f^{\times k})^*(y) \prod_{i=1}^{k-1} (f^* f_!(1) - i e) \tag{2–16}$$

leading to

$$\sigma(\Delta_k(f)) = \frac{1}{k!} \left\langle L(M) L(\nu)^{-(k-1)} \prod_{i=1}^{k-1} (f^* f_!(1) - i e), [M] \right\rangle \tag{2–17}$$

$$p_J[\Delta_k(f)] = \frac{1}{k!} \left\langle \left(P(M) P(\nu)^{-(k-1)} \right)_J \prod_{i=1}^{k-1} (f^* f_!(1) - i e), [M] \right\rangle \tag{2–18}$$

$$c_J[\Delta_k(f)] = \frac{1}{k!} \left\langle \left(C(M) C(\nu)^{-(k-1)} \right)_J \prod_{i=1}^{k-1} (f^* f_!(1) - i e), [M] \right\rangle. \tag{2–19}$$

$e = 0$ In case $e = 0$, the only nonzero summand in (2–6) corresponds to $\alpha = 0$.

$$j_{k!} \circ i_k^* = \Gamma_k^* \circ (1 \times f^{\times (k-1)})_!$$

leading to

$$
\begin{aligned}
\sigma(\Delta_k(f)) &= \frac{1}{k!}\Big\langle L(M)(f^* f_! L(\nu)^{-1})^k, [M]\Big\rangle \\
&= \frac{1}{k!}\Big\langle L(N)\big(f_!(L(\nu)^{-1})\big)^k, [N]\Big\rangle \\
p_J[\Delta_k(f)] &= \frac{1}{k!}\Big\langle \Delta_k^* f_!^{\times k}\Big(P(M)\times \bigtimes_{i=2}^{k} P(\nu)^{-1}\Big)_J, [N]\Big\rangle \\
c_J[\Delta_k(f)] &= \frac{1}{k!}\Big\langle \Delta_k^* f_!^{\times k}\Big(C(M)\times \bigtimes_{i=2}^{k} C(\nu)^{-1}\Big)_J, [N]\Big\rangle.
\end{aligned}
$$

$\boldsymbol{f^* f_! = 0}$ If $f^* f_! = 0$ (this is the case if f is nullhomotopic and $n > 0$) then the only nonzero summand in (2–8) corresponds to $\alpha = 1$. Hence the formulas reduce to simple products:

$$
j_{k!} i_k^*(x_1 \times \cdots \times x_k) = A_k e^{k-1}(x_1 \cdots x_k) = (-1)^{k-1}(k-1)! e^{k-1} x_1 \cdots x_k
$$

leading to

$$
\begin{aligned}
\sigma(\Delta_k(f)) &= \frac{(-1)^{k-1}}{k}\Big\langle e^{k-1} L(M) L(\nu)^{1-k}, [M]\Big\rangle \\
p_J[\Delta_k(f)] &= \frac{(-1)^{k-1}}{k}\Big\langle e^{k-1}\Big(P(M)P(\nu)^{1-k}\Big)_J, [M]\Big\rangle \\
c_J[\Delta_k(f)] &= \frac{(-1)^{k-1}}{k}\Big\langle e^{k-1}\Big(C(M)C(\nu)^{1-k}\Big)_J, [M]\Big\rangle.
\end{aligned}
$$

In particular, if f is nullhomotopic, then $f^*(TN)$ is a trivial bundle and hence $L(\nu)^{-1} = L(M)$ and $P(\nu)^{-1} = P(M)$, so the formulas reduce to those in [3, Theorems 4 and 5] (up to minor notational differences), where these are claimed only when N is $\mathbb{R}^n$ (which we can replace by the sphere S^n if we want N to be compact):

$$
\sigma(\Delta_k(f)) = \frac{(-1)^{k-1}}{k}\Big\langle e^{k-1} L(M)^k, [M]\Big\rangle \tag{2–20}
$$

$$
p_J[\Delta_k(f)] = \frac{(-1)^{k-1}}{k}\Big\langle e^{k-1}(P(M)^k)_J, [M]\Big\rangle. \tag{2–21}
$$

This also corrects a typo (missing sign) in [3, Theorem 4].

3 Sketch of proof

From a technical point of view, the main result is formula (2–6). The other results easily follow from it, as we show in Section 6. In this section, we sketch the proof of formula (2–6).

Briefly, the proof splits into two parts: The first part (Section 4) studies the geometric situation to obtain formula (4–1). The second part is an algebraic rewrite of this formula to achieve our goal: (2–6).

In more detail, as in [3], the geometric idea is the description of the preimage of the (narrow) diagonal $\Delta_k(N)$ under $f^{\times k}$. Its components are parametrized by the equivalence relations on k elements. The component belonging to an equivalence relation with l equivalence classes is canonically isomorphic to the manifold of l–tuple points of f. Ronga's Clean Intersection Theorem translates this geometric decomposition into formula (4–1).

The algebraic manipulation of (4–1) is guided by an interpretation of this formula as a power series equation: $G = F \circ H$, where F collects the unknowns $j_{k!} i_k^*$. The power series H turns out to be invertible so we can rewrite the formula as $F = G \circ H^{-1}$, which is just (2–6).

4 From topology to algebra

In this section we derive the recursion (4–1) on $j_{k!} i_k^*$.

Subcartesian diagram We will use Ronga's theorem on clean intersections [2, Proposition 2.2]. We recall the notion of clean intersection:

Definition 4.1 Two smooth functions $f\colon A \to M$ and $g\colon B \to M$ *intersect cleanly* if for every $a \in A$ and $b \in B$ such that $f(a) = g(b)$, there are local maps around a of A, around b of B and around $f(a) = g(b)$ of M such that both f and g are linear in these maps. It follows that

$$Z := \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

is a submanifold of $A \times B$, which we shall call the *clean intersection* of f and g. The projections of Z to A and B form a so called *subcartesian diagram* together with f and g, see Figure 1. The *excess vector bundle* is the bundle $TM/(TA + TB)$ over Z, where we have omitted the obvious pullback functions in the notation as an abuse of language.

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & M \end{array}$$

Figure 1: A subcartesian diagram

Ronga's theorem states a cohomological identity for subcartesian diagrams:

Theorem 4.2 (Clean Intersection Theorem [2, Proposition 2.2]) *For every subcartesian diagram we have, using the notation of the above definition:*

$$g^*(f_!(x)) = \beta_!(e \cdot \alpha^*(x)) \quad (x \in H^*(A)),$$

where e is the Euler class of the excess bundle.

Main argument In this paragraph we apply the Clean Intersection Theorem to the maps $1 \times f^{\times(k-1)}$ and Γ_k to obtain (4–1).

First, the clean intersection of the maps is the preimage of the image of Γ_k under $1 \times f^{\times(k-1)}$, or, equivalently, the preimage of the diagonal $\Delta = \Delta^k(N)$ of $N^{\times k}$ under $f^{\times k}$. As in [3], this preimage is the disjoint union of closed submanifolds

$$M_\alpha := \big\{(x_1, \dots, x_k) \in (f^{\times k})^{-1}(\Delta) \big| x_i = x_j \iff i \;\alpha\; j\big\},$$

where α runs over the equivalence relations on $\{1, \dots, k\}$. The manifold M_α is canonically isomorphic to $\widetilde{\Delta}_{|\alpha|}(f)$, and its inclusion into $M^{\times k}$ factors as

$$M_\alpha \xrightarrow{i_{|\alpha|}} M^{\times|\alpha|} \xrightarrow{\Delta^\alpha} M^{\times k}.$$

Among these, $M_{0(k)} = \widetilde{\Delta}_k(f)$ is the manifold whose characteristic numbers we want to compute.

Second, we determine the maps in the subcartesian diagram. The map from M_α into $M^{\times k}$ is just the canonical embedding. The map from M_α to the factor M of Γ_k is the projection to the first coordinate.

All in all, the subcartesian diagram of $1 \times f^{\times(k-1)}$ and Γ_k looks as the outer square of Figure 2. The inner square just explains some maps of the outer square.

We are going to compute the excess vector bundle. We will omit the pullback maps to simplify our notation since the context will always make it clear which map is missing.

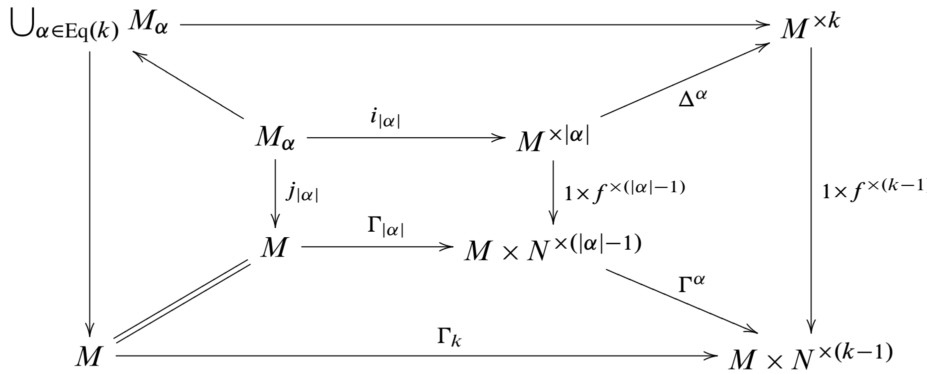

Figure 2: Multiple point manifolds of a generic immersion (top left) fitted into a subcartesian diagram (the outer square). The inner square is just for giving the maps of the outer square.

We fix an equivalence relation α on $\{1,\dots,k\}$ and determine the excess vector bundle restricted to M_α. Therefore we consider all vector bundles pulled back to M_α.

Recall that the excess vector bundle is the factor of $T(M\times N^{\times(k-1)})$ by $TM^{\times k}$ and TM. Notice that the inner square of the diagram is a transverse intersection because f is generic, so the sum of $TM^{\times|\alpha|}$ (which is contained in $TM^{\times k}$) and TM is $T(M\times N^{\times(|\alpha|-1)})$. Thus the excess vector bundle is the factor of $T(M\times N^{\times(k-1)})$ by $TM^{\times k}$ and $T(M\times N^{\times(|\alpha|-1)})$.

At this point, we notice that the factor makes sense even on $M^{\times|\alpha|}$. Hence from now on we consider all vector bundles pulled back to $M^{\times|\alpha|}$.

Let Θ_i denote the i–th equivalence class of α. We will write $l\xi$ for the direct sum of l copies of a vector bundle ξ.

The embeddings of $TM^{\times k}$ and $T(M\times N^{\times(|\alpha|-1)})$ into $T(M\times N^{\times(k-1)})$ factor into the components of $M^{\times|\alpha|}$. For $i>1$ on the i–th component we have $|\Theta_i|TM$ and TN embedded into $|\Theta_i|TN$. The bundle TN is embedded diagonally, and the embedding of $|\Theta_i|TM$ is induced by f. Hence the factor is $(|\Theta_i|-1)\nu$, where ν is the normal bundle of f. The case of $i=1$ is similar.

Thus, the factor on $M^{\times|\alpha|}$ is $(|\Theta_1|-1)\nu\times\cdots\times(|\Theta_l|-1)\nu$ where $l:=|\alpha|$. Last, the excess vector bundle restricted to M_α is the restriction of this bundle from $M^{\times|\alpha|}$ to M_α.

Finally, applying the Clean Intersection Theorem (Theorem 4.2) to the diagram, one obtains:

$$\Gamma_k^*(1\times f^{\times(k-1)})_!(x)=\sum_{\alpha\in\mathrm{Eq}(k)} j_{|\alpha|!}\left(i_{|\alpha|}^*\left(\bigtimes_{\Theta\in\alpha} e^{|\Theta|-1}\right)\cdot i_{|\alpha|}^*\Delta^{\alpha *}x\right). \tag{4–1}$$

Recall that $e=e(\nu)$ is the Euler class of ν.

5 The power series identity

Now we have the recursion formula (4–1) on $j_{k!}i_k^*$. To make this recurrence relation transparent, we interpret it as a power series equality. Then our theorems will be reduced to routine calculations.

5.1 Power series

The general definition *Power series* are morphisms of the following category. Objects are sequences $A=(A_k)_{k=1}^\infty$ of modules. Let $A=(A_k)_{k=1}^\infty$ and $B=(B_k)_{k=1}^\infty$ be two sequences of modules. A morphism or *power series* F from A to B is a collection of homomorphisms $(F_\alpha\colon A_k\to B_{|\alpha|}\mid \alpha\in\mathrm{Eq}(k), k=1,\dots,\infty)$. Given two power series $F\colon A\to B$ and $G\colon B\to C$, we define their *composite* $G\circ F$ by

$$(G\circ F)_\alpha=\sum_{\beta\le\alpha} G_{\alpha/\beta}\circ F_\beta. \tag{5–1}$$

Here β and α are equivalence relations on the same set. The notation $\beta\le\alpha$ means that every class of the equivalence relation α is a union of some classes of the equivalence relation β (this is the usual ordering of equivalence relations). Thus α induces an equivalence relation α/β on the classes of β: namely, those classes of β are equivalent which belong to the same class of α. There is a unique identification between the ordered set of classes of β and the ordered set $\{1,\dots,|\beta|\}$. Thus we may regard α/β as an equivalence relation on the latter set. This explains the notation in the above formula.

We leave the easy verifications of the axioms of category to the reader. The unit elements are of the form $E\colon A\to A$ defined as $E_\alpha=1$ if $\alpha=0\in\mathrm{Eq}(k)$ for some k, and $E_\alpha=0$ for all other α.

Classical examples Now we shall see that this definition is an extension of the usual definition of formal power series. Classically, given an analytic function $f\colon U\to V$ between real vector spaces, its (exponential) power series is the sequence of its derivatives at a point u of U. Let us denote by $(f_k\colon U^{\times k} \to V : k = 1, \dots, \infty)$ the k–th derivative of f at u, it is a (symmetric) k–linear map. In our setting, this corresponds to $F\colon (U^{\otimes k})_{k=1}^{\infty} \to (V^{\otimes k})_{k=1}^{\infty}$ defined by

$$F_\alpha(u_1 \otimes \cdots \otimes u_k) := \bigotimes_{\Theta \in \alpha} f_{|\Theta|}(u_i : i \in \Theta). \tag{5–2}$$

(The arguments of $f_{|\Theta|}$ are the elements $u_i : i \in \Theta$ in some order. The order does not matter since the function $f_{|\Theta|}$ is symmetric.) Our definition of composition generalizes the composition of usual power series, since Equation (5–1) is the generalization of the formula for the derivatives of a composite function.

By the above formula (5–2), we can define for all modules U and V and every sequence $(f_k\colon U^{\times k} \to V : k = 1, \dots, \infty)$ of symmetric multilinear maps a power series $F\colon (U^{\otimes k})_{k=1}^{\infty} \to (V^{\otimes k})_{k=1}^{\infty}$. Let us call power series of this form *classical*. They are clearly closed under composition. We will use classical power series for the $\mathbb{Z}[x]$–modules $U = V = \mathbb{Z}[x]$. In this special case, every k–linear function is of the form $f_k(x_1, \dots, x_k) = a x_1 \cdots x_k$ for some constant $a \in \mathbb{Z}[x]$.

5.2 Power series in cohomology

We are only interested in power series from the sequence of cohomology groups $(H^*(M^{\times k}))_{k=1}^{\infty}$ to itself. We will denote this sequence by $\bar{H}^*(M)$.

Special series We want to map the monoid of classical power series of the $\mathbb{Z}[x]$–module $\mathbb{Z}[x]$ to the monoid of power series of $\bar{H}^*(M)$. Substitution of an element $e \in H^*(M)$ for x defines a ring homomorphism $\mathbb{Z}[x] \to H^*(M)$, which maps a polynomial a into the cohomology class $a(e)$. Note that a classical power series of $\mathbb{Z}[x]$ is just a sequence $(a_k)_{k=1}^{\infty}$ of elements of $\mathbb{Z}[x]$. We map such a sequence to the power series F defined by

$$F_\alpha(y) := \Bigl(\bigtimes_{\Theta \in \alpha} a_{|\Theta|}(e)\Bigr) \cup \Delta^{\alpha *}(y) \quad (y \in H^*(M^{\times k}), \alpha \in \mathrm{Eq}(k)). \tag{5–3}$$

Note that the exponential power series of many real functions, like the exponential function exp and the natural logarithm ln, have integer coefficients and hence are $\mathbb{Z} \to \mathbb{Z}$ and $\mathbb{Z}[x] \to \mathbb{Z}[x]$ series.

Solving the recursion We are now ready to analyze our recursion (4–1), which we repeat here in a slightly simpler form:

$$\Gamma_k^*(1 \times f^{\times(k-1)})_!(x) = \sum_{\alpha \in \mathrm{Eq}(k)} (j_{|\alpha|!} \circ i_{|\alpha|}^*)\Bigl(\bigtimes_{\Theta \in \alpha} e^{|\Theta|-1} \cdot \Delta^{\alpha *} x\Bigr). \tag{5–4}$$

Our main observation is that the right-hand side is a special case of the composition formula (5–1). Let us form a power series from the unknown functions $j_{k!} \circ i_k^*$:

$$F_\alpha := \begin{cases} j_{k!} \circ i_k^* & \text{if } \alpha = 1(k) \text{ for some } k \\ 0 & \text{otherwise.} \end{cases}$$

Note that in the exponential power series expansion of the function

$$H(x) := \frac{\exp(ex) - 1}{e} = \sum_{k=1}^{\infty} \frac{e^{k-1}}{k!} x^k$$

the coefficient of the k–th term is e^{k-1}, a polynomial in e. Therefore we may treat H as a power series via (5–3) with $a_k = e^{k-1}$. So the right-hand side of (5–4) is just $(F \circ H)_{1(k)}$. Clearly, $(F \circ H)_\alpha = 0$ for $\alpha \neq 1(k)$. Therefore, similarly to the definition of F, we can define a power series G from the left-hand side of (5–4):

$$G_\alpha := \begin{cases} \Gamma_k^*(1 \times f^{\times(k-1)})_! & \text{if } \alpha = 1(k) \text{ for some } k \\ 0 & \text{otherwise,} \end{cases}$$

so that our recursion (5–4) simply means

$$G = F \circ H.$$

The power series H comes from an invertible function, and hence is invertible. The inverse is induced by the inverse function

$$H^{-1}(y) = \frac{\ln(1 + ey)}{e} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!e^{k-1}}{k!} y^k = \sum_{k=1}^{\infty} \frac{A_k e^{k-1}}{k!},$$

where $A_k := (-1)^{k-1}(k-1)!$ as declared in (2–1). We see that the coefficients are polynomials in e with integer coefficients and hence we may treat H^{-1} as a power series from $\bar{H}^*(M)$ to itself via (5–3) with $a_k = A_k e^{k-1}$. Hence the solution of our recursion is $F = G \circ H^{-1}$, and this means

$$j_{k!}(i_k^*(x)) = \sum_{\alpha \in \mathrm{Eq}(k)} \Gamma_{|\alpha|}^*(1 \times f^{\times(|\alpha|-1)})_! \left(\bigtimes_{\Theta \in \alpha} A_{|\Theta|} e^{|\Theta|-1} \cdot \Delta^{\alpha *} x \right).$$

This is exactly Equation (2–6). Applying $f_!$ to it and using the identity

$$f_! \Gamma_{|\alpha|}^*(1 \times f^{\times(|\alpha|-1)})_! = \Delta^{\alpha *} f_!^{\times|\alpha|},$$

we obtain (2–7). Substituting $x_1 \times \cdots \times x_k$ for x into these two formulas yield (2–8) and (2–9). We indicate below only how one can deduce (2–9) from (2–7). Recall that the x_i are assumed to be even dimensional in Theorem 2.2, so no sign appears when we permute them.

$$\Delta^{\alpha *}(x_1 \times \cdots \times x_k) = \bigtimes_{\Theta \in \alpha} \prod_{i \in \Theta} x_i \tag{5–5}$$

$$\Big(\bigtimes_{\Theta \in \alpha} A_{|\Theta|} e^{|\Theta|-1}\Big)\Big(\bigtimes_{\Theta \in \alpha} \prod_{i \in \Theta} x_i\Big) = \bigtimes_{\Theta \in \alpha} \Big(A_{|\Theta|} e^{|\Theta|-1} \prod_{i \in \Theta} x_i\Big) \tag{5–6}$$

$$f_!^{\times |\alpha|}\Big(\bigtimes_{\Theta \in \alpha} A_{|\Theta|} e^{|\Theta|-1} \prod_{i \in \Theta} x_i\Big) = \bigtimes_{\Theta \in \alpha} A_{|\Theta|} f_!\Big(e^{|\Theta|-1} \prod_{i \in \Theta} x_i\Big) \tag{5–7}$$

6 Finishing the computation

We have done the hard job in the previous sections. Now we derive the other results in Section 2 from Theorem 2.2 by direct computation.

6.1 Proof of Theorem 2.1

Recall, eg from [1, 5.1 and Theorem 8.2.2], that for every manifold X

$$\sigma(X) = \langle L(X), [X] \rangle, \tag{6–1}$$

$$p_J[X] = \langle P(X)_J, [X] \rangle. \tag{6–2}$$

We start by determining the normal bundle of $\widetilde{\Delta}_k(f)$ in $M^{\times k}$ using the diagram in Figure 3, where $p_k := f \circ j_k$.

$$\begin{array}{ccc} \widetilde{\Delta}_k(f) & \xrightarrow{i_k} & M^{\times k} \\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle f^{\times k}} \\ N & \xrightarrow{\Delta_k} & N^{\times k} \end{array}$$

Figure 3: Multiple point manifold as part of a transverse intersection

Note that $\widetilde{\Delta}_k(f)$ is the transverse preimage of the diagonal of $N^{\times k}$ under $f^{\times k}$ at least in a neighbourhood of $\widetilde{\Delta}_k(f)$. So the normal bundle of $\widetilde{\Delta}_k(f)$ in $M^{\times k}$ is the pullback of the normal bundle of the diagonal in $N^{\times k}$:

$$\nu(i_k) = p_k^* \nu(\Delta) = p_k^*\Big(\bigoplus^{k-1} TN\Big) = i_k^*(1 \times \underbrace{f^*TN \times \cdots \times f^*TN}_{k-1}).$$

Hence one obtains for the Hirzebruch class of $\widetilde{\Delta}_k(f)$:

$$\begin{aligned}2L(\widetilde{\Delta}_k(f)) &= 2i_k^*(L(M^{\times k}))\cdot L(\nu(i_k))^{-1}\\ &= 2i_k^*\big((L(M)\times\cdots\times L(M))\cdot(1\times L(f^*TN)^{-1}\times\cdots\times L(f^*TN)^{-1})\big)\\ &= 2i_k^*\big(L(M)\times L(\nu)^{-1}\times\cdots\times L(\nu)^{-1}\big).\end{aligned}$$

We have multiplied everything with 2 to get rid of eventual torsion parts. This has no consequence when computing the signature. We get by (6–1)

$$\begin{aligned}\sigma(\widetilde{\Delta}_k(f)) &= \langle i_k^*(L(M)\times L(\nu)^{-1}\times\cdots\times L(\nu)^{-1}),[\widetilde{\Delta}_k(f)]\rangle\\ &= \langle j_{k!}i_k^*(L(M)\times L(\nu)^{-1}\times\cdots\times L(\nu)^{-1}),[M]\rangle.\end{aligned}$$

This gives (2–2) of Theorem 2.1 since $\sigma(\Delta_k(f)) = \sigma(\widetilde{\Delta}_k(f))/k!$. Formula (2–3) is obtained by using $2L(M) = 2f^*(L(N))\cdot L(\nu)^{-1}$. The formulas (2–4) and (2–5) are proved similarly.

6.2 Hirzebruch's virtual signature formula

Proof of Corollary 2.3 The corollary is obtained by simply plugging Equation (2–8) directly into (2–2) and plugging (2–9) into (2–3). Substituting $x_i = L(\nu)^{-1}$ for all i into (2–9), the summand corresponding to α will depend only on the number of elements of the classes of α. Hence we can collect those summands together which are shown equal by this observation. Adding the collected terms, we obtain a new summation whose index will run through all tuples of non-negative integers $l_1,\dots,l_k$ for which $\sum_{i=1}^k il_i = k$, corresponding to the equivalence relations α with exactly l_i pieces of i–element classes. There are exactly $k!/(\prod_{i=1}^k i!^{l_i}\cdot l_i!)$ many such equivalence relations. Hence

$$\begin{aligned}f_! j_{k!} i_k^*\Big(\bigtimes_{i=1}^k L(\nu)^{-1}\Big) &= \sum_{\sum_{i=1}^k il_i=k}\frac{k!}{\prod_{i=1}^k i!^{l_i}\cdot l_i!}\prod_{i=1}^k\big(A_i f_!(e^{i-1}L(\nu)^{-i})\big)^{l_i}\\ &= \sum_{\sum_{i=1}^k il_i=k}\frac{k!(-1)^{k-\sum_{i=1}^k l_i}}{\prod_{i=1}^k i^{l_i}\cdot l_i!}\prod_{i=1}^k\big(f_!(e^{i-1}L(\nu)^{-i})\big)^{l_i}.\end{aligned}$$

Recall from (2–1) that $A_i = (-1)^{i-1}(i-1)!$, which is used in the second equation above.

This gives the second formula of Corollary 2.3. The first formula is obtained in a similar way using (2–8) but now the equivalence class of 1 is special. Therefore the summation runs through the tuples $(l,l_1,\dots,l_{k-1})$ corresponding to those equivalence relations for which the class of 1 has l elements and there are exactly l_i classes with i elements which does not contain 1. The number of such equivalence relations is $(k-1)!/((l-1)!\prod_{i=1}^{k} i!^{l_i} \cdot l_i!)$. Therefore

$$
\begin{aligned}
j_{k!}i_k^*\Big(L(M) \times \bigtimes_{i=2}^{k} L(\nu)^{-1}\Big) &= \sum_{l+\sum_{i=1}^{k-1} il_i = k} \frac{(k-1)!}{(l-1)!\prod_{i=1}^{k-1} i!^{l_i} \cdot l_i!} A_l e^{l-1} L(M) \\
&\qquad \cdot L(\nu)^{1-l} \prod_{i=1}^{k-1} \big(A_i f^* f_!(e^{i-1} L(\nu)^{-i})\big)^{l_i} \\
&= \sum_{l+\sum_{i=1}^{k-1} il_i = k} \frac{(k-1)!(-1)^{k-1-\sum_{i=1}^{k} l_i}}{\prod_{i=1}^{k-1} i^{l_i} \cdot l_i!} L(M) e^{l-1} \\
&\qquad \cdot L(\nu)^{1-l} \prod_{i=1}^{k-1} \big(f^* f_!(e^{i-1} L(\nu)^{-i})\big)^{l_i}.
\end{aligned}
\tag{6–3}
$$

The exponent of (-1) in both formulas of the Corollary is the difference between k and the number of equivalence classes.

Proof of Theorem 2.4 Now we examine the case when M is a disjoint union of manifolds $M_1, \dots, M_l$. Then the cartesian power $M^{\times k}$ is the disjoint union of products $M_{l_1} \times \dots \times M_{l_k}$ for all $1 \le l_1, \dots, l_k \le l$. This decomposition also decomposes $\widetilde{\Delta}_k(f)$, which leads to a decomposition of $j_{k!}i_k^*$, and hence $B_k(f)$, into a sum. We are gong to determine the summands.

Therefore, let us fix a tuple $(l_1, \dots, l_k)$. Let $\widetilde{j_k}$ denote the restriction of j_k to $\widetilde{\Delta}_k(f) \cap M_{l_1} \times \dots \times M_{l_k}$, and, similarly, let $\widetilde{i_k}$ denote the restriction of i_k.

Let k_t be the multiplicity of t in the tuple, so that $M_{l_1} \times \dots \times M_{l_k} \cong \bigtimes_{t=1}^{l} M_t^{\times k_t}$ by a permutation of coordinates. Let s denote the inclusion of this space into $M^{\times k}$.

Under this identification, $\widetilde{\Delta}_k(f) \cap M_{l_1} \times \dots \times M_{l_k}$ is clearly a subspace of the product $\bigtimes_{t=1}^{l} \widetilde{\Delta}_{k_t}(f_t)$. Actually, it is the preimage of the diagonal of $N^{\times l}$ under $\bigtimes_{t=1}^{l}(f \circ j_{k_t}^{(f_t)})$, so the square in Figure 4 is a transverse intersection since f is generic. In the diagram p denotes the obvious inclusion map.

By transversality, we have (eg as a special case of Theorem 4.2):

$$
f_! \widetilde{j}_{k!} p^* = \Delta_l^* \Big(\bigtimes_{t=1}^{l} f \circ j_{k_t}^{(f_t)} \Big)_! . \tag{6–4}
$$

$$\begin{array}{ccccccc}
\widetilde{\Delta}_k(f)\cap \bigtimes_{t=1}^{l} M_{l_t} & \xrightarrow{p} & \bigtimes_{t=1}^{l} \widetilde{\Delta}_{k_t}(f_t) & \xrightarrow{\bigtimes_{t=1}^{l} i_{k_t}^{(f_t)}} & \bigtimes_{t=1}^{l} M_t^{\times k_t} & \xrightarrow{s} & M^{\times k} \\
\downarrow{\scriptstyle f\circ \widetilde{j}_k} & & \downarrow{\scriptstyle \bigtimes_{t=1}^{l} f\circ j_{k_t}^{(f_t)}} & & & & \\
N & \xrightarrow{\Delta_l} & \bigtimes_{t=1}^{l} N & & & &
\end{array}$$

Figure 4: Decompositions of multiple point manifold for immersion of a nonconnected manifold

The composition of the top row is $\widetilde{i_k}$, so we obtain

$$f_!\widetilde{j}_{k!}\widetilde{i}_k^{\,*} = f_!\widetilde{j}_{k!}p^*\Big(\bigtimes_{t=1}^{l} i_{k_t}^{(f_t)}\Big)^* s^* = \Delta_l^*\Big(\bigtimes_{t=1}^{l} f\circ j_{k_t}^{(f_t)}\Big)_!\Big(\bigtimes_{t=1}^{l} i_{k_t}^{(f_t)}\Big)^* s^*.$$

Evaluating the expression at $\bigtimes_{i=1}^{k} L(\nu)^{-1}$, we obtain the summand of $B_k(f)$ corresponding to $(l_1,\dots,l_k)$. We write ν_t for the normal bundle of f_t, which is the restriction of ν to M_t.

$$\begin{aligned}
\Delta_l^*\Big(\bigtimes_{t=1}^{l} f\circ j_{k_t}^{(f_t)}\Big)_!\Big(\bigtimes_{t=1}^{l} i_{k_t}^{(f_t)}\Big)^* s^*\Big(\bigtimes_{i=1}^{k} L(\nu)^{-1}\Big)& \\
= \Delta_l^*\Big(\bigtimes_{t=1}^{l} f\circ j_{k_t}^{(f_t)}\Big)_!\Big(\bigtimes_{t=1}^{l} i_{k_t}^{(f_t)}\Big)^*\Big(\bigtimes_{t=1}^{l}(L(\nu_t)^{-1})^{\times k_t}\Big)& \\
= \prod_{t=1}^{l} f_{t!}j_{k_t!}^{(f_t)}(i_{k_t}^{(f_t)})^*(L(\nu_t)^{-1})^{\times k_t} = \prod_{t=1}^{l} B_{k_t}(f_t).&
\end{aligned}$$

The class $B_k(f)$ is the sum of the last expression for all tuples $(l_1,\dots,l_k)$, which leads to (2–13). Note that a tuple $k_1,\dots,k_l$ appears exactly for $k!/(k_1!\cdots k_l!)$ many tuples $(l_1,\dots,l_k)$.

6.3 Proof of the special cases

Most of the special cases in Subsection 2.3 follow easily from the general formula, since almost all summands become zero in (2–6). Let us examine Equation (2–6) in more detail. If $e=0$ then all summands containing e become zero and hence only the summand corresponding to $\alpha=1$ can be nonzero. If $f^*f_!=0$ then we use (2–8) and see that the only summand which does not contain $f^*f_!$ corresponds to $\alpha=0$, and hence the other summands are zero.

Finally, the case "e, $L(v)$ comes from N" requires more calculations. We will use power series again to save some computations. Let us evaluate (2–6) at $x = 1$.

(6–5)
$$j_{k!}(1) = \sum_{\alpha \in \mathrm{Eq}(k)} \left(\prod_{\Theta \in \alpha} A_{|\Theta|} e^{|\Theta|-1} \right) (f^* f_!(1))^{|\alpha|-1} = \partial_3^k q(e, f^* f_!(1), 0),$$

where the sum is again a special case of the formula of composition of power series (or higher order derivatives). Namely, the two functions we compose are

$$g(y,z) := \frac{\exp yz - 1}{y}$$
$$h(x,z) := \frac{\ln(1+xz)}{x}$$

and q is their composition:

$$q(x,y,z) := g(y, h(x,z)) = \frac{1}{y} \left(\exp \left(y \frac{\ln(1+xz)}{x} \right) - 1 \right).$$

Now we find the terms of the power series q using ordinary power series. Recall that

$$\exp(t \ln(1+x)) = (1+x)^t = \sum_{n=0}^{\infty} \binom{t}{n} x^n.$$

Substituting y/x for t and xz for x this becomes

$$1 + yq(x,y,z) = \exp \left(y \frac{\ln(1+xz)}{x} \right) = \sum_{n=0}^{\infty} \binom{y/x}{n} x^n z^n.$$

Thus the n–th partial derivative of q in its third variable z is

$$\partial_3^n q(x,y,0) = \frac{n!}{y} \binom{y/x}{n} x^n = \prod_{i=1}^{n-1} (y - ix).$$

Plugging this into (6–5), we obtain

$$j_{k!}(1) = \prod_{i=1}^{k-1} (f^* f_!(1) - ie),$$

$$j_{k!} i_k^* (f^{\times k})^*(y) = \Delta_k^* (f^{\times k})^*(y) j_{k!}(1) = \Delta_k^* (f^{\times k})^*(y) \prod_{i=1}^{k-1} (f^* f_!(1) - ie).$$

The last formula is exactly (2–16), from which (2–17), (2–18) and (2–19) are straightforward.

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Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
Budapest, Reáltanoda u 13–15, 1053, Hungary

braung at renyi.hu

http://www.renyi.hu/~braung/

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