Knot Floer homology and Seifert surfaces

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Let \( K \) be a knot in \( S^3 \) of genus \( g \) and let \( n > 0 \). We show that if \( \text{rk} HFK(K, g) < 2^{n+1} \) (where \( HFK \) denotes knot Floer homology), in particular if \( K \) is an alternating knot such that the leading coefficient \( a_g \) of its Alexander polynomial satisfies \( |a_g| < 2^{n+1} \), then \( K \) has at most \( n \) pairwise disjoint nonisotopic genus \( g \) Seifert surfaces.

For \( n = 1 \) this implies that \( K \) has a unique minimal genus Seifert surface up to isotopy.

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1 Introduction and preliminaries

If \( S_1 \) and \( S_2 \) are Seifert surfaces of a knot \( K \subset S^3 \) then \( S_1 \) and \( S_2 \) are said to be equivalent if \( S_1 \cap X(K) \) and \( S_2 \cap X(K) \) are ambient isotopic in the knot exterior \( X(K) = S^3 \setminus N(K) \), where \( N(K) \) is a regular neighborhood of \( K \). In [4] Kakimizu assigned a simplicial complex \( MS(K) \) to every knot \( K \) in \( S^3 \) as follows.

**Definition 1.1** \( MS(K) \) is a simplicial complex whose vertices are the equivalence classes of the minimal genus Seifert surfaces of \( K \). The equivalence classes \( \sigma_0, \ldots, \sigma_n \) span an \( n \)–simplex if and only if for each \( 0 \leq i \leq n \) there is a representative \( S_i \) of \( \sigma_i \) such that the surfaces \( S_0, \ldots, S_n \) are pairwise disjoint.

Scharlemann and Thompson [10] showed that the complex \( MS(K) \) is always connected. In other words, if \( S \) and \( T \) are minimal genus Seifert surfaces for a knot \( K \) then there is a sequence \( S = S_1, S_2, \ldots, S_k = T \) of minimal genus Seifert surfaces such that \( S_i \cap S_{i+1} = \emptyset \) for \( 0 \leq i \leq k - 1 \).

The main goal of this short note is to show that for a genus \( g \) knot \( K \) and for \( n > 0 \) the condition \( \text{rk} HFK(K, g) < 2^{n+1} \) implies \( \dim MS(K) < n \), consequently for \( n = 1 \) the knot \( K \) has a unique Seifert surface up to equivalence. This condition involves the use of knot Floer homology introduced by Ozsváth and Szabó in [8] and independently by Rasmussen in [9]. However, when \( K \) is alternating then this condition is equivalent to \( |a_g| < 2^{n+1} \), where \( a_g \) is the leading coefficient of the Alexander polynomial of \( K \). The alternating case is already a new result whose statement doesn’t involve knot Floer homology.
homology. On the other hand, the proof of this particular case seems to need sutured Floer homology techniques, which is a generalization of knot Floer homology that was introduced by the author in [2].

The above statement does not hold for \( n = 0 \) since every knot has at least one minimal genus Seifert surface. However, it was shown by Ni [6] and the author [3] that \( \text{rk } \widehat{HF}_K(K, g) < 2 \) implies that the knot \( K \) is fibred, and hence \( MS(K) \) is a single point.

To a knot \( K \) in \( S^3 \) and every \( j \in \mathbb{Z} \) knot Floer homology assigns a graded abelian group \( \widehat{HF}_K(K, j) \) whose Euler characteristic is the coefficient \( a_j \) of the Alexander polynomial \( \Delta_K(t) \). Ozsváth and Szabó [7] have shown that if \( K \) is alternating then \( \text{rk } \widehat{HF}_K(K, j) \) is nonzero in at most one grading, thus \( \text{rk } \widehat{HF}_K(K, j) = |a_j| \).

Next we are going to review some necessary definitions and results from the theory of sutured manifolds and sutured Floer homology. Sutured manifolds were introduced by Gabai in [1].

**Definition 1.2** A sutured manifold \((M, \gamma)\) is a compact oriented 3–manifold \( M \) with boundary together with a set \( \gamma \subset \partial M \) of pairwise disjoint annuli \( A(\gamma) \) and tori \( T(\gamma) \). Furthermore, the interior of each component of \( A(\gamma) \) contains a suture, ie, a homologically nontrivial oriented simple closed curve. We denote the union of the sutures by \( s(\gamma) \).

Finally every component of \( R(\gamma) = \partial M \setminus \text{Int}(\gamma) \) is oriented. Define \( R_+(\gamma) \) (or \( R_-(\gamma) \)) to be those components of \( \partial M \setminus \text{Int}(\gamma) \) whose normal vectors point out of (into) \( M \). The orientation on \( R(\gamma) \) must be coherent with respect to \( s(\gamma) \), ie, if \( \delta \) is a component of \( \partial R(\gamma) \) and is given the boundary orientation, then \( \delta \) must represent the same homology class in \( H_1(\gamma) \) as some suture.

A sutured manifold is called **taut** if \( R(\gamma) \) is incompressible and Thurston norm minimizing in \( H_2(M, \gamma) \).

The following definition was introduced in [2].

**Definition 1.3** A sutured manifold \((M, \gamma)\) is called **balanced** if \( M \) has no closed components, \( \chi(R_+(\gamma)) = \chi(R_-(\gamma)) \), and the map \( \pi_0(A(\gamma)) \to \pi_0(\partial M) \) is surjective.

**Example 1.4** If \( R \) is a Seifert surface of a knot \( K \) in \( S^3 \) then we can associate to it a balanced sutured manifold \( S^3(R) = (M, \gamma) \) such that \( M = S^3 \setminus (R \times I) \) and \( \gamma = K \times I \). Observe that \( R_-(\gamma) = R \times \{0\} \) and \( R_+(\gamma) = R \times \{1\} \). Furthermore, \( S^3(R) \) is taut if and only if \( R \) is of minimal genus.
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Sutured Floer homology is an invariant of balanced sutured manifolds defined by the author in [2], and is a common generalization of the invariants $\widehat{HF}$ and $\widehat{HFK}$. It assigns an abelian group $SFH(M, \gamma)$ to each balanced sutured manifold $(M, \gamma)$. The following theorem is a special case of [3, Theorem 1.5].

**Theorem 1.5** Let $K$ be a genus $g$ knot in $S^3$ and suppose that $R$ is a minimal genus Seifert surface for $K$. Then

$$SFH(S^3(R)) \cong \widehat{HFK}(K, g).$$

A sutured manifold $(M, \gamma)$ is called a product if it is homeomorphic to $(\Sigma \times I, \partial \Sigma \times I)$, where $\Sigma$ is an oriented surface with boundary. If $(M, \gamma)$ is a product, $SFH(M, \gamma) \cong \mathbb{Z}$. Let us recall [3, Theorem 1.4] and [3, Theorem 9.3].

**Theorem 1.6** If $(M, \gamma)$ is a taut balanced sutured manifold then $\text{rk} \ SFH(M, \gamma) \geq 1$. Furthermore, if $(M, \gamma)$ is not a product then $\text{rk} \ SFH(M, \gamma) \geq 2$.

**Definition 1.7** Let $(M, \gamma)$ be a balanced sutured manifold. An oriented surface $S \subset M$ is called a horizontal surface if $S$ is open, $\partial S = s(\gamma)$ in an oriented sense; moreover, $[S] = [R_+(\gamma)]$ in $H_2(M, \gamma)$, and $\chi(S) = \chi(R_+(\gamma))$.

A horizontal surface $S$ defines a horizontal decomposition

$$(M, \gamma) \hookrightarrow S (M_-, \gamma_-) \bigsqcup (M_+, \gamma_+)$$

as follows. Let $M_\pm$ be the union of the components of $M \setminus \text{Int}(N(S))$ that intersect $R_\pm(\gamma)$. Similarly, let $\gamma_\pm$ be the union of the components of $\gamma \setminus \text{Int}(N(S))$ that intersect $R_\pm(\gamma)$.

The following proposition is a special case of [3, Proposition 8.6].

**Proposition 1.8** Suppose that $(M, \gamma)$ is a taut balanced sutured manifold and let $S$ be a horizontal surface in it. Then

$$\text{rk} \ SFH(M, \gamma) = \text{rk} \ SFH(M_-, \gamma_-) \cdot \text{rk} \ SFH(M_+, \gamma_+).$$

The following definition can be found for example in [6].

**Definition 1.9** A balanced sutured manifold $(M, \gamma)$ is called horizontally prime if every horizontal surface $S$ in $(M, \gamma)$ is isotopic to either $R_+(\gamma)$ or $R_-(\gamma)$ rel $\gamma$. 

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2 The results

**Theorem 2.1** Let \((M, \gamma)\) be a taut balanced sutured manifold such that both \(R_+ (\gamma)\) and \(R_- (\gamma)\) are connected. Suppose that there is a sequence of pairwise disjoint nonisotopic connected horizontal surfaces \(R_- (\gamma) = S_0, S_1, \ldots, S_n = R_+ (\gamma)\). Then

\[
\text{rk} SFH (M, \gamma) \geq 2^n.
\]

**Proof** We prove the theorem using induction on \(n\). If \(n = 1\) then \((M, \gamma)\) is not a product since \(R_- (\gamma)\) and \(R_+ (\gamma)\) are nonisotopic. Thus Theorem 1.6 implies that \(\text{rk} SFH (M, \gamma) \geq 2\).

Now suppose that the theorem is true for \(n \geq 1\). Since each \(S_k\) is connected we can suppose without loss of generality that \(S_1\) separates \(S_i\) and \(S_0\) for every \(i \geq 2\). Let \((M_-, \gamma_-)\) and \((M_+, \gamma_+)\) be the sutured manifolds obtained after horizontally decomposing \((M, \gamma)\) along \(S_1\). Note that both \((M_-, \gamma_-)\) and \((M_+, \gamma_+)\) are taut. As \(S_0\) and \(S_1\) are nonisotopic, \((M_-, \gamma_-)\) is not a product so as before \(\text{rk} SFH (M_-, \gamma_-) \geq 2\). Applying the induction hypothesis to \((M_+, \gamma_+)\) and to the surfaces \(R_- (\gamma_+), S_2, \ldots, S_n = R_+ (\gamma_+)\) we get that \(\text{rk} SFH (M_+, \gamma_+) \geq 2^{n-1}\). So using Proposition 1.8 we see that \(\text{rk} SFH (M, \gamma) \geq 2^n\).

**Corollary 2.2** If \((M, \gamma)\) is a taut balanced sutured manifold and \(\text{rk} SFH (M, \gamma) < 4\) then \((M, \gamma)\) is horizontally prime. More generally, if \(n > 0\) and \(\text{rk} SFH (M, \gamma) < 2^{n+1}\) then \((M, \gamma)\) can be cut into horizontally prime pieces by less than \(n\) horizontal decompositions.

**Proof** Suppose that \(\text{rk} SFH (M, \gamma) < 2^{n+1}\). If \((M, \gamma)\) is not horizontally prime then there is a surface \(S_1\) in \((M, \gamma)\) which is not isotopic to \(R_\pm (\gamma)\). Decomposing \((M, \gamma)\) along \(S_1\) we get two sutured manifolds \((M_-, \gamma_-)\) and \((M_+, \gamma_+)\). If they are not both horizontally prime then repeat the above process with a nonprime piece and obtain a horizontal surface \(S_2\), etc. This process has to end in less than \(n\) steps according to Theorem 2.1.

**Theorem 2.3** Let \(K\) be a knot in \(S^3\) of genus \(g\) and let \(n > 0\). If \(\text{rk} HFK (K, g) < 2^{n+1}\) then \(K\) has at most \(n\) pairwise disjoint nonisotopic genus \(g\) Seifert surfaces, in other words, \(\dim MS (K) < n\). If \(n = 1\) then \(K\) has a unique Seifert surface up to equivalence.

**Proof** Suppose that \(R, S_1, \ldots, S_n\) are pairwise disjoint nonisotopic Seifert surfaces for \(K\). According to Theorem 1.5 we have \(HFK (K, g) \cong SFH (S^3 (R))\). Let \(S^3 (R) = \)
(M, γ). If R+(γ) and R−(γ) were isotopic then (M, γ) would be a product and S1 and R would be equivalent. So the surfaces R−(γ) = S0, S1, . . . , Sn, Sn+1 = R+(γ) satisfy the conditions of Theorem 2.1, thus \( \text{rk} \, SFH(S^3(R)) \geq 2^{n+1} \), a contradiction.

In particular, if n = 1 then \( \dim \, MS(K) = 0 \). But according to [10] the complex MS(K) is connected, so it consists of a single point.

**Corollary 2.4** Suppose that \( K \) is an alternating knot in \( S^3 \) of genus \( g \) and let \( n > 0 \). If the leading coefficient \( a_g \) of its Alexander polynomial satisfies \( |a_g| < 2^{n+1} \) then \( \dim \, MS(K) < n \). If \( |a_g| < 4 \) then \( K \) has a unique Seifert surface up to equivalence.

**Proof** This follows from Theorem 2.3 and the fact that for alternating knots the equality \( \text{rk} \, \HFK(K, g) = |a_g| \) holds.

**Remark** In [5] Kakimizu classified the minimal genus Seifert surfaces of all the prime knots with at most 10 crossings. The \( n = 1 \) case of Corollary 2.4 is sharp since the knot 7_4 is alternating, the leading coefficient of its Alexander polynomial is 4, and has 2 inequivalent minimal genus Seifert surfaces. On the other hand, the Alexander polynomial of the alternating knot 9_2 is also 4, but has a unique minimal genus Seifert surface up to equivalence.

Also note that [3, Theorem 1.7] implies that if the leading coefficient \( a_g \) of the Alexander polynomial of an alternating knot \( K \) satisfies \( |a_g| < 4 \) then the knot exterior \( X(K) \) admits a depth \( \leq 2 \) taut foliation transversal to \( \partial X(K) \). Indeed, for alternating knots \( g = g(K) \) and \( |a_g| = \text{rk} \, \HFK(K, g) \neq 0 \), so the conditions of [3, Thorem 1.7] are satisfied.

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**References**


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