Nonsmoothable, locally indicable group actions on the interval

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By the Thurston Stability Theorem, a group of C^1 orientation-preserving diffeomorphisms of the closed unit interval is locally indicable. We show that the local order structure of orbits gives a stronger criterion for nonsmoothability that can be used to produce new examples of locally indicable groups of homeomorphisms of the interval that are not conjugate to groups of C^1 diffeomorphisms.

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1 Nonsmoothable actions

1.1 Thurston Stability Theorem

A simple, but important case of the Thurston Stability Theorem is usually stated in the following way:

Theorem 1.1 (Thurston Stability Theorem [8]) *Let G be a group of orientation-preserving C^1 diffeomorphisms of the closed interval I. Then G is locally indicable; ie every nontrivial finitely generated subgroup H of G admits a surjective homomorphism to $\mathbb{Z}$.*

The proof is nonconstructive, and uses the axiom of choice. The idea is to "blow up" the action of H near one of the endpoints at a sequence of points that are moved a definite distance, but not too far. Some subsequence of blow-ups converges to an action by translations.

Note that it is only *finitely* generated subgroups that admit surjective homomorphisms to $\mathbb{Z}$, as the following example of Sergeraert shows.

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Example 1.2 (Sergeraert [7]) Let G be the group of C^∞ orientation-preserving diffeomorphisms of I that are infinitely tangent to the identity at the endpoints. Then G is perfect.

Another countable example comes from Thompson's group.

Example 1.3 (Navas [6], Ghys–Sergiescu [3]) Thompson's group F of dyadic rational piecewise linear homeomorphisms of I is known to be conjugate to a group of C^∞ diffeomorphisms. On the other hand, the commutator subgroup $[F,F]$ is simple; since it is non-Abelian, it is perfect.

Given a group $G \subset \mathrm{Homeo}_+(I)$, Theorem 1.1 gives a criterion to show that the action of G is not conjugate into $\mathrm{Diff}^1_+(I)$. It is natural to ask whether Thurston's criterion is sharp. That is, suppose G is locally indicable. Is it true that every homomorphism from G into $\mathrm{Homeo}_+(I)$ is conjugate into $\mathrm{Diff}^1_+(I)$? It turns out that the answer to this question is no. However, apart from Thurston's criterion, very few obstructions to conjugating a subgroup of $\mathrm{Homeo}_+(I)$ into $\mathrm{Diff}^1_+(I)$ are known. Most significant are dynamical obstructions concerning the existence of elements with hyperbolic fixed points when the action has positive topological entropy by Hurder [4], or when there is no invariant probability measure for some sub-pseudogroup by Deroin, Kleptsyn and Navas [2] (also, see Cantwell and Conlon [1]).

In this note we give some new examples of actions of locally indicable groups on I that are not conjugate to C^1 actions.

Example 1.4 ($\mathbb{Z}^\mathbb{Z}$) Let $T\colon I \to I$ act freely on the interior, so that T is conjugate to a translation. Let $I_0 \subset \mathrm{int}(I)$ be a closed fundamental domain for T, and let $S\colon I_0 \to I_0$ act freely on the interior. Extend S by the identity outside I_0 to an element of $\mathrm{Homeo}_+(I)$. For each $i \in \mathbb{Z}$ let $I_i = T^i(I_0)$ and let $S_i\colon I_i \to I_i$ be the conjugate T^iST^{-i}. For each $f \in \mathbb{Z}^\mathbb{Z}$ define Z_f to be the product:

$$Z_f = \prod_{i\in\mathbb{Z}} S_i^{f(i)}$$

Let G be the group consisting of all elements of the form Z_f. Then G is isomorphic to $\mathbb{Z}^\mathbb{Z}$ and is therefore abelian.

However, G is not conjugate into $\mathrm{Diff}^1_+(I)$. For, suppose otherwise, so that there is some homeomorphism $\varphi\colon I \to I$ so that the conjugate $G^\varphi \subset \mathrm{Diff}^1_+(I)$. We suppose by abuse of notation that S_i denotes the conjugate S_i^φ. For each i, let p_i be the midpoint of I_i. Since for each fixed i the sequence $S_i^n(p_i)$ converges to an endpoint of I_i as

n goes to infinity, it follows that for each i there is some n_i so that $dS_i^{n_i}(p_i) < 1/2$. Let $F \in \mathbb{Z}^{\mathbb{Z}}$ satisfy $F(i) = n_i$. Then $dZ_F(p_i) < 1/2$ for all i. However, Z_F fixes the endpoints of I_i for all i, so Z_F has a sequence of fixed points converging to 1. It follows that $dZ_F(1) = 1$. But $p_i \to 1$, so if Z_F is C^1 we must have $dZ_F(1) \le 1/2$. This contradiction shows that no such conjugacy exists.

Remark 1.5 The group $\mathbb{Z}^{\mathbb{Z}}$ is locally indicable, but uncountable. Note in fact that this group action is not even conjugate to a *bi-Lipschitz* action. On the other hand, Theorem D from [2] says that every countable group of homeomorphisms of the circle or interval is conjugate to a group of bi-Lipschitz homeomorphisms.

1.2 Order structure of orbits

In this section we describe a new criterion for nonsmoothability, depending on the local order structure of orbits.

Definition 1.6 Let G act on I by $\rho\colon G \to \mathrm{Homeo}_+(I)$. A point $p \in I$ determines an order $<_p$ on G by

$$a <_p b$$

if and only if $a(p) < b(p)$ in I.

Note that with this definition, $<_p$ is really an order on the left G–space G/G_p, where G_p denotes the stabilizer of p.

Lemma 1.7 *Suppose $\rho\colon G \to \mathrm{Diff}^1_+(I)$ is injective. Let H be a finitely generated subgroup of G, with generators $S = \{h_1, \cdots, h_n\}$. Let $p \in I$ be in the frontier of $\mathrm{fix}(H)$ (ie the set of common fixed points of all elements of H) and let $p_i \to p$ be a sequence contained in $I - \mathrm{fix}(H)$. Then there is a sequence $k_m \in \{1, \ldots, n\}$ and $e_m \in \{-1, +1\}$ such that for any $h \in [H, H]$, and for all sufficiently large m (depending on h), there is an inequality:*

$$h <_{p_m} h_{k_m}^{e_m}$$

Proof There is a homomorphism $\rho\colon H \to \mathbb{R}$ defined by the formula $\rho(h) = \log h'(p)$. Of course this homomorphism vanishes on $[H, H]$. If h_i is such that $\rho(h_i) \neq 0$ then (after replacing h_i by h_i^{-1} if necessary) it is clear that for any $h \in [H, H]$, there is an inequality $h <_{p_m} h_i$ for all p_m sufficiently close to p. Therefore in the sequel we assume ρ is trivial.

For each i, let U_i be the smallest (closed) interval containing $p_i \cup Sp_i$. Given a bigger open interval V_i containing U_i, one can rescale V_i linearly by $1/\mathrm{length}(U_i)$ and move

p_i to the origin thereby obtaining an interval $\overline{V}_i$ on which H has a partially defined action as a pseudogroup.

The argument of the Thurston Stability Theorem implies that one can choose a sequence V_i such that any sequence of indices $\to \infty$ contains a subsequence for which $\overline{V}_i \to \mathbb{R}$, and the pseudogroup actions converge, in the compact-open topology, to a (nontrivial) action of H on $\mathbb{R}$ by *translations*. In an action by translations, some generator or its inverse moves 0 a positive distance, but every element of $[H,H]$ acts trivially. The proof follows. □

Example 1.8 Let T be a hyperbolic once-punctured torus with a cusp. The hyperbolic structure determines up to conjugacy a faithful homomorphism $\rho\colon \pi_1(T) \to \mathrm{PSL}(2,\mathbb{R})$.

The group $\mathrm{PSL}(2,\mathbb{R})$ acts by real analytic homeomorphisms on $\mathbb{RP}^1 = S^1$. Since $\pi_1(T)$ is free on two generators (say a,b) the homomorphism ρ lifts to an action $\widetilde{\rho}$ on the universal cover $\mathbb{R}$. We choose a lift so that both a and b have fixed points. If we choose coordinates on $\mathbb{R}$ so that a fixes x, then a also fixes $x+n$ for every integer n. Similarly, if b fixes y, then b fixes $y+n$ for every n. On the other hand, if $p \in S^1$ is the parabolic fixed point of $[a,b]$, and $\widetilde{p}$ is a lift of p to $\mathbb{R}$, then the commutator $[a,b]$ takes $\widetilde{p}$ to $\widetilde{p}+1$. Since the action of every element on $\mathbb{R}$ commutes with the generator of the deck group $x \to x+1$, the element $[a,b]$ acts on $\mathbb{R}$ without fixed points, and moves every point in the positive direction, satisfying $[a,b]^n(z) > z+n-1$ for every $z \in \mathbb{R}$ and every positive integer n. See Figure 1.

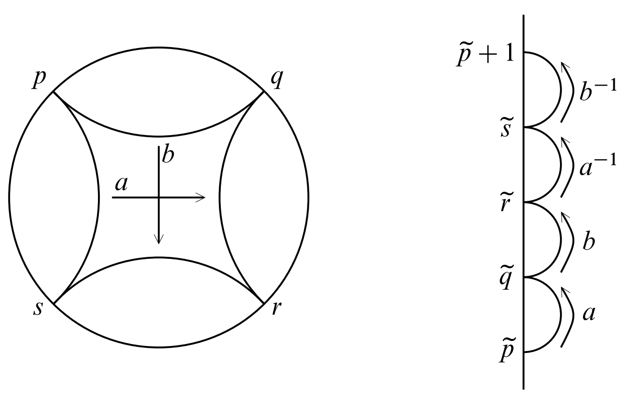

Figure 1: In the lifted action, a and b have fixed points, but $[a,b]$ takes $\widetilde{p}$ to $\widetilde{p}+1$.

This action on $\mathbb{R}$ can be made into an action on I by homeomorphisms, by including $\mathbb{R}$ in I as the interior. Then the points $\widetilde{p}+n \to \infty$ in $\mathbb{R}$ map to points $p_n \to 1$ in I. Note that for each n, the elements a and b have fixed points q_n, r_n respectively satisfying $p_n < q_n < p_{n+1}$ and $p_n < r_n < p_{n+1}$. Moreover, $[a,b](p_n) = p_{n+1}$ for all n. It follows that

$$a, a^{-1} <_{p_n} [a,b]^2, \quad b, b^{-1} <_{p_n} [a,b]^2$$

for every n, so by Lemma 1.7, this action is not topologically conjugate into $\mathrm{Diff}^1_+(I)$. On the other hand, this is a faithful action of the free group on two generators. A free group is locally indicable, since every subgroup of a free group is free.

Remark 1.9 The relationship between order structures and dynamics of subgroups of homeomorphisms of the interval is subtle and deep. For an introduction to this subject, see eg Navas [5].

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