Algebraic & Geometric Topology 8 (2008) 849–868

Co-contractions of graphs and right-angled Artin groups

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We define an operation on finite graphs, called *co-contraction*. Then we show that for any co-contraction $\widehat{\Gamma}$ of a finite graph Γ, the right-angled Artin group on Γ contains a subgroup which is isomorphic to the right-angled Artin group on $\widehat{\Gamma}$. As a corollary, we exhibit a family of graphs, without any induced cycle of length at least 5, such that the right-angled Artin groups on those graphs contain hyperbolic surface groups. This gives the negative answer to a question raised by Gordon, Long and Reid.

20F65, 20F36; 05C25

1 Introduction

In this paper, by a *graph* we mean a finite graph without loops and without multi-edges. A *right-angled Artin group* is a group defined by a presentation with a finite generating set, where the relators are certain commutators between the generators. Such a presentation naturally determines the *underlying graph*, where the vertices correspond to the generators and the edges to the pairs of commuting generators. It is known that the isomorphism type of a right-angled Artin group uniquely determines the isomorphism type of the underlying graph by Droms [6] and Kim, Makar-Limanov, Neggers and Roush [13]. Also, right-angled Artin groups possess various group theoretic properties. To name a few, right-angled Artin groups are linear by Humphries [12], Hsu and Wise [11] and Davis and Januszkiewicz [4], biorderable by Duchamp and Thibon [8], biautomatic by Van Wyk [20] and moreover, admitting free and cocompact actions on finite-dimensional CAT(0) cube complexes by Charney and Davis [1], Meier and Van Wyk [15] and Niblo and Reeves [17].

On the other hand, it is interesting to ask what we can say about the isomorphism type of the underlying graph, if a right-angled Artin group satisfies a given group theoretic property. Let Γ be a graph. We denote the vertex set and the edge set of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The *complement graph* of Γ is the graph $\overline{\Gamma}$ defined by $V(\overline{\Gamma}) = V(\Gamma)$ and $E(\overline{\Gamma}) = \{\{u,v\} : \{u,v\} \notin E(\Gamma)\}$. For a subset S of $V(\Gamma)$ the *induced subgraph* on S, denoted by Γ_S, is defined to be the maximal subgraph of Γ with the vertex set S. This implies that $V(\Gamma_S) = S$ and $E(\Gamma_S) = \{\{u,v\} : u, v \in S \text{ and } \{u,v\} \in E(\Gamma)\}$. If Λ is another graph, an *induced* Λ in Γ means an induced

Published: 3 June 2008 DOI: 10.2140/agt.2008.8.849

subgraph isomorphic to Λ in Γ. We denote by C_n the cycle of length n. That is, $V(C_n)$ is a set of n vertices, say $\{v_1, v_2, \ldots, v_n\}$, and $E(C_n)$ consists of the edges $\{v_i, v_j\}$ where $|i-j| \equiv 1 \pmod{n}$. Let $A(\Gamma)$ be the right-angled Artin group with its underlying graph Γ. Then, the following are true.

- $A(\Gamma)$ is coherent, if and only if Γ is *chordal*, ie Γ does not contain an induced C_n for any $n \geq 4$; see Droms [5]. This happens if and only if $[A(\Gamma), A(\Gamma)]$ is free; see H Servatius, Droms and B Servatius [19].
- $A(\Gamma)$ is a virtually 3–manifold group, if and only if each connected component of Γ is a tree or a triangle; see Droms [5] and Gordon [9]
- $A(\Gamma)$ is subgroup separable, if and only if no induced subgraph of Γ is a square or a path of length 3 by Metaftsis and Raptis [16]. This happens if and only if every subgroup of $A(\Gamma)$ is also a right-angled Artin group, again by Droms [7].
- $A(\Gamma)$ contains a *hyperbolic surface group*, ie the fundamental group of a closed, hyperbolic surface, if there exists an induced C_n for some $n \geq 5$ in Γ; see Crisp and Wiest [3] and again Servatius, Droms and Servatius [19].

In [10], Gordon, Long and Reid proved that a word-hyperbolic (not necessarily right-angled) Coxeter group either is virtually free or contains a hyperbolic surface group. They also showed that certain (again, not necessarily right-angled) Artin groups do not contain a hyperbolic surface group, raising the following question.

Question 1.1 Does $A(\Gamma)$ contain a hyperbolic surface group if and only if Γ contains an induced C_n for some $n \geq 5$?

In this paper, we give the negative answer to the above question. Let Γ be a graph and B be a set of vertices of Γ such that Γ_B is connected. The *contraction* of Γ relative to B is the graph $\mathrm{CO}(\Gamma, B)$ obtained from Γ by collapsing Γ_B to a vertex, and deleting loops or multi-edges. We define the *co-contraction* $\overline{\mathrm{CO}}(\Gamma, B)$ of Γ relative B, such that

$$\overline{\mathrm{CO}}(\Gamma, B) = \overline{\mathrm{CO}(\overline{\Gamma}, B)}.$$

Then we prove the following theorem, which will imply that $A(\overline{C_n})$ contains $A(\overline{C_5}) = A(C_5)$ and hence a hyperbolic surface subgroup, for $n \geq 5$ (see Figure 3). An easy combinatorial argument shows that $\overline{C_n}$ does not contain an induced cycle of length at least 5, for $n > 5$.

Theorem *Let Γ be a graph and B be a set of vertices in Γ, such that $\overline{\Gamma_B}$ is connected. Then $A(\Gamma)$ contains a subgroup isomorphic to $A(\overline{\mathrm{CO}}(\Gamma, B))$.*

In this paper, the above theorem is proved in the following steps.

In Section 2, we recall basic facts on right-angled Artin groups and HNN extensions. A *dual van Kampen diagram* is described. We owe the notation to Crisp and Wiest [3] where a closely related concept, a *dissection*, was defined and used with great clarity.

In Section 3, we define *co-contraction* of a graph, and examine its properties.

In Section 4, we prove the theorem by exhibiting an embedding of $A(\overline{\mathrm{CO}}(\Gamma, B))$ into $A(\Gamma)$. The main tool for the proof is a dual van Kampen diagram.

In Section 5, we compute intersections of certain subgroups of right-angled Artin groups. From this, we deduce a more detailed version of the theorem describing some other choices of the embeddings.

Acknowledgements I am deeply grateful to my thesis advisor, Andrew Casson, for his insights and guidance. I would like to also thank Daniel Spielman, for helpful comments.

2 Preliminaries on right-angled Artin groups

Let Γ be a graph. The *right-angled Artin group on* Γ is the group presented as

$$A(\Gamma) = \langle v \in V(\Gamma) \mid [a,b] = 1 \text{ if and only if } \{a,b\} \in E(\Gamma)\rangle.$$

Each element of $A(\Gamma)$ can be expressed as $w = \prod_{i=1}^{k} c_i^{e_i}$, where $c_i \in V(\Gamma)$ and $e_i = \pm 1$. Such an expression is called a *word (of length k)* and each $c_i^{e_i}$ is called a *letter* of the word w. We say the word w is *reduced*, if the length is minimal among the words representing the same element. For each $i_0 = 1, 2, \dots, k$, the word $w_1 = \prod_{i=i_0}^{k} c_i^{e_i} \cdot \prod_{i=1}^{i_0-1} c_i^{e_i}$ is called a *cyclic conjugation* of $w = \prod_{i=1}^{k} c_i^{e_i}$. By a *subword* of w, we mean a word $w' = \prod_{i=i_0}^{i_1} c_i^{e_i}$ for some $1 \le i_0 < i_1 \le k$. A letter or a subword w' of w is on the *left* of a letter or a subword w'' of w, if $w' = \prod_{i=i_0}^{i_1} c_i^{e_i}$ and $w'' = \prod_{i=j_0}^{j_1} c_i^{e_i}$ for some $i_1 < j_0$.

The expression $w_1 = w_2$ shall mean that w_1 and w_2 are equal as words (letter by letter). On the other hand, $w_1 =_{A(\Gamma)} w_2$ means that the words w_1 and w_2 represent the same element in $A(\Gamma)$. For an element $g \in A(\Gamma)$ and a word w, $w =_{A(\Gamma)} g$ means that the word w is representing the group element g. 1 denotes both the trivial element in $A(\Gamma)$ and the empty word, depending on the context.

Let w be a word representing the trivial element in $A(\Gamma)$. A *dual van Kampen diagram* Δ for w in $A(\Gamma)$ is a pair $(\mathcal{H}, \lambda)$ satisfying the following (Figure 1 (c)):

(i) $\mathcal{H}$ is a set of transversely oriented simple closed curves and transversely oriented properly embedded arcs in general position, in an oriented disk $D \subseteq \mathbb{R}^2$.

(ii) λ is a map from $\mathcal{H}$ to $V(\Gamma)$ such that γ and γ' in $\mathcal{H}$ are intersecting *only if* $\lambda(\gamma)$ and $\lambda(\gamma')$ are adjacent in Γ.

(iii) Enumerate the boundary points of the arcs in $\mathcal{H}$ as $v_1, v_2, \ldots, v_m$ so that v_i and v_j are adjacent on ∂D if and only if $|i-j| \equiv 1 \pmod{n}$. For each i, let a_i be the label of the arc that intersects with v_i. Put $e_i = 1$ if, at v_i, the orientation of ∂D coincides with the transverse orientation of the arc that v_i is intersecting, and $e_i = -1$ otherwise. Then w is a cyclic conjugation of $v_1^{e_1} v_2^{e_2} \cdots v_m^{e_m}$.

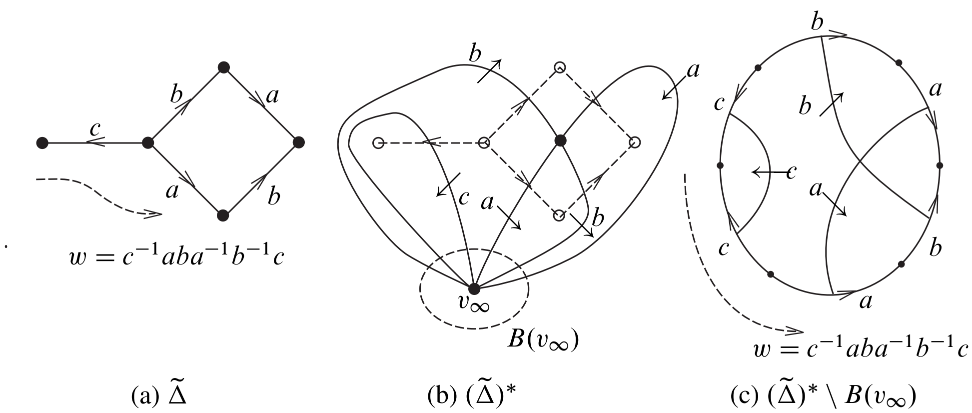


Figure 1: Constructing a dual van Kampen diagram from a van Kampen diagram, for $w = c^{-1}aba^{-1}b^{-1}$ in $\langle a, b, c \mid [a, b] = 1 \rangle$

Note that simple closed curves in a dual van Kampen diagram can always be assumed to be removed. Also, we may assume that two curves in Δ are minimally intersecting, in the sense that there does not exist any bigon formed by arcs in $\mathcal{H}$. See Crisp and Wiest [3] for more details, as well as generalization of this definition to arbitrary compact surfaces, rather than a disk.

Let $\widetilde{\Delta} \subseteq S^2$ be a (standard) van Kampen diagram for w, with respect to a standard presentation $A(\Gamma) = \langle V(\Gamma) \mid [u, v] = 1 \text{ if and only if } \{u, v\} \in E(\Gamma) \rangle$ (Figure 1). Consider $\widetilde{\Delta}^*$, the dual of $\widetilde{\Delta}$ in S^2, and name the vertex which is dual to the face $S^2 \setminus \widetilde{\Delta}$ as v_∞. Then for a sufficiently small ball $B(v_\infty)$ around v_∞, $\widetilde{\Delta}^* \setminus B(v_\infty)$ can be considered as a dual van Kampen diagram with a suitable choice of the labeling map. Therefore a dual van Kampen diagram exists for any word w representing the trivial element in $A(\Gamma)$. Conversely, a van Kampen diagram $\widetilde{\Delta}$ for a word can be obtained from a dual van Kampen diagram Δ by considering the dual complex again. So, the existence of a dual van Kampen diagram for a word w implies that $w =_{A(\Gamma)} 1$.

Given a dual van Kampen diagram Δ, divide ∂D into segments so that each segment intersects with exactly one arc in $\mathcal{H}$. Let the label and the orientation of each segment be induced from those of the arc that intersects with the segment. The resulting labeled and directed graph structure on ∂D is called the *boundary* of Δ and denoted by $\partial\Delta$.

We call each arc in $\mathcal{H}$ labeled by $q \in V(\Gamma)$ as a *q–arc*, and each segment in $\partial\Delta$ labeled by q as a *q–segment*. Sometimes we identify the letter $q^{\pm 1}$ of w with the corresponding q–segment. A connected union of segments on $\partial\Delta$ is called an *interval*. By convention, a subword w_1 of w shall also denote the corresponding interval (called *w_1–interval*) on $\partial\Delta$.

Now let $\Delta = (\mathcal{H}, \lambda)$ be a dual van Kampen diagram on $D \subseteq \mathbb{R}^2$. Suppose γ is a properly embedded arc in D, which is either an element in $\mathcal{H}$ or in general position with $\mathcal{H}$. Then one can cut Δ along γ in the following sense. First, cut D along γ to get two disks D' and D''. Consider the intersections of the disks with the curves in $\mathcal{H}$. Then, let those curves in D' and D'' inherit the transverse orientations and the labeling maps from Δ. We obtain two dual van Kampen diagrams, one for each of D' and D''. Conversely, we can glue two dual van Kampen diagrams along identical words. γ is called an *innermost q–arc* if the interior of D' or D'' does not intersect any q–arc.

Definition 2.1 Let Γ be a graph. Let w be a word representing the trivial element in $A(\Gamma)$, and Δ be a dual van Kampen diagram for w. Two segments on the boundary of Δ are called a *canceling q–pair* if there exists a q–arc joining the segments. For *any* word w_1, two letters of w_1 are called a *canceling q–pair* if there exist another word $w_1' =_{A(\Gamma)} w_1$ and a dual van Kampen diagram Δ for $w_1 w_1'^{-1}$, such that the two letters are a canceling q–pair with respect to Δ. A canceling q–pair is also called as a *q–pair* for abbreviation. A *canceling pair* is a canceling q–pair for some $q \in V(\Gamma)$.

For a group G and its subset P, $\langle P \rangle$ denotes the subgroup generated by P. For a subgroup H of $A(\Gamma)$, $w \in H$ shall mean that w represents an element in H.

Lemma 2.2 *Let Γ be a graph and q be a vertex of Γ. If a word w in $A(\Gamma)$ has a q–pair, then $w = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$ for some subwords w_1, w_2 and w_3 such that $w_2 \in \langle \mathrm{link}_\Gamma(q) \rangle$. In this case, w is not reduced.*

Proof There exists a word $w' =_{A(\Gamma)} w$ and a dual van Kampen diagram Δ for $w w'^{-1}$, such that a q–arc joins two segments of w.

Write $w = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$, where the letters $q^{\pm 1}$ and $q^{\mp 1}$ (identified with the corresponding segments on $\partial\Delta$) are joined by a q–arc γ as in Figure 2.

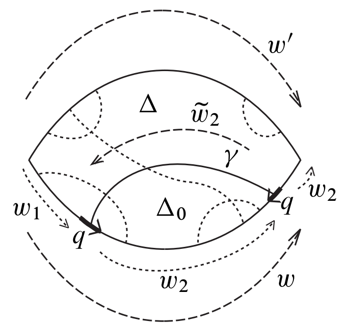


Figure 2: Cutting Δ along γ

Cut Δ along γ, to get a dual van Kampen diagram Δ_0, which contains w_2 on its boundary. Give Δ_0 the orientation that coincides with the orientation of Δ on w_2. Let $\widetilde{w}_2$ be the word, read off by following γ in the orientation of Δ_0. $\widetilde{w}_2 \in \langle \mathrm{link}_\Gamma(q) \rangle$, for the arcs intersecting with γ are labeld by vertices in $\mathrm{link}_\Gamma(q)$. Since Δ_0 is a dual van Kampen diagram for the word $w_2 \widetilde{w}_2$, we have $w_2 =_{A(\Gamma)} \widetilde{w}_2^{-1} \in \langle \mathrm{link}_\Gamma(q) \rangle$. □

For $S \subseteq V(\Gamma)$, we let $S^{-1} = \{q^{-1} : q \in S\}$ and $S^{\pm 1} = S \cup S^{-1}$. The following lemma is standard, and we briefly sketch the proof.

Lemma 2.3 *Let Γ be a graph and S be a subset of $V(\Gamma)$. Then the following are true.*

(1) *$\langle S \rangle$ is isomorphic to $A(\Gamma_S)$.*

(2) *Each letter of any reduced word in $\langle S \rangle$ is in $S^{\pm 1}$.*

Proof (1) The inclusion $V(\Gamma_S) \subseteq V(\Gamma)$ induces a map $f\colon A(\Gamma_S) \to A(\Gamma)$. Let w be a word representing an element in $\ker f$. Since $w =_{A(\Gamma)} 1$, there exists a dual van Kampen diagram Δ for the word w in $A(\Gamma)$. Remove simple closed curves labeled by $V(\Gamma) \setminus V(\Gamma_S)$, if there is any. Since the boundary of Δ is labeled by vertices in $V(S)$, Δ can be considered as a dual van Kampen diagram for the word w in $A(\Gamma_S)$. So we get $w =_{A(\Gamma_S)} 1$.

(2) $w =_{A(\Gamma)} w'$ for some word w' such that the letters of w' are in S. Let Δ be a dual van Kampen diagram for ww'^{-1}. If w contains a q–segment for some $q \notin S$, then a q–arc joins two segments in Δ, and these segments must be in w. This is impossible by Lemma 2.2. □

From this point on, $A(\Gamma_S)$ is considered as a subgroup of $A(\Gamma)$, for $S \subseteq V(\Gamma)$. Let H be a group and $\phi\colon C \to D$ be an isomorphism between subgroups of H. Then we define $H*_\phi = \langle H, t \mid t^{-1}ct = \phi(c),\ \text{for } c \in C\rangle$, which is the HNN extension of H with the amalgamating map ϕ and the stable letter t. Sometimes, we explicitly state what the stable letter is. If $C = D$ and ϕ is the identity map, then we let $H*_C = \langle H, t \mid t^{-1}ct = t \text{ for } c \in C\rangle$.

For a vertex v of a graph Γ, the *link of* v is the set

$$\mathrm{link}_\Gamma(v) = \{u \in V(\Gamma) : u \text{ is adjacent to } v\}.$$

Lemma 2.4 *Let Γ be a graph. Suppose Γ' is an induced subgraph of Γ such that $V(\Gamma') = V(\Gamma) \setminus \{v\}$ for some $v \in V(\Gamma)$. Let C be the subgroup of $A(\Gamma')$ generated by $\mathrm{link}_\Gamma(v)$. Then the inclusion $A(\Gamma') \hookrightarrow A(\Gamma)$ extends to the isomorphism $f\colon A(\Gamma')*_C \to A(\Gamma)$ such that $f(t) = v$.*

Proof Immediate from the definition of right-angled Artin groups. □

We first note the following general lemma.

Lemma 2.5 *Let H be a group and $\phi\colon C \to D$ be an isomorphism between subgroups C and D. Let K be a subgroup of H and $J = \langle K, t\rangle \le H*_\phi$. We let $\psi\colon J\cap C \to J\cap D$ be the restriction of ϕ. Then the inclusion $J \cap H \hookrightarrow J$ extends to the isomorphism $f\colon (J \cap H)*_\psi \to J$ such that $f(\hat{t}) = t$, where $\hat{t}$ and t denote the stable letters of $(J \cap H)*_\psi$ and $H*_\phi$, respectively.*

Proof Note that $G = H*_\phi$ acts on a tree T, with a vertex v_0 and an edge $e_0 = \{v_0, t.v_0\}$ satisfying $\mathrm{Stab}(v_0) = H$ and $\mathrm{Stab}(e_0) = C$ [18]. Let T_0 be the induced subgraph on $\{j.v_0 : j \in J\}$. For each vertex $j.v_0$ of T_0, write $j = k_1t^{\epsilon_1}k_2t^{\epsilon_2}\cdots k_mt^{\epsilon_m}$, where $k_i \in K$ and $\epsilon_i = \pm 1$ for each i. Then the following sequence in $V(T_0)$

$$\begin{aligned}
v_0 &= k_1.v_0,\\
k_1t^{\epsilon_1}.v_0 &= k_1t^{\epsilon_1}k_2.v_0,\\
k_1t^{\epsilon_1}k_2t^{\epsilon_2}.v_0 &= k_1t^{\epsilon_1}k_2t^{\epsilon_2}k_3.v_0,\\
&\cdots\\
k_1t^{\epsilon_1}k_2t^{\epsilon_2}k_3\cdots t^{\epsilon_m}.v_0 &= j.v_0
\end{aligned}$$

gives rise to a path in T_0 from v_0 to $j.v_0$. Hence T_0 is connected. Note that $\psi\colon J\cap C = \mathrm{Stab}_J(e_0) \to J \cap D = \mathrm{Stab}_J(e_0)^t$. Since J acts on a tree T_0, we have an isomorphism $J \cong \mathrm{Stab}_J(v_0)*_\psi = (J \cap H)*_\psi$. □

3 Co-contraction of graphs

Let Γ be a graph and $B \subseteq V(\Gamma)$. We say B is *connected*, if Γ_B is connected. B is *anticonnected*, if $\overline{\Gamma_B}$ is connected.

Definition 3.1 Let Γ be a graph and $B \subseteq V(\Gamma)$.

(i) If B is connected, the *contraction of* Γ *relative to* B is the graph $\mathrm{CO}(\Gamma, B)$ defined by:

$$V(\mathrm{CO}(\Gamma, B)) = (V(\Gamma) \setminus B) \cup \{v_B\}$$
$$E(\mathrm{CO}(\Gamma, B)) = E(\Gamma_{V(\Gamma)\setminus B}) \cup \{\{v_B, q\} : q \in V(\Gamma) \setminus B \text{ and } \mathrm{link}_\Gamma(q) \cap B \neq \varnothing\}$$

(ii) If B is anticonnected, the *co-contraction of* Γ *relative to* B is the graph $\overline{\mathrm{CO}}(\Gamma, B)$ defined by:

$$V(\overline{\mathrm{CO}}(\Gamma, B)) = (V(\Gamma) \setminus B) \cup \{v_B\}$$
$$E(\overline{\mathrm{CO}}(\Gamma, B)) = E(\Gamma_{V(\Gamma)\setminus B}) \cup \{\{v_B, q\} : q \in V(\Gamma) \setminus B \text{ and } \mathrm{link}_\Gamma(q) \supseteq B\}$$

(iii) More generally, if $B_1, B_2, \ldots, B_m$ are disjoint connected subsets of $V(\Gamma)$, then inductively define

$$\mathrm{CO}(\Gamma, (B_1, B_2, \ldots, B_m)) = \mathrm{CO}(\mathrm{CO}(\Gamma, (B_1, B_2, \ldots, B_{m-1})), B_m)$$

and if $B_1, B_2, \ldots, B_m$ are disjoint anticonnected subsets, then similarly,

$$\overline{\mathrm{CO}}(\Gamma, (B_1, B_2, \ldots, B_m)) = \overline{\mathrm{CO}}(\overline{\mathrm{CO}}(\Gamma, (B_1, B_2, \ldots, B_{m-1})), B_m).$$

In a graph Γ, if B is connected, then $\mathrm{CO}(\Gamma, B)$ is obtained by (homotopically) collapsing Γ_B onto one vertex and removing any loops or multi-edges. If B is anticonnected, one has (see Figure 3)

$$\overline{\mathrm{CO}}(\Gamma, B) = \overline{\mathrm{CO}(\overline{\Gamma}, B)}.$$

If $B \subseteq V(\Gamma)$ and $\mathrm{link}_\Gamma(q) \supseteq B$, then we say that q is a *common neighbor of* B.

The following lemma states that the co-contraction of a set of anticonnected vertices can be obtained by considering a sequence of co-contractions of two nonadjacent vertices. The proof is immediate by considering the complement graphs.

Lemma 3.2 *Let Γ be a graph and $B \subseteq V(\Gamma)$ be anticonnected. Then there exists a sequence of graphs*

$$\Gamma_0 = \Gamma, \Gamma_1, \Gamma_2, \ldots, \Gamma_p = \overline{\mathrm{CO}}(\Gamma, B)$$

such that for each $i = 0, 1, \ldots, p-1$, Γ_{i+1} is a co-contraction of Γ_i relative to a pair of nonadjacent vertices of Γ_i. □

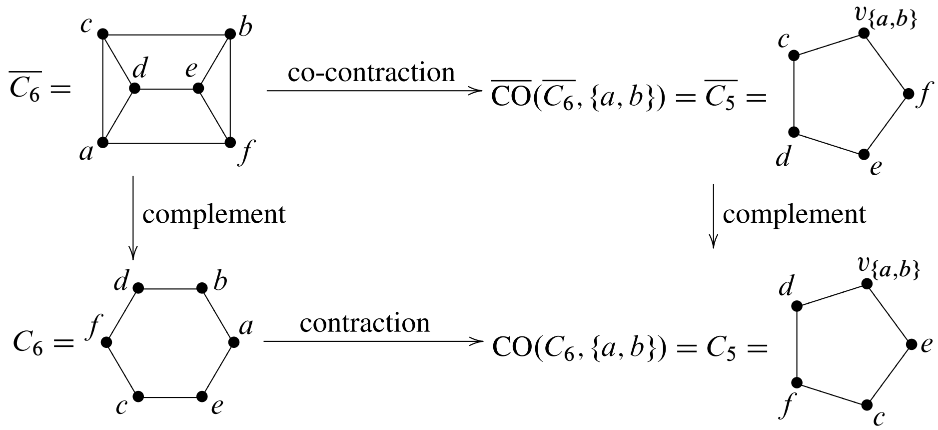


Figure 3: Note that $\{q : \mathrm{link}_{\overline{C_6}}(q) \supseteq \{a,b\}\} = \{c, f\}$, ie c and f are common neighbors of $\{a,b\}$. Hence in $\overline{\mathrm{CO}}(\overline{C_6}, \{a,b\})$, $v_{\{a,b\}}$ is adjacent to c and f. This can be also viewed by looking at the complement graph of $\overline{C_6}$, namely C_6, and collapsing the edge $\{a,b\}$.

Lemma 3.3 (i) *If B is a connected subset of p vertices of C_n, then* $\mathrm{CO}(C_n, B) \cong C_{n-p+1}$.

(ii) *If B is an anticonnected subset of p vertices of $\overline{C_n}$, then* $\overline{\mathrm{CO}}(\overline{C_n}, B) \cong \overline{C_{n-p+1}}$.

Proof (1) is obvious. Considering the complement graphs, (2) follows from (1). □

4 Co-contraction of graphs and right-angled Artin groups

Let Γ be a graph and B be an anticonnected subset of $V(\Gamma)$. Fix a word $\widetilde{w} \in \langle B \rangle$ in $A(\Gamma)$. If a vertex x of $\overline{\mathrm{CO}}(\Gamma, B)$ is adjacent to v_B, then x is a common neighbor of B in Γ, and so, $[x, \widetilde{w}] =_{A(\Gamma)} 1$. This implies that there exists a map ϕ: $A(\overline{\mathrm{CO}}(\Gamma, B)) \to A(\Gamma)$ satisfying:

$$\phi(x) = \begin{cases} \widetilde{w} & \text{if } x = v_B \\ x & \text{if } x \in V(\overline{\mathrm{CO}}(\Gamma, B)) \setminus \{v_B\} = V(\Gamma) \setminus B \end{cases}$$

In this section, we show that this map ϕ is injective for a suitable choice of the word $\widetilde{w}$. First, we prove the injectivity for the case when $B = \{a, b\}$ and $\widetilde{w} = b^{-1}ab$.

Lemma 4.1 *Let Γ be a graph. Suppose a and b are nonadjacent vertices of Γ. Then there exists an injective map $\phi\colon A(\overline{\mathrm{CO}}(\Gamma,\{a,b\}))\to A(\Gamma)$ satisfying:*

$$\phi(x)=\begin{cases} b^{-1}ab & \text{if } x=v_{\{a,b\}} \\ x & \text{if } x\in V(\Gamma)\setminus\{a,b\}\end{cases}$$

Proof Let $\widehat{\Gamma}=\overline{\mathrm{CO}}(\Gamma,\{a,b\}), \widehat{v}=v_{\{a,b\}}$ and $A=\{q : q\in V(\Gamma)\setminus\{a,b\}\}$. For $q\in A$, let $\widehat{q}$ denote the corresponding vertex in $\widehat{\Gamma}$, and $\widehat{A}=\{\widehat{q} : q\in A\}$.

Define $\phi\colon A(\widehat{\Gamma})\to A(\Gamma)$ by:

$$\phi(x)=\begin{cases} b^{-1}ab & \text{if } x=\widehat{v} \\ q & \text{if } x=\widehat{q}\in\widehat{A}\end{cases}$$

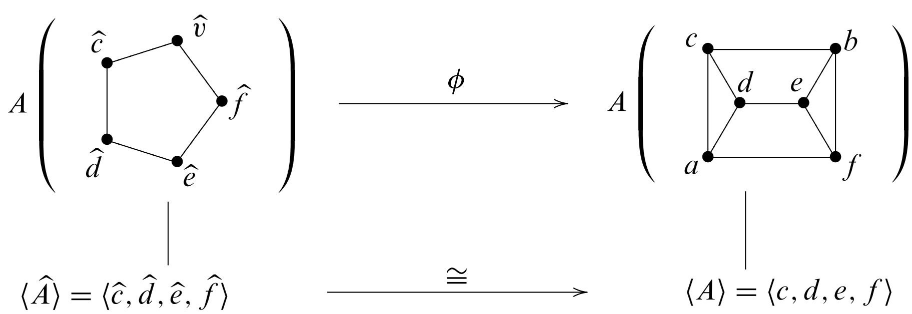
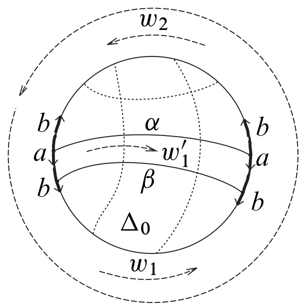


Figure 4:

Suppose ϕ is not injective. Choose a word $\widehat{w}$ of the minimal length in $\ker\phi\setminus\{1\}$. Write $\widehat{w}=\prod_{i=1}^{k}\widehat{c}_i^{\,e_i}$, where $\widehat{c}_i\in\widehat{A}\cup\{\widehat{v}\}$ and $e_i=\pm1$. As $\widehat{\Gamma}_{\widehat{A}}$ is isomorphic to Γ_A, ϕ maps $\langle\widehat{A}\rangle$ isomorphically onto $\langle A\rangle$ (Figure 4). So $\widehat{c}_i=\widehat{v}$ for some i.

Let $w=\prod_{i=1}^{k}\phi(\widehat{c}_i)^{e_i}$. Since $w=_{A(\Gamma)}1$, there exists a dual van Kampen diagram $\Delta=(\mathcal{H},\lambda)$ for w in $A(\Gamma)$. In Δ, choose an innermost a–arc α. By considering a cyclic conjugation of $\widehat{w}$ if necessary, one may write $\widehat{w}=\widehat{v}^{\pm1}\cdot\widehat{w}_1\cdot\widehat{v}^{\mp1}\cdot\widehat{w}_2$ and $w=b^{-1}a^{\pm1}b\cdot w_1\cdot b^{-1}a^{\mp1}b\cdot w_2$, so that $w_1=\phi(\widehat{w}_1), w_2=\phi(\widehat{w}_2)$ and α joins the leftmost $a^{\pm1}$ of w and the $a^{\mp1}$ between w_1 and w_2 (Figure 5). Then the interval w_1 does not contain any a–segment. Since each b–segment in w is adjacent to some a–segment, one sees that there does not exist any b–segment in w_1, either. Hence, $w_1\in\langle A\rangle=A(\Gamma_A)$ and $\widehat{w}_1\in\langle\widehat{A}\rangle=A(\widehat{\Gamma}_{\widehat{A}})$. Note that $\Gamma_A\cong\widehat{\Gamma}_{\widehat{A}}$. Since $\widehat{w}_1$ is reduced, so is w_1.



$$w = b^{-1}a^{\pm 1}b \cdot w_1 \cdot b^{-1}a^{\mp 1}b \cdot w_2$$

Figure 5: Δ in the proof of Lemma 4.1

Let β be the b–arc that meets the letter b, following $a^{\pm 1}$ on the left of w_1 in w. β does not intersect α, for $[a,b] \neq 1$. Since w_1 does not contain any b–segment, β intersects with the letter b^{-1} between w_1 and w_2.

w_1 does not contain any canceling pair, for w_1 is reduced. So each segment of w_1 is joined to a segment in w_2 by an arc in $\mathcal{H}$. Such an arc must intersect both α and β. This implies that the segments in w_1 are labeled by vertices in $\mathrm{link}_\Gamma(a) \cap \mathrm{link}_\Gamma(b) = \phi(\mathrm{link}_{\widehat{\Gamma}}(\widehat{v}))$. It follows that $\widehat{w}_1 \in \langle \mathrm{link}_{\widehat{\Gamma}}(\widehat{v}) \rangle$, from the following diagram.

$$\begin{array}{ccccc} \widehat{w}_1 \in \langle \widehat{A} \rangle & & \langle \mathrm{link}_{\widehat{\Gamma}}(\widehat{v}) \rangle & \leq & \langle \widehat{A} \rangle \\ \downarrow & & \downarrow \cong & & \phi \downarrow \cong \\ w_1 & \in & \langle \mathrm{link}_\Gamma(a) \cap \mathrm{link}_\Gamma(b) \rangle & \leq & \langle A \rangle \end{array}$$

But then, $\widehat{w} = \widehat{v}^{\pm 1}\widehat{w}_1\widehat{v}^{\mp 1}\widehat{w}_2 =_{A(\widehat{\Gamma})} \widehat{w}_1\widehat{w}_2$, which contradicts to the minimality of the length of $\widehat{w}$. □

Theorem 4.2 *Let Γ be a graph and B be an anticonnected subset of $V(\Gamma)$. Then $A(\Gamma)$ contains a subgroup isomorphic to $A(\overline{\mathrm{CO}}(\Gamma, B))$.*

Proof Proof is immediate from Lemma 3.2 and Lemma 4.1. □

Figure 4 and Lemma 4.1 show the existence of an isomorphism:

$$\phi\colon A(C_5) \to \langle b^{-1}ab, c, d, e, f \rangle \leq A(\overline{C_6})$$

More generally, we have the following corollary.

Corollary 4.3 (1) *$A(\overline{C_n})$ contains a subgroup isomorphic to $A(\overline{C_{n-p+1}})$ for each $1 \leq p \leq n$.*

(2) *If* Γ *contains an induced* C_n *or* $\overline{C_n}$ *for some* $n \geq 5$*, then* $A(\Gamma)$ *contains a hyperbolic surface group.*

Proof (1) Immediate from Lemma 3.3 and Theorem 4.2.

(2) $A(C_n)$ contains a hyperbolic surface group for $n \geq 5$ [19]. One has an embedding $\phi\colon A(C_5) = A(\overline{C_5}) \hookrightarrow A(\overline{C_n})$, for $n \geq 5$. □

A simple combinatorial argument shows that for $n > 5$, the induced subgraph of $\overline{C_n}$ on any five vertices contains a triangle. So $\overline{C_n}$ does not contain an induced C_m for any $m \geq 5$. From the Corollary 4.3 (2), we deduce the negative answer to Question 1.1 as follows.

Corollary 4.4 *There exists an infinite family* $\mathcal{F}$ *of graphs satisfying the following.*

(i) *Each element in* $\mathcal{F}$ *does not contain an induced* C_n *for* $n \geq 5$*.*

(ii) *Each element in* $\mathcal{F}$ *is not an induced subgraph of another element in* $\mathcal{F}$*.*

(iii) *For each* $\Gamma \in \mathcal{F}$*,* $A(\Gamma)$ *contains a hyperbolic surface group.*

Proof Set $\mathcal{F} = \{\overline{C_n} : n > 5\}$. □

5 Contraction words

In Lemma 4.1, the word $b^{-1}ab$ was used to construct an injective map from the group $A(\overline{\mathrm{CO}}(\Gamma, \{a, b\}))$ into $A(\Gamma)$. This can be generalized by considering a *contraction word*, defined as follows.

Definition 5.1 (1) Let Γ_0 be an anticonnected graph. A sequence $b_1, b_2, \ldots, b_p$ of vertices of Γ_0 is a *contraction sequence of* Γ_0, if the following holds: for any $(b, b') \in V(\Gamma_0) \times V(\Gamma_0)$, there exists $l \geq 1$ and $1 \leq k_1 < k_2 < \cdots < k_l \leq p$ such that, $b_{k_1}, b_{k_2}, \ldots, b_{k_l}$ is a path from b to b' in $\overline{\Gamma}$.

(2) Let Γ be a graph and B be an anticonnected set of vertices of Γ. A reduced word $w = \prod_{i=1}^{p} b_p^{e_i}$ is called a *contraction word of* B if $b_i \in B$, $e_i = \pm 1$ for each i, and $b_1, b_2, \ldots, b_p$ is a contraction sequence of Γ_B. An element of $A(\Gamma)$ is called a *contraction element*, if it can be represented by a contraction word.

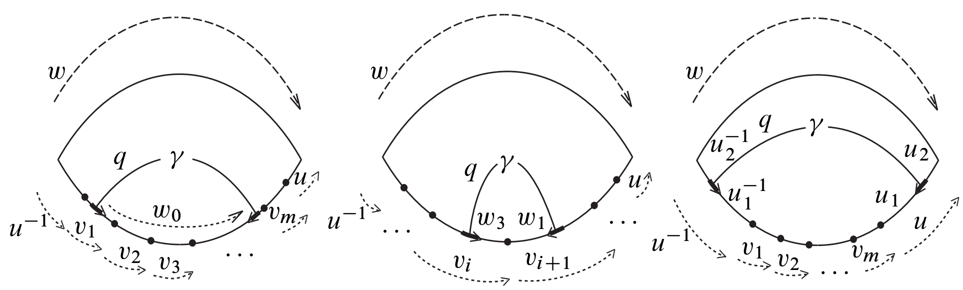
Remark 5.2 If a and b are nonadjacent vertices in Γ, then any word in $\langle a, b\rangle \backslash \{a^m b^n : m, n \in \mathbb{Z}\}^{\pm 1}$ is a contraction word of $\{a, b\}$.

We first note the following general lemma.

Lemma 5.3 *Let* Γ *be a graph and* $g \in A(\Gamma)$. *Then* $g =_{A(\Gamma)} u^{-1}vu$ *for some words* u, v *such that* $u^{-1}v^m u$ *is reduced for each* $m \neq 0$.

Proof Choose words u, v such that $u^{-1}vu$ is a reduced word representing g and the length of u is maximal. We will show that $u^{-1}v^m u$ is reduced for any $m \neq 0$.

Assume that $u^{-1}v^m u$ is not reduced for some $m \neq 0$. We may assume that $m > 0$. Let w be a reduced word for $u^{-1}v^m u$. Draw a dual van Kampen diagram Δ for $u^{-1}v^m uw^{-1}$. Let v_i denote the v–interval on $\partial\Delta$ corresponding to the i–th occurrence of v from the left in $u^{-1}v^m u$ (Figure 6 (a)).



(a) (b) Case 1 (c) Case 2

Figure 6: Proof of Lemma 5.3

By Lemma 2.2, there exists a q–arc γ joining two q–segments of $u^{-1}v^m u$ for some $q \in V(\Gamma)$. Let w_0 denote the interval between those two q–segments. We may choose q and γ so that the number of the segments in w_0 is minimal. Then any arc intersecting with a segment in w_0 must intersect γ. It follows that any letter in w_0 should commute with q. Moreover, w_0 does not contain any q–segment.

Case 1 *The intervals* u^{-1} *and* u *do not intersect with* γ.

Because w_0 does not contain any q–segment, γ joins v_i and v_{i+1} for some i (Figure 6 (b)). So one can write $v = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$ for some subwords w_1, w_2, w_3 of v such that and $w_0 = w_3 w_1$. $[w_3, q] =_{A(\Gamma)} 1 =_{A(\Gamma)} [w_1, q]$. So $u^{-1}vu =_{A(\Gamma)} u^{-1}q^{\pm 1}w_1 w_2 w_3 q^{\mp 1}u$, which contradicts to the maximality of u.

Case 2 γ *intersects* u*- or* u^{-1}*–interval.*

Suppose u^{-1} intersects γ. Since $u^{-1}v$ is reduced, γ cannot intersect v_1. So, w_0 contains v_1. Since w_0 does not contain any q–segment, v does not contain the letters

q or q^{-1} and so, γ cannot intersect any v_i for $i = 1, \dots, m$. γ should intersect with the u–interval of $u^{-1}v^m u$ (Figure 6 (c)). This implies that γ intersects with the leftmost q–segment in the u–interval of $u^{-1}v^m u$. One can write $u^{-1}v^m u = {u_2}^{-1}q^{\pm 1}{u_1}^{-1}v^m u_1 q^{\mp 1}u_2$ such that any letter in $w_0 = {u_1}^{-1}v^m u_1$ commutes with q, ie $[q, u_1] =_{A(\Gamma)} 1 =_{A(\Gamma)} [q, v]$. But then $u^{-1}vu =_{A(\Gamma)} {u_2}^{-1}{u_1}^{-1}vu_1u_2$, which is a contradiction to the assumption that $u^{-1}vu$ is reduced. □

Lemma 5.4 (1) *Any reduced word for a contraction element is a contraction word.*

(2) *Any nontrivial power of a contraction element is a contraction element.*

Proof (1) Let $w = \prod_{i=1}^{p} b_p^{e_i}$ be a contraction word of an anticonnected set B in $V(\Gamma)$. Here, $b_i \in B$ and $e_i = \pm 1$ for each i. Suppose w' is a reduced word, such that $w' =_{A(\Gamma)} w$. There exists a dual van Kampen diagram Δ for ww'^{-1}. Note that any properly embedded arc of Δ meets both of the intervals w and w', since w and w' are reduced (Lemma 2.2). Now let $b, b' \in B$. w is a contraction word, so one can find $l \geq 1$ and $1 \leq k_1 < k_2 < \cdots < k_l \leq p$ such that, b_{k_i} and $b_{k_{i+1}}$ are nonadjacent for each $i = 1, \dots, l-1$, and $b = b_{k_1}, b' = b_{k_l}$. Let γ_i be the arc that intersects with the segment b_{k_i} of w. Since $\gamma_1, \gamma_2, \dots, \gamma_l$ are all disjoint, the boundary points of those arcs on w' will yield the desired subsequence of the letters of w'.

(2) Let $u^{-1}vu$ be a reduced word for g as in Lemma 5.3. Note that a sequence, containing a contraction sequence as a monotonic subsequence, is again a contraction sequence. So the reduced word $u^{-1}v^m u$ is a contraction word of B, for each $m \neq 0$. □

Definition 5.5 Let Γ be a graph, and P and Q be disjoint subsets of $V(\Gamma)$. Suppose P_1 is a set of words in $\langle P \rangle \leq A(\Gamma)$. A *canonical expression for* $g \in \langle P_1, Q \rangle$ *with respect to* $\{P_1, Q\}$ is a word $\prod_{i=1}^{k} c_i^{e_i}$, where

(i) $c_i \in P_1 \cup Q$

(ii) $e_i = 1$ or -1

(iii) $\prod_{i=1}^{k} c_i^{e_i} =_{A(\Gamma)} g$

such that k is minimal. k is called the *length* of the canonical expression.

Remark 5.6 In the above definition, a canonical expression exists for any element in $\langle P_1, Q \rangle$. In the case when $P_1 \subseteq P$, a word is a canonical expression with respect to $\{P_1, Q\}$, if and only if it is reduced in $A(\Gamma)$.

Now we compute intersections of certain subgroups of $A(\Gamma)$.

Lemma 5.7 *Let Γ be a graph, P, Q be disjoint subsets of $V(\Gamma)$ and P_1 be a set of words in $\langle P\rangle \le A(\Gamma)$. Let R be any subset of $V(\Gamma)$.*

1. *If w is a canonical expression with respect to $\{P_1, Q\}$, then there does not exist a q–pair of w for any $q \in Q$.*
2. *$\langle P_1, Q\rangle \cap \langle R\rangle \subseteq \langle P_1, Q\cap R\rangle$. Moreover, the equality holds if $P \subseteq R$.*
3. *Let $\widetilde{w}$ be a contraction word of P, and $P_1 = \{\widetilde{w}\}$. Assume $P \not\subseteq R$. Then $\langle P_1, Q\rangle \cap \langle R\rangle = \langle Q\cap R\rangle$.*

Proof (1) Let w be a canonical expression, Suppose there exists a q–pair of w for some $q \in Q$. Then by Lemma 2.2, one can write $w = w_1 q^{\pm 1} w_2 q^{\mp 1} w_3$ for some subwords w_1, w_2 and w_3 such that $w_2 \in \langle \mathrm{link}_\Gamma(q)\rangle$. It follows that $w =_{A(\Gamma)} w'' = w_1 w_2 w_3$. Since $P \cap Q = \varnothing$, w_1, w_2 and w_3 are also canonical expressions with respect to $\{P_1, Q\}$. This contradicts to the minimality of k.

(2) Let w be a canonical expression of an element in $\langle P_1, Q\rangle \cap \langle R\rangle$, and $w' =_{A(\Gamma)} w$ be a reduced word. Consider a dual van Kampen diagram Δ for ww'^{-1}.

Suppose that there exists a q–segment in w, for some $q \in Q$. Then by (1), the q–segment should be joined, by a q–arc, to another q–segment of w'. Since w' is a reduced word representing an element in $\langle R\rangle$, each segment of w' is labeled by $R^{\pm 1}$ (Lemma 2.3 (2)). Therefore, $q \in Q \cap R$.

If $P \subseteq R$, then $\langle P_1, Q\cap R\rangle \subseteq \langle P_1, Q\rangle \cap \langle R\rangle$ is obvious.

(3) $\langle Q\cap R\rangle \subseteq \langle P_1, Q\rangle \cap \langle R\rangle$ is obvious.

To prove the converse, suppose $w \in (\langle P_1, Q\rangle \cap \langle R\rangle) \setminus \langle Q\cap R\rangle$. w is chosen so that w is a canonical expression with respect to $\{P_1, Q\}$, and the length (as a canonical expression) is minimal.

Let $w = \prod_{i=1}^{k} c_i^{e_i}$ ($c_i \in \{P_1, Q\}$, $e_i = \pm 1$), w' be a reduced word satisfying $w' =_{A(\Gamma)} w$, and $\Delta = (\mathcal{H}, \lambda)$ be a dual van Kampen diagram for ww'^{-1} (Figure 7). From the proof of (2), $c_i \in P_1 \cup (Q\cap R) = \{\widetilde{w}\} \cup (Q\cap R)$ for each i. Also, any shorter canonical expression than w, for an element in $\langle P_1, Q\rangle \cap \langle R\rangle$, is in $\langle Q\cap R\rangle$. This implies that $c_1 = \widetilde{w} = c_k$. Note that each segment of w' is labeled by $R^{\pm 1}$.

Now suppose $c_i = \widetilde{w}$ for some i. Fix $b \in P \setminus R$. Choose the b–arc β that intersects with the leftmost b–segment in w on $\partial\Delta$. Note that this b–segment is contained in the leftmost $\widetilde{w}$–interval in w.

Write $w = \widetilde{w}^m w_1 \widetilde{w}^e w_2$ for some subwords w_1, w_2 of w, $m \in \mathbb{Z}\setminus\{0\}$ and $e \in \{1, -1\}$. Here, w_1 and w_2 are chosen so that the letters of w_1 are in $(Q\cap R)^{\pm 1}$ and β intersects with a segment in the interval $\widetilde{w}^e w_2$. Without loss of generality, we may assume $m > 0$.

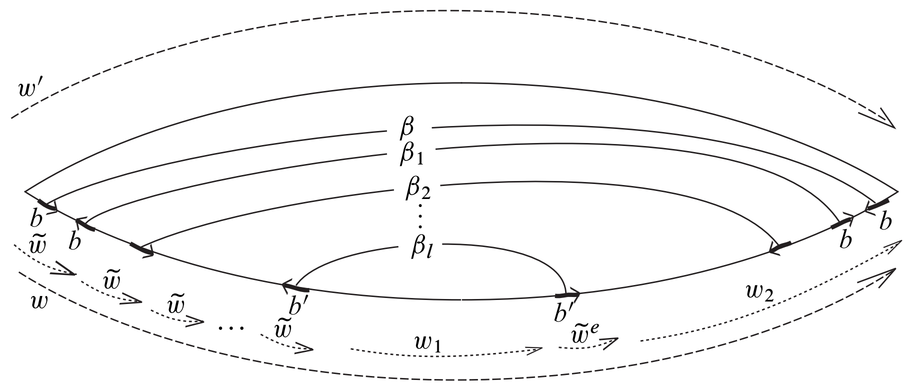


Figure 7: Δ in the proof of Lemma 5.7

Let b' be any element in P. By Lemma 5.4, any reduced word for $\widetilde{w}^m$ is a contraction word of P. So, one can find a sequence of arcs $\beta_1, \beta_2, \dots, \beta_l \in \mathcal{H}$ such that

(i) $\lambda(\beta_1) = b, \lambda(\beta_l) = b'$,

(ii) $\lambda(\beta_i)$ and $\lambda(\beta_{i+1})$ are nonadjacent in Γ, for each $i = 1, 2, \dots, l-1$, and

(iii) each β_i intersects with a segment in the interval $\widetilde{w}^e w_2$.

Note that (iii) comes from the assumptions that β_i does not join two segments from $\widetilde{w}^m$ (by reducing $\widetilde{w}^m$ first), and that the letters of w_1 are in $(Q \cap R)^{\pm 1}$, which is disjoint from P.

As in the proof of (2), each segment of w_1 is joined to a segment in w'. In particular, $[b', w_1] = [\lambda(\beta_l), w_1] =_{A(\Gamma)} 1$. Since this is true for any $b' \in P$, $w =_{A(\Gamma)} w_1 \widetilde{w}^{m+e} w_2$. One has $\widetilde{w}^{m+e} w_2 \in (\langle P_1, Q\rangle \cap \langle R\rangle) \setminus \langle Q \cap R\rangle$, for $w \notin \langle Q \cap R\rangle$ and $w_1 \in \langle Q \cap R\rangle$. By the minimality of w, we have $w_1 = 1$. This argument continues, and finally one can write $w = \widetilde{w}^{m'}$ for some $m' \neq 0$. In particular, any reduced word for w is a contraction word of P (Lemma 5.4). This is impossible since $w \in \langle R\rangle$ and $P \not\subseteq R$. □

Lemma 5.8 *Let Γ be a graph, B be an anticonnected set of vertices of Γ and g be a contraction element of B. Then there exists an injective map $\phi\colon A(\overline{\mathrm{CO}}(\Gamma, B)) \to A(\Gamma)$ satisfying:*

$$\phi(x) = \begin{cases} g & \text{if } x = v_B \\ x & \text{if } x \in V(\Gamma) \setminus B \end{cases}$$

Proof As in the proof of Lemma 4.1, let $\widehat{\Gamma} = \overline{\mathrm{CO}}(\Gamma, B)$, $\widehat{v} = v_B$ and $A = \{q : q \in V(\Gamma) \setminus B\}$. For $q \in A$, let $\widehat{q}$ denote the corresponding vertex in $\widehat{\Gamma}$, and $\widehat{A} = \{\widehat{q} : q \in A\}$.

There exists a map $\phi\colon A(\widehat{\Gamma}) \to A(\Gamma)$ satisfying:

$$\phi(x) = \begin{cases} g & \text{if } x = \hat{v} \\ q & \text{if } x = \hat{q} \in \widehat{A} \end{cases}$$

To prove that ϕ is injective, we use an induction on $|A|$.

If $A = \varnothing$, then $V(\Gamma) = B$ and $\widehat{\Gamma}$ is the graph with one vertex $\hat{v}$. So, ϕ maps $\langle \hat{v} \rangle = A(\widehat{\Gamma}) \cong \mathbb{Z}$ isomorphically onto $\mathbb{Z} \cong \langle g \rangle \le A(\Gamma)$.

Assume the injectivity of ϕ for the case when $|A| = k$, and now let $|A| = k+1$.

Choose any $t \in A$. Let $A_0 = A \setminus \{t\}$ and $\widehat{A}_0 = \{\hat{q} : q \in A_0\}$. Let Γ_0 be the induced subgraph on $A_0 \cup B$ in Γ, and $\widehat{\Gamma}_0$ be the induced subgraph on $\widehat{A}_0 \cup \{\hat{v}\}$ in $\widehat{\Gamma}$. We consider $A(\Gamma_0)$ and $A(\widehat{\Gamma}_0)$ as subgroups of $A(\Gamma)$ and $A(\widehat{\Gamma})$, respectively, so that $A(\Gamma_0) = \langle A_0, B \rangle$ and $A(\widehat{\Gamma}_0) = \langle \widehat{A}_0, \hat{v} \rangle$. Let $K = \langle A_0, g \rangle = \phi(A(\widehat{\Gamma}_0))$ and $J = \langle A, g \rangle = \phi(A(\widehat{\Gamma}))$. By the inductive hypothesis, ϕ maps $A(\widehat{\Gamma}_0)$ isomorphically onto K (Figure 8).

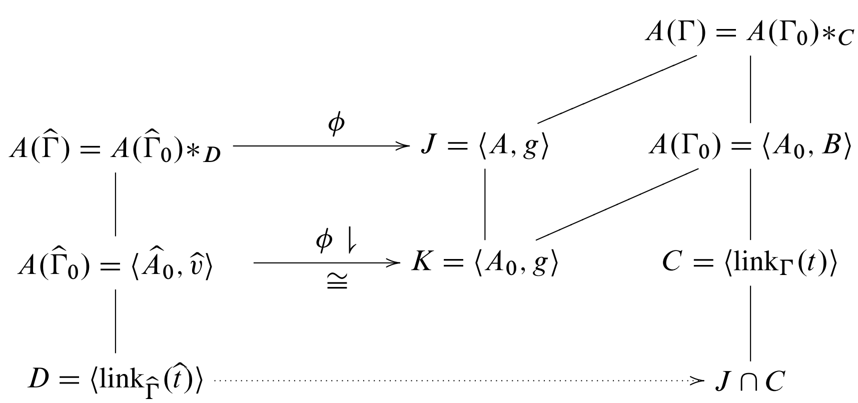


Figure 8: Proof of Lemma 5.8. Note that $V(\Gamma) = A \sqcup B = A_0 \cup \{t\} \cup B$ and $V(\widehat{\Gamma}) = \widehat{A} \sqcup \{\hat{v}\} = \widehat{A}_0 \cup \{\hat{t}\} \cup \{\hat{v}\}$.

From Lemma 2.4, we can identify $A(\Gamma) = A(\Gamma_0)*_C$, where $C = \langle \mathrm{link}_\Gamma(t) \rangle$ and t is the stable letter. Since $J = \langle A_0, g, t \rangle = \langle K, t \rangle$, Lemma 2.5 implies that we can also identify $J = (J \cap A(\Gamma_0))*_{J \cap C}$, where t is the stable letter again. Also, we identify $A(\widehat{\Gamma}) = A(\widehat{\Gamma}_0)*_D$, where $D = \langle \mathrm{link}_{\widehat{\Gamma}}(\hat{t}) \rangle$ and $\hat{t}$ is the stable letter.

By Lemma 5.7 (2), $J \cap A(\Gamma_0) = \langle g, A \rangle \cap \langle A_0, B \rangle = \langle g, A \cap (A_0 \cup B) \rangle = \langle g, A_0 \rangle = \phi(A(\widehat{\Gamma}_0))$.

Applying Lemma 5.7 (2) and (3) for the case when $R = \mathrm{link}_\Gamma(t)$, we have:

$$J \cap C = \langle g, A\rangle \cap \langle \mathrm{link}_\Gamma(t)\rangle = \begin{cases} \langle \mathrm{link}_\Gamma(t) \cap A, g\rangle & \text{if } B \subseteq \mathrm{link}_\Gamma(t) \\ \langle \mathrm{link}_\Gamma(t) \cap A\rangle & \text{otherwise} \end{cases}$$

From the definition of a co-contraction, we note that:

$$D = \mathrm{link}_{\widehat{\Gamma}}(\widehat{t}) = \begin{cases} \{\widehat{q} : q \in \mathrm{link}_\Gamma(t) \cap A\} \cup \{\widehat{v}\} & \text{if } B \subseteq \mathrm{link}_\Gamma(t) \\ \{\widehat{q} : q \in \mathrm{link}_\Gamma(t) \cap A\} & \text{otherwise} \end{cases}$$

Hence, $J \cap C = \phi(D)$. This implies that $\phi\colon A(\widehat{\Gamma}) \to J$ is an isomorphism, as follows.

$$\begin{array}{ccccccc} D & \leq & A(\widehat{\Gamma}_0) & \leq & A(\widehat{\Gamma}_0) *_D & = & A(\widehat{\Gamma}) \\ \downarrow \cong & & \downarrow \cong & & & & \phi \downarrow \\ J \cap C & \leq & K = J \cap A(\Gamma_0) & \leq & (J \cap A(\Gamma_0)) *_{J \cap C} & = & J \end{array}$$

□

Now the following theorem is immediate by an induction on m.

Theorem 5.9 *Let Γ be a graph and $B_1, B_2, \ldots, B_m$ be disjoint subsets of $V(\Gamma)$ such that each B_i is anticonnected. For each i, let v_{B_i} denote the vertex corresponding to B_i in $\overline{\mathrm{CO}}(\Gamma, (B_1, B_2, \ldots, B_m))$, and g_i be a contraction element of B_i. Then there exists an injective map $\phi\colon A(\overline{\mathrm{CO}}(\Gamma, (B_1, B_2, \ldots, B_m))) \to A(\Gamma)$ satisfying:*

$$\phi(x) = \begin{cases} g_i & \text{if } x = v_{B_i}, \text{ for some } i \\ x & \text{if } x \in V(\Gamma) \setminus \bigcup_{i=1}^m B_i \end{cases}$$

We conclude this article by noting that there is another partial answer to the question of which right-angled Artin groups contain hyperbolic surface groups. Namely, if Γ does not contain an induced cycle of length ≥ 5, and either Γ does not contain an induced C_4 (hence chordal), or Γ is triangle-free (hence bipartite), then $A(\Gamma)$ does not contain a hyperbolic surface group [14]. In [2], an independent study by Crisp, Sapir and Sageev proves a similar result, as well as the complete classification of graphs with up to eight vertices, on which the corresponding right-angled Artin groups contain hyperbolic surface subgroups.

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Received: 4 January 2008