Knot exteriors with additive Heegaard genus and Morimoto’s Conjecture

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Given integers \( g \geq 2, n \geq 1 \) we prove that there exist a collection of knots, denoted by \( \mathcal{K}_{g,n} \), fulfilling the following two conditions:

1) For any integer \( 2 \leq h \leq g \), there exist infinitely many knots \( K \in \mathcal{K}_{g,n} \) with \( g(E(K)) = h \).

2) For any \( m \leq n \), and for any collection of knots \( K_1, \ldots, K_m \in \mathcal{K}_{g,n} \), the Heegaard genus is additive:

\[
g(E(\#_{i=1}^{m} K_i)) = \sum_{i=1}^{m} g(E(K_i)).
\]

This implies the existence of counterexamples to Morimoto’s Conjecture [17].

57M25; 57M27

1 Introduction and statements of results

Let \( K_i \ (i = 1, 2) \) be knots in the 3–sphere \( S^3 \), and let \( K_1 \# K_2 \) be their connected sum. We use the notation \( t(\cdot) \), \( E(\cdot) \), and \( g(\cdot) \) to denote tunnel number, exterior, and Heegaard genus respectively. It is well known that the union of a tunnel system for \( K_1 \), a tunnel system for \( K_2 \) and a tunnel on a decomposing annulus for \( K_1 \# K_2 \) forms a tunnel system for \( K_1 \# K_2 \). Therefore:

\[
t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1.
\]

Since \( t(K) = g(E(K)) - 1 \), this gives:

\[
g(E(K_1 \# K_2)) \leq g(E(K_1)) + g(E(K_2)).
\]

Given integers \( g \geq 0 \) and \( n \geq 1 \), we say that a knot \( K \) in a closed orientable manifold \( M \) admits a \( (g,n) \) position if there exists a genus \( g \) Heegaard surface \( \Sigma \) for \( M \), separating \( M \) into the handlebodies \( H_1 \) and \( H_2 \), so that \( H_i \cap K \ (i = 1, 2) \) consists of \( n \) arcs that are simultaneously parallel into \( \partial H_i \). We say that \( K \) admits a \( (g,0) \) position if \( g(E(K)) \leq g \). Note that if \( K \) admits a \( (g,n) \) position, then \( K \) admits both a \( (g,n+1) \) position and a \( (g+1,n) \) position.
From Morimoto [17, Proposition 1.3], it is known that if $K_i$ ($i = 1$ or 2) admits a $(t(K_i), 1)$ position, then Inequality (1) is strict:

\[(2) \quad g(E(K_1\#K_2)) < g(E(K_1)) + g(E(K_2)).\]

Morimoto proved that if $K_1$ and $K_2$ are m-small knots\(^1\) in $S^3$, then the converse holds [17, Theorem 1.6]. This result was generalized to arbitrarily many m-small knots in general manifolds by the authors [9]. Morimoto conjectured that the converse holds in general [17, Conjecture 1.5]:

**Morimoto’s Conjecture**  Given knots $K_1, K_2 \subset S^3$,
\[g(E(K_1\#K_2)) < g(E(K_1)) + g(E(K_2))\]

if and only if $K_i$ admits a $(t(K_i), 1)$ position (for $i = 1$ or $i = 2$).

**Remark 1.1**  Morimoto stated the above conjecture in terms of 1–bridge genus $g_1(K)$. It is easy to see that Conjecture 1.5 of [17] is equivalent to the statement above.

In [10] the authors showed that the existence of a knot $K$ satisfying the two conditions below implies the existence of counterexamples to **Morimoto’s Conjecture**:

- $K$ does not admit a $(t(K), 2)$ position.
- $K$ is m-small.

We asked [10, Question 1.9] if there exists a knot $K$ with $g(E(K)) = 2$ that does not admit a $(1, 2)$ position; this question was answered affirmatively by Johnson and Thompson. In fact, in [5, Lemma 4] Johnson showed the existence of knots $K$ with $g(E(K)) = 2$ admitting Heegaard splittings with arbitrarily high distance (see Definition 2.4), and in [6, Corollary 2] Johnson and Thompson showed that (for any $n$) infinitely many of these knots do not admit a $(1, n)$ position. At about the same time Minsky, Moriah and Schleimer [11, Theorem 3.1] proved a more general result, showing that for any integer $g \geq 2$, there exist infinitely many knots $K$ with $g(E(K)) = g$ admitting a minimal genus Heegaard splitting with arbitrarily high distance. By Proposition 2.6 (for any $n$) infinitely many of these knots do not admit a $(t(K), n)$ position. However, at the time of writing, the existence of an m-small knot $K$ not admitting a $(t(K), 2)$ position is not known.

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\(^1\)A knot $K$ is called m-small if its exterior does not admit an essential surface whose boundary consists of a nonempty collection of meridians of $K$. 

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Given \( n \geq 1 \), consider the following conditions:

1. \( K \) does not admit a \((t(K), n)\) position.
2. \( E(K) \) does not admit an essential surface \( S \) with \( \chi(S) \geq 4 - 2ng(E(K)) \).

Our main result is Theorem 1.2 below, which implies the existence of knots fulfilling Conditions (1) and (2) for each \( n \geq 1 \); specifically, in the proof of Theorem 1.2 we show that some of the knots whose existence was proved in [5] and [11] fulfill these conditions. In Corollary 1.5, we show that this implies the existence of counterexamples to Morimoto’s Conjecture.

**Theorem 1.2** Given integers \( g \geq 2 \) and \( n \geq 1 \), let \( \mathcal{K}_{g,n} \) be the set of all knots \( K \subset S^3 \) with the following three properties:

(a) \( g(E(K)) \leq g \).
(b) \( K \) does not admit a \((t(K), n)\) position.
(c) \( E(K) \) does not admit an essential surface \( S \) with \( \chi(S) \geq 4 - 2gn \).

Then \( \mathcal{K}_{g,n} \) has the following properties:

1. For each \( h \), \( 2 \leq h \leq g \), there exists infinitely many knots \( K \in \mathcal{K}_{g,n} \) with \( g(E(K)) = h \).
2. For each \( m \leq n \) and for any collection of knots \( K_1, \ldots, K_m \in \mathcal{K}_{g,n} \) (possibly, \( K_i = K_j \) for \( i \neq j \)) we have:
\[
g(E(\#_{i=1}^m K_i)) = \sum_{i=1}^m g(E(K_i)).
\]

Moreover, for each \( g \), we have:
\[
\bigcap_{n=1}^{\infty} \mathcal{K}_{g,n} = \emptyset.
\]

**Remark 1.3** The existence of knots \( K_1, K_2 \) with \( g(E(K_1 \# K_2)) = g(E(K_1)) + g(E(K_2)) \) is known from Moriah and Rubinstein [15] and Morimoto, Sakuma and Yokota [18]. Theorem 1.2 is new in the following ways:

1. It is the first time that the connected sum of more than two knots is shown to have additive Heegaard genus.
A knot $K \subset M$ is called \textit{admissible} if $g(E(K)) > g(M)$. Thus any knot $K \subset S^3$ is admissible. We denote the connected sum of $m$ copies of $K$ by $mK$. By [10, Theorem 1.2] for any admissible knot $K$, there exists $N$ so that if $m > N$ then $g(E(mK)) < mg(E(K))$. In contrast to this, as an obvious consequence of Theorem 1.2 we have:

**Corollary 1.4** Given integers $g \geq 2$ and $n \geq 1$, there exist infinitely many knots $K \subset S^3$ so that $g(E(K)) = g$ and for any $m \leq n$, $g(E(mK)) = mg$.

A consequence of Corollary 1.4 is:

**Corollary 1.5** There exists a counterexample to Morimoto’s Conjecture. Specifically, there exist knots $K_1, K_2 \subset S^3$ such that the following two conditions hold:

1. $K_1$ does not admit a $(t(K_1), 1)$ position ($i = 1, 2$).
2. There exists an integer $m_0 \geq 4$ such that:
   a. $g(E(K_1)) = 4$.
   b. $g(E(K_2)) = 2(m_0 - 2)$.
   c. $g(E(K_1 \# K_2)) < 2m_0$.

The argument of the proof of Corollary 1.5 was originally given in [10, Theorem 1.4]. We outline it here for completeness.

**Proof of Corollary 1.5** Let $K$ be a knot as in Corollary 1.4, for $g = 2$ and $n = 3$. By [10, Theorem 1.2], for some $m > 1$, $g(E(mK)) < mg(E(K)) = 2m$. Let $m_0$ be the minimal number with that property. Since we chose $K$ for $n = 3$, $m_0 \geq 4$. Hence $g(E(2K)) = 2g(E(K)) = 4$. By the minimality of $m_0$, $g(E((m_0 - 2)K)) = (m_0 - 2)g(E(K)) = 2(m_0 - 2)$. Let $K_1 = 2K$ and $K_2 = (m_0 - 2)K$. Note that $K_1 \# K_2 = m_0 K$. Thus:

a. $g(E(K_1)) = 4$.

b. $g(E(K_2)) = 2(m_0 - 2)$.

c. $g(E(K_1 \# K_2)) < 2m_0$.

We claim that $K_1$ does not admit a $(t(K_1), 1)$ position. Assume for a contradiction it does. By Inequality (2) and the above (a), $g(E(3K)) = g(E(K_1 \# K)) < g(E(K_1)) + g(E(K)) = 6$. Since $m_0 \geq 4$, $g(E(3K)) = 3g(E(K)) = 6$, which is a contradiction.

We claim that $K_2$ does not admit a $(t(K_2), 1)$ position. Assume for a contradiction it does. By Inequality (2) and the above (b), $g(E((m_0 - 1)K)) < g(E((m_0 - 2)K)) + g(E(K)) = (m_0 - 1)g(E(K))$. By the minimality of $m_0$, $g(E((m_0 - 1)K)) = (m_0 - 1)g(E(K))$, which is a contradiction.  

\[\square\]
We note that $K_1$ and $K_2$ are composite knots. This led Moriah to conjecture [13, Conjecture 7.14] that if $K_1$ and $K_2$ are prime then Morimoto’s Conjecture holds.

Outline Section 2 is devoted to three propositions necessary for the proof of Theorem 1.2: Proposition 2.2 relates strongly irreducible Heegaard splittings and bridge position, Proposition 2.5 relates essential surfaces and the distance of Heegaard splitting (Proposition 2.5 is exactly Theorem 3.1 of Scharlemann [22]), and Proposition 2.6 relates bridge position and distance of Heegaard splittings (Proposition 2.6 is exactly Theorem 1 of Johnson and Thompson [6] except for knots $K \subset M$ that admit a $t(K)$, 1) position and are isotopic onto a Heegaard surface for $M$ of genus $t(K)$). In Section 3 we calculate the genera of certain manifolds that we denote by $X^{(e)}$ (see Notation 2.1). In Section 4 we prove Theorem 1.2.

Remarks 1.6 (1) Tomova, independently and using different techniques, obtained a stronger result than Proposition 2.6 [28, Theorem 1.3].

(2) We refer the reader to our paper [7], that can be used as an introduction to the ideas in the current paper. In [7] an easy argument is given for a special case of Corollary 1.4, namely, $g = 2$ and $n = 3$. Note that this special case is sufficient for Corollary 1.5.

2 Decomposing $X^{(e)}$

In this and the following sections, we adopt the following notation.

Notation 2.1 Let $K$ be a knot in a closed orientable connected manifold $M$ and $X$ its exterior. For an integer $c \geq 0$ we denote by $X^{(e)}$ the manifold obtained by drilling $c$ curves out of $X$ that are simultaneously parallel to meridians of $K$. Note that $X^{(0)} = X$.

Proposition 2.2 Let $X$, $X^{(e)}$ be as above and $g \geq 0$ an integer. Suppose that for some integer $c > 0$, $X^{(e)}$ admits a strongly irreducible Heegaard surface of genus $g$. Then one of the following holds:

1. $X$ admits an essential surface $S$ with $\chi(S) \geq 4 - 2g$.

2. (a) $c \leq g$, and
   (b) for some $b$, $c \leq b \leq g$, $K$ admits a $(g - b, b)$ position.
Proof of Proposition 2.2 Assume Conclusion (1) does not hold.

Let $C_1 \cup \Sigma C_2$ be a genus $g$ strongly irreducible Heegaard splitting of $X^{(c)}$. Since $c > 0$, $X^{(c)}$ admits an essential torus $T$ that gives the decomposition $X^{(c)} = X' \cup T Q^{(c)}$, where $X' \cong X$ and $Q^{(c)}$ is a $c$–times punctured annulus cross $S^1$. Since $T$ is incompressible and $\Sigma$ is strongly irreducible, we may isotope $\Sigma$ so that every component of $\Sigma \cap T$ is essential in both surfaces (see, for example, Schultens [26, Lemma 6]). Isotope $\Sigma$ to minimize $|\Sigma \cap T|$ subject to this constraint. Denote $\Sigma \cap X'$ by $\Sigma_X$, and $\Sigma \cap Q^{(c)}$ by $\Sigma_Q$. Note that, since $T$ is essential, $\Sigma \cap T \neq \emptyset$. By the minimality of $|\Sigma \cap T|$ no component of $\Sigma_X$ (resp. $\Sigma_Q$) is boundary parallel in $X'$ (resp. $Q^{(c)}$).

We claim that $\Sigma_X$ is connected and compresses into both sides in $X'$, and that $\Sigma_Q$ is incompressible in $Q^{(c)}$. We sketch this argument here (see [9, Claim 4.5]). By the minimality of $|\Sigma \cap T|$, for $i = 1, 2$, the components of $T \cap C_i$ are incompressible, non–boundary parallel annuli in $C_i$. It follows that there is a meridian disk $D_i \subset C_i$ which is disjoint from $T$. Hence there is some component of $\Sigma$ cut open along $T$ that compresses into $C_1$ and some component that compresses into $C_2$. By strong irreducibility of $\Sigma$, the same component compresses into both sides; moreover, all other components are incompressible. As remarked above no component of $\Sigma$ cut open along $T$ is boundary parallel; hence any incompressible component is essential. If some such component is in $X'$ then Conclusion (1) holds, contradicting our assumption. Hence $\Sigma_X$ is connected and compresses into both sides, and every component of $\Sigma_Q$ is essential. This completes the proof of the claim.

Since $Q^{(c)}$ is a punctured annulus cross $S^1$ and $\Sigma_Q$ is incompressible and has no boundary parallel or closed component, every component of $\Sigma_Q$ is a vertical annulus (see, for example, Jaco [4, VI.34]). Hence $\partial \Sigma_X$ consists of meridians of $K$. For $i = 1, 2$, let $\Sigma_i$ be the surface obtained by simultaneously compressing $\Sigma_X$ maximally into $C_i \cap X'$. (By simultaneous compression, we mean compressing $\Sigma_X$ once along a collection of mutually disjoint disks, without iterations.) Then the argument of Claim 6 of [8, page 248] shows that every component of $\Sigma_i$ is incompressible. Hence, every component of $\Sigma_i$ is a boundary parallel annulus in $X'$ or a 2–sphere, for otherwise Conclusion (1) holds, contradicting our assumption. Denote the number of boundary parallel annuli by $b$ (note that $b = \frac{1}{2} |\partial \Sigma_X|$ and is the same for $\Sigma_1$ and $\Sigma_2$). Denote the solid tori that define the boundary parallelism of the annular components of $\Sigma_i$ by $N_{i,1}, \ldots, N_{i,b}$ ($i = 1, 2$).

Claim 1 For each $i$ ($i = 1, 2$), $N_{i,1}, \ldots, N_{i,b}$ are mutually disjoint.

Proof of Claim 1 Assume, for a contradiction, that two components (say $N_{i,1}$ and $N_{i,2}$) intersect, say $N_{i,2} \subset N_{i,1}$. Note that $\Sigma_X$ is retrieved from $\Sigma_i$ by tubing. Since
To complete the proof we need to show that \( c \) vertical annuli that separate \( Q \). For each \( c \) holds. Suppose, for a contradiction, that \( b < c \). Note that \( \Sigma_Q \) consists of \( b \) vertical annuli that separate \( Q^{(c)} \) into \( b + 1 \) components. Note that \( \partial X^{(c)} \) consists of \( c + 1 \) tori; thus if \( b < c \) then two components of \( \partial Q^{(c)} \) are in the same component of \( Q^{(c)} \) cut open along \( \Sigma_Q \). It is easy to see that there is a vertical annulus connecting

\[ \Sigma_i \] is obtained from \( \Sigma_X \) by simultaneously compressing into the \( C_i \) side only and \( \Sigma_X \) is connected, all the tubes are contained in \( N_{i,1} \). This implies that \( N_{i,j} \subset N_{i,1} \) for all \( j \). This shows that \( \Sigma \) is isotopic into \( Q^{(c)} \), hence \( T \) is isotopic into \( C_1 \) or \( C_2 \). Since \( T \) is essential, this is impossible. This proves Claim 1.

\[ \square \]

Remark 2.3 As a part of the proof of Proposition 2.2, we analyze the intersection of \( \Sigma \) with \( Q^{(c)} \). When \( K \) is a hyperbolic knot, \( Q^{(c)} \) is a component of the characteristic subvariety. We point the reader to [23, Theorem 3.8], where Scharlemann and Schultens treat the intersection of a strongly irreducible Heegaard surface with the characteristic subvariety in general. Our setting is more limited, and this allows us to obtain more detailed information, e.g. Claim 2 below.

Claim 2 \( K \) admits a \((g - b, b)\) position.

Proof of Claim 2 For each \( i \) \((i = 1, 2)\), let \( A_{i,j} \) be the annulus \( N_{i,j} \cap T \) \((j = 1, \ldots, b)\). Note that \( A_{i,j} \) is a longitudinal annulus in \( N_{i,j} \). By Claim 1, \( C_i \cap X' \) is obtained from \( N_{i,1}, \ldots, N_{i,b} \) and a (possibly empty) collection of 3–balls by attaching 1–handles. Hence \( C_i \cap X' \) is a handlebody and \( \{A_{i,j}\}_{j=1}^b \) is a primitive system of annuli in \( \partial(C_i \cap X') \), i.e. there exists a system of properly embedded disjoint disks \( \{\Delta_{i,j}\}_{j=1}^b \) such that \( \Delta_{i,j} \cap A_{i,k} = \emptyset \) for \( j \neq k \), and \( \Delta_{i,j} \cap A_{i,j} \) is a spanning arc for \( A_{i,j} \).

Since \( X' \) is homeomorphic to \( X \), we may perform the trivial Dehn filling on \( X' \) to obtain \( M \). In \( M \) we cap \( \Sigma_X \) off by attaching \( 2b \) disks to obtain a genus \( g - b \) closed surface, say \( S \). Then \( S \) separates \( M \) into two parts, denoted \( H_1 \) and \( H_2 \), so that \( H_i \) is obtained from \( C_i \cap X' \) by attaching \( b \) 2–handles along \( A_{i,1}, \ldots, A_{i,b} \). Since the system \( \{A_{i,j}\}_{j=1}^b \) is primitive, \( H_i \) is a handlebody. Hence \( H_1 \cup_S H_2 \) is a Heegaard splitting of \( M \).

Up to isotopy, the knot \( K \) is the core of the attached solid torus. Thus \( K \cap H_i \) \((i = 1, 2)\) is the union of the co-cores of the 2–handles, and each co-core is isotopic into \( \partial H_i \) via one of the disks \( \Delta_{i,j} \). Since the disks \( \Delta_{i,j} \) are disjoint, we see that \( K \cap H_i \) consists of \( b \) simultaneously boundary parallel arcs. Hence \( H_1 \cup H_2 \) induces a \((g - b, b)\) position of \( K \). This proves Claim 2.

\[ \square \]

To complete the proof we need to show that \( c \leq b \leq g \). Since \( g - b \geq 0 \), it is obvious that \( b \leq g \) holds. Suppose, for a contradiction, that \( b < c \). Note that \( \Sigma_Q \) consists of \( b \) vertical annuli that separate \( Q^{(c)} \) into \( b + 1 \) components. Note that \( \partial X^{(c)} \) consists of \( c + 1 \) tori; thus if \( b < c \) then two components of \( \partial Q^{(c)} \) are in the same component of \( Q^{(c)} \) cut open along \( \Sigma_Q \). It is easy to see that there is a vertical annulus connecting
these tori, which is disjoint from $\Sigma$. Hence this annulus is contained in a compression body $C_i$ and connects components of $\partial C_i \setminus \Sigma$. This contradiction completes the proof of Proposition 2.2.

\textbf{Definition 2.4} (Hempel [3]) Let $H_1 \cup_\Sigma H_2$ be a Heegaard splitting. The distance of $\Sigma$, denoted $d(\Sigma)$, is the least integer $d$ so that there exist meridian disks $D_i \subset H_i$ ($i = 1, 2$) and essential curves $\gamma_0, \ldots, \gamma_d \subset \Sigma$ so that $\gamma_0 = \partial D_1$, $\gamma_d = \partial D_2$, and $\gamma_{i-1} \cap \gamma_i = \emptyset$ ($i = 1, \ldots, d$). There are three cases where this definition does not apply: $M \cong S^3$ and $g(\Sigma) = 0$, $M$ is a genus $g$ handlebody and $g(\Sigma) = g$, and $M$ is a lens space and $g(\Sigma) = 1$. In the first two cases on at least one side there are no meridian disks, and in the last case there is no sequence of curves on $\Sigma$ as required in the definition. In all three cases, we define $d(\Sigma)$ to be zero.

We need two properties of knots whose exteriors admit a Heegaard splittings of high distance. The first is Theorem 3.1 of [22] (for closed surfaces this was shown by Hartshorn [2]):

\textbf{Proposition 2.5} [22] Let $K$ be a knot and $d \geq 0$ an integer. Suppose $X$ admits a Heegaard splitting with distance greater than $d$. Then $X$ does not admit a connected essential surface $S$ with $\chi(S) \geq 2 - d$.

Proposition 2.6 below was first stated as Theorem 4.1 of [11]. Our proof is a combination of Theorem 1 of [6] and Corollary 4.7 of [24]. The statements of Theorem 1 of [6] and of Proposition 2.6 are very similar; however, the definitions of $(p, 0)$ position used in [6] and here are distinct. In [6] $K$ is said to admit a $(p, 0)$ position\footnote{The term used in [6] is “$K$ is $(p, 0)$”, rather than “$K$ admits a $(p, 0)$ position”.} if and only if $K$ is isotopic into a genus $p$ Heegaard splitting. Recall that by our definition, $K$ admits a $(p, 0)$ position if and only if $g(X) \leq p$. Thus, if $p < g(X)$ and $K$ is isotopic into a genus $p$ Heegaard surface, then $K$ admits a $(p, 0)$ in the sense of [6], and does not admit a $(p, 0)$ position in our sense; note that in that case $K$ admits a $(p, 1)$ position in our sense. In all other cases, $K$ admits a $(p, q)$ position in the sense of [6] if and only if it admits a $(p, q)$ position in our sense.

Shortly after our paper was posted, Tomova proved a stronger version of Proposition 2.6 using different techniques [28, Theorem 1.3].

\textbf{Proposition 2.6} Let $K \subset S^3$ be a knot and $p$, $q$ integers so that $K$ admits a $(p, q)$ position.

If $p < g(X)$ then any Heegaard splitting for $X$ has distance at most $2(p + q)$.
We claim that $A$ admits a $(p, q)$ position with $p < g(X)$. By tubing the surface that gives the bridge position $r$ times ($0 \leq r \leq q$) we obtain a $(p + r, q - r)$ position. We take $r = g(X) - p - 1$; thus $p + r = g(X) - 1 = t(K)$. Let $n$ be the minimal number so that $A$ admits a $(t(K), n)$ position in our sense. We see that $n \leq q - r$. Since $t(K) = p + r$, this implies that $t(K) + n \leq p + q$. Hence, for the proof of Proposition 2.6, it suffices to show that any Heegaard splitting of $X$ has distance at most $2(t(K) + n)$.

Claim 1  The knot exterior $X$ admits a minimal genus Heegaard surface with distance at most $2(t(K) + n)$.

Proof of Claim 1  Let $n'$ be the minimal integer so that $K$ admits a $(t(K), n')$ position according to the definition given in [6]. Assume first that $K$ is not isotopic onto any genus $t(K)$ Heegaard surface of $S^3$. Then $n = n'$, and the claim then follows directly from [6, Theorem 1].

Thus we may assume that $S^3$ admits a genus $t(K)$ Heegaard splitting, say $H_1 \cup_{\Sigma} H_2$, so that $K \subset \Sigma$, i.e., $n' = 0$. On the other hand, as explained above $n = 1$. We base our analysis on [19; 20; 21]. We perform a tiny isotopy of $K$ in $H_2$, pushing it off $\Sigma$. Denote the knot obtained by $\tilde{K} \subset H_2$. The image of the isotopy is an annulus (say $A$) embedded in $H_2$ so that one boundary component of $A$ is $\tilde{K}$ and the other is $K \subset \Sigma$. Let $\alpha$ be a spanning arc for $A$. Let $\tilde{H}_1 = H_1 \cup N_{H_2}(\alpha \cup \tilde{K})$ and let $\tilde{H}_2 = \text{cl}(M \setminus \tilde{H}_1)$. It is easy to see that $\tilde{H}_1$ and $\tilde{H}_2$ are handlebodies (with $\tilde{K} \subset \tilde{H}_1$) and therefore $\partial \tilde{H}_1 = \partial \tilde{H}_2$ is a Heegaard surface for $S^3$, denoted $S_{\tilde{K}}(\Sigma)$. Denote the exterior of $\tilde{K}$ by $\tilde{X}$. Note that $\tilde{X} \cong X$. In [19] it was shown that $S_{\tilde{K}}(\Sigma)$ is a Heegaard surface for $\tilde{X}$. Since $g(S_{\tilde{K}}(\Sigma)) = g(\Sigma) + 1 = t(K) + 1 = g(\tilde{X})$, we have that $S_{\tilde{K}}(\Sigma)$ is a minimal genus Heegaard surface for $\tilde{X}$.

We claim $d(S_{\tilde{K}}(\Sigma)) \leq 2$. Let $\tilde{D}_1 \subset \tilde{H}_1$ be the disk $\text{cl}(\Sigma \setminus S_{\tilde{K}}(\Sigma))$ and let $\gamma_0 = \partial \tilde{D}_1$. Since $t(K) > 0$, $\gamma_0$ is essential in $S_{\tilde{K}}(\Sigma)$. Let $\tilde{D}_2 \subset \tilde{H}_2$ be the disk $A \setminus \tilde{H}_1$, and let $\gamma_2$ be $\partial \tilde{D}_2$. Since $\gamma_2$ is nonseparating it is essential in $S_{\tilde{K}}(\Sigma)$. Let $\gamma_1$ be a longitude of $\partial N_{H_2}(\alpha \cup K)$ chosen so that $\gamma_0 \cap \gamma_1 = \emptyset$ and $\gamma_1 \cap \gamma_2 = \emptyset$. Then $\gamma_1$ is essential in $S_{\tilde{K}}(\Sigma)$. Hence by Definition 2.4, $d(S_{\tilde{K}}(\Sigma)) \leq 2 < 2(t(K) + n)$.

This proves Claim 1.

Claim 2  Any Heegaard surface for $X$ has distance at most $2(t(K) + n)$.

$S_{\tilde{K}}(\Sigma)$ is called stabilization of $\Sigma$ along $\tilde{K}$ [19, Definition 2.1]. For a detailed description see also Subsection 4.2 of [16].

The referee interprets the proof above as follows: first, we show that $S_{\tilde{K}}(\Sigma)$ is so-called $\mu$–primitive, and then we show that all $\mu$–primitive Heegaard surfaces have distance at most 2.
Proof of Claim 2. Let \( \Sigma \) be a Heegaard surface as in Claim 1, i.e., \( \Sigma \) is minimal genus and \( d(\Sigma) \leq 2(t(K)+n) \). Let \( \tilde{\Sigma} \) be any Heegaard surface for \( X \). By [24, Corollary 4.7] (with \( \Sigma \) corresponding to \( Q \) and \( \tilde{\Sigma} \) to \( P \)) one of the following holds:

1. Either \( \Sigma \) is isotopic to \( \tilde{\Sigma} \), or \( \Sigma \) is obtained from \( \tilde{\Sigma} \) by stabilizations or boundary stabilizations.
2. \( d(\tilde{\Sigma}) \leq 2g(\Sigma) \).

We treat the cases in order:

1. Since \( \Sigma \) is a minimal genus Heegaard splitting, \( \Sigma \) is isotopic to \( \tilde{\Sigma} \). Therefore \( d(\tilde{\Sigma}) = d(\Sigma) \leq 2(t(K)+n) \).
2. In this case, \( d(\tilde{\Sigma}) \leq 2g(\Sigma) = 2(t(K)+1) \leq 2(t(K)+n) \).

This proves Claim 2. \( \square \)

Claim 2 establishes Proposition 2.6. \( \square \)

3 Calculating \( g(X^{(c)}) \)

For \( X^{(c)} \), recall Notation 2.1. The following lemma is an easy application of the concept of stabilizing along a knot [19, Definition 2.1] that is described in the proof of Proposition 2.6.

Lemma 3.1 Let \( K \subset M \) be a knot, \( X \) the exterior of \( K \), and \( c \geq 0 \) an integer. Denote the genus of \( X \) by \( g \). Then

\[
g(X^{(c)}) \leq g + c.
\]

Proof The proof is an induction on \( c \). For \( c = 0 \) there is nothing to prove.

Fix \( c > 0 \). We obtain \( X^{(c-1)} \) by Dehn filling a component of \( \partial X^{(c)} \) and the core of the attached solid torus (say \( \gamma \)) is isotopic into \( \partial X \). Any Heegaard surface for \( X^{(c-1)} \) is obtained from a torus parallel to \( \partial X \) and a (possibly empty) collection of tori parallel to other components of \( \partial X^{(c-1)} \) by tubing. Hence \( \gamma \) is isotopic onto any Heegaard surface for \( X^{(c-1)} \). By stabilizing a minimal genus Heegaard surface for \( X^{(c-1)} \) along \( \gamma \) we obtain a Heegaard surface for \( X^{(c)} \) of genus \( g(X^{(c-1)}) + 1 \). Hence \( g(X^{(c)}) \leq g(X^{(c-1)}) + 1 \).

By the induction hypothesis, \( g(X^{(c-1)}) \leq g + (c-1) \); hence we get: \( g(X^{(c)}) \leq g(X^{(c-1)}) + 1 \leq g + (c-1) + 1 = g + c \). \( \square \)
Proposition 3.2 Let $M$ be a compact orientable manifold that does not admit a nonseparating surface. Let $K \subset M$ be a knot, and $X$ its exterior. Let $c \geq 0$ be an integer. Denote the genus of $X$ by $g$. Suppose that $X$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2(g + c)$, and that $K$ does not admit a $(g - 1, c)$ position. Then
\[ g(X^{(c)}) = g + c. \]

Proof The proof is an induction on $c$. For $c = 0$ there is nothing to prove.

Fix $c > 0$ and let $\Sigma \subset X^{(c)}$ be a minimal genus Heegaard surface. It follows from the assumptions that $X$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2(g + c)$, and that $K$ does not admit a $(g - 1, c)$ position; hence the induction hypothesis applies to $X^{(c)}$, giving that $g(X^{(c)}) = g + c$.

The proof is divided into the following two cases:

Case 1 $\Sigma$ is strongly irreducible.

By Proposition 2.2 one of the following holds:

1. $X$ admits an essential surface $S$ with $\chi(S) \geq 4 - 2g(X^{(c)})$.
2. $c \leq g(X^{(c)})$, and for some $b$ ($c \leq b \leq g(X^{(c)}))$, $K$ admits a $(g(X^{(c)}) - b, b)$ position.

By Lemma 3.1, we have $4 - 2g(X^{(c)}) \geq 4 - 2(g + c)$. By assumption $X$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2(g + c)$, so Case 1 above cannot happen and we may assume that we are in Case 2. Since $b - c \geq 0$, we can tube the Heegaard surface giving the $(g(X^{(c)}) - b, b)$ position $b - c$ times to obtain a $(g(X^{(c)}) - b + (b - c), b - (b - c)) = (g(X^{(c)}) - c, c)$ position.

By assumption $K$ does not admit a $(g - 1, c)$ position; this implies that if $K$ admits a $(p, c)$ position for some $p$, then $p > g - 1$. Thus $g(X^{(c)}) - c > g - 1$. Together with Lemma 3.1, this implies that $g(X^{(c)}) = g + c$.

Case 2 $\Sigma$ is weakly reducible.

In [27] Sedgwick proved a relative version of Casson and Gordon’s seminal theorem [1], proving that an appropriately chosen weak reduction of a minimal genus Heegaard surface yields an essential surface (see the statement and the proof of Theorem 1.1 of [27], cf [14, Theorem 3.1]). Denote by $\hat{F}$ the essential surface obtained by weakly
reducing $\Sigma$. Let $F$ be a connected component of $\tilde{F}$. Since $F \subset X^{(c)} \subset M$, it separates. Hence by [9, Proposition 2.13], $\Sigma$ weakly reduces to $F$. Note that $\chi(F) \geq \chi(\Sigma) + 4$.

**Claim** $F$ can be isotoped into $Q^{(c)}$.

**Proof of Claim** Recall the definitions of $T$, $X'$ and $Q^{(c)}$ from the proof of Proposition 2.2. Assume, for a contradiction, that $F$ cannot be isotoped into $Q^{(c)}$. Since $X$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2(g + c)$, $X$ is irreducible. Minimize $|F \cap T|$. Since $F$ and $T$ are essential and $X$ and $Q^{(c)}$ are irreducible, $F \cap T$ consists of a (possibly empty) collection of curves that are essential in both surfaces. If $F \cap X'$ compresses, then, since the curves of $F \cap T$ are essential in $F$, so does $F$, contradiction. Since $T$ is a torus, boundary compression of $F \cap X'$ implies a compression (see, for example, [8, Lemma 2.7]). Finally, minimality of $|F \cap T|$ implies that no component of $F \cap X'$ is boundary parallel. Thus, every component of $F \cap X'$ is essential (including the case $F \subset X'$). Since no component of $F \cap Q^{(c)}$ is a disk or a sphere, $\chi(F \cap X') \geq \chi(F) \geq \chi(\Sigma) + 4$. By Lemma 3.1, $\chi(\Sigma) \geq 2 - 2(g + c)$, thus $\chi(\Sigma) + 4 \geq 6 - 2(g + c)$. Hence $\chi(F \cap X') \geq 6 - 2(g + c)$. Since $X' \cong X$, this contradicts the assumption of Proposition 3.2. This proves the claim. □

Since $F$ is a closed incompressible surface in $Q^{(c)}$, and $Q^{(c)}$ is a punctured annulus cross $S^1$, $F$ is a vertical torus (see, for example, [4, VI.34]).

First, suppose that $F$ is not boundary parallel in $Q^{(c)}$. Then $F$ decomposes $X^{(c)}$ as $X^{(p+1)} \cup_F D(c-p)$, where $0 \leq p \leq c$ is an integer and $D(c-p)$ is a disk with $c - p$ holes cross $S^1$. Note that since $F$ is not parallel to a component of $\partial Q^{(c)}$, $c - p \geq 2$. Therefore $p + 1 < c$. This, together with the assumption of the proposition, implies that $X$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2(g + (p + 1))$, and that $K$ does not admit a $(g - 1, p)$ position; hence the induction hypothesis applies to $X^{(p+1)}$, giving that $g(X^{(p+1)}) = g + p + 1$. By Schultens [25], $g(D(c-p)) = c - p$.

Since $F$ was obtained by weakly reducing a minimal genus Heegaard surface [9, Proposition 2.9] (see also [25, Remark 2.7]) gives:

$$g(X^{(c)}) = g(X^{(p+1)}) + g(D(c-p)) - g(F)$$

$$= (g + p + 1) + (c - p) - 1$$

$$= g + c.$$

Next, suppose that $F$ is boundary parallel in $Q^{(c)}$. Since $F$ is essential in $X^{(c)}$, it cannot be isotopic to a component of $\partial X^{(c)}$ and must therefore be isotopic to $\partial Q^{(c)} \setminus \partial X^{(c)} = T$. This gives the decomposition $X^{(c)} = X' \cup_F Q^{(c)}$. Since $X' \cong X$,
\[ g(X') = g \]. By [25] \( g(Q^{(c)}) = c + 1 \). We get, as above:
\[
g(X^{(c)}) = g(X') + g(Q^{(c)}) - g(F)
\]
\[
= g + (c + 1) - 1
\]
\[
= g + c.
\]
This completes the proof of Proposition 3.2. \( \square \)

**Proposition 3.3** Let \( m \geq 1 \) and \( c \geq 0 \) be integers, and let \( \{ K_i \subset M_i \}_{i=1}^m \) be knots in closed orientable manifolds. Suppose that \( M_i \) does not admit a nonseparating surface \((1 \leq i \leq m)\). Denote the exterior of \( K_i \) by \( X_i \), and the exterior of \( \#_{i=1}^m K_i \) by \( X \). Let \( g \) be an integer so that \( g(X_i) \leq g \) \((1 \leq i \leq m)\).

Suppose that no \( X_i \) admits an essential surface \( S \) with \( \chi(S) \geq 4 - 2g(m + c) \), and that no \( K_i \) admit a \((g(X_i) - 1, m + c - 1)\) position. Then we have:
\[
g(X^{(c)}) = \sum_{i=1}^m g(X_i) + c.
\]

**Proof** Suppose first that \( m = 1 \). Note that \( 4 - 2g(1 + c) \leq 4 - 2(c + g) \); therefore Proposition 3.3 follows from Proposition 3.2 in this case. Assume from now on \( m \geq 2 \).

We induct on \((m, c)\) ordered lexicographically, where \( m \) is the number of summands and \( c \) is the number of curves drilled. Note that by Miyazaki [12], \( m \) is well defined (see [9, Claim 1]).

By Lemma 3.1, Inequality (1) in Section 1, and the assumption that \( g(X_i) \leq g \) for all \( i \), we get:
\[
g(X^{(c)}) \leq g(X) + c \leq \sum_{i=1}^m g(X_i) + c \leq mg + c.
\]
Since \( g \geq 2 \), we have that \( g(X^{(c)}) \leq g(m + c) \).

By assumption, for all \( i \), \( X_i \) does not admit an essential surface \( S \) with \( \chi(S) \geq 4 - 2g(m + c) \). Hence by the Swallow Follow Torus Theorem [9, Theorem 4.1], any minimal genus Heegaard surface for \( X^{(c)} \) weakly reduces to a swallow follow torus \( F \) giving the decomposition \( X^{(c)} = X_I^{(c_1)} \cup_F X_J^{(c_2)} \), where \( I \subset \{ 1, \ldots, m \} \), \( K_I = \#_{i \in I} K_i \), \( K_J = \#_{i \not\in I} K_i \), \( X_I = E(K_I) \), \( X_J = E(K_J) \), and \( c_1 + c_2 = c + 1 \) (for details see the first paragraph of Section 4 of [9]). Denote the number of factors of \( K_I \), \(|I|\), by \( m_1 \), and the number of factors of \( K_J \), \( m - |I| \), by \( m_2 \). Note that \( m_1 = 0 \) or \( m_2 = 0 \) are possible. However, at least one of \( m_1 \) or \( m_2 \) is not zero so by symmetry we may assume \( m_1 \neq 0 \).

First assume that \( m_1 = m \). Then \( m_2 = 0 \) and \( X_J^{(c_2)} \) is a disk with \( c_2 \) holes cross \( S^1 \).

Since \( F \) is essential [27, Theorem 1.1], \( c_2 \geq 2 \). Then \( c_1 = c - c_2 + 1 \leq c - 1 \). Since \( m_1 = m \), we see that \( m_1 + c_1 \leq c + m - 1 \). By assumption, no \( X_i \) \((1 \leq i \leq m)\) admits
an essential surface $S$ with $\chi(S) \geq 4 - 2g(m_1 + c_1) > 4 - 2g(m + c)$. Hence, the induction hypotheses applies to $X_{I}^{(c_1)} \cong X^{(c_1)}$, showing that
\[ g(X_{I}^{(c_1)}) = \sum_{i=1}^{m} g(X_i) + c_1. \]
Since $X_{J}^{(c_2)}$ is homeomorphic to a disk with $c_2$ holes cross $S^1$, $g(X_{J}^{(c_2)}) = c_2$ by [25]. Since $F$ was obtained by weakly reducing a minimal genus Heegaard surface, Proposition 2.9 of [9] and the fact that $c_1 + c_2 = c + 1$, we get:
\[ g(X^{(c)}) = g(X_{I}^{(c_1)}) + g(X_{J}^{(c_2)}) - g(F) = (\sum_{i=1}^{m} g(X_i) + c_1) + c_2 - 1 = \sum_{i=1}^{m} g(X_i) + c. \]
This proves Proposition 3.3 when $m_1 = m$.

Next assume that $m_1 < m$. By assumption $m_1 > 0$, hence $m_2 < m$. By construction $c_1 \leq c + 1$, and $c_2 \leq c + 1$. Hence $m_1 + c_1 \leq m + c$, and $m_2 + c_2 \leq m + c$. By assumption, no $X_i$ ($1 \leq i \leq m$) admits an essential surface $S$ with $\chi(S) \geq 4 - 2(m_j + c_j)g \geq 4 - 2(m + c)g$ ($j = 1, 2$). Hence the induction hypothesis applies to $X_{I}^{(c_1)}$ and $X_{J}^{(c_2)}$, giving $g(X_{I}^{(c_1)}) = \sum_{i \in I} g(X_i) + c_1$, and $g(X_{J}^{(c_2)}) = \sum_{i \notin I} g(X_i) + c_2$. We get, as above:
\[ g(X^{(c)}) = g(X_{I}^{(c_1)}) + g(X_{J}^{(c_2)}) - g(F) = (\sum_{i \in I} g(X_i) + c_1) + (\sum_{i \notin I} g(X_i) + c_2) - 1 = \sum_{i=1}^{m} g(X_i) + c_1 + c_2 - 1 = \sum_{i=1}^{m} g(X_i) + c. \]
This completes the proof of Proposition 3.3. \hfill \Box

**Remark 3.4** For $m \geq 2$, the proof is an application of the Swallow Follow Torus Theorem [9, Theorem 4.1]. In [9, Remark 4.2] it was shown by means of a counterexample that the Swallow Follow Torus Theorem does not apply to $X^{(c)}$ when $m = 1$. Hence the argument of the proof of Proposition 3.3 cannot be used to simplify the proof of Proposition 3.2.

### 4 Proof of Theorem 1.2

Fix $g \geq 2$ and $n \geq 1$. Let $K_{g,n}$ be the set of all knots $K \subset S^3$ with the following three properties:
(a) $g(E(K)) \leq g$. 

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(b) \( K \) does not admit a \((t(K), n)\) position.

c) \( E(K) \) does not admit an essential surface \( S \) with \( \chi(S) \geq 4 - 2gn \).

Fix \( h \) satisfying \( 2 \leq h \leq g \). There exist infinitely many knots in \( S^3 \), each admitting a genus \( h \) Heegaard splitting of distance greater than \( \max\{2gn - 2, 2(h + n - 1)\} \), by [11, Theorem 3.1]. Let \( K_h \) be such a knot, and \( X_h \) its exterior.

Since \( X_h \) admits a genus \( h \) Heegaard splitting with distance greater than \( 2(h + n - 1) \), \( 2h \) (as \( n \geq 1 \)), by [24, Corollary 4.7] this splitting must be minimal genus; in particular, \( g(E(K_h)) = h \). Since \( X_h \) admits a Heegaard splitting with distance greater than \( 2(h + n - 1) \), by Proposition 2.6, \( K_h \) does not admit a \((h - 1, n) = (t(K), n)\) position. Since \( X_h \) admits a Heegaard splitting with distance greater than \( 2gn - 2 \), by Proposition 2.5, \( X_h \) does not admit an essential surface \( S \) with \( \chi(S) \geq 4 - 2gn \). We see that \( K_h \in \mathcal{K}_{g,n} \) and hence \( \mathcal{K}_{g,n} \) contains infinitely many knots \( K \) with \( g(X) = h \). This proves that \( \mathcal{K}_{g,n} \) fulfills Conclusion (1) of Theorem 1.2.

Since (for any \( K \in \mathcal{K}_{g,n} \)) \( X \) does not admit an essential surface \( S \) with \( \chi(S) \geq 4 - 2gn \), and \( K \) does not admit a \((t(K), n)\) position, applying Proposition 3.3 with \( m \leq n \) and \( c = 0 \), we see that the knots in \( \mathcal{K}_{g,n} \) fulfill Conclusion (2) of Theorem 1.2.

By [10, Theorem 1.2] for any knot \( K' \subset S^3 \), there exists \( N \) so that if \( n > N \), then \( g(E(nK')) < ng(E(K')) \). This shows that \( K' \not\in \mathcal{K}_{g,n} \) for \( n > N \). Hence \( K' \not\in \bigcap_{n=1}^{\infty} \mathcal{K}_{g,n} \). As \( K' \) was arbitrary, \( \bigcap_{n=1}^{\infty} \mathcal{K}_{g,n} = \emptyset \).

This completes the proof of Theorem 1.2.

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Knot exteriors with additive Heegaard genus


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